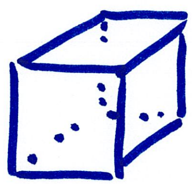
16.7 The Divergence Theorem

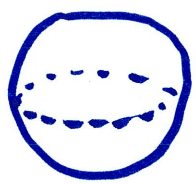
The Divergence Theorem relates a surface integral over a closed surface to a volume integral over the enclosed volume.

$$\boxed{\iiint_E \operatorname{div} \vec{F}\, dV = \iint_S \vec{F} \cdot d\vec{S} = \iint_{\partial E} \vec{F} \cdot d\vec{S}}$$

S : closed surface (boundary of enclosed volume)

E : enclosed volume

for example, E : cube  then S is the combination of the 6 faces

E : sphere  then S is the surface of the sphere

Divergence Theorem is also Gauss' Theorem or Ostrogradsky's Theorem

Karl Friedrich Gauss
(1777 - 1858)

Mikhail Ostrogradsky
(1801 - 1892)

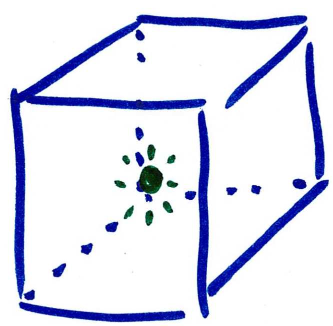
but really was first discovered by Joseph Louis Lagrange
in 1762

why is $\underbrace{\iint_S \vec{F} \cdot d\vec{S}}$ = $\iiint_E \underbrace{\text{div}\,\vec{F}}\, dV$?

flux integral measures the amount of stuff flowing through the surface

↳ ~~change in~~ the quantity of the vector field coming out of a source or into a sink

lightbulb in cube



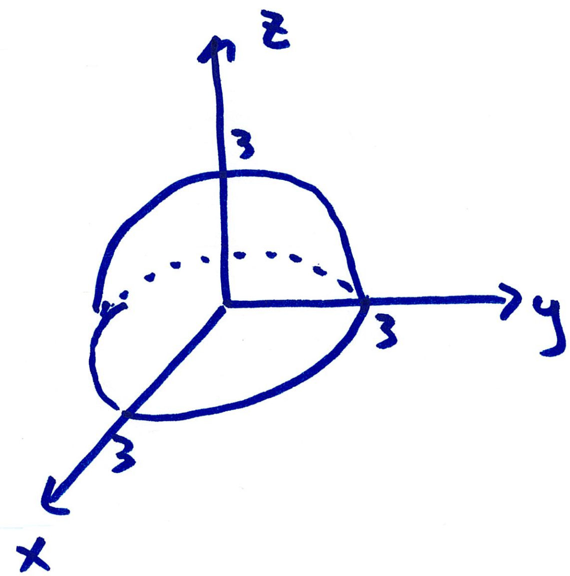
$\iint_S \vec{F} \cdot d\vec{S}$: how many photons come out ~~to~~ through all 6 faces?

$\text{div}\,\vec{F}$: # of photons coming out of the bulb

so $\iiint_E \text{div}\,\vec{F}\, dV$ tells us the same thing as $\iint_S \vec{F} \cdot d\vec{S}$

example $\vec{F} = \langle xy^2, yz^2, x^2z \rangle$

$$E: \underbrace{x^2+y^2+z^2 \le 9 \ , \ z \ge 0}_{\text{upper half of sphere of radius 3,}}$$



look at $\iint_S \vec{F} \cdot d\vec{S}$ first

as a surface integral, there a Two surfaces: top half of sphere
bottom disk

as usually, assume normal points <u>out</u>

$$\iint_{S_1} \vec{F} \cdot d\vec{S}_1 + \iint_{S_2} \vec{F} \cdot d\vec{S}_2$$

$S_1 \hookrightarrow$ sphere $\qquad S_2 \hookrightarrow$ bottom disk

we need: $S \to \vec{r}(u,v)$ (time-consuming)

then $d\vec{S} = (\vec{r}_u \times \vec{r}_v)\,du\,dv$

or $\vec{r}_v \times \vec{r}_u$

Divergence Theorem: $\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV$

as a volume integral:

$\vec{F} = \langle xy^2, yz^2, x^2z \rangle$

$\operatorname{div}\vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle xy^2, yz^2, x^2z \rangle$

$= y^2 + z^2 + x^2$

E: ~~sp~~ hemisphere (upper), radius 3

spherical coord: $0 \le \rho \le 3$, $0 \le \phi \le \pi/2$, $0 \le \theta \le 2\pi$

