

SUBCONVEXITY IN THE INHOMOGENEOUS CUBIC VINOGRADOV SYSTEM

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ABSTRACT. When $\mathbf{h} \in \mathbb{Z}^3$, denote by $B(X; \mathbf{h})$ the number of integral solutions to the system

$$\sum_{i=1}^6 (x_i^j - y_i^j) = h_j \quad (1 \leq j \leq 3),$$

with $1 \leq x_i, y_i \leq X$ ($1 \leq i \leq 6$). When $h_1 \neq 0$ and appropriate local solubility conditions on \mathbf{h} are met, we obtain an asymptotic formula for $B(X; \mathbf{h})$, thereby establishing a subconvex local-global principle in the inhomogeneous cubic Vinogradov system. We obtain similar conclusions also when $h_1 = 0$, $h_2 \neq 0$ and X is sufficiently large in terms of h_2 . Our arguments involve minor arc estimates going beyond square-root cancellation.

1. INTRODUCTION

The application of the Hardy-Littlewood (circle) method in the asymptotic analysis of the number of integral solutions of a Diophantine system is, with few exceptions, limited to scenarios in which the number of variables is larger than twice the total degree of the system. This *convexity barrier* arises from the relative sizes of the putative main term, given by the product of local densities associated with the system, and the most optimistic bound anticipated for the error term, namely the square-root of the number of choices for the variables. Almost all of the exceptions to this rule are inherently linear [10] or quadratic [9, 11, 12] in nature. There is work on pairs of diagonal cubic forms of special shape in 11 or more variables [6], and also an asymptotic formula for a special system consisting of one diagonal cubic and two linear equations in 10 variables [7]. Recently, the author [20] succeeded in breaking the convexity barrier for the Hilbert-Kamke problem of degree k , establishing an asymptotic formula for the number of solutions when the number of variables is at least $k(k+1)$. We turn our attention in this memoir to a related problem in which latent translation-dilation invariance obstructs the method of [20].

In order to describe our conclusions we must introduce some notation. Let s be a positive number and $\mathbf{h} = (h_1, h_2, h_3)$ a triple of integers. When X is a

2010 *Mathematics Subject Classification.* 11P55, 11L07, 11D72.

Key words and phrases. Subconvexity, Vinogradov's mean value theorem, Hasse principle. The author's work is supported by NSF grants DMS-2001549 and DMS-1854398.

large real number, write

$$f(\boldsymbol{\alpha}; X) = \sum_{1 \leq x \leq X} e(\alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3), \quad (1.1)$$

where $e(z)$ denotes $e^{2\pi iz}$. We consider the twisted mean value

$$B_s(X; \mathbf{h}) = \int_{[0,1]^3} |f(\boldsymbol{\alpha}; X)|^{2s} e(-\boldsymbol{\alpha} \cdot \mathbf{h}) \, d\boldsymbol{\alpha}, \quad (1.2)$$

in which we write $\boldsymbol{\alpha} \cdot \mathbf{h}$ for $\alpha_1 h_1 + \alpha_2 h_2 + \alpha_3 h_3$. Note that when $s \in \mathbb{N}$, it follows via orthogonality that the mean value $B_s(X; \mathbf{h})$ counts the number of integral solutions of the system of equations

$$\sum_{i=1}^s (x_i^j - y_i^j) = h_j \quad (1 \leq j \leq 3), \quad (1.3)$$

with $1 \leq x_i, y_i \leq X$ ($1 \leq i \leq s$). This is the inhomogeneous cubic Vinogradov system of the title.

In order to describe asymptotic formulae associated with $B_s(X; \mathbf{h})$, we introduce the generating functions

$$I(\boldsymbol{\beta}) = \int_0^1 e(\beta_1 \gamma + \beta_2 \gamma^2 + \beta_3 \gamma^3) \, d\gamma \quad (1.4)$$

and

$$S(q, \mathbf{a}) = \sum_{r=1}^q e_q(a_1 r + a_2 r^2 + a_3 r^3), \quad (1.5)$$

in which $e_q(u)$ denotes $e^{2\pi i u/q}$. Next, put $n_j = h_j X^{-j}$ ($1 \leq j \leq 3$), and define

$$\mathfrak{J}_s(\mathbf{h}) = \int_{\mathbb{R}^3} |I(\boldsymbol{\beta})|^{2s} e(-\boldsymbol{\beta} \cdot \mathbf{n}) \, d\boldsymbol{\beta} \quad (1.6)$$

and

$$\mathfrak{S}_s(\mathbf{h}) = \sum_{q=1}^{\infty} \sum_{\substack{1 \leq a_1, a_2, a_3 \leq q \\ (q, a_1, a_2, a_3) = 1}} |q^{-1} S(q, \mathbf{a})|^{2s} e_q(-\mathbf{a} \cdot \mathbf{h}). \quad (1.7)$$

We note that the *singular integral* $\mathfrak{J}_s(\mathbf{h})$, and *singular series* $\mathfrak{S}_s(\mathbf{h})$, are known to converge absolutely for $s > 7/2$, and $s > 4$, respectively (see [1, Theorem 1] or [2, Theorem 3.7]).

Theorem 1.1. *Suppose that $\mathbf{h} \in \mathbb{Z}^3$ and $h_1 \neq 0$. Let s be a natural number with $s \geq 6$. Then whenever X is sufficiently large in terms of s , one has*

$$B_s(X; \mathbf{h}) = \mathfrak{J}_s(\mathbf{h}) \mathfrak{S}_s(\mathbf{h}) X^{2s-6} + o(X^{2s-6}), \quad (1.8)$$

in which $0 \leq \mathfrak{J}_s(\mathbf{h}) \ll 1$ and $0 \leq \mathfrak{S}_s(\mathbf{h}) \ll 1$. If the system (1.3) possesses a non-singular real solution with positive coordinates, moreover, then $\mathfrak{J}_s(\mathbf{h}) \gg 1$. Likewise, if the system (1.3) possesses primitive non-singular p -adic solutions for each prime p , then $\mathfrak{S}_s(\mathbf{h}) \gg 1$.

Theorem 1.1 delivers a conclusion tantamount to a quantitative form of the Hasse principle for the system (1.3) at the convexity barrier when $s = 6$, and in that situation applies a minor arc estimate going beyond square-root cancellation. Thus, provided that the latter system admits appropriate non-singular solutions in every completion of \mathbb{Q} , one has an asymptotic formula of the shape $B_6(X; \mathbf{h}) \sim CX^6$ for a suitable positive number C . When $s \geq 7$, the conclusion of Theorem 1.1 is a routine consequence of the resolution [17] of the cubic case of the main conjecture in Vinogradov's mean value theorem, as the reader may confirm by applying the methods of Arkhipov [1]. Establishing such a conclusion when $s \leq 6$, however, requires something of a breakthrough in order that the familiar square-root barrier in the circle method be surmounted.

Brandes and Hughes [4] have recently investigated the inhomogeneous case of Vinogradov's mean value theorem of degree k in the subcritical regime. While this work shows, inter alia, that $B_s(X; \mathbf{h}) = o(X^s)$ for $s \leq 5$, their methods fall short of providing conclusions for the critical exponent $s = 6$ addressed by Theorem 1.1. We remark that quantitative aspects of their conclusions have been sharpened in work [19] of the author.

Theorem 1.1 addresses no scenario in which $h_1 = 0$. Although we are unable to obtain uniform conclusions in such a situation, we do obtain asymptotic formulae when $h_2 \neq 0$ and X is sufficiently large in terms of h_2 .

