COMPLEXITY OF CYLINDRICAL DECOMPOSITIONS OF SUB-PFAFFIAN SETS

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ABSTRACT. We construct an algorithm for a cylindrical cell decomposition of a closed cube $I^n \subset \mathbb{R}^n$ compatible with a "restricted" sub-Pfaffian subset $Y \subset I^n$, provided an oracle deciding consistency of a system of Pfaffian equations and inequalities is given. In particular, the algorithm produces the complement $\tilde{Y} = I^n \setminus Y$. The complexity bound of the algorithm, the number and formats of cells are doubly exponential in n^3.

INTRODUCTION

Subanalytic sets are defined as images of relatively proper real analytic maps of semianalytic sets. In [8] Gabrielov proved that the complement of any subanalytic set is also subanalytic. This *complement theorem*, being a natural extension of the Tarski-Seidenberg principle for semialgebraic sets, plays a key role in real analytic geometry (see [2, 5]) as well as in model-theoretic study of o-minimality [7, 6, 17].

The complement theorem immediately follows from the existence of a *cylindrical decomposition* of the ambient space compatible with a subanalytic set. The existence was proved in [9] by means of a quasi-constructive process of manipulating with symbols of real analytic functions and their derivatives.

In the present paper we modify the method from [9] so that being applied to subanalytic sets defined by *Pfaffian* functions it yields an algorithm (a *real numbers machine* [3] with an oracle) producing a cylindrical decomposition. Pfaffian functions are solutions of triangular systems of first order partial differential equations with polynomial coefficients. *Semi-Pfaffian* sets, defined by systems of equations and inequalities between these functions, are characterized by global finiteness properties [13, 14]. This means that their basic geometric and topological characteristics can be explicitly estimated in terms of formats of their defining formulae. In the paper we prove some global finiteness properties for *sub-Pfaffian* sets (relatively proper images of semi-Pfaffian sets) as a consequence of an explicit complexity bound of our algorithm for a cylindrical decomposition. The bound is doubly exponential in a polynomial in the number of variables.

Note that for a special case of *semialgebraic* sets similar or better complexity results are well known [4, 18, 12, 1].

1. PFAFFIAN FUNCTIONS AND SUB-PFAFFIAN SETS

Definition 1.1. (See [13, 14] , and [11].) A *Pfaffian chain* of the order $r \geq 0$ and degree $\alpha \geq 1$ in an open domain $G \subset \mathbb{R}^n$ is a sequence of real analytic functions

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$f_1, \dots, f_r$ in G satisfying Pfaffian equations

$$df_j(x) = \sum_{1 \le i \le n} g_{ij}(x, f_1(x), \dots, f_j(x))dx_i$$

for $1 \le j \le r$. Here $g_{ij}(x, y)$ are polynomials in $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_j)$ of degree not exceeding α. A function

$$f(x) = P(x, f_1(x), \dots, f_r(x))$$

where $P(x, y_1, \dots, y_r)$ is a polynomial of degree not exceeding $\beta \ge 1$ is a *Pfaffian function* of order r and degree (α, β).

For examples of Pfaffian functions see [14, 11].

Lemma 1.2. (*See* [14, 11].)

1. *The sum (resp. product) of two Pfaffian functions, f_1 and f_2, of orders r_1 and r_2 and degrees (α_1, β_1) and (α_2, β_2), is a Pfaffian function of the order $r_1 + r_2$ and degree $(\alpha, \max(\beta_1, \beta_2))$ (resp. $\alpha, \beta_1 + \beta_2$) where $\alpha = \max(\alpha_1, \alpha_2)$. If the two Pfaffian functions are defined by the same Pfaffian chain of order r, then the order of the sum and product is also r.*
2. *A partial derivative of a Pfaffian function of order r and degree (α, β) is a Pfaffian function of the order r and degree $(\alpha, \alpha + \beta - 1)$.*

The following definitions are slightly more restrictive than the usual ones. In particular, we only consider the "restricted" case in which Pfaffian functions are defined also on the boundary of the domain.

Definition 1.3. (*Semi- and sub-Pfaffian set.*)

1. A set $X \subset \mathbb{R}^s$ is called *semi-Pfaffian* in an open domain $G \subset \mathbb{R}^s$ if it consists of points from G satisfying a Boolean combination of atomic equations and inequalities $f = 0, g > 0$, where f, g are Pfaffian functions having a common Pfaffian chain defined in the domain G. A semi-Pfaffian set is called *basic* if the Boolean combination is just a system of equations and inequalities.
2. Consider the closed unit cube $I^{m+n} \subset G$, where $G \subset \mathbb{R}^{m+n}$ is an open domain, and the projection map

   $$\pi : \quad \mathbb{R}^{m+n} \longrightarrow \mathbb{R}^n.$$

   A subset $Y \subset \mathbb{R}^n$ is called *(restricted) sub-Pfaffian* if $Y = \pi(X)$ for semi-Pfaffian set $X \subset I^{m+n}$.

Definition 1.4. (*Format.*) For a semi-Pfaffian set

$$X = \bigcup_{1 \le l \le M'} \{f_{l1} = \dots = f_{lI_l} = 0, g_{l1} > 0, \dots, g_{lJ_l} > 0\} \subset G \subset \mathbb{R}^s, \qquad (1)$$

where f_{ij}, g_{ij} are Pfaffian functions with a common Pfaffian chain, of order r and degree (α, β), defined in an open domain G, its *format* is a triple (N, D, s), where $N \ge \sum_{1 \le l \le M'}(I_l + J_l)$, $D \ge \alpha + \beta$. For $s = m + n$ and a sub-Pfaffian set $Y \subset \mathbb{R}^n$ such that $Y = \pi(X)$, its *format* is the format of X.

Definition 1.5. For a set of differentiable functions $h = (h_1, \dots, h_k)$ in variables $x_1, \dots, x_n$, a set of distinct indices $\mathbf{i} = (i_1, \dots, i_k)$ with $1 \le i_\nu \le n$, and an index

j, $1 \leq j \leq n$, different from all i_ν, we define a partial differential operator

$$\partial_{h,\mathbf{i},j} = \det \begin{pmatrix} \frac{\partial h_1}{\partial x_{i_1}} & \cdots & \frac{\partial h_1}{\partial x_{i_k}} & \frac{\partial h_1}{\partial x_j} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial h_k}{\partial x_{i_1}} & \cdots & \frac{\partial h_k}{\partial x_{i_k}} & \frac{\partial h_k}{\partial x_j} \\ \frac{\partial}{\partial x_{i_1}} & \cdots & \frac{\partial}{\partial x_{i_k}} & \frac{\partial}{\partial x_j} \end{pmatrix}.$$

When $k = 0$, the corresponding operator is simply $\partial_j = \frac{\partial}{\partial x_j}$.

