Exam of Lie Algebras and Lie Groups

DMIST - 23 Mar. 2011

1. (a) Let L be a nilpotent Lie algebra. Show that its Killing form is identically zero. (b) Is the Killing form of $\mathsf{gl}(V)$ non-degenerate?

2. Let L be a Lie algebra and $Z(L)$ be its center. Suppose that $L \cong Z(L) \oplus [L, L]$ and let $\varphi : L \to \mathsf{gl}(V)$ be a representation such that $\varphi(x)$ is diagonalizable for every $x \in Z(L)$. Show that φ is completely reducible. [Hint: use the fact that commuting matrices can be simultaneously diagonalizable, and compare L-submodules with $[L, L]$-submodules]

3. Let R be a root system and $\alpha, \beta \in R$ be non-proportional roots. Prove that the subgroup of the Weyl group generated by the reflections $\sigma_\alpha, \sigma_\beta$ is a dihedral group with $2m$ elements, and that $\sigma_\alpha\sigma_\beta$ is an element of order m.

4. Show that the exponential map for the group $SL(2, \mathbb{R})$ is not surjective. Determine the image of the exponential map and the possible values of $\mathsf{tr}(exp(A))$ for matrices $A \in \mathsf{sl}(2, \mathbb{R})$.

5. Let G be a compact connected Lie group, T a maximal torus of G and let $N_T = \{g \in G : gTg^{-1} = T\}$ be its normalizer. (a) Show that the isomorphism class of $W = N_T/T$ is independent of the choice of T. (W is called the Weyl group of G) (b) Determine the group W in the case when $G = SU(n)$.

6. Let $\rho : G \to GL(V)$ be a representation of a compact connected Lie group G, and $f : G \to \mathbb{C}$ be a continuous class function, that is $f(hgh^{-1}) = f(g)$ for all $g, h \in G$. Define, using the normalized Haar measure dg,

$$\psi(f) := \int_G f(g)\rho(g)dg \in End(V).$$

   (a) Show that $\rho(h) \circ \psi(f) = \psi(f) \circ \rho(h)$ for all $h \in G$. (b) If ρ is irreducible, show that $\psi(f)$ is a multiple of the identity in $GL(V)$, and compute this multiple in terms of f and the character of ρ.