Theorem 1.2. *Let s be a natural number with $s \geq 6$. Then the asymptotic formula (1.8) holds when $h_2 \neq 0$ provided that X is sufficiently large in terms of s and h_2 .*

As a consequence of Fermat's theorem, the system (1.3) has solutions only when $h_3 \equiv h_1 \pmod{3}$ and $h_3 \equiv h_2 \equiv h_1 \pmod{2}$. Theorems 1.1 and 1.2 offer local-global principles incorporating such conditions. More significant is the proof of an asymptotic formula at the convexity barrier, wherein we have 12 variables available and the sum of the degrees of the underlying equations is 6. Hitherto, no such conclusion has been available for inhomogeneous Vinogradov systems of degree exceeding 2. Unfortunately, our methods yield no conclusion analogous to Theorem 1.1 for Vinogradov systems of degree exceeding 3.

With additional effort, the methods of this paper would deliver a refinement of the asymptotic formula (1.8) in which a power of X is saved in the error term. Thus, in the conclusions of both Theorem 1.1 and 1.2, it would be possible to obtain the asymptotic relation

$$B_s(X; \mathbf{h}) = \mathfrak{J}_s(\mathbf{h})\mathfrak{S}_s(\mathbf{h})X^{2s-6} + O(X^{2s-6-\delta}),$$

for a positive value of δ . Here, in the case of Theorem 1.1, it would be permissible to take any value of δ with $\delta < 1/12$. In the case of Theorem 1.2, meanwhile, so long as one makes accommodation for an implicit constant depending on h_2 , a modest refinement of our work would allow for any choice of δ with $\delta < 1/2$.

We prove Theorem 1.1 by applying the circle method, a key ingredient in our argument being an estimate for the contribution of the minor arcs beyond

square-root cancellation. This we achieve in §§2, 3 and 4 by adapting the author's work on the asymptotic formula in Waring's problem (see [15]). Ignoring for now the restriction to minor arcs, we observe that an integral shift z , with $1 \leq z \leq X$, in every variable in the system (1.3) generates the related system

$$\left. \begin{aligned} \sum_{i=1}^s (u_i^3 - v_i^3) &= h_3 + 3h_2z + 3h_1z^2 \\ \sum_{i=1}^s (u_i^2 - v_i^2) &= h_2 + 2h_1z \\ \sum_{i=1}^s (u_i - v_i) &= h_1 \end{aligned} \right\} \quad (1.9)$$

in which $1 \leq u_i, v_i \leq 2X$. There is now the potential for additional averaging using this new variable z . Were the polynomials on the right hand side of (1.9) to have respective degrees 3, 2 and 1, then an appropriate minor arc estimate would follow at once via Weyl's inequality. However, the degrees of the polynomials are too small for such a simple treatment to apply, and instead we must relate the system to auxiliary mixed systems. It is critical here that available Weyl estimates for cubic polynomials are relatively strong. Weaker estimates available for larger degrees are insufficient for our purposes. It is vital, moreover, that in these auxiliary mixed systems the degree of the polynomial $h_3 + 3h_2z + 3h_1z^2$ be at least 2. Indeed, were h_1 to be 0, the resulting linear polynomial would offer insufficient scope for obtaining minor arc estimates of sufficient strength for application in the proof of Theorem 1.1.

Having prepared the auxiliary lemma exploiting shifts in §2, we prepare in §3 the auxiliary mean value estimates required in §4 for the derivation of our basic minor arc estimate breaking the classical convexity barrier. In §5 we describe the Hardy-Littlewood dissection required in our proof of Theorem 1.1, and we reinterpret the conclusion of §4 as a minor arc estimate in a form convenient for the application at hand. Some pruning manoeuvres convert this bound into two estimates more classically associated with minor arcs in §6. From here, it remains in §7 to analyse the contribution of the major arcs, and thereby we complete the proof of Theorem 1.1 drawing heavily on the work of Arkhipov [1]. We devote §8 to the discussion of the scenario in which $h_1 = 0$, and the proof of Theorem 1.2. Here, at the cost of sacrificing uniformity with respect to \mathbf{h} in our conclusions, it transpires that one may make use of recent work on small cap decouplings [8] in order to salvage a viable analysis.

Our basic parameter is X , a sufficiently large positive number. Whenever ε appears in a statement, either implicitly or explicitly, we assert that the statement holds for each $\varepsilon > 0$. Implicit constants in Vinogradov's notation \ll and \gg may depend on ε . Vector notation in the form $\mathbf{x} = (x_1, \dots, x_r)$ is used with the dimension r depending on the course of the argument. Also, we write

(a_1, \dots, a_s) for the greatest common divisor of the integers a_1, \dots, a_s , ambiguity between ordered s -tuples and corresponding greatest common divisors being easily resolved by context. Finally, we write $\|\theta\|$ for $\min\{|\theta - m| : m \in \mathbb{Z}\}$.

The author is grateful to the referee for useful comments.

2. AN AUXILIARY MEAN VALUE ESTIMATE VIA SHIFTS

Our starting point is a method applied in the proof of [15, Theorem 2.1], whereby the latent translation-dilation invariance of the system (1.3) is applied to generate additional cancellation. Define $f(\boldsymbol{\alpha}; X)$ as in (1.1). Then, when $\mathbf{h} \in \mathbb{Z}^3$ and $\mathfrak{B} \subseteq \mathbb{R}$ is measurable, we put

$$I_s(\mathfrak{B}; X; \mathbf{h}) = \int_{\mathfrak{B}} \int_0^1 \int_0^1 |f(\boldsymbol{\alpha}; X)|^{2s} e(-\boldsymbol{\alpha} \cdot \mathbf{h}) \, d\boldsymbol{\alpha}, \quad (2.1)$$

in which $\boldsymbol{\alpha} \cdot \mathbf{h} = \alpha_1 h_1 + \alpha_2 h_2 + \alpha_3 h_3$ and $d\boldsymbol{\alpha}$ denotes $d\alpha_1 d\alpha_2 d\alpha_3$. Thus, in particular, we find from (1.2) that $I_s([0, 1]; X; \mathbf{h}) = B_s(X; \mathbf{h})$. We also make use of the auxiliary generating function

$$g(\boldsymbol{\alpha}, \theta; X) = \sum_{1 \leq y \leq X} e(y\theta + 2h_1 y \alpha_2 + (3h_2 y + 3h_1 y^2) \alpha_3). \quad (2.2)$$

Lemma 2.1. *Suppose that $s \in \mathbb{N}$, $\mathbf{h} \in \mathbb{Z}^3$ and $\mathfrak{B} \subseteq \mathbb{R}$ is measurable. Then*

$$I_s(\mathfrak{B}; X; \mathbf{h}) \ll X^{-1} (\log X)^{2s} \sup_{\Gamma \in [0, 1]} \int_{\mathfrak{B}} \int_0^1 \int_0^1 |f(\boldsymbol{\alpha}; 2X)^{2s} g(\boldsymbol{\alpha}, \Gamma; X)| \, d\boldsymbol{\alpha}.$$

Proof. For every integral shift y with $1 \leq y \leq X$, one has

$$f(\boldsymbol{\alpha}; X) = \sum_{1+y \leq x \leq X+y} e(\alpha_3(x-y)^3 + \alpha_2(x-y)^2 + \alpha_1(x-y)). \quad (2.3)$$

Write

$$\mathfrak{f}_y(\boldsymbol{\alpha}; \gamma) = \sum_{1 \leq x \leq 2X} e(\alpha_3(x-y)^3 + \alpha_2(x-y)^2 + (\alpha_1 + \gamma)(x-y)) \quad (2.4)$$

and

$$K(\gamma) = \sum_{1 \leq z \leq X} e(-\gamma z).$$

Then it follows from (2.3) via orthogonality that when $1 \leq y \leq X$, one has

$$f(\boldsymbol{\alpha}; X) = \int_0^1 \mathfrak{f}_y(\boldsymbol{\alpha}; \gamma) K(\gamma) \, d\gamma. \quad (2.5)$$