Lemma 1.6. *For a Pfaffian function g of the order r and degree α, for a set $h = (h_1, \ldots, h_k)$ of Pfaffian functions of the order r and degrees α defined by the same Pfaffian chain as g, and for the set of distinct indices $\mathbf{i} = (i_1, \ldots, i_k), j$, the function $\partial_{h,\mathbf{i},j} g$ is a Pfaffian function of the order r and the degree $O(k\alpha)$.*

Proof. This statement follows from Lemma 1.2. □

Definition 1.5 implies the following statement.

Lemma 1.7. *Let a system of equations and inequalities defining a semi-Pfaffian set $X \subset \mathbb{R}^n$ of codimension k include a set of k Pfaffian functions $h_1, \ldots, h_k$ such that the restriction $h_j|_X \equiv 0$ for each $1 \leq j \leq k$, and $dh_1 \wedge \cdots \wedge dh_k \neq 0$ at every point of X. Let $x \in X$ be a critical point of a Pfaffian function $g : X \longrightarrow \mathbb{R}$. Define the matrix*

$$\mathcal{M}_{h,g} = (\partial_{h,\mathbf{i},j} \partial_{h,\mathbf{r},s} g)_{\mathbf{i},j,\mathbf{r},s}.$$

Then x is a non-degenerate *critical point of $g|_X$ if and only if* $\operatorname{rank}(\mathcal{M}_{h,g}(x)) = n - k$.

Lemma 1.8. *Let a sub-Pfaffian set $Y = \pi(X) \subset \mathbb{R}^n$, where $X \subset \mathbb{R}^{n+m}$ is a semi-Pfaffian set defined by Pfaffian functions of order r and $\pi : \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^n$ is a projection map, have a format $(L, D, n+m)$. Let $\dim(Y) < n$ and $\mathbb{R}^n$ have coordinates $x_1, \ldots, x_n$. Denote $M = 2^{r^2}(n+m)^r (LD)^{n+m+r}$. Then there is an integer vector $\gamma = (\gamma_1, \ldots, \gamma_n)$ such that $0 \leq \gamma_i \leq M$ and $\gamma \notin Y$.*

Proof. We conduct the proof by induction on n. If $n = 1$, then $\dim(Y) \leq 0$ and therefore the number of points in Y does not exceed the number of connected components in X. According to [13, 14], the latter is at most M, so for at least one $0 \leq \gamma_1 \leq M$ the intersection $X \cap \{x_1 = \gamma_1\} = \emptyset$.

For $n > 1$ suppose that for all $0 \leq \gamma_n \leq M$ the dimension $\dim(Y \cap \{x_n = \gamma_n\}) = n - 1$. Then for a vector $(a_1, \ldots, a_{n-1})$ the intersection

$$Y \cap \{x_1 = a_1, \ldots, x_{n-1} = a_{n-1}\} = \bigcup_{0 \leq \gamma_n \leq M} \{x_n = \gamma_n\}$$

consists of $M + 1$ points. On the other hand, the number of points in the intersection does not exceed the number of all connected components of $X \cap \{x_1 = a_1, \ldots, x_{n-1} = a_{n-1}\}$ which is at most M, according to [13, 14]. This contradiction shows that there exists $0 \leq \gamma_n \leq M$ such that $\dim(Y \cap \{x_n = \gamma_n\}) < n - 1$. Applying the inductive hypothesis to $Y \cap \{x_n = \gamma_n\}$ we conclude the proof. □

Definition 1.9. (*Weak stratification.*) A *weak stratification* of a semi-Pfaffian set X is a subdivision of X into a disjoint union of smooth, not necessarily connected, possibly empty, basic semi-Pfaffian subsets X_α, called *strata*. Each stratum X_α is *effectively non-singular*, that is the system of equations and inequalities for X_α of codimension k includes a set of k Pfaffian functions $h_{\alpha,1}, \ldots, h_{\alpha,k}$ such that the

restriction $h_{\alpha,j}|_{X_\alpha} \equiv 0$ for each $1 \le j \le k$, and $dh_{\alpha,1} \wedge \cdots \wedge dh_{\alpha,k} \ne 0$ at every point of X_α.

As a model of computation we use a *real numbers machine* (Blum-Shub-Smale model) [3] equipped with an *oracle* for deciding the feasibility of any system of Pfaffian equations and inequalities. An oracle is a subroutine which can be used by the algorithm any time the latter needs to check feasibility. We assume that this procedure always gives the correct answer though we do not specify how it actually works. For some classes of Pfaffian functions the feasibility problem is decidable on real numbers machines or Turing machines with explicit (singly-exponential) complexity bounds. Apart from polynomials, such class form, for example, terms of the kind $P(e^h, x_1, \dots, x_n)$ where h is a fixed polynomial in $x_1, \dots, x_n$ and P is an arbitrary polynomial in $x_0, x_1, \dots, x_n$ (see [16]). For such classes the oracle can be replaced by a deciding procedure, and we get an algorithm in the usual sense. As far as the computational complexity is concerned, we assume that each oracle call has the unit cost.

Proposition 1.10. ([11], *Theorem 3*) *There is an algorithm which for a semi-Pfaffian set X in an open domain $G \subset \mathbb{R}^s$ of format (L, D, s) and defined by (1) produces a finite stratification of X. The number of strata is less than $L^{s+r} D^{r^{O(s)}}$. The format of each stratum is*

$$(LD^{r^{O(s)}}, D^{r^{O(s)}}, s).$$

All functions defining a stratum have the same Pfaffian chain as the input functions. The complexity of the algorithm does not exceed $L^{s+r} D^{r^{O(s)}}$.

Definition 1.11. The *closure* $\overline{X}$ of a semi-Pfaffian set X in an open domain G is an intersection with G of the usual topological closure of X:

$$\overline{X} = \{x \in G : \forall \varepsilon > 0 \, \exists z \in X \, (|x - z| < \varepsilon)\}.$$

The *frontier* ∂X of X is $\partial X = \overline{X} \setminus X$.

Proposition 1.12. ([10], *Theorem 1.1*) *Let X be a semi-Pfaffian set in an open domain $G \subset \mathbb{R}^s$, of format (L, D, s) and defined by (1). Then the closure $\overline{X}$ and frontier ∂X are* semi-Pfaffian *sets. Moreover, there is an algorithm which produces $\overline{X}$ and ∂X with formats*

$$((LD)^{O((s+r)s)}, D^{O(s)}, s).$$

The complexity of the algorithm does not exceed $(LD)^{O((s+r)s)}$.

