Conductivity Imaging from Minimal Current Density Data

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Motivation: Current density impedance imaging

Goal: Determine the conductivity of human tissue by combining

- electrical (voltage/current) measurements on the boundary (EIT)
- magnitude of one current density field inside (CDI)

Current Density Imaging (Scott& Joy '91)

Very low frequency/ direct current $\Rightarrow$ stationary Maxwell Current Density Field $J := \nabla \times H$ (two rotations of the object)

MR measurements $\Rightarrow$ Magnetic field H produced by the applied current can be identified from the total field produced by the coils+fixed magnet

1-Laplacian in the conformal metric $g_{ij} = |J|^{2/(n-1)}\delta_{ij}$

$\sigma_- \leq \sigma(x) \leq \sigma_+$= isotropic conductivity of a body

- Ohm's Law: $J = -\sigma \nabla u \Rightarrow \sigma = |J|/|\nabla u|$.
- Conservation of charge (absence of sources/sinks inside): $\nabla \cdot J = 0$.

1-Laplacian (Seo et al., '02):

$$\nabla \cdot (\frac{|J|}{|\nabla u|} \nabla u) = 0.$$

Level sets of smooth, regular solutions are minimal surfaces in the metric $g = \left(|J|^{2/(n-1)}\delta_{ij}\right)$.

Admissible Data: $(f, a) \in H^{1/2}(\partial\Omega) \times L^2(\Omega)$

$\exists \sigma(x)$ with $0 < c_- \leq \sigma(x) \leq \sigma_+$, such that, if u_σ is weak solution of

$$\nabla \cdot \sigma \nabla u_\sigma = 0, \quad u_\sigma|_{\partial\Omega} = f,$$

then

$$a = |\sigma \nabla u_\sigma|.$$

σ = **generating conductivity** for the pair (f, a),

u = **corresponding potential**.

Sternberg-Ziemer example (for Dirichlet data)

Sternberg& Ziemer

$$\nabla \cdot \left(\frac{1}{|\nabla u(x)|} \nabla u(x) \right) = 0, \; x \in D \equiv unit \;\; disk,$$
$$u(x) = (x_1)^2 - (x_2)^2, \; x \in \partial D.$$

has a one parameter family of viscosity solutions u^λ, $\lambda \in (-1, 1)$, with

$$u^\lambda \equiv \lambda$$

in inscribed rectangles.

Remark: u^λs are NOT voltage potentials of some $\sigma \in L^\infty_+(\Omega)$:

$$1 \equiv |J| \neq \sigma |\nabla u^\lambda| \equiv 0.$$

Admissibility and the minimum weighted gradient problem

- If (f, a) is admissible, say generated by some conductivity σ_0 then the corresponding voltage potential

$$u_0 \in argmin \left\{ \int_\Omega a|\nabla u|dx : \; u \in H^1(\Omega), \; u|_{\partial\Omega} = f \right\}.$$

- If $u_0 \in argmin \left\{ \int_\Omega a|\nabla u|dx : \; u \in H^1(\Omega), \; u|_{\partial\Omega} = f \right\}$ and $|J|/|\nabla u_0| \in L^\infty_+(\Omega)$, then (f, a) is admissible.

Notes:

- Formally (not smooth) the Euler-Lagrange for $\int_\Omega a|\nabla u|dx$ is the 1-Laplacian.
- In the example before only u^0 (for $\lambda = 0$) is a minimizer of $\int_\Omega |\nabla u(x)|dx$.

Unique determination

Theorem (Nachman-T-Timonov '09, Moradifam-Nachman-T' 11)

$(f, |J|) \in C^{1,\alpha}(\partial\Omega) \times C^{\alpha}(\overline{\Omega})$ = *admissible pair,* $|J| > 0$ *a.e. in* Ω.

Then $\quad \min \int_\Omega |J||\nabla u| dx$

$$over \ \left\{ u \in W^{1,1}(\Omega) \bigcap C(\overline{\Omega}),\ |\nabla u| > 0 \ a.e.,\ u|_{\partial\Omega} = f \right\}$$

has a unique solution, say u_0*;*

$\sigma = |J|/|\nabla u_0|$ *is the unique conductivity generating* $(f, |J|)$.