Next we substitute (2.5) into (2.1). Define

$$\mathfrak{F}_y(\boldsymbol{\alpha}; \boldsymbol{\gamma}) = \prod_{i=1}^s \mathfrak{f}_y(\boldsymbol{\alpha}; \gamma_i) \mathfrak{f}_y(-\boldsymbol{\alpha}; -\gamma_{s+i}), \quad (2.6)$$

$$\tilde{K}(\boldsymbol{\gamma}) = \prod_{i=1}^s K(\gamma_i) K(-\gamma_{s+i})$$

and

$$\mathcal{I}(\boldsymbol{\gamma}; y; \mathbf{h}) = \int_{\mathfrak{B}} \int_0^1 \int_0^1 \mathfrak{F}_y(\boldsymbol{\alpha}; \boldsymbol{\gamma}) e(-\boldsymbol{\alpha} \cdot \mathbf{h}) \, d\boldsymbol{\alpha}. \quad (2.7)$$

Then, when $1 \leq y \leq X$, we see that

$$I_s(\mathfrak{B}; X; \mathbf{h}) = \int_{[0,1]^{2s}} \mathcal{I}(\boldsymbol{\gamma}; y; \mathbf{h}) \tilde{K}(\boldsymbol{\gamma}) \, d\boldsymbol{\gamma}. \quad (2.8)$$

By orthogonality, it is apparent from (2.6) that

$$\int_0^1 \int_0^1 \mathfrak{F}_y(\boldsymbol{\alpha}; \boldsymbol{\gamma}) e(-\boldsymbol{\alpha} \cdot \mathbf{h}) \, d\alpha_1 \, d\alpha_2 = \sum_{1 \leq \mathbf{x} \leq 2X} \Delta(\alpha_3, \boldsymbol{\gamma}; \mathbf{h}, y), \quad (2.9)$$

where $\Delta(\alpha_3, \boldsymbol{\gamma}; \mathbf{h}, y)$ is equal to

$$e\left(\sum_{i=1}^s (\alpha_3 ((x_i - y)^3 - (x_{s+i} - y)^3) + (\gamma_i(x_i - y) - \gamma_{s+i}(x_{s+i} - y))) - \alpha_3 h_3\right),$$

when

$$\sum_{i=1}^s ((x_i - y)^j - (x_{s+i} - y)^j) = h_j \quad (j = 1, 2), \quad (2.10)$$

and otherwise $\Delta(\alpha_3, \boldsymbol{\gamma}; \mathbf{h}, y)$ is equal to 0.

By applying the binomial theorem within (2.10), we obtain the relations

$$\begin{aligned} \sum_{i=1}^s (x_i - x_{s+i}) &= h_1, \\ \sum_{i=1}^s (x_i^2 - x_{s+i}^2) &= h_2 + 2yh_1, \\ \sum_{i=1}^s (x_i^3 - x_{s+i}^3) &= 3yh_2 + 3y^2h_1 + \sum_{i=1}^s ((x_i - y)^3 - (x_{s+i} - y)^3). \end{aligned}$$

Therefore, if we define $\Gamma = \Gamma(\boldsymbol{\gamma})$ by taking

$$\Gamma(\boldsymbol{\gamma}) = \sum_{i=1}^s (\gamma_i - \gamma_{s+i}),$$

and then write

$$\mathfrak{g}_y(\boldsymbol{\alpha}; \mathbf{h}; \boldsymbol{\gamma}) = e\left(-\sum_{j=1}^3 \alpha_j \sum_{l=0}^{j-1} \binom{j}{l} h_{j-l} y^l - y\Gamma(\boldsymbol{\gamma})\right),$$

then we deduce from (2.6) and (2.9) that

$$\int_0^1 \int_0^1 \mathfrak{F}_y(\boldsymbol{\alpha}; \boldsymbol{\gamma}) e(-\boldsymbol{\alpha} \cdot \mathbf{h}) \, d\alpha_1 \, d\alpha_2 = \int_0^1 \int_0^1 \mathfrak{F}_0(\boldsymbol{\alpha}; \boldsymbol{\gamma}) \mathfrak{g}_y(\boldsymbol{\alpha}; \mathbf{h}; \boldsymbol{\gamma}) \, d\alpha_1 \, d\alpha_2. \quad (2.11)$$

Referring next to (2.8), we see that when $X \in \mathbb{N}$ one obtains the relation

$$I_s(\mathfrak{B}; X; \mathbf{h}) = X^{-1} \sum_{1 \leq y \leq X} \int_{[0,1]^{2s}} \mathcal{I}(\gamma; y; \mathbf{h}) \tilde{K}(\gamma) d\gamma.$$

Thus, we infer from (2.7) and (2.11) that

$$I_s(\mathfrak{B}; X; \mathbf{h}) \ll X^{-1} \int_{[0,1]^{2s}} |H(\gamma) \tilde{K}(\gamma)| d\gamma, \quad (2.12)$$

where

$$H(\gamma) = \int_{\mathfrak{B}} \int_0^1 \int_0^1 \mathfrak{F}_0(\boldsymbol{\alpha}; \gamma) G(\boldsymbol{\alpha}; \mathbf{h}; \gamma) d\boldsymbol{\alpha}, \quad (2.13)$$

and

$$G(\boldsymbol{\alpha}; \mathbf{h}; \gamma) = \sum_{1 \leq y \leq X} \mathfrak{g}_y(\boldsymbol{\alpha}; \mathbf{h}; \gamma). \quad (2.14)$$

We now aim to simplify the upper bound (2.12). Observe first that by reference to (2.6), it follows from the elementary inequality

$$|z_1 \cdots z_n| \leq |z_1|^n + \dots + |z_n|^n$$

that

$$|\mathfrak{F}_0(\boldsymbol{\alpha}; \gamma)| \leq \sum_{i=1}^{2s} |\mathfrak{f}_0(\boldsymbol{\alpha}; \gamma_i)|^{2s} = \sum_{i=1}^{2s} |\mathfrak{f}_0(\alpha_3, \alpha_2, \alpha_1 + \gamma_i; 0)|^{2s}.$$

Also, from (2.2) and (2.14) we have

$$\begin{aligned} |G(\boldsymbol{\alpha}; \mathbf{h}; \gamma)| &= \left| \sum_{1 \leq y \leq X} e(y\Gamma(\gamma) + 2h_1y\alpha_2 + (3h_2y + 3h_1y^2)\alpha_3) \right| \\ &= |g(\boldsymbol{\alpha}, \Gamma(\gamma); X)|. \end{aligned}$$

Therefore, since $|G(\boldsymbol{\alpha}; \mathbf{h}; \gamma)|$ does not depend on α_1 , it follows from (2.13) via a change of variable that

$$|H(\gamma)| \ll \int_{\mathfrak{B}} \int_0^1 \int_0^1 |\mathfrak{f}_0(\boldsymbol{\alpha}; 0)^{2s} g(\boldsymbol{\alpha}, \Gamma(\gamma); X)| d\boldsymbol{\alpha}. \quad (2.15)$$

Define

$$U_s(\mathfrak{B}) = \sup_{\Gamma \in [0,1]} \int_{\mathfrak{B}} \int_0^1 \int_0^1 |f(\boldsymbol{\alpha}; 2X)^{2s} g(\boldsymbol{\alpha}, \Gamma; X)| d\boldsymbol{\alpha}.$$

Also, recall that

$$\int_0^1 |K(\gamma)| d\gamma \ll \int_0^1 \min\{X, \|\gamma\|^{-1}\} d\gamma \ll \log(2X),$$

and note from (1.1) and (2.4) that $\mathfrak{f}_0(\boldsymbol{\alpha}; 0) = f(\boldsymbol{\alpha}; 2X)$. Then we deduce from (2.15) that $|H(\gamma)| \leq U_s(\mathfrak{B})$, and hence (2.12) yields the bound

$$\begin{aligned} I_s(\mathfrak{B}; X; \mathbf{h}) &\ll X^{-1} U_s(\mathfrak{B}) \left(\int_0^1 |K(\gamma)| d\gamma \right)^{2s} \\ &\ll X^{-1} (\log(2X))^{2s} U_s(\mathfrak{B}). \end{aligned}$$