Definition 1.13. ([6, 17]) Cylindrical cell is defined as follows.

1. Cylindrical 0-cell in $\mathbb{R}^n$ is an isolated point.
2. Cylindrical 1-cell in $\mathbb{R}$ is an open interval (a, b), $a, b \in \mathbb{R}$.
3. For $n \ge 2$ and $0 \le k < n$, a cylindrical $(k+1)$-cell in $\mathbb{R}^n$ is either a graph of a continuous bounded function $f : \quad C \longrightarrow \mathbb{R}$, where C is a cylindrical $(k+1)$-cell in $\mathbb{R}^{n-1}$, or else a set of the form

$$(f, g) \equiv \{(x_1, \dots, x_n) \in \mathbb{R}^n : (x_1, \dots, x_{n-1}) \in C \quad \text{and}$$
$$f(x_1, \dots, x_{n-1}) < x_n < g(x_1, \dots, x_{n-1})\},$$

where C is a cylindrical k-cell in $\mathbb{R}^{n-1}$, and

$$f, g : \quad C \longrightarrow \mathbb{R}$$

are continuous bounded functions satisfying

$$f(x_1, \dots, x_{n-1}) < g(x_1, \dots, x_{n-1})$$

for all points $(x_1, \dots x_{n-1}) \in C$.

Clearly, a cylindrical k-cell is a topological cell, i.e. a homeomorphic image of an open k-dimensional ball.

Definition 1.14. Cylindrical cell decomposition, say $\mathcal{D}$, of a subset $A \subset \mathbb{R}^n$ is defined as follows.

1. If $n = 1$, then $\mathcal{D}$ is a finite family of pairwise disjoint cylindrical cells (i.e., isolated points and intervals) whose union is A.
2. If $n \geq 2$, then $\mathcal{D}$ is a finite family of pairwise disjoint cylindrical cells in $\mathbb{R}^n$ whose union is A and there is a cell decomposition of $\pi(A)$ such that $\pi(C)$ is its cell for each cell C of $\mathcal{D}$, where $\pi : \quad \mathbb{R}^n \longrightarrow \mathbb{R}^{n-1}$ is the projection map onto the coordinate subspace of $x_1, \dots, x_{n-1}$.

Definition 1.15. If $B \subset A \subset \mathbb{R}^n$ and $\mathcal{D}$ is a cylindrical cell decomposition of A, then $\mathcal{D}$ is *compatible* with B if for all $C \in \mathcal{D}$ either $C \subset B$ or $C \cap B = \emptyset$ (i.e., some $\mathcal{D}' \subset \mathcal{D}$ is a cylindrical cell decomposition of B).

Lemma 1.16. 1. *Let X be a smooth manifold in $\mathbb{R}^n$. Let $f_c(x) = f(x) - \sum_\alpha c_\alpha g_\alpha(x)$ be a family of smooth functions on X, depending on parameters $c \in \mathbb{R}^m$. Suppose that, for any $x \in X$, differentials of g_α generate cotangent space to X at x. Then, for a generic c, $f_c(x)$, has only non-degenerate critical points. More precisely, the set*

$$S = \{c : f_c(x) \text{ has a degenerate critical point}\}$$

has zero measure in $\mathbb{R}^m$.

2. *Let X be a smooth manifold in $\mathbb{R}^n$, and $f(x)$ a smooth non-vanishing function on X. For a generic $c = (c_1, \dots, c_n)$, all critical points of a function $f(x)(1 + (c, x))$ are non-degenerate. More precisely, the set*

$$V = \{c : f(x)(1 + (c, x)) \text{ has a degenerate critical point}\}$$

has zero measure in $\mathbb{R}^n$.

3. *Consider a smooth manifold X in $\mathbb{R}^{n+d}$ and the projection Y of X onto $\mathbb{R}^d$. Let for any fixed $y \in Y$ the set $X_y = X \cap \{y = \text{const}\}$ be smooth. Let $F(x, y)$ be a smooth non-vanishing function on X. For a fixed $y \in Y$, consider $f_y(x) = F(x, y)$ as a function on X_y. For a generic c, the set*

$$W_c = \{y : f_y(x)(1 + (c, x)) \text{ has a degenerate critical point}\}$$

has zero measure in Y.

Proof. 1. This is a variant of Thom's transversality theorem. For convenience, we give a proof here. Let $d = \dim X$. Fix $x^0 \in X$. One can renumber g_α so that differentials of $g_1, \dots, g_d$ generate cotangent space to X at x^0. Let us change coordinates in the neighborhood U of x^0 so that $g_i(x) = x_i - a_i$, for $i = 1, \dots, d$. Consider the mapping $df : U \to \mathbb{R}^d$ in these coordinates. The set of critical points of f_c in U coincides with $df^{-1}(c)$, and all these points are non-degenerate when c is not a critical value of df. From Sard's theorem, the set S_U of critical values of df has zero measure. Since the sets U selected for different points x^0 cover X, a countable covering of X by these sets can be found. Accordingly, the set S, a countable union of the sets S_U, has zero measure.

2. Consider the following family: $f_{a,c} = f(x) - af(x) + (c,x)f(x)$. It is easy to see that differentials of $f(x)$ and $x_i f(x)$ generate cotangent space to X at each point $x^0 \in X$. Part 1 of this lemma implies that the set

$$S = \{(a,c) : f_{a,c} \text{ has a degenerate critical point}\}$$

has zero measure in $\mathbb{R}^{n+1}$. Since multiplication by a constant does not change critical points and their degeneracy, $S \cap \{a \neq 1\}$ is a cylinder over the set V. Hence V has zero measure in $\mathbb{R}^n$.

3. Part 2 of this lemma implies that, for each $y \in Y$, the set

$$S_y = \{c : f_y(x)(1 + (c,x)) \text{ has a degenerate critical point}\}$$

has zero measure in $\mathbb{R}^n$. Let $S = \cup_y (S_y, y) \subset \mathbb{R}^n \times Y$. Due to Fubini theorem, S has measure zero in $\mathbb{R}^n \times Y$. This implies, again due to Fubini theorem, that, for a generic c, the set $W_c = S \cap \{c = \text{const}\}$ has zero measure in Y. □

2. The main result

The aim of this paper is to describe an algorithm for producing a cylindrical decomposition of a sub-Pfaffian set Y in the unit cube $I^n \subset \mathbb{R}^n$. More precisely, an input of the algorithm is a semi-Pfaffian set X in an open domain $G \subset \mathbb{R}^{m+n}$ defined by (1) with $s = m+n$. Assume that X is contained in the closed unit cube I^{m+n}. Let for the projection function

$$\pi : \quad \mathbb{R}^{m+n} \longrightarrow \mathbb{R}^n,$$

$\pi(X) = Y$, and $\dim(Y) = d$.