Note (joint with A. Moradifam and A. Nachman): Uniqueness carries to

$$over \ \{u \in BV(\Omega), \quad u|_{\partial\Omega} = f\}$$

Implies stability in the minimization problem!

Equipotential surfaces are (globally) area minimizing

Theorem (Nachman-T-Timonov '11) *Let* $\Omega \subset \mathbb{R}^n$, $n \geq 2$, *be Lipschitz domain,* $\sigma \in C^{1,\delta}(\Omega)$, *and* $f \in C^{2,\delta}(\partial\Omega)$. *Let* $|J| = \sigma|\nabla u_\sigma|$, *where* u_σ *solves* $\nabla \cdot \sigma \nabla u_\sigma = 0$ *with* $u|_{\partial\Omega} = f$. *Assume* $|J| > 0$ *in* $\overline{\Omega}$.

Then, for a.e. $\lambda \in \mathbb{R}$ *and any* $v \in C^2(\overline{\Omega})$ *with* $v|_{\partial\Omega} = f$ *and* $|\nabla v| > 0$,

$$\int_{u^{-1}(\lambda)\cap\Omega} |J(x)| dS_x \leq \int_{v^{-1}(\lambda)\cap\Omega} |J(x)| dS_x;$$

dS = *induced Euclidean surface measure.*

Note: the integrals also represent the surface area induced from the metric $g = |J|^{(n-1)/2}\delta_{ij}$.

Insulating and perfectly conductive embeddings

V = Insulating "$\sigma = 0$",

U=perfectly conductive "$\sigma = \infty$".

Let $k \to \infty$ in the equation:

$$\nabla \cdot (\chi_U(k\tilde{\sigma} - \sigma) + \sigma)\nabla u = 0 \text{ in } \Omega,$$
$$\partial_\nu u|_{\partial V} = 0,$$
$$u|_{\partial\Omega} = f.$$

Still get

$$\nabla \cdot \sigma \nabla u = 0 \text{ in } \Omega \setminus \overline{(U \cup V)}$$
$$\nabla u = 0 \text{ in } U, \quad \text{but} \quad |J| \neq 0!$$

Further complications: In 3D+ and σ rough $\Rightarrow$ Non-unique continuation for solutions of elliptic

Admissibility in the presence of insulating/infinitely conductive embeddings

Admissibility of the data (a, f)

- On $\Omega \setminus \overline{(U \cup V)}$ same as before (with u_σ a solution of the limiting equation)
- On U:

$$\inf_{v \in W^{1,1}(U)} \int_U a|\nabla v|dx - \int_{\partial U} \sigma \left(\frac{\partial u_\sigma}{\partial \nu} \right)\Big|_{U+} v dx = 0$$

- $\{x : a(x) = 0\} = V \cup \Gamma \cup E$, where
  - V = one insulating connected component
  - Γ-negligible
  - E = Exotic = conductive region where $\nabla u = 0$

Admisibility is physical for infinitly conductive inclusions

$U \subset \Omega$ open, $\sigma \in L^{\infty}\Omega \setminus U$ and $a \in L^{\infty}(\Omega)$. Assume there exists $J \in Lip(U; \mathbb{R}^n)$ with

$$\nabla \cdot J = 0, \text{ in } U,$$

$$|J| \leq a, \text{in } U,$$

$$J|_{\partial U} = \sigma \frac{\partial u_\sigma}{\partial \nu}\bigg|_{\partial U}.$$

Then

$$\inf_{v \in W^{1,1}(U)} \int_U a|\nabla v| dx - \int_{\partial U} \sigma \left(\frac{\partial u_\sigma}{\partial \nu}\right)\bigg|_{U+} v dx = 0$$

$$\int_{\partial U} \sigma \frac{\partial u_\sigma}{\partial \nu} ds = 0.$$

What can be determined via the minimization problem?

Step1: From minimization determine u outside the zero set of a.

Step 2: Regions where $u \equiv const. \Rightarrow$ PERFECT CONDUCTORS.