This completes the proof of the lemma. \square

3. FURTHER AUXILIARY MEAN VALUE ESTIMATES

We now prepare mean value estimates of use in bounding a minor arc contribution of utility in an application of the Hardy-Littlewood method. Recalling the exponential sum $g(\boldsymbol{\alpha}, \theta; X)$ defined in (2.2), and writing $g(\boldsymbol{\alpha}; X) = g(\boldsymbol{\alpha}, 0; X)$, these mixed mean values take the shape

$$\Theta_m(X; \mathbf{h}) = \int_{[0,1]^3} |f(\boldsymbol{\alpha}; 2X)^{2m} g(\boldsymbol{\alpha}; X)^6| d\boldsymbol{\alpha} \quad (m \in \mathbb{N}). \quad (3.1)$$

Lemma 3.1. *When $\mathbf{h} \in \mathbb{Z}^3$ and $h_1 \neq 0$, one has $\Theta_1(X; \mathbf{h}) \ll X^4 \log(2X)$.*

Proof. By orthogonality, one has

$$\int_0^1 |f(\boldsymbol{\alpha}; 2X)|^2 d\alpha_1 \leq 2X.$$

Then since $g(\boldsymbol{\alpha}; X)$ is independent of α_1 , we deduce from (3.1) that

$$\Theta_1(X; \mathbf{h}) \leq 2X \int_{[0,1]^2} |g(0, \alpha_2, \alpha_3; X)|^6 d\alpha_2 d\alpha_3. \quad (3.2)$$

A second application of orthogonality reveals that the integral on the right hand side here counts the number of integral solutions $T_0(X)$ of the system

$$\begin{aligned} 3h_1 \sum_{i=1}^3 (x_i^2 - y_i^2) + 3h_2 \sum_{i=1}^3 (x_i - y_i) &= 0, \\ 2h_1 \sum_{i=1}^3 (x_i - y_i) &= 0, \end{aligned}$$

with $1 \leq x_i, y_i \leq X$ ($1 \leq i \leq 3$). Since, by hypothesis, one has $h_1 \neq 0$, we see that $T_0(X)$ counts the integral solutions of the Vinogradov system of equations

$$\sum_{i=1}^3 (x_i^j - y_i^j) = 0 \quad (j = 1, 2),$$

with the same conditions on \mathbf{x} and \mathbf{y} . Thus $T_0(X) \ll X^3 \log(2X)$ (a precise asymptotic formula can be found in [3]), and the conclusion of the lemma follows by substituting this upper bound into (3.2). \square

We next consider mean values in which the multiplicity of the generating functions $f(\boldsymbol{\alpha}; 2X)$ is increased by appealing to the Hardy-Littlewood method. With this goal in mind, we introduce a Hardy-Littlewood dissection. When Q is a real parameter with $1 \leq Q \leq X$, we define the set of major arcs $\mathfrak{M}(Q)$ to be the union of the arcs

$$\mathfrak{M}(q, a) = \{\alpha \in [0, 1) : |q\alpha - a| \leq QX^{-3}\},$$

with $0 \leq a \leq q \leq Q$ and $(a, q) = 1$. We then define the complementary set of minor arcs $\mathfrak{m}(Q) = [0, 1) \setminus \mathfrak{M}(Q)$.

We begin with a familiar auxiliary bound for $f(\boldsymbol{\alpha}; 2X)$. In this context, it is useful to define the function $\Psi(\alpha)$ for $\alpha \in [0, 1)$ by putting

$$\Psi(\alpha) = (q + X^3|q\alpha - a|)^{-1},$$

when $\alpha \in \mathfrak{M}(q, a) \subseteq \mathfrak{M}(X)$, and otherwise by taking $\Psi(\alpha) = 0$.

Lemma 3.2. *One has $f(\boldsymbol{\alpha}; 2X)^4 \ll X^{3+\varepsilon} + X^{4+\varepsilon}\Psi(\alpha_3)$.*

Proof. Suppose that $\alpha_3 \in \mathbb{R}$, and $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a, q) = 1$ and $|\alpha_3 - a/q| \leq q^{-2}$. Then from Weyl's inequality (see [13, Lemma 2.4]), we have

$$|f(\boldsymbol{\alpha}; 2X)| \ll X^{1+\varepsilon}(q^{-1} + X^{-1} + qX^{-3})^{1/4}. \quad (3.3)$$

Hence, by a standard transference principle (see [16, Lemma 14.1]), whenever $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a, q) = 1$, one has

$$|f(\boldsymbol{\alpha}; 2X)| \ll X^{1+\varepsilon}(\lambda^{-1} + X^{-1} + \lambda X^{-3})^{1/4}, \quad (3.4)$$

where $\lambda = q + X^3|q\alpha_3 - a|$.

When $\alpha_3 \in [0, 1)$, an application of Dirichlet's approximation theorem shows that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $0 \leq a \leq q \leq X^2$, $(a, q) = 1$ and $|q\alpha_3 - a| \leq X^{-2}$. Thus $\lambda = q + X^3|q\alpha_3 - a| \ll X^2$. Note that when $\alpha_3 \in \mathfrak{m}(X)$ we must have $\lambda \geq q > X$, whilst for $\alpha_3 \in \mathfrak{M}(X)$ one has $\lambda^{-1} = \Psi(\alpha_3)$. Then in any case we find from (3.4) that

$$|f(\boldsymbol{\alpha}; 2X)|^4 \ll X^{3+\varepsilon} + X^{4+\varepsilon}\lambda^{-1} \ll X^{3+\varepsilon} + X^{4+\varepsilon}\Psi(\alpha_3),$$

and the conclusion of the lemma follows. \square

Lemma 3.3. *Suppose that $\mathbf{h} \in \mathbb{Z}^3$ and $h_1 \neq 0$. Then one has*

$$\Theta_3(X; \mathbf{h}) \ll X^{7+\varepsilon} \quad \text{and} \quad \Theta_5(X; \mathbf{h}) \ll X^{10+\varepsilon}.$$

Proof. By applying Lemma 3.2 to (3.1), one obtains

$$\Theta_3(X; \mathbf{h}) \ll X^{3+\varepsilon}\Theta_1(X; \mathbf{h}) + X^{4+\varepsilon}T_1, \quad (3.5)$$

where

$$T_1 = \int_{[0,1]^3} \Psi(\alpha_3)|f(\boldsymbol{\alpha}; 2X)^2g(\boldsymbol{\alpha}; X)^6| d\boldsymbol{\alpha}.$$

Moreover, as a consequence of [5, Lemma 2], we have

$$T_1 \ll X^{\varepsilon-3}(X\Theta_1(X; \mathbf{h}) + T_2), \quad (3.6)$$

where

$$T_2 = \int_{[0,1]^2} |f(\alpha_1, \alpha_2, 0; 2X)^2g(\alpha_1, \alpha_2, 0; X)^6| d\alpha_1 d\alpha_2.$$

By orthogonality, the mean value T_2 counts the integral solutions of the simultaneous equations

$$\begin{aligned} x_1^2 - x_2^2 &= 2h_1(y_1 + y_2 + y_3 - y_4 - y_5 - y_6), \\ x_1 - x_2 &= 0, \end{aligned}$$

with $1 \leq x_1, x_2 \leq 2X$ and $1 \leq y_i \leq X$ ($1 \leq i \leq 6$). Plainly, in any such solution one has $x_1 = x_2$, and so there are at most $O(X)$ possible choices for x_1 and x_2 . Meanwhile, given y_1, \dots, y_5 , the variable y_6 is determined uniquely

from the first of these equations, so there are $O(X^5)$ possible choices for \mathbf{y} . We therefore see that $T_2 = O(X^6)$, and hence (3.6) delivers the bound

$$T_1 \ll X^{\varepsilon-2} \Theta_1(X; \mathbf{h}) + X^{3+\varepsilon}.$$

The first bound of the lemma follows by substituting this estimate into (3.5), noting the bound $\Theta_1(X; \mathbf{h}) \ll X^{4+\varepsilon}$ available from Lemma 3.1.