The output of the algorithm is a cell decomposition (i.e. subdivision into finite disjoint family of topological cells) of $I^n = \pi(I^{m+n})$ compatible with Y. The decomposition is *cylindrical* after some linear change of coordinates. Each cell is described by a formula of the type

$$\pi'\Big(\bigcup_{1 \le i \le M'} \bigcap_{1 \le j \le M''} \{h_{ij} *_{ij} 0\}\Big),$$

where h_{ij} are Pfaffian functions in $n' \ge m+n$ variables, π' is the projection function

$$\pi' : \quad \mathbb{R}^{n'} \longrightarrow \mathbb{R}^n,$$

$*_{ij} \in \{=, >\}$, and M', M'', are certain integers.

Using an oracle the algorithm can then decide which cells belong to Y and which to its complement $\tilde{Y} = I^n \setminus Y$.

We prove that the number of cells in the decomposition is less than

$$N^{(d!)^2 (m+2n)^d (r+m+2n)^d} (\alpha+\beta)^{r^{O(d(m+dn))}},$$

the format of each cell is

$$(N^{((d-1)!)^2 (m+2n)^d (r+m+2n)^d} (\alpha+\beta)^{r^{O(d(m+dn))}}, (\alpha+\beta)^{r^{O(d(m+dn))}},$$

$$N^{((d-1)!)^2 (m+2n)^d (r+m+2n)^d} (\alpha+\beta)^{r^{O(d(m+dn))}}).$$

The complexity of the algorithm is

$$N^{(r+m+n)^{O(d)}} (\alpha+\beta)^{r^{O(d(m+dn))}}.$$

3. Algorithm

3.1. **Computing the dimension of Y.** The algorithm applies the weak stratification subroutine from Proposition 1.10 to X. Consider a stratum $X_\alpha = \{f = 0, g > 0\}$ of X, with $\dim(X_\alpha) = d'$, $\dim(\pi(X_\alpha)) = d$, where $f = (f_1, \dots, f_k), g = (g_1, \dots, g_{k'})$ are vectors of Pfaffian functions and relations $=, >$ are understood component-wise. The stratum X_α is effectively non-singular, i.e the list $f_1, \dots, f_k$ includes $m + n - d'$ Pfaffian functions $f_{i_1}, \dots, f_{i_{m+n-d'}}$ such that the restriction $f_{i_j}|_{X_\alpha} \equiv 0$ for each $1 \leq j \leq m + n - d'$, and $df_{i_1} \wedge \dots \wedge df_{i_{m+n-d'}} \neq 0$ at every point of X_α. Let for a subspace $\mathbb{R}^n$ its coordinates be $y = (x_1, \dots, x_n)$, while the coordinates of the complement space $\mathbb{R}^m$ be $x = (x_{n+1}, \dots, x_{n+m})$. Then the fiber $\pi^{-1}(y)$ for any $y \in \pi(X_\alpha)$ is at least $(d' - d)$-dimensional, so

$$\operatorname{rank}(\frac{\partial f}{\partial x}) \leq m - d' + d$$

at any $y \in \pi(X_\alpha)$. The algorithm chooses among all values of d from 0 the maximal such that the set

$$\hat{X}_\alpha = X_\alpha \cap \{\operatorname{rank}(\frac{\partial f}{\partial x}) = m - d' + d\}$$

is non-empty. This value is $\dim(\pi(X_\alpha))$. Then the algorithm selects the maximum of these dimensions over all strata of X.

3.2. **The "down" procedure.** After determining $d = \dim(Y)$ the algorithm uses one after another two procedures: "down" and "up". We start with the description of the "down" procedure.

The input of the lth recursion step is a pair X_l, Y_l where $X_l \subset \mathbb{R}_l^{n_l}$ for some $n_l \geq n + m$, $Y_l \subset \mathbb{R}_l^n$ (the sub-index l in $\mathbb{R}_l^{n_l}$ and $\mathbb{R}_l^n$ indicates that the coordinate systems linearly change depending on l), $\dim(Y_l) = d_l$ and $\pi_l(X_l) = Y_l$ for a projection

$$\pi_l : \quad \mathbb{R}_l^{n_l} \longrightarrow \mathbb{R}_l^n,$$

$$\pi_l(x_1, \dots, x_n, x_{n+1}, \dots, x_{n_l}) = (x_1, \dots, x_n).$$

Here $X_1 = X$, $Y_1 = Y$, $\mathbb{R}_1 = \mathbb{R}$, $n_1 = n + m$, $d_1 = d$, $\pi_1 = \pi$. Let $d_0 = n$, and $\rho_0 : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the identity map.

In the description of a recursion step we drop for brevity the sub-indices in X_l and Y_l, i.e. write X and Y respectively.

The algorithm applies the weak stratification subroutine from Proposition 1.10 to X. Let $y = (x_1, \dots, x_n)$, $x = (x_{n+1}, \dots, x_{n_l})$. For each stratum $X_\alpha = \{f = 0, h > 0\}$ of the dimension $r \geq d_l$ the algorithm performs a further decomposition into semi-Pfaffian sets

$$X'_\alpha = X_\alpha \cap \{\operatorname{rank}(\frac{\partial f}{\partial x}) = n_l - n - r + d_l\}$$

and

$$V'_\alpha = X_\alpha \cap \{\operatorname{rank}(\frac{\partial f}{\partial x}) < n_l - n - r + d_l\},$$

i.e. into sets of regular and critical points of the projection $\pi_l|_{X_\alpha}$. By Sard's theorem, $\dim(\pi_l(V'_\alpha)) < d_l$. Note that for any $y \in Y \setminus \pi_l(V'_\alpha)$ the intersection $\pi_l^{-1}(y) \cap X'_\alpha$ is smooth.