Step 3: Determine σ outside the zeros of a and perfect conductors

Step 4: Identify maximal open connected components within zeros of a. If at the boundary of such a set

- u varies $\Rightarrow$ INSULATOR
- $u = const. \Rightarrow$ Fake perfectly conductive (EXOTIC =only happen in 3D when data is rough than Lipschitz).

The least weighted total variation problem

Would like solve:

$$min\{\int_\Omega a|\nabla u|dx : \ u \in H^1(\Omega), \ u|_{\partial\Omega} = f\}$$

Difficulties:

- minimizing sequence $\{u_n\}$ is not necessarily bounded in H^1 (but merely in $W^{1,1}$).
- Although u_n converges in $L^1_{loc}(\Omega)$, the limit is only BV.

$$min\{\int_\Omega a|Du| : \ u \in BV(\Omega), \ u|_{\partial\Omega} = f\}$$

New problem: if a solution lies in $BV \setminus W^{1,1}$ cannot be automatically approximated (in BV-norm) by smooth maps (otherwise they would be in $W^{1,1}$).

A regularized well-posed problem for the admissible case

Theorem (Nashed-T'11)Consider

$$u_n \in \operatorname{argmin}_{u \in H_0^1} F_{\epsilon_n}[u : a_n] := \int_\Omega a_n |\nabla h_f + \nabla u| dx + \epsilon_n \int_\Omega |\nabla u|^2 dx,$$

where $a_n \to a$ in $L^2(\Omega)$, and $\|a_n - a\| = o(\epsilon_n)$. Then

$$\liminf [F_{\epsilon_n}[u_n : a_n]] = \min_{v \in BV(\Omega), v|_{\partial\Omega} = f} \int_\Omega a|Dv|$$

If, in addition $0 < \inf(a) \le a \le \sup(a) < \infty$, then on a subsequence $u_n \to v^*$ in L^1, and $v^* \in BV(\Omega)$ is a minimizer.