The second bound of the lemma is obtained by applying Lemma 3.2 to (3.1) again, yielding

$$\Theta_5(X; \mathbf{h}) \ll X^{3+\varepsilon} \Theta_3(X; \mathbf{h}) + X^{4+\varepsilon} T_3, \quad (3.7)$$

where

$$T_3 = \int_{[0,1]^3} \Psi(\alpha_3) |f(\boldsymbol{\alpha}; 2X)^6 g(\boldsymbol{\alpha}; X)^6| d\boldsymbol{\alpha}.$$

Again utilising [5, Lemma 2], we deduce that

$$T_3 \ll X^{\varepsilon-3} (X \Theta_3(X; \mathbf{h}) + T_4), \quad (3.8)$$

where

$$T_4 = \int_{[0,1]^2} |f(\alpha_1, \alpha_2, 0; 2X)^6 g(\alpha_2, \alpha_2, 0; X)^6| d\alpha_1 d\alpha_2.$$

By applying the trivial estimate $|g(\alpha_1, \alpha_2, 0; X)| = O(X)$, we see that

$$T_4 \ll X^6 \int_{[0,1]^2} |f(\alpha_1, \alpha_2, 0; 2X)|^6 d\alpha_1 d\alpha_2 \ll X^6 \cdot X^{3+\varepsilon}.$$

Here, applying orthogonality, we recognised that the last integral is equal to our acquaintance $T_0(2X)$ introduced in the proof of Lemma 3.1, and shown therein to be $O(X^{3+\varepsilon})$. We thus deduce from (3.8) that

$$T_3 \ll X^{\varepsilon-2} \Theta_3(X; \mathbf{h}) + X^{6+\varepsilon}.$$

The second bound of the lemma follows by substituting this estimate into (3.7), noting the first bound $\Theta_3(X; \mathbf{h}) \ll X^{7+\varepsilon}$ already obtained. \square

We convert the second bound of Lemma 3.3 into one suitable for later use. In this context, it is convenient to introduce the mean value

$$V(\Gamma; X; \mathbf{h}) = \int_{[0,1]^3} |f(\boldsymbol{\alpha}; 2X)^{10} g(\boldsymbol{\alpha}, \Gamma; X)^6| d\boldsymbol{\alpha}.$$

Lemma 3.4. *Suppose that $\mathbf{h} \in \mathbb{Z}^3$ and $h_1 \neq 0$. Then one has*

$$\sup_{\Gamma \in [0,1]} V(\Gamma; X; \mathbf{h}) \ll X^{10+\varepsilon}.$$

Proof. By orthogonality, the mean value $V(\Gamma; X; \mathbf{h})$ counts the integral solutions of the system

$$\begin{aligned} \sum_{i=1}^5 (x_i^3 - y_i^3) &= 3h_2 \sum_{j=1}^3 (u_j - v_j) + 3h_1 \sum_{j=1}^3 (u_j^2 - v_j^2) \\ \sum_{i=1}^5 (x_i^2 - y_i^2) &= 2h_1 \sum_{j=1}^3 (u_j - v_j) \\ \sum_{i=1}^5 (x_i - y_i) &= 0, \end{aligned}$$

with $1 \leq x_i, y_i \leq 2X$ ($1 \leq i \leq 5$) and $1 \leq u_j, v_j \leq X$ ($1 \leq j \leq 3$), and with each solution $\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}$ being counted with weight

$$e(-\Gamma(u_1 + u_2 + u_3 - v_1 - v_2 - v_3)).$$

Since the latter weight is unimodular, we obtain an upper bound for $V(\Gamma; X; \mathbf{h})$ by replacing that weight with 1, or equivalently, by setting Γ to be 0. Thus, by reference to (3.1), we conclude that

$$\sup_{\Gamma \in [0,1]} V(\Gamma; X; \mathbf{h}) \leq V(0; X; \mathbf{h}) = \Theta_5(X; \mathbf{h}).$$

The conclusion of the lemma is therefore immediate from Lemma 3.3. \square

4. A FIRST MINOR ARC BOUND

Before announcing our first estimate of minor arc type, we recall a standard consequence of Weyl's inequality. Recall the set of minor arcs $\mathbf{m}(Q)$ defined in the preamble to Lemma 3.2, and suppose that $\alpha_3 \in \mathbf{m}(Q)$. By Dirichlet's approximation theorem, there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $0 \leq a \leq q \leq Q^{-1}X^3$, $(a, q) = 1$ and $|q\alpha_3 - a| \leq QX^{-3}$. Since $\alpha_3 \in \mathbf{m}(Q)$ one has $q > Q$, and thus we deduce from Weyl's inequality (3.3) that

$$\sup_{\alpha_3 \in \mathbf{m}(Q)} \sup_{(\alpha_1, \alpha_2) \in [0,1]^2} |f(\boldsymbol{\alpha}; 2X)| \ll X^{1+\varepsilon} Q^{-1/4}. \quad (4.1)$$

Lemma 4.1. *Let s be a natural number with $s \geq 6$, and put $\delta = 2s - 35/3$. Then whenever $\mathbf{h} \in \mathbb{Z}^3$ and $h_1 \neq 0$, one has*

$$I_s(\mathbf{m}(Q); X; \mathbf{h}) \ll X^{2s-6+\varepsilon} Q^{-\delta/4}.$$

Proof. We find from Lemma 2.1 that

$$I_s(\mathbf{m}(Q); X; \mathbf{h}) \ll X^{\varepsilon-1} \sup_{\Gamma \in [0,1]} W_s(\Gamma; X; \mathbf{h}), \quad (4.2)$$

where

$$W_s(\Gamma; X; \mathbf{h}) = \int_{\mathbf{m}(Q)} \int_0^1 \int_0^1 |f(\boldsymbol{\alpha}; 2X)^{2s} g(\boldsymbol{\alpha}, \Gamma; X)| d\boldsymbol{\alpha}.$$

An application of Hölder's inequality reveals that whenever $s \geq 6$, one has

$$W_s(\Gamma; X; \mathbf{h}) \leq \left(\sup_{\alpha_3 \in \mathfrak{m}(Q)} \sup_{(\alpha_1, \alpha_2) \in [0,1]^2} |f(\boldsymbol{\alpha}; 2X)| \right)^{2s-35/3} U_1^{5/6} U_2^{1/6}, \quad (4.3)$$

in which we write

$$U_1 = \int_{[0,1]^3} |f(\boldsymbol{\alpha}; 2X)|^{12} d\boldsymbol{\alpha} \quad \text{and} \quad U_2 = \int_{[0,1]^3} |f(\boldsymbol{\alpha}; 2X)|^{10} |g(\boldsymbol{\alpha}, \Gamma; X)|^6 d\boldsymbol{\alpha}.$$

The cubic case of the main conjecture in Vinogradov's mean value theorem established by the author [17] shows that $U_1 \ll X^{6+\varepsilon}$. Meanwhile, the bound $U_2 \ll X^{10+\varepsilon}$ is confirmed in Lemma 3.4. By substituting these bounds together with (4.1) into (4.3), we obtain the estimate

$$W_s(\Gamma; X; \mathbf{h}) \ll X^\varepsilon (XQ^{-1/4})^{2s-35/3} (X^6)^{5/6} (X^{10})^{1/6} \ll X^{2s-5+\varepsilon} Q^{-\delta/4}.$$