The algorithm finds an integer vector c and a semi-Pfaffian set $V''_\alpha \subset X'_\alpha$ such that $\dim(\pi_l(V''_\alpha)) < d_l$, and for any $y \in Y \setminus \pi_l(V'_\alpha \cup V''_\alpha)$ the critical points of the function

$$g = (\prod_j h_j)(1 + (c, x))$$

(the product of zero factors is assumed to be 1) on $\pi_l^{-1}(y) \cap X'_\alpha$ are non-degenerate, in particular isolated. More precisely, introduce a function

$$g(x, y, z) = (\prod_j h_j(x, y))(1 + (z, x)),$$

which for fixed y, z is considered as a function on $\pi_l^{-1}(y) \cap X'_\alpha$, and a set

$$A(x, y, z) = \{(x, y, z) \in (X \setminus V'_\alpha) \times \mathbb{R}_l^{n_l - n} :$$
$$g(x, y, z) \text{ has a degenerate critical point at } x\}.$$

Due to Lemma 1.7,

$$A(x, y, z) = \{(x, y, z) \in (X \setminus V'_\alpha) \times \mathbb{R}_l^{n_l - n} : \quad \operatorname{rank}(\mathcal{M}_{f,g}) < r - d_l\}.$$

Consider two projection functions

$$\tau_1 : \quad \mathbb{R}_l^{2n_l - n} \longrightarrow \mathbb{R}_l^{n_l}$$
$$(x, y, z) \longmapsto (y, z),$$

and

$$\tau_2 : \quad \mathbb{R}_l^{n_l} \longrightarrow \mathbb{R}_l^{n_l - n}$$
$$(y, z) \longmapsto z.$$

Then according to part 3 of Lemma 1.16, for the set

$$B(y, z) = \{(y, z) \in \tau_1(A(x, y, z)) : \quad \dim((\tau_2^{-1}\tau_2(y, z)) \cap \tau_1(A(x, y, z)) = d_l\},$$

the dimension $\dim(\tau_2(B(y, z))) < n_l - n$.

Let

$$A(x, y, z) = \bigcup_\beta A_\beta(x, y, z)$$

be a weak stratification of $A(x, y, z)$, and $C(x, y, z)$ be the union of critical sets of $\tau_2\tau_1|_{A_\beta(x,y,z)}$ for all strata $A_\beta(x, y, z)$ of dimensions at least d_l. Then $B(y, z)$ is a subset of $\tau_1(C(x, y, z))$. The set $C(x, y, z)$ is semi-Pfaffian and

$$\dim(\tau_2\tau_1(C(x, y, z))) < n_l - n$$

due to Sard's theorem. According to Lemma 1.8, there is an integer vector

$$\lambda = (\lambda_1, \dots, \lambda_{n_l - n}) \notin \tau_2\tau_1(C(x, y, z))$$

such that $0 \le \lambda_i \le M$, where M is a certain explicit function of the format of $C(x, y, z)$.

The algorithm computes the set $C(x, y, z)$ by applying the procedure from Proposition 1.10 to $A(x, y, z)$ and writing out the conditions on the rank (cf. the definition of V'_α). The algorithm tests each vector λ for membership to $\tau_2\tau_1(C(x, y, z))$ by checking (with a use of the oracle) whether $\tau_1^{-1}\tau_2^{-1}(\lambda) \cap C(x, y, z) = \emptyset$. If yes, then $\lambda \notin \tau_2(B(y, z)) \subset \tau_2\tau_1(C(x, y, z))$, and the algorithm takes λ as c. According to Lemma 1.7, we also define

$$V''_\alpha = X'_\alpha \cap \{\operatorname{rank}(\mathcal{M}_{f,g}) < r - d_l\}.$$

Thus, the vector c and the set V''_α are constructed.

Observe that each connected component of $\pi_l^{-1}(y) \cap X'_\alpha$ contains a critical point of $g|_{\pi_l^{-1}(y)\cap X'_\alpha}$. Note that $\dim(\pi_l(V'_\alpha \cup V''_\alpha)) < d_l$.

Denote

$$X''_\alpha = X'_\alpha \cap \{\mathrm{rank}(\frac{\partial(f,g)}{\partial x}) = n_l - n - r + d_l\} \setminus (V'_\alpha \cup V''_\alpha),$$

thus for each $y \in Y \setminus \pi_l(V'_\alpha \cup V''_\alpha))$ the intersection $X''_\alpha \cap \pi_l^{-1}(y)$ is the finite set of all critical points of g on $\pi_l^{-1}(y) \cap X'_\alpha$. It follows that $\dim(X''_\alpha) = d_l$.

The algorithm applies the weak stratification subroutine from Proposition 1.10 to X''_α. For each stratum $X_{\alpha\beta} = \{\hat{f} = 0, \hat{h} > 0\}$ of the (maximal) dimension d_l the algorithm performs a further decomposition into semi-Pfaffian sets

$$X'_{\alpha\beta} = X_{\alpha\beta} \cap \{\mathrm{rank}(\frac{\partial \hat{f}}{\partial x}) = n_l - n\}$$

and

$$V'_{\alpha\beta} = X_{\alpha\beta} \cap \{\mathrm{rank}(\frac{\partial \hat{f}}{\partial x}) < n_l - n\},$$

i.e. into sets of regular and critical points of the projection $\pi_l|_{X_{\alpha\beta}}$. Note that $\dim(\pi_l(V'_{\alpha\beta})) < d_l$.

Let $V_{\alpha\beta} = V'_\alpha \cup V''_\alpha \cup V'_{\alpha\beta}$, and T be the union of all strata of X of the dimension less than d_l. The following properties are true.

1. $Y = \bigcup_{\alpha,\beta} \pi_l(X'_{\alpha\beta} \cup V_{\alpha\beta}) \cup \pi_l(T)$;
2. $X'_{\alpha\beta}$ is effectively non-singular, $\dim(X'_{\alpha\beta}) = d_l$, and $\pi_l|_{X'_{\alpha\beta}}$ has rank $n_l - n$ at every point of $X'_{\alpha\beta}$, for each α, β;
3. $\dim(\pi_l(\bigcup_{\alpha,\beta} V_{\alpha\beta})) < d_l$;
4. $X'_{\alpha\beta} \cap X'_{\alpha'\beta'} = \emptyset$, for $(\alpha,\beta) \neq (\alpha',\beta')$.

If $d_l = n$ (this can only happen when $l = 1$, so $n = d$), then setting

$$Z = \bigcup_{\alpha,\beta} \pi_l(\partial X'_{\alpha\beta} \cup V_{\alpha\beta}) \cup \pi_l(T)$$

(note that $\dim(Z) < d$), the algorithm uses the subroutine from Proposition 1.12 to find the semi-Pfaffian set $\partial X'_{\alpha\beta}$ and then sets $Y_2 = Z$, $X_2 = \bigcup_\alpha (\partial X'_{\alpha\beta} \cup V_\alpha) \cup T$.