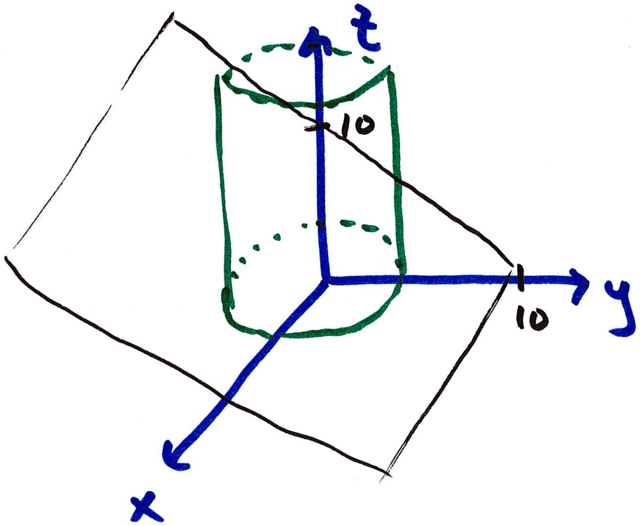
$$\iiint_E \operatorname{div}\vec{F}\, dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^3 \rho^2 \cdot \overbrace{\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta}^{dV \text{ in spherical}}$$

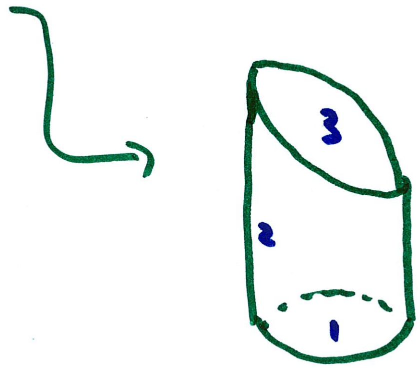
$\operatorname{div}\vec{F} = x^2+y^2+z^2$ in spherical coord

$= \cdots = \frac{486}{5}\pi$

example $\vec{F} = \langle x^4, -x^3 z^2, 4xy^2 z \rangle$

S : surface of a solid bounded by $\underbrace{x^2+y^2=1}_{\text{cylinder, radius 1, along } z\text{-axis}}$, $\underbrace{z=10-y}_{\text{plane}}$, $\underbrace{z=0}_{xy\text{-plane}}$



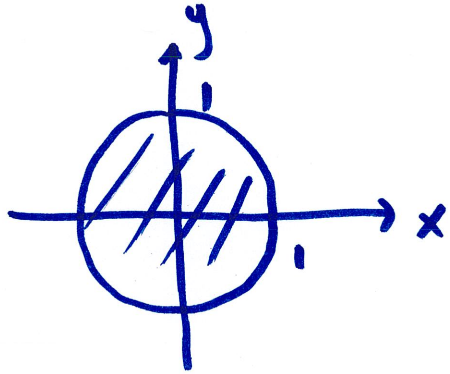


$$\underbrace{\iint_S \vec{F}\cdot d\vec{S}}_{\substack{\text{3 surfaces} \\ \text{3 parametrizations} \\ 3\ \underbrace{\vec{r_u}\times\vec{r_v}}_{\text{point out}}}} = \iiint_E \operatorname{div}\vec{F}\, dV$$

once again, it's easier as $\iiint_E \operatorname{div}\vec{F}\, dV$

$\operatorname{div}\vec{F} = 4x^3 + 4xy^2$

E : cylindrical is best

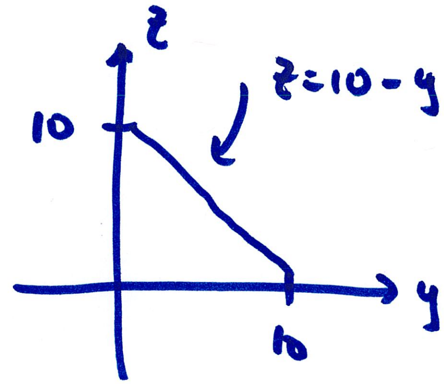


$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$0 \le z \le 10 - y$

r sin θ in cylindrical

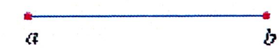
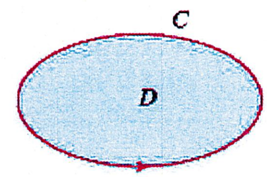
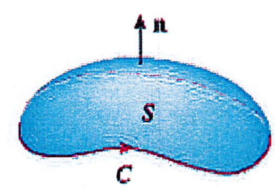


$$\operatorname{div} \vec{F} = 4x^3 + 4xy^2 = 4(r\cos\theta)^3 + 4(r\cos\theta)(r\sin\theta)^2$$

$$\iiint_E \operatorname{div} \vec{F}\, dV$$

$$= \int_0^{2\pi} \int_0^1 \int_0^{10 - r\sin\theta} \left(4r^3\cos^3\theta + 4r^3\cos\theta\sin^2\theta\right) \overbrace{r\,dz\,dr\,d\theta}^{dV \text{ in cylindrical}}$$

$$= \cdots = \frac{2\pi}{3}$$

Fundamental Theorem of Calculus	$\int_a^b F'(x)\,dx = F(b) - F(a)$	
Fundamental Theorem for Line Integrals	$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$	
Green's Theorem	$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C P\,dx + Q\,dy$	
Stokes' Theorem	$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$	
Divergence Theorem	$\iiint_E \operatorname{div} \mathbf{F}\,dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$