The conclusion of the lemma follows by substituting this bound into (4.2). \square

5. THE HARDY-LITTLEWOOD DISSECTION

Our application of the Hardy-Littlewood method follows the strategy pursued in our recent work on the Hilbert-Kamke problem (see [20]), though equipped in this instance with the minor arc estimate prepared in §4. We begin our discussion by introducing a close relative of the mean value $I_s(\mathfrak{B}; X; \mathbf{h})$ introduced in (2.1). Thus, when $\mathfrak{A} \subseteq [0, 1]^3$ is measurable, we define the mean value $T_s(\mathfrak{A}; \mathbf{h}) = T_s(\mathfrak{A}; X; \mathbf{h})$ by putting

$$T_s(\mathfrak{A}; X; \mathbf{h}) = \int_{\mathfrak{A}} |f(\boldsymbol{\alpha}; X)|^{2s} e(-\boldsymbol{\alpha} \cdot \mathbf{h}) d\boldsymbol{\alpha}. \quad (5.1)$$

We require an appropriate Hardy-Littlewood dissection of the unit cube $[0, 1]^3$ into major and minor arcs. When Z is a real parameter with $1 \leq Z \leq X$, we define the set of major arcs $\mathfrak{K}(Z)$ to be the union of the arcs

$$\mathfrak{K}(q, \mathbf{a}; Z) = \{\boldsymbol{\alpha} \in [0, 1]^3 : |\alpha_j - a_j/q| \leq ZX^{-j} \ (1 \leq j \leq 3)\},$$

with $1 \leq q \leq Z$, $0 \leq a_j \leq q$ ($1 \leq j \leq 3$) and $(q, a_1, a_2, a_3) = 1$. We then define the complementary set of minor arcs $\mathfrak{k}(Z) = [0, 1]^3 \setminus \mathfrak{K}(Z)$.

We have already defined the one-dimensional Hardy-Littlewood dissection of $[0, 1]$ into the sets of arcs $\mathfrak{M}(Q)$ and $\mathfrak{m}(Q)$. We now fix $L = X^{1/72}$ and $Q = L^3$, and abbreviate our notation by writing $\mathfrak{M} = \mathfrak{M}(Q)$ and $\mathfrak{m} = \mathfrak{m}(Q)$. Next, we define intermediate sets of 3-dimensional arcs $\mathfrak{N} = \mathfrak{K}(Q^2)$ and $\mathfrak{n} = \mathfrak{k}(Q^2)$. We also need a narrow set of major arcs $\mathfrak{P} = \mathfrak{K}(L)$ and a corresponding set of minor arcs $\mathfrak{p} = \mathfrak{k}(L)$. It is convenient, in this context, to write $\mathfrak{P}(q, \mathbf{a}) = \mathfrak{K}(q, \mathbf{a}; L)$. As is easily verified, one has $\mathfrak{P} \subseteq [0, 1]^2 \times \mathfrak{M}$. Hence, the set of points $(\alpha_1, \alpha_2, \alpha_3)$

lying in $[0, 1]^3$ may be partitioned into the four disjoint subsets

$$\begin{aligned}\mathfrak{W}_1 &= [0, 1]^2 \times \mathfrak{m}, \\ \mathfrak{W}_2 &= ([0, 1]^2 \times \mathfrak{M}) \cap \mathfrak{n}, \\ \mathfrak{W}_3 &= ([0, 1]^2 \times \mathfrak{M}) \cap (\mathfrak{N} \setminus \mathfrak{P}), \\ \mathfrak{W}_4 &= \mathfrak{P}.\end{aligned}$$

Thus, on comparing (1.2) and (5.1), we find that

$$B_s(X; \mathbf{h}) = T_s([0, 1]^3; X; \mathbf{h}) = \sum_{i=1}^4 T_s(\mathfrak{W}_i; X; \mathbf{h}). \quad (5.2)$$

Our work in §§2, 3 and 4 bounds $T_s(\mathfrak{W}_1; \mathbf{h})$.

Lemma 5.1. *Let s be a positive integer with $s \geq 6$, and put $\delta = 2s - 35/3$. Then, whenever $\mathbf{h} \in \mathbb{Z}^3$ and $h_1 \neq 0$, one has*

$$T_s(\mathfrak{W}_1; \mathbf{h}) \ll X^{2s-6-\delta/100}.$$

Proof. By substituting $Q = X^{1/24}$ into Lemma 4.1, we find that

$$T_s(\mathfrak{W}_1; \mathbf{h}) = I_s(\mathfrak{m}(Q); X; \mathbf{h}) \ll X^{2s-6+\varepsilon} \cdot X^{-\delta/96},$$

and the conclusion of the lemma follows at once. \square

6. FURTHER MINOR ARC ESTIMATES

Our analysis of the sets of arcs \mathfrak{W}_2 and \mathfrak{W}_3 within (5.2) involves standard tools from the theory of Vinogradov's mean value theorem. We begin by recording an estimate of Weyl type for the exponential sum $f(\boldsymbol{\alpha}; X)$.

Lemma 6.1. *One has*

$$\sup_{\boldsymbol{\alpha} \in \mathfrak{n}} |f(\boldsymbol{\alpha}; X)| \ll X^{1-1/54} \quad \text{and} \quad \sup_{\boldsymbol{\alpha} \in \mathfrak{p}} |f(\boldsymbol{\alpha}; X)| \ll X^{1-1/324}.$$

Proof. This is the case $k = 3$ of [20, Lemma 4.1]. \square

Lemma 6.2. *Suppose that $\mathbf{h} \in \mathbb{Z}^3$ and $2s \geq 11$. Then one has*

$$T_s(\mathfrak{W}_2; \mathbf{h}) \ll X^{2s-6-1/110}.$$

Proof. By applying the triangle inequality to (5.1), we find that

$$T_s(\mathfrak{W}_2; \mathbf{h}) \ll \left(\sup_{\boldsymbol{\alpha} \in \mathfrak{n}} |f(\boldsymbol{\alpha}; X)| \right)^{2s-6} \int_{\mathfrak{M}} \int_0^1 \int_0^1 |f(\boldsymbol{\alpha}; X)|^6 d\boldsymbol{\alpha}. \quad (6.1)$$

Here, by orthogonality, the inner mean value

$$\int_{[0,1]^2} |f(\boldsymbol{\alpha}; X)|^6 d\alpha_1 d\alpha_2$$

counts the integral solutions of the system of equations

$$\sum_{i=1}^3 (x_i^j - y_i^j) = 0 \quad (j = 1, 2),$$

with $1 \leq x_i, y_i \leq X$ ($1 \leq i \leq 3$), and with each solution \mathbf{x}, \mathbf{y} being counted with the unimodular weight

$$e\left(\alpha_3 \sum_{i=1}^3 (x_i^3 - y_i^3)\right).$$

Making use of the familiar bound in the quadratic case of Vinogradov's mean value theorem, and observing that $\text{mes}(\mathfrak{M}) \ll Q^2 X^{-3}$, we thus conclude that

$$\int_{\mathfrak{M}} \int_0^1 \int_0^1 |f(\boldsymbol{\alpha}; X)|^6 d\boldsymbol{\alpha} \ll X^{3+\varepsilon} \text{mes}(\mathfrak{M}) \ll Q^2 X^\varepsilon. \quad (6.2)$$

Substituting the bound (6.2) into (6.1), and invoking Lemma 6.1, we obtain

$$T_s(\mathfrak{W}_2; \mathbf{h}) \ll (X^{1-1/54})^{2s-6} Q^2 X^\varepsilon \ll X^{2s-6} (X^{\varepsilon-5/54} Q^2).$$

Since $Q = X^{1/24}$, the conclusion of the lemma follows at once. \square

The analysis of the set of arcs \mathfrak{W}_3 is accomplished via the standard literature.