If $d_l < n$, the algorithm computes integer coefficients of a linear $(d_{l-1} - d_l)$-dimensional subspace L in $\mathbb{R}_l^{d_{l-1}}$ such that for the map $\rho_l = \rho\rho_{l-1}$, where ρ is the projection map along L, and for each $y \in \overline{Y}$ the set $\rho_l^{-1}(\rho_l y)$ is finite. According to the Koopman-Brown theorem [15], a generic subspace satisfies this requirement. More precisely, observe that because X is bounded (contained in a cube), the closure $\overline{Y}$ coincides with the projection $\pi(\overline{X})$. Using the procedure from Proposition 1.12, the algorithm computes the closure $\overline{X}$. Then the algorithm considers the sub-Pfaffian set

$$A_{\varepsilon,z} = \{(\varepsilon, z) \in \mathbb{R}^{n(n-d_{l-1}+d_l)+1} : \quad \exists y \in \overline{Y},$$

$$\exists x' \in \rho_l^{-1}(\rho_l y), \exists x'' \in \rho_l^{-1}(\rho_l y), \|x' - x''\| = \varepsilon, \varepsilon > 0\}.$$

Fix a sufficiently small positive value $\hat{\varepsilon}$ of ε, then $A_{\hat{\varepsilon},z}$ is the set of all coefficient vectors of subspaces L for which this finiteness condition is not valid. Due to Lemma 1.8, there is an integer vector

$$\gamma = (\gamma_1, \ldots, \gamma_{n(n-d_{l-1}+d_l)}) \notin A_{\hat{\varepsilon},z}$$

such that $0 \le \gamma_i \le M$, where M is a certain explicit function of the format of $A_{\hat{\varepsilon},z}$. The algorithm computes the closure $\overline{A_{\varepsilon,\gamma}}$ by applying the procedure from Proposition 1.12 to the quantifier-free part of the formula defining $A_{\varepsilon,\gamma}$. The algorithm tests each vector γ in this range for membership to $A_{\hat{\varepsilon},z}$ by checking whether $0 \notin \overline{A_{\varepsilon,\gamma}}$. If yes, then $\gamma \notin A_{\hat{\varepsilon},z}$, and the algorithm takes γ as the vector of coefficients of L.

Let $\mathbb{R}^n_{l+1}$ denote the result of a linear coordinate change in $\mathbb{R}^n_l$ such that L becomes a $(d_{l-1} - d_l)$-dimensional coordinate subspace. Accordingly, all the sub- and super-spaces of $\mathbb{R}^n_{l+1}$ get the sub-index $l+1$.

Consider the set

$$S_{\alpha\beta} = X'_{\alpha\beta} \cap \{\mathrm{rank}(\frac{\partial \hat{f}}{\partial(x,z)}) < n_l - n + d_{l-1} - d_l\},$$

where z are coordinates in L, i.e. the set of critical points of the projection $\rho_l\pi_l|_{X'_{\alpha\beta}}$ on $\mathbb{R}^{d_l}_{l+1}$. Observe that $\dim(\rho_l\pi_l(S_\alpha)) < d_l$ by Sard's theorem, and $\dim(\pi_l(S_\alpha)) < d_l$ due to the definition of L.

Introduce the sets

$$W'_i = \{y, \varepsilon :\ y = (z_1, \dots, z_{n-d_l}, y_{n-d_l+1}, \dots, y_n) \in Y, \varepsilon \in \mathbb{R}^1,$$
$$\exists y' = (z'_1, \dots, z'_{n-d_l}, y_{n-d_l+1}, \dots, y_n) \in Y,$$
$$\rho_l(y') = \rho_l(y), z'_1 = z_1, \dots, z'_{i-1} = z_{i-1}, z'_i \ne z_i, |z'_i - z_i| < \varepsilon\},$$
$$W_i = \overline{W'_i} \cap \{\varepsilon = 0\},$$
$$W = \bigcup_{1 \le i \le n-d_l} W_i,$$
$$Z = \rho_l\pi_l(\bigcup_{\alpha,\beta}(\partial X'_{\alpha\beta} \cup V_{\alpha\beta} \cup S_{\alpha\beta})) \cup \rho_l(W) \cup \rho_l\pi_l(T).$$

Then $\dim(Z) = \dim(Y \cap \rho_l^{-1}(Z)) < d_l$. Taking into the account Proposition 1.12, observe that W is a sub-Pfaffian set, more precisely, there exist an integer n_{l+1}, $n_l \le n_{l+1} \le n_l + n$, and a semi-Pfaffian set $U' \subset \mathbb{R}^{n_{l+1}}_{l+1}$ such that for the projection

$$\pi_{l+1} :\quad \mathbb{R}^{n_{l+1}}_{l+1} \longrightarrow \mathbb{R}^n_{l+1},$$

$\pi_{l+1}(U') = W$.

Let U'' denote the semi-Pfaffian set defined in $\mathbb{R}^{n_{l+1}}_{l+1}$ by the same formula as

$$\bigcup_{\alpha,\beta}(\partial X'_{\alpha\beta} \cup V_{\alpha\beta} \cup S_{\alpha\beta}) \cup T \subset \mathbb{R}^{n_l}_{l+1}.$$

The algorithm finds Z and sets

$$Y_{l+1} = Y \cap \rho_l^{-1}(Z) = \pi_{l+1}(U' \cup U''),\ Z_l = Z$$

and

$$X_{l+1} = \pi_{l+1}^{-1}(Y \cap \rho_l^{-1}(Z)) = U' \cup U''.$$

Observe that X_{l+1} is defined by an explicit quantifier-free formula with Pfaffian functions in n_{l+1} variables. The algorithm determines $d_{l+1} = \dim(Y_{l+1}) = \dim(Z)$ using the subroutine from Section 3.1.

On the last step $\hat{l} \le d$ of the "down" procedure the dimension $\dim(Y_{\hat{l}}) = 0$ and $Z_{\hat{l}} = \emptyset$.

3.3. **The "up" procedure.** Then the algorithm starts the "up" recursion procedure. An input of the rth recursion step is a pair $Y_{\hat{l}-r+1}$, $Z_{\hat{l}-r+1} \subset \mathbb{R}^n_{\hat{l}}$ of sub-Pfaffian sets constructed in the "down" process (in the description of the step we drop for brevity the sub-index in Y, Z, ρ and $d = \dim(Y)$), and a cylindrical cell decomposition $\mathcal{D}$ of $I^n \subset \mathbb{R}^n_{\hat{l}}$ compatible with $Y \cap \rho^{-1}(Z)$. The decomposition $\mathcal{D}$, being cylindrical, induces a cell decomposition D of $\rho(I^n) = I^d \subset \mathbb{R}^d_{\hat{l}}$ compatible with Z, namely the elements of D are exactly the ρ-projections of the elements of $\mathcal{D}$. By the definition of Z, for any d-dimensional cell C of the decomposition D, for any $y \in C$ the cardinality of the set $\rho^{-1}(y) \cap Y$ is a constant, say M. Moreover, the union