Moreover, provided $v^* \in W^{1,1}(\Omega)$,

$$\sigma = \frac{a}{|\nabla(v^* + h_f)|}.$$

Mixed Boundary Value Problem

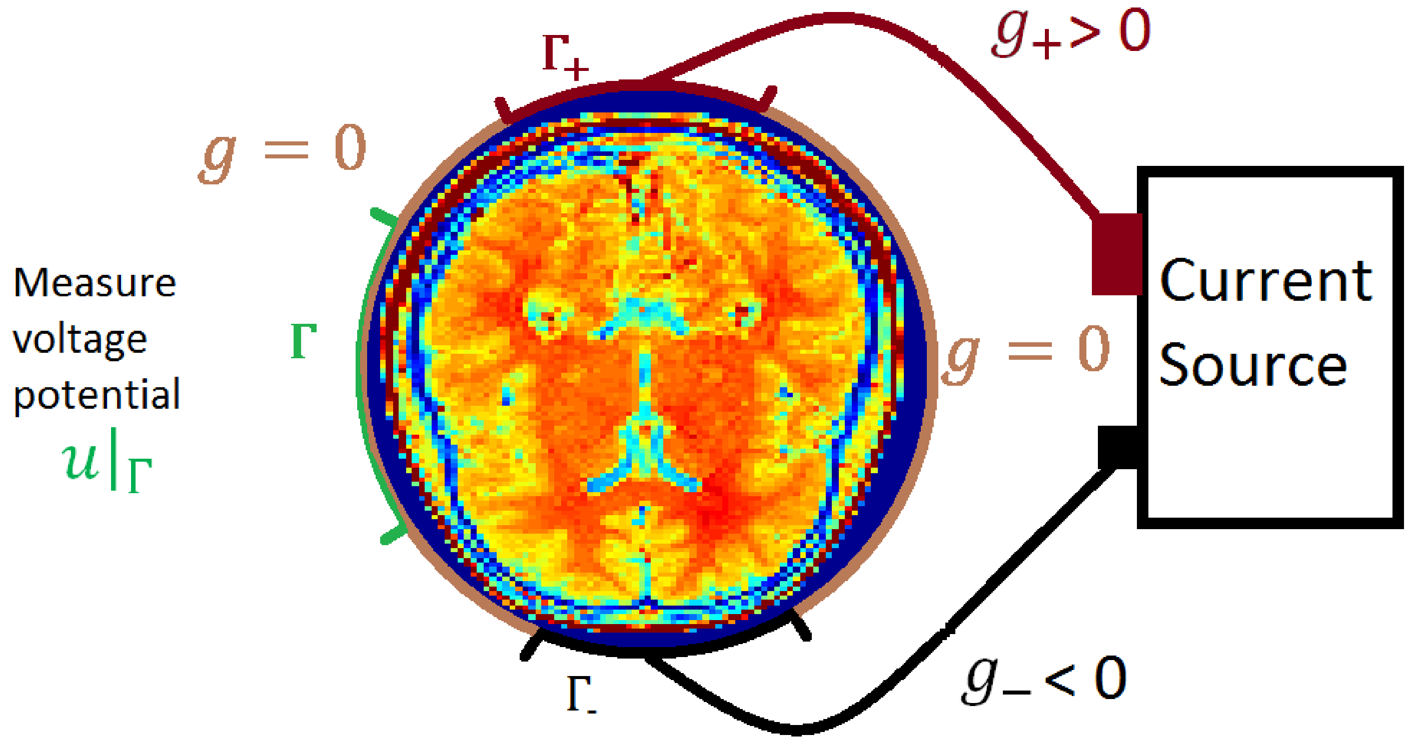


Figure 1:

$$\nabla \cdot \frac{|J|}{|\nabla u|} \nabla u = 0, u|_{\Gamma = f}, \partial_\nu u|_{\Gamma_\pm} = g$$