Lemma 6.3. *When $u > 8$, one has*

$$\int_{\mathfrak{N}} |f(\boldsymbol{\alpha}; X)|^u d\boldsymbol{\alpha} \ll_u X^{u-6}.$$

Proof. This is essentially [18, Lemma 7.1], following by the methods of [2]. \square

We may now announce our estimate for $T_s(\mathfrak{W}_3; \mathbf{h})$.

Lemma 6.4. *Suppose that $\mathbf{h} \in \mathbb{Z}^3$ and $s \geq 5$. Then one has*

$$T_s(\mathfrak{W}_3; \mathbf{h}) \ll X^{2s-6-1/324}.$$

Proof. Since $\mathfrak{W}_3 \subseteq \mathfrak{N} \setminus \mathfrak{P}$, we have

$$\sup_{\boldsymbol{\alpha} \in \mathfrak{W}_3} |f(\boldsymbol{\alpha}; X)| \leq \sup_{\boldsymbol{\alpha} \in \mathfrak{P}} |f(\boldsymbol{\alpha}; X)|.$$

Thus, by the triangle inequality,

$$T_s(\mathfrak{W}_3; \mathbf{h}) \ll X^{2s-10} \left(\sup_{\boldsymbol{\alpha} \in \mathfrak{P}} |f(\boldsymbol{\alpha}; X)| \right) \int_{\mathfrak{N}} |f(\boldsymbol{\alpha}; X)|^9 d\boldsymbol{\alpha}.$$

Then as a consequence of Lemmata 6.1 and 6.3, we obtain the bound

$$T_s(\mathfrak{W}_3; \mathbf{h}) \ll X^{2s-10} \cdot X^{1-1/324} \cdot X^3 \ll X^{2s-6-1/324}.$$

This completes the proof of the lemma. \square

7. THE MAJOR ARC CONTRIBUTION

By substituting the estimates supplied by Lemmata 5.1, 6.2 and 6.4 into (5.2), noting also that $\mathfrak{W}_4 = \mathfrak{P}$, we find that whenever $s \geq 6$, one has

$$B_s(X; \mathbf{h}) = T_s(\mathfrak{P}; \mathbf{h}) + o(X^{2s-6}). \quad (7.1)$$

In this section we analyse the major arc contribution $T_s(\mathfrak{P}; \mathbf{h})$. This is routine, though the small number of available variables requires appropriate recourse to the literature.

Recall the notation (1.4) and (1.5). When $\alpha \in \mathfrak{P}(q, \mathbf{a}) \subseteq \mathfrak{P}$, put

$$V(\alpha; q, \mathbf{a}) = q^{-1}S(q, \mathbf{a})I(\alpha - \mathbf{a}/q; X),$$

where we write

$$I(\beta; X) = \int_0^X e(\beta_1\gamma + \beta_2\gamma^2 + \beta_3\gamma^3) d\gamma = XI(\beta_1X, \beta_2X^2, \beta_3X^3).$$

We then define the function $V(\alpha)$ to be $V(\alpha; q, \mathbf{a})$ when $\alpha \in \mathfrak{P}(q, \mathbf{a}) \subseteq \mathfrak{P}$, and to be zero otherwise. It now follows from [13, Theorem 7.2] that when $\alpha \in \mathfrak{P}(q, \mathbf{a}) \subseteq \mathfrak{P}$, one has

$$\begin{aligned} f(\alpha; X) - V(\alpha) &\ll q + X|q\alpha_1 - a_1| + X^2|q\alpha_2 - a_2| + X^3|q\alpha_3 - a_3| \\ &\ll L^2. \end{aligned}$$

Thus, uniformly in $\alpha \in \mathfrak{P}$, we have

$$|f(\alpha; X)|^{2s} - |V(\alpha)|^{2s} \ll X^{2s-1}L^2.$$

Since $\text{mes}(\mathfrak{P}) \ll L^7X^{-6}$, we deduce from (5.1) that

$$T_s(\mathfrak{P}; \mathbf{h}) = \int_{\mathfrak{P}} |V(\alpha)|^{2s} e(-\alpha \cdot \mathbf{h}) d\alpha + O(L^9X^{2s-7}). \quad (7.2)$$

Next, applying the definition of \mathfrak{P} in the familiar manner, we see that

$$\int_{\mathfrak{P}} |V(\alpha)|^{2s} e(-\alpha \cdot \mathbf{h}) d\alpha = \mathfrak{S}_s(X; \mathbf{h}) \mathfrak{J}_s(X; \mathbf{h}), \quad (7.3)$$

where

$$\mathfrak{J}_s(X; \mathbf{h}) = \int_{\mathfrak{X}} |I(\beta; X)|^{2s} e(-\beta \cdot \mathbf{h}) d\beta$$

and

$$\mathfrak{S}_s(X; \mathbf{h}) = \sum_{1 \leq q \leq L} \sum_{\substack{1 \leq \mathbf{a} \leq q \\ (q, a_1, a_2, a_3) = 1}} q^{-2s} |S(q, \mathbf{a})|^{2s} e_q(-\mathbf{a} \cdot \mathbf{h}),$$

in which we write

$$\mathfrak{X} = [-LX^{-1}, LX^{-1}] \times [-LX^{-2}, LX^{-2}] \times [-LX^{-3}, LX^{-3}].$$

The singular integral (1.6) converges absolutely for $s > 7/2$ (see [2, Theorem 1.3] or [1, Theorem 1]), and moreover [13, Theorem 7.3] supplies the bound

$$I(\beta; X) \ll X(1 + |\beta_1|X + |\beta_2|X^2 + |\beta_3|X^3)^{-1/3}.$$

Then we infer via two changes of variable that

$$\begin{aligned}\mathfrak{J}_s(X; \mathbf{h}) &= X^{2s-6} \int_{\mathbb{R}^3} |I(\boldsymbol{\beta})|^{2s} e\left(-\beta_1 \frac{h_1}{X} - \beta_2 \frac{h_2}{X^2} - \beta_3 \frac{h_3}{X^3}\right) d\boldsymbol{\beta} + o(X^{2s-6}) \\ &= (\mathfrak{J}_s(\mathbf{h}) + o(1)) X^{2s-6} \ll X^{2s-6}.\end{aligned}\tag{7.4}$$

Similarly, the singular series (1.7) converges absolutely for $s > 4$ (see [2, Theorem 2.4] or [1, Theorem 1]), and in addition [13, Theorem 7.1] shows that when $(q, a_1, a_2, a_3) = 1$, one has $S(q, \mathbf{a}) \ll q^{2/3+\varepsilon}$. Thus, it follows that

$$\mathfrak{S}_s(X; \mathbf{h}) = \mathfrak{S}_s(\mathbf{h}) + o(1) \ll 1.\tag{7.5}$$

By substituting (7.4) and (7.5) into (7.3), and thence into (7.2), we obtain

$$\begin{aligned}T_s(\mathfrak{P}; \mathbf{h}) &= (\mathfrak{S}_s(\mathbf{h}) + o(1)) (\mathfrak{J}_s(\mathbf{h}) + o(1)) X^{2s-6} + o(X^{2s-6}) \\ &= \mathfrak{S}_s(\mathbf{h}) \mathfrak{J}_s(\mathbf{h}) X^{2s-6} + o(X^{2s-6}).\end{aligned}$$

By substituting this relation into (7.1), we conclude that

$$B_s(X; \mathbf{h}) = \mathfrak{S}_s(\mathbf{h}) \mathfrak{J}_s(\mathbf{h}) X^{2s-6} + o(X^{2s-6}).$$

We note that the absolute convergence of the integral $\mathfrak{J}_s(\mathbf{h})$ and of the series $\mathfrak{S}_s(\mathbf{h})$ shows, via familiar technology from the circle method, that

$$0 \leq \mathfrak{J}_s(\mathbf{h}) \ll 1 \quad \text{and} \quad 0 \leq \mathfrak{S}_s(\mathbf{h}) \ll 1.$$

This standard technology also shows that the singular series may be written in the form

$$\mathfrak{S}_s(\mathbf{h}) = \prod_p \varpi_p(s, \mathbf{h}),$$

where for each prime number p , the p -adic density $\varpi_p(s, \mathbf{h})$ is defined by

$$\varpi_p(s, \mathbf{h}) = \sum_{h=0}^{\infty} \sum_{\substack{1 \leq \mathbf{a} \leq p^h \\ (p, a_1, a_2, a_3) = 1}} p^{-2sh} |S(p^h, \mathbf{a})|^{2s} e_{p^h}(-\mathbf{a} \cdot \mathbf{h}).$$

The positivity of $\mathfrak{J}_s(\mathbf{h})$ and $\mathfrak{S}_s(\mathbf{h})$ corresponds to the existence of non-singular real and p -adic solutions to the system (1.3). Granted the existence of primitive such solutions, the standard theory shows that $\mathfrak{J}_s(\mathbf{h}) \gg 1$ and $\mathfrak{S}_s(\mathbf{h}) \gg 1$. This confirms the conclusion of Theorem 1.1.

8. NON-UNIFORM CONCLUSIONS: THE PROOF OF THEOREM 1.2

In our proof of Theorem 1.2, we abandon the uniformity in \mathbf{h} implicit in the error term of Theorem 1.1, though now we require only that $h_2 \neq 0$. The case $h_1 \neq 0$ having already been handled in Theorem 1.1, we assume that $h_1 = 0$ and $h_2 \neq 0$. In such circumstances, it now follows from (2.2) that

$$g(\boldsymbol{\alpha}; X) = \sum_{1 \leq y \leq X} e(3h_2 y \alpha_3).$$

We begin by deriving an analogue of Lemma 3.3.

Lemma 8.1. *Suppose that $\mathbf{h} \in \mathbb{Z}^3$ and $h_1 = 0$, $h_2 \neq 0$. Then one has*

$$\int_{[0,1]^3} |f(\boldsymbol{\alpha}; 2X)^{10} g(\boldsymbol{\alpha}; X)^2| d\boldsymbol{\alpha} \ll_{h_2} X^{6+\varepsilon}. \quad (8.1)$$

Proof. By orthogonality, the mean value in (8.1) counts the integral solutions of the system of equations

$$\begin{aligned} \sum_{i=1}^5 (x_i^3 - y_i^3) &= 3h_2(z_1 - z_2), \\ \sum_{i=1}^5 (x_i^2 - y_i^2) &= 0, \\ \sum_{i=1}^5 (x_i - y_i) &= 0, \end{aligned} \quad (8.2)$$

with $1 \leq x_i, y_i \leq 2X$ ($1 \leq i \leq 5$) and $1 \leq z_1, z_2 \leq X$. Thus, we find that

$$\int_{[0,1]^3} |f(\boldsymbol{\alpha}; 2X)^{10} g(\boldsymbol{\alpha}; X)^2| d\boldsymbol{\alpha} \ll XU_1, \quad (8.3)$$

where U_1 counts the number of integral solutions of the system

$$\begin{aligned} \left| \sum_{i=1}^5 (x_i^3 - y_i^3) \right| &< 3|h_2|X, \\ \sum_{i=1}^5 (x_i^2 - y_i^2) &= 0, \\ \sum_{i=1}^5 (x_i - y_i) &= 0, \end{aligned}$$

with $1 \leq x_i, y_i \leq 2X$ ($1 \leq i \leq 5$).

A standard argument (see [14, Lemma 2.1]) shows from here that

$$U_1 \ll |h_2|X \int_{-1/X}^{1/X} \int_0^1 \int_0^1 |f(\boldsymbol{\alpha}; 2X)|^{10} d\boldsymbol{\alpha}.$$

As a direct consequence of [8, Theorem 3.3], however, we have the bound

$$\int_{-1/X}^{1/X} \int_0^1 \int_0^1 |f(\boldsymbol{\alpha}; 2X)|^{10} d\boldsymbol{\alpha} \ll X^{4+\varepsilon}. \quad (8.4)$$

Thus we deduce that $U_1 \ll |h_2|X^{5+\varepsilon}$, and the upper bound of the lemma follows at once from (8.3). \square

By substituting the estimate (8.1) within the argument of the proof of Lemma 3.4, we readily deduce the bound contained in the following lemma.

Lemma 8.2. *Suppose that $\mathbf{h} \in \mathbb{Z}^3$ and $h_1 = 0$, $h_2 \neq 0$. Then one has*

$$\sup_{\Gamma \in [0,1)} \int_{[0,1)^3} |f(\boldsymbol{\alpha}; 2X)^{10} g(\boldsymbol{\alpha}, \Gamma; X)^2| d\boldsymbol{\alpha} \ll_{h_2} X^{6+\varepsilon}.$$

Proof. We may proceed as in the proof of Lemma 3.4, adopting the notation therein. Thus, the mean value

$$\int_{[0,1)^3} |f(\boldsymbol{\alpha}; 2X)^{10} g(\boldsymbol{\alpha}, \Gamma; X)^2| d\boldsymbol{\alpha}$$

counts the integral solutions of the system (8.2) with each solution $\mathbf{x}, \mathbf{y}, \mathbf{z}$ being counted with weight $e(-\Gamma(z_1 - z_2))$. Since this weight is unimodular, it follows via orthogonality that

$$\sup_{\Gamma \in [0,1)} \int_{[0,1)^3} |f(\boldsymbol{\alpha}; 2X)^{10} g(\boldsymbol{\alpha}, \Gamma; X)^2| d\boldsymbol{\alpha} \leq \int_{[0,1)^3} |f(\boldsymbol{\alpha}; 2X)^{10} g(\boldsymbol{\alpha}; X)^2| d\boldsymbol{\alpha}.$$

The conclusion of the lemma therefore follows at once from Lemma 8.1. \square

In the interests of concision, we extract from Lemma 8.2 the estimate

$$\sup_{\Gamma \in [0,1)} \int_{[0,1)^3} |f(\boldsymbol{\alpha}; 2X)^{10} g(\boldsymbol{\alpha}, \Gamma; X)^6| d\boldsymbol{\alpha} \ll_{h_2} X^{10+\varepsilon} \quad (8.5)$$

by means of the trivial bound $|g(\boldsymbol{\alpha}, \Gamma; X)| = O(X)$. Equipped with this as a direct substitute for the estimate delivered by Lemma 3.4, we see that no modification whatsoever is required in the discussion of §§4 to 7 in order to deliver the asymptotic formula (1.8) provided that $s \geq 6$, and X is sufficiently large in terms of h_2 . This completes the proof of Theorem 1.2. The reason that the latter condition concerning h_2 must be imposed is simply that the minor arc bound

$$I_s(\mathbf{m}(Q); X; \mathbf{h}) \ll_{h_2} X^{2s-6+\varepsilon} Q^{-\delta/4}$$

derived in the analogue of Lemma 4.1 must now have dependence on h_2 in the implicit constant, as a consequence of this same dependence in (8.5). It would not be difficult to ensure that the asymptotic formula (1.8) remains valid, with an acceptable error term, for values of h_2 satisfying $|h_2| \leq X^{1/2}$, or indeed a little larger still. However, in order to permit values of h_2 having absolute value nearly as large as X^2 , in order to accommodate the most general situation, one would need to obtain sharp variants of the cap estimate (8.4). We shall have more to say on such matters in a future communication.

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