$$\bigcup_{1\le\nu\le M+1} \{y \in \rho^{-1}(C) \cap I^n : \exists y_1 \in Y, \dots, \exists y_M \in Y,$$

$$y_1 \prec \cdots \prec y_{\nu-1} \prec y \prec y_\nu \prec \cdots \prec y_M, \rho(y_1) = \cdots = \rho(y_M) = \rho(y)\},$$

where the relation $u \prec v$ for $u = (u_1, \dots, u_n), v = (v_1, \dots, v_n) \in \mathbb{R}^n_{\hat{l}}$ stands for the disjunction

$$\bigvee_{d+1\le i\le n+1} \{u_1 = v_1, \dots, u_{i-1} = v_{i-1}, u_i < v_i\},$$

represents a cylindrical cell decomposition of $\rho^{-1}(C) \cap I^n$ compatible with $Y \cap \rho^{-1}(C)$. Thus, if

$$y_{\nu-1,1} = y_{\nu,1}, \dots, y_{\nu-1,i-1} = y_{\nu,i-1}, y_{\nu-1,i} < y_{\nu,i}$$

for a certain i, $d+1 \le i \le n+1$, then the decomposition contains the cells
$\rho^{-1}(C) \cap I^n \cap \{y : y_{\nu-1,i} < y_i < y_{\nu,i}\}$,
$\rho^{-1}(C) \cap I^n \cap \{y : y_{\nu-1,i} = y_i, y_{\nu-1,i+1} < y_{i+1}\}, \dots$,
$\rho^{-1}(C) \cap I^n \cap \{y : y_{\nu-1,i} = y_i, \dots, y_{\nu-1,n} = y_n\}$,
$\rho^{-1}(C) \cap I^n \cap \{y : y_i = y_{\nu,i}, y_{i+1} < y_{\nu,i+1}\}, \dots$,
$\rho^{-1}(C) \cap I^n \cap \{y : y_i = y_{\nu,i}, \dots, y_n = y_{\nu,n}\}$.

The algorithm finds M and computes the cell decomposition.

Combining the cell decompositions for $\rho^{-1}(C) \cap I^n$ for all d-dimensional cells C of D, with the cell decomposition $\mathcal{D}$, the algorithm gets a cylindrical cell decomposition of I^n compatible with Y. This finishes the description of the recursive step of the "up" procedure.

On the last step of the "up" process the algorithm produces a cylindrical decomposition of I^n compatible with $Y_1 = Y$.

4. Complexity

We first estimate the complexity of *computing the dimension of* Y. Recall that X has a format $(N, \alpha+\beta, n+m)$. According to Proposition 1.10, the format of each stratum X_α is

$$(N(\alpha+\beta)^{r^{O(n+m)}}, (\alpha+\beta)^{r^{O(n+m)}}, n+m),$$

the number of strata does not exceed $N^{n+m+r}(\alpha+\beta)^{r^{O(n+m)}}$, the complexity of the stratification is bounded by $N^{n+m+r}(\alpha+\beta)^{r^{O(n+m)}}$. For each of $\binom{n}{d'}$ subspaces of $\mathbb{R}^n$ the algorithm constructs the set $\hat{X}_\alpha$. The matrix $(\frac{\partial f}{\partial x})$ is $(m+n-d') \times (n-r)$-matrix, so the number of all minors is less than $2^{2(n+m)}$. The degrees of maximal

minors are less than $(n+m)(\alpha+\beta)^{r^{O(n+m)}}$. It follows that the format of each $\hat{X}_\alpha$ is

$$(N(\alpha+\beta)^{r^{O(n+m)}}, (\alpha+\beta)^{r^{O(n+m)}}, n+m).$$

Thus, the complexity of computing $\dim(Y)$ is $N^{n+m+r}(\alpha+\beta)^{r^{O(n+m)}}$.

Now we estimate the complexity of constructing, and the formats of the sets Z in a recursive step of the "down" procedure. At the input of a step we have sets X and Y both of format (L, D, s) (i.e. $n_l = s$). On the first step $L = N$, $D = \alpha + \beta$ and $s = n + m$.

1. *Computing sets X_α (stratification).* According to Proposition 1.10, the format of each stratum X_α is
$$(LD^{r^{O(s)}}, D^{r^{O(s)}}, s), \tag{2}$$
the number of strata does not exceed $L^{s+r}D^{r^{O(s)}}$, the complexity up to this stage is bounded by $L^{s+r}D^{r^{O(s)}}$.
2. *Computing sets X'_α and V'_α.* The number of all minors in the matrix $(\frac{\partial f}{\partial x})$ is less than 2^{2s}. The degrees of maximal minors are less than $D^{r^{O(s)}}$. It follows that the format of each X'_α or V'_α is(2), the number of these sets is less than $L^{s+r}D^{r^{O(s)}}$, the complexity up to this stage is bounded by $L^{s+r}D^{r^{O(s)}}$.
3. *Computing sets X''_α and V''_α.* The matrix $\mathcal{M}_{f,g}$ has the order less than $\binom{s}{s-r}$, so the number of all minors is less than $2^{2\cdot 2^s}$. According to Lemma 1.6, the degrees of minors do not exceed $O(s2^sD)$. It follows that the format of $A(x, y, z)$ is
$$(LD^{r^{O(s)}}, D^{r^{O(s)}}, 2s-n).$$
According to Proposition 1.10, the format of each stratum $A_\beta(x, y, z)$ is the same, the number of strata is less than $L^{O(s+r)}D^{r^{O(s)}}$, the complexity up to this stage is bounded by $L^{O(s+r)}D^{r^{O(s)}}$. These bounds imply that the format of $C(x, y, z)$ is
$$(LD^{r^{O(s)}}, D^{r^{O(s)}}, 2s-n).$$
It follows from Lemma 1.8 that the range limit M for λ_i can be taken as $LD^{r^{O(s)}}$. Thus, the number of vectors λ for which the condition $\tau_1^{-1}\tau_2^{-1}(\lambda) \cap C(x, y, z) = \emptyset$ is tested is less than $M^{O(s)}$, which is $LD^{r^{O(s)}}$. It follows that the complexity of computing c is $L^{O(s+r)}D^{r^{O(s)}}$.

   The format of each set V''_α is (2), the number of these sets is less than $L^{s+r}D^{r^{O(s)}}$. Computing of sets X''_α is similar to step 2, the format of each X''_α is (2), the number of these sets is less than $L^{s+r}D^{r^{O(s)}}$. The complexity up to this stage is bounded by $L^{O(s+r)}D^{r^{O(s)}}$.
4. *Computing sets $X_{\alpha,\beta}$ (stratification)* is similar to step 1. Due to Proposition 1.10, the format of each stratum $X_{\alpha\beta}$ is (2), the number of strata does not exceed $L^{s+r}D^{r^{O(s)}}$, the complexity up to this stage is bounded by $L^{O(s+r)}D^{r^{O(s)}}$.
5. *Computing sets $X'_{\alpha\beta}$ and $V'_{\alpha\beta}$* is similar to step 2. The format of each $X'_{\alpha\beta}$ or $V'_{\alpha\beta}$ is (2), the number of these sets is less than $L^{s+r}D^{r^{O(s)}}$, the complexity up to this stage is bounded by $L^{O(s+r)}D^{r^{O(s)}}$.

6. *Computing the subspace L.* Due to Proposition 1.12, the format of the semi-Pfaffian set $\overline{X}$ is
$$((LD)^{O((s+r)s)}, D^{O(s)}, s).$$
It follows that the format of $A_{\varepsilon,z}$ is,
$$((LD)^{O((s+r)s)}, D^{O(s)}, O(s+n^2)).$$
Lemma 1.8 now implies that the range limit M for integers γ_i can be taken as
$$(s+n^2)^{O(r)}(LD)^{O(s(s+r)(s+r+n^2}.$$
Thus, the number of vectors γ for which the membership to $A_{\hat{\varepsilon},z}$ is tested is less than $M^{O(n^2)}$, which is
$$(s+n^2)^{O(rn^2)}(LD)^{O(s(s+r)(s+r+n^2)n^2)}.$$
Taking into the account the complexity of the procedure from Proposition 1.12, we conclude that the complexity of computing the subspace L is bounded by
$$(s+n^2)^{O(rn^2)}(LD)^{O(s(s+r)(s+r+n^2)n^2)}.$$
7. *Computing sets $S_{\alpha\beta}$* is similar to steps 2 and 3. The format of each S_α is (2), the number of these sets is less than $L^{s+r}D^{r^{O(s)}}$. Taking step 6 into the account, we conclude that the complexity up to this stage is bounded by
$$L^{O(s(s+r)(s+r+n^2)n^2)}D^{r^{O(s)}}. \tag{3}$$
8. *Computing sets $\partial X'_{\alpha\beta}$.* According to Proposition 1.12, the format of each $\partial X'_{\alpha\beta}$ is (2), the number of these sets is less than $L^{s+r}D^{r^{O(s)}}$, the complexity up to this stage is bounded by (3).
9. *Computing the set W.*
   (a) *Sets W'_i.* From the formula defining W'_i it follows that the format of each W'_i is
   $$(L+n, D, s+n),$$
   the number of these sets is less than n, the complexity of computing them is $O((L+n)D^s)$.
   (b) *Sets W_i.* According to Proposition 1.12, the format of each W_i is
   $$(LD)^{O((s+r)s)}, D^{O(s)}, s+n),$$
   the number of these sets is less than n, the complexity of computing them is $(LD)^{O((s+r)s)}$.
   (c) *Set W.* From (b) it follows that the format of W is
   $$(LD)^{O((s+r)s)}, D^{O(s)}, s+n),$$
   the complexity of computing W is $(LD)^{O((s+r)s)}$.
10. *Computing the set Z.* Combining steps 5 and 6(c), we get that the format of Z is
$$(L^{O((s+r)s)}D^{r^{O(s)}}, D^{r^{O(s)}}, s+n),$$
the complexity up to this stage is (3).

11. *Computing the sets* X_{l+1}, Y_{l+1}. According to the defining formulae for sets X_{l+1}, Y_{l+1}, their formats are

$$(L^{O((s+r)s)}D^{r^{O(s)}}, D^{r^{O(s)}}, s+n),$$

and the total complexity of the recursive step of the "down" procedure is (3).

The "down" procedure consists of at most d recursion stages each of which includes steps 1–11. Iterating the bounds from (11) d times we conclude that for all l, $1 \le l \le d$ formats of the sets X_l, Y_l, Z_l are

$$(N^{((d-1)!)^2(m+2n)^d(r+m+2n)^d}(\alpha+\beta)^{r^{O(d(m+dn))}}, (\alpha+\beta)^{r^{O(d(m+dn))}},$$

$$O(d(m+dn))), \tag{4}$$

and the complexity of the "down" procedure is bounded by

$$N^{(r+m+n)^{O(d)}}(\alpha+\beta)^{r^{O(d(m+dn))}}. \tag{5}$$

Now we estimate the complexity of constructing and the formats of the cell decompositions of I^n compatible with Y in a recursion step of the "up" procedure.

The upper bound (4) on the format of Z implies that on each recursion step the cardinality M of $\rho^{-1}(y) \cap Y$ for any y in a d-dimensional cell of the decomposition on I^d is less than

$$(N^{((d-1)!)^2(m+2n)^d(r+m+2n)^d}(\alpha+\beta)^{r^{O(d(m+dn))}}.$$

On the first step $Z = \emptyset$, $\dim(Y) = 0$, and the decomposition is described by the formula

$$\bigcup_{1\le\nu\le M+1} \{y \in I^n : \exists y_1 \in Y, \dots, \exists y_M \in Y,\ y_1 \prec \cdots \prec y_{\nu-1} \prec y \prec y_\nu \prec \cdots \prec y_M\}.$$

The number of cells is less than $2nM+1$, the number of variables in formulae describing each cell is less than $O(d(m+dn))$, thus the format of each cell is bounded by

$$(N^{((d-1)!)^2(m+2n)^d(r+m+2n)^d}(\alpha+\beta)^{r^{O(d(m+dn))}}, (\alpha+\beta)^{r^{O(d(m+dn))}},$$

$$N^{((d-1)!)^2(m+2n)^d(r+m+2n)^d}(\alpha+\beta)^{r^{O(d(m+dn))}}). \tag{6}$$

The complexity of constructing the decomposition is bounded by (5).

On a general recursion step, let the number of cells in the decomposition $\mathcal{D}$ compatible with $Y \cap \rho^{-1}(Z)$ will be less than T. Then the number of cells in the induced decomposition D, compatible with Z is also less than T (in particular, the number of cells C of the maximal dimension is less than T). It follows that there are less than $2nM+1$ cells in $\rho^{-1}(C) \cap I^n$ and thus less than $T(2nM+1)$ cells in the decomposition of I^n compatible with Y. As a result, on the last step d of the recursion, the algorithm produces a cell decomposition having less than

$$(2nM+1)^d < N^{(d!)^2(m+2n)^d(r+m+2n)^d}(\alpha+\beta)^{r^{O(d(m+dn))}}$$

cells. The formats of all intermediate cell decompositions continue to be (6). It follows that the complexity of the whole algorithm is bounded by (5).

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