Interior Data $|J|$ and computed equipotential lines

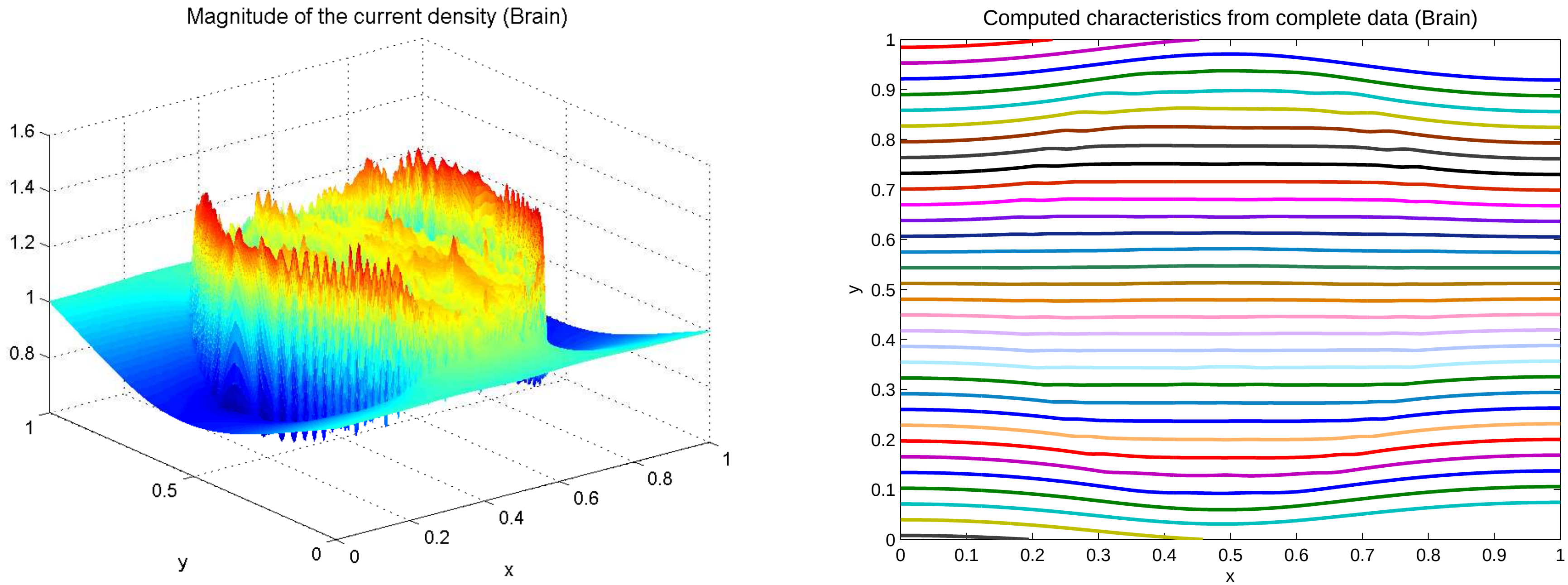
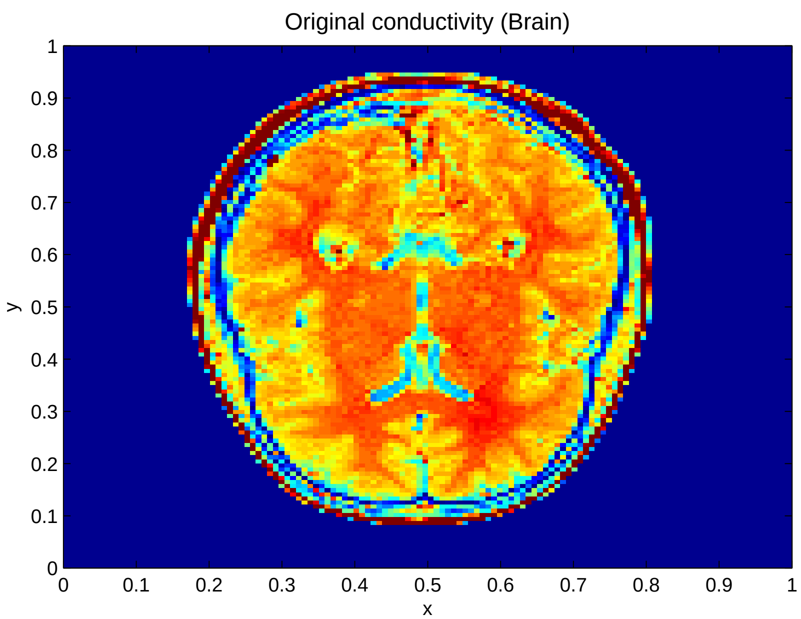


Figure 2:

Original and reconstructed conductivities via the equipotential lines



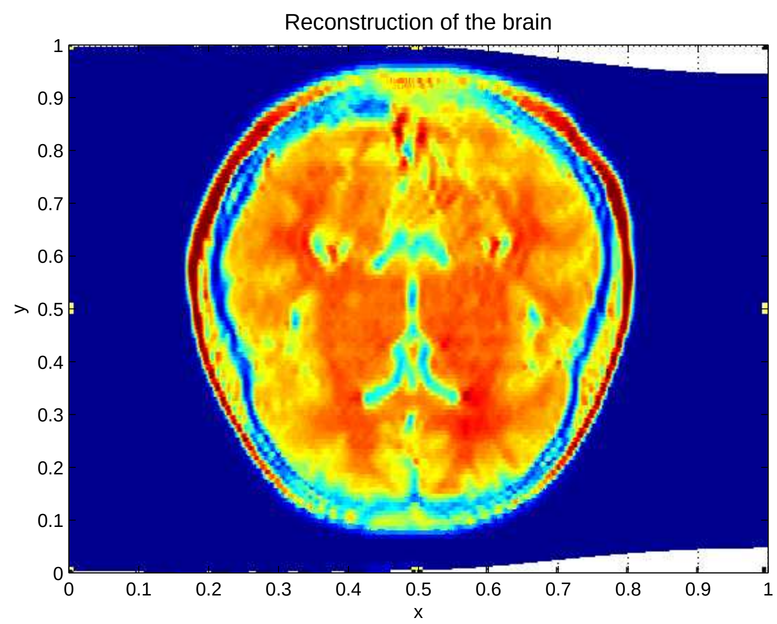


Figure 3:

Original and reconstructed conductivities via the minimization approach

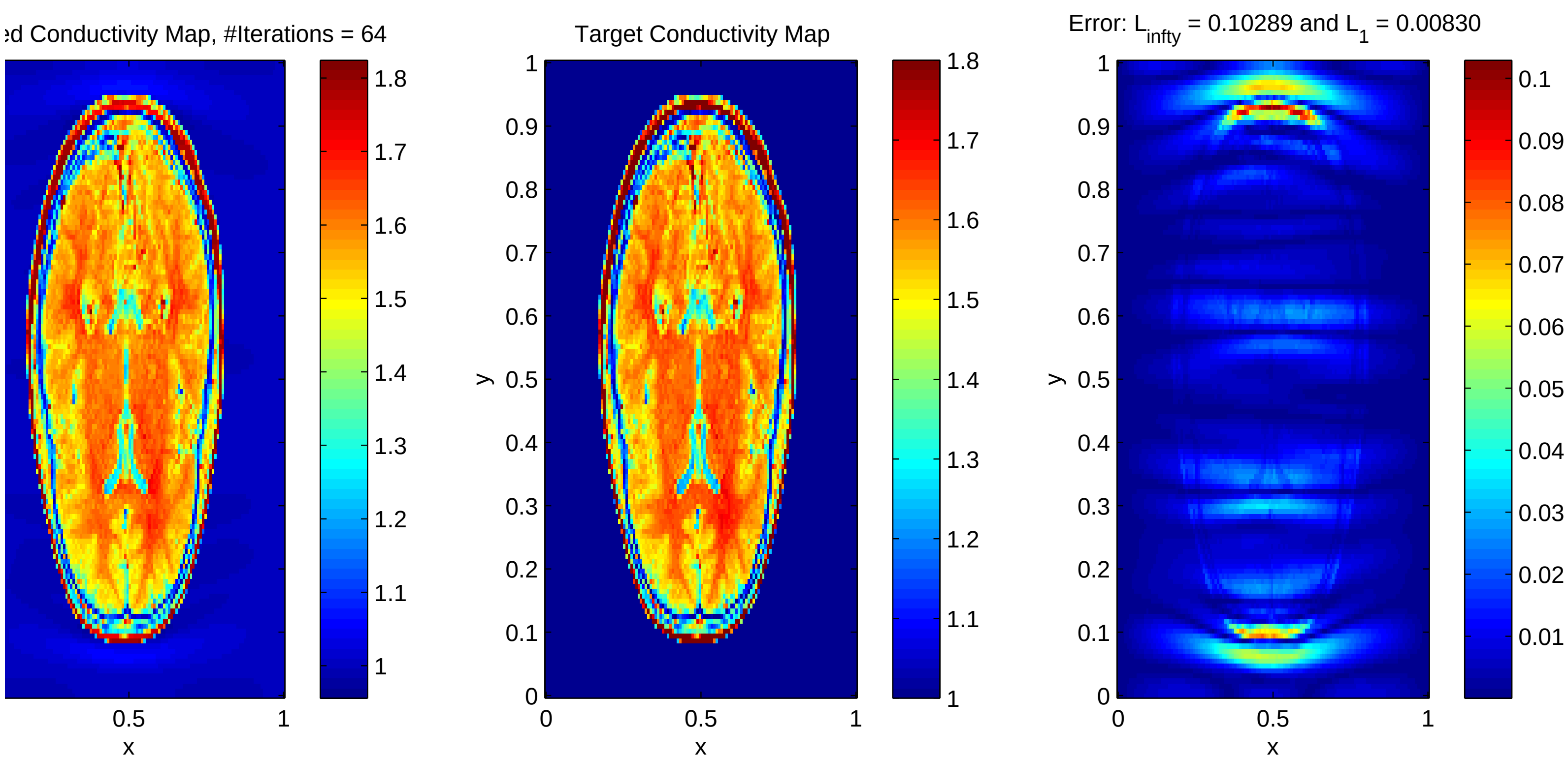


Figure 4: