Question 1

Problem 1(a) (Baire category theorem)

Let X be a complete metric space or a locally compact Hausdorff space. Show that every countable collection of dense open sets has a dense intersection. Equivalently, show that the union of every countable collection of closed nowhere dense sets has empty interior.
Remark: A subset of X is nowhere dense if its closure has empty interior.

Solution

This proof is standard. Different expositions can be found, for example, on Wikipedia, in Chapter 48 of Munkres' *Topology* (second edition), and in Chapter 5.3 of Folland's *Real Analysis: Modern Techniques and Their Applications* (second edition).

Problem 1(b)

Let $\mathcal{F}$ be a collection of continuous real-valued functions on a locally compact Hausdorff space or a complete metric space X. Suppose that $\mathcal{F}$ is piecewise bounded in the sense that for every $x \in X$, there exists $M > 0$ such that $|f(x)| \leq M$ for all $f \in \mathcal{F}$. Show that $\mathcal{F}$ is uniformly bounded on some open set $U \subseteq X$, meaning that there exists $M > 0$ such that $\sup_{x \in U} |f(x)| \leq M$ for all $f \in \mathcal{F}$.
Hint: Consider $A_n := \{x \in X \mid |f(x)| \leq n \text{ for all } f \in \mathcal{F}\}$.

Solution

Observe that A_n as defined in the hint is equal to $\bigcap_{f \in \mathcal{F}} f^{-1}([-n, n])$, and is hence closed. Moreover, $X = \bigcup_{n=1}^{\infty} A_n$ by the assumption that $\mathcal{F}$ is piecewise bounded. We claim that some A_n has non-empty interior. By way of contradiction, suppose not; then $\{A_n\}_{n=1}^{\infty}$ is a countable collection of closed nowhere dense sets, and hence $X = \bigcup_{n=1}^{\infty} A_n$ has empty interior by the Baire category theorem. This is impossible (unless $X = \varnothing$, in which case there is nothing to prove).
Thus, there exists some $n \in \mathbb{N}$ with $\operatorname{Int}(A_n) \neq \varnothing$. Setting $U = \operatorname{Int}(A_n)$ yields the desired open set.

Problem 1(c)

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with the property that for every $x \in \mathbb{R}$, $\lim_{n \to \infty} f(nx) = 0$. Show that $\lim_{x \to \infty} f(x) = 0$.

Solution

Fix $\varepsilon > 0$. For each $n \in \mathbb{N}$, define $A_n = \{x \in [0, \infty) \mid |f(nx)| \leq \varepsilon\}$ and $B_n = \bigcap_{m=n}^{\infty} A_m$. Then A_n is closed (as the preimage of the closed set $[0, \varepsilon]$ under the continuous function $x \mapsto |f(nx)|$), hence so too is B_n. The assumption that $\lim_{n \to \infty} f(nx) = 0$ for every $x \in \mathbb{R}$ implies that $\bigcup_{n=1}^{\infty} B_n = [0, \infty)$. By the Baire category theorem, there exists an open interval (a, b) contained in some B_N. Given $n > N$ and $x \in (a, b)$, it follows that $|f(nx)| \leq \varepsilon$; said differently, $|f(x)| \leq \varepsilon$ for all $x \in (na, nb)$. These intervals are of length $n(b - a)$, which grows linearly in n; it follows that the union of all such intervals contains all sufficiently large x. Thus $|f(x)| \leq \varepsilon$ for all sufficiently large x. This holds for all $\varepsilon > 0$, whence the conclusion.

Problem 1(d)

Let $f : \mathbb{R} \to \mathbb{R}$ be a function and let D be the set of all points in $\mathbb{R}$ at which f is continuous. Show that D cannot be a countable dense set.
Hint: Show that D is a G_δ set.

Solution

Define

$$D_n = \left\{ x \in \mathbb{R} \mid \exists \delta > 0 \text{ such that } |f(x_1) - f(x_2)| < \frac{1}{n} \text{ whenever } x_1, x_2 \in (x - \delta, x + \delta) \right\}.$$

The condition defining membership in D_n is an open condition, so D_n is open. We claim that $D = \bigcap_{n=1}^{\infty} D_n$, whence it follows that D is a G_δ set. Indeed, if $x \in D$ and $n \in \mathbb{N}$, then there exists a $\delta > 0$ satisfying the ε-δ definition of continuity of f at x for $\varepsilon = \frac{1}{2n}$; then for all $x_1, x_2 \in (x - \delta, x + \delta)$, the triangle inequality yields

$$|f(x_1) - f(x_2)| \leq |f(x_1) - f(x)| + |f(x) - f(x_2)| < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n},$$

so $x \in D_n$. This holds for all n and x, hence $D \subseteq \bigcap_{n=1}^{\infty} D_n$. Conversely, suppose $x \in \bigcap_{n=1}^{\infty} D_n$. Given $\varepsilon > 0$, choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$; then taking $x_2 = x$ in the definition of D_n shows that f is continuous at x, hence $x \in D$.
Suppose D is dense. Then $D^c = \bigcup_{n=1}^{\infty} D_n^c$ has empty interior; in particular, each D_n^c has empty interior, and is thus nowhere dense (as D_n^c is also closed). From this it follows that D must be uncountable: if it were the case that D were countable, then

$$\mathbb{R} = D \cup D^c = \left(\bigcup_{x \in D} \{x\} \right) \cup \bigcup_{n=1}^{\infty} D_n^c,$$

which would express $\mathbb{R}$ as a countable union of nowhere dense sets. This is impossible by the Baire category theorem.

Problem 1(e) (bonus)

Equip $C([0,1])$ with the supremum metric making it a complete metric space. For each $n \in \mathbb{N}$, let F_n be the subset of $(C([0,1], d_{\sup})$ consisting of functions for which there is a point $x_0 \in [0,1]$ such that $|f(x) - f(x_0)| \leq n|x - x_0|$ for all $x \in [0,1]$. Show that F_n is a closed nowhere dense set. Using the Baire category theorem, conclude that the set of continuous nowhere differentiable functions is dense in $C([0,1])$.

Solution

First, we introduce some terminology and notation:

- The *uniform norm* $\|\cdot\|_u$ on $C([0,1])$ is defined by $\|f\|_u = \sup_{x \in [0,1]} |f(x)|$, so that

$d_{\sup}(f, g) = \|f - g\|_u$.
- A subset F of a topological space X is called *meagre* if it is a countable union of nowhere dense sets.
- The complement of a meagre set is called *residual*. If X is a complete metric space or locally compact Hausdorff space (so that the Baire category theorem holds), then every residual subset of X is dense in X.

F_n is closed: We show that every convergent sequence $(f_k)_{k=1}^{\infty}$ in F_n with limit f satisfies $f \in F_n$ (note that f is automatically continuous as it is the uniform limit of a sequence of continuous functions). For each $k \in \mathbb{N}$, choose $x_k \in [0,1]$ such that

$$|f_k(x) - f_k(x_k)| \le n|x - x_k|, \qquad \forall x \in [0,1].$$

Since $[0,1]$ is compact, the sequence $(x_k)_{k=1}^{\infty}$ has a subsequence converging to some $x_0 \in [0,1]$. By replacing the sequence $(f_k)_{k=1}^{\infty}$ with the corresponding subsequence, we may assume that $x_k \to x_0$. Then $f_k \to f$ uniformly and $x_k \to x_0$. Note that $|f_k(x_k) - f(x_0)| \le |f_k(x_k) - f(x_k)| + |f(x_k) - f(x_0)|$; the first term tends to 0 since $f_k \to f$ uniformly, and the second term tends to 0 since f is continuous and $x_k \to x_0$. It follows that $f_k(x_k) \to f(x_0)$. Then for all $x \in [0,1]$, we have

$$|f(x) - f(x_0)| = \lim_{k\to\infty} |f_k(x) - f_k(x_k)| \le \lim_{k\to\infty} n|x - x_k| = n|x - x_0|.$$

Thus $f \in F_n$.
F_n is nowhere dense: We need a few results before proceeding.

PROPOSITION 1.1. *Fix $n \in \mathbb{N}$.*

(1) If $\psi \in C([0,1])$ is piecewise linear and each linear piece of ψ has slope greater than or equal to $2n$ in absolute value, then $\psi \notin F_n$.
(2) For all $f \in C([0,1])$ and $\varepsilon > 0$, there exists $\psi \in C([0,1])$ as in (1) which additionally satisfies $\|f - \psi\|_u < \varepsilon$.

For a detailed proof of Proposition 1.1, see the following section.
Since F_n is closed, it is nowhere dense if and only if it has empty interior, if and only if every $f \in C([0,1])$ is the (uniform) limit of some sequence $(f_k)_{k=1}^{\infty}$ in $C([0,1]) \setminus F_n$. By Proposition 1.1(2), there exists a sequence of piecewise linear functions f_k such that $\|f_k - f\|_u < \frac{1}{k}$ and the linear pieces of each f_k have slopes greater than or equal to $2n$ in absolute value. Each f_k is in $C([0,1]) \setminus F_n$ by Proposition 1.1(1), so we are done.
Continuous, nowhere differentiable functions are dense in $C([0,1])$:
Suppose $f \in C([0,1])$ is differentiable at a point $x_0 \in [0,1]$. Then there exists $\delta > 0$ such that

$$\left|\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0)\right| \le 1 \text{ whenever } 0 < |x - x_0| < \delta.$$

Applying the reverse triangle inequality and rearranging, we obtain

$$|f(x) - f(x_0)| \le (1 + |f'(x_0)|)|x - x_0| \text{ whenever } |x - x_0| < \delta.$$

If instead $|x - x_0| \ge \delta$, then

$$|f(x) - f(x_0)| \le 2\|f\|_u = \frac{2\|f\|_u}{|x - x_0|}|x - x_0| \le \frac{2\|f\|_u}{\delta}|x - x_0|.$$

Thus, if $n \geq \max(1 + |f'(x_0)|, 2\|f\|_u/\delta)$, then $f \in F_n$.
It follows that the set $F \subset C([0,1])$ of functions differentiable at some point is contained inside $\bigcup_{n=1}^{\infty} F_n$. Each F_n is nowhere dense, so $\bigcup_{n=1}^{\infty} F_n$ is meagre. A subset of a meagre set is itself meagre, so F is meagre. Thus, F^c is residual in $C([0,1])$ (and in particular dense), and F^c is precisely the nowhere differentiable continuous functions.

Proof of Proposition 1.1

Proof of Proposition 1.1(1). Fix $x_0 \in [0,1]$. For sufficiently small $\varepsilon > 0$, we have that ψ is affine linear on at least one of the intervals $[x_0, x_0+\varepsilon]$ or $[x_0-\varepsilon, x_0]$, with slope m satisfying $|m| \geq 2n$. Set $x = x_0 \pm \frac{\varepsilon}{2}$, the sign chosen so that x lies in such an interval. Then

$$|\psi(x) - \psi(x_0)| = |m||x - x_0| \geq 2n|x - x_0| > n|x - x_0|,$$

the strict inequality holding because $|x - x_0| \neq 0$. ■

We prove Proposition 1.1(2) in a two-step process: first, show that every $f \in C([0,1])$ can be uniformly approximated by some piecewise linear function. Then, show that every piecewise linear function can be uniformly approximated by a piecewise linear function with slopes greater than or equal to $2n$ in absolute value.

LEMMA 1.2. *For all $f \in C([0,1])$ and $\varepsilon > 0$, there exists a piecewise linear $\phi \in C([0,1])$ such that $\|f - \phi\|_u < \varepsilon$.*

Proof. Since $[0,1]$ is compact, f is uniformly continuous on $[0,1]$; hence there exists $N \in \mathbb{N}$ such that $|f(x) - f(y)| < \frac{\varepsilon}{4}$ whenever $|x - y| \leq \frac{1}{N}$. Set $x_k = \frac{k}{N}$, so that $0 = x_0 < x_1 < \cdots < x_N = 1$. We define $\phi \in C([0,1])$ by $\phi(x_k) = f(x_k)$ and linearly interpolating between the x_k's, i.e.

$$\phi(x) := \frac{f(x_k) - f(x_{k-1})}{1/N}(x - x_{k-1}) + f(x_{k-1}), \qquad \forall x \in (x_{k-1}, x_k).$$

Fix $x \in [0,1]$. If $x = x_k$ for some k, then $f(x) = \phi(x)$; otherwise, $x \in (x_{k-1}, x_k)$ for some k, and

$$\begin{aligned} |f(x) - \phi(x)| &\leq |f(x) - f(x_{k-1})| + |f(x_k) - f(x_{k-1})| \frac{x - x_{k-1}}{1/N} \\ &\leq |f(x) - f(x_{k-1})| + |f(x_k) - f(x_{k-1})| \\ &< \varepsilon/4 + \varepsilon/4. \end{aligned}$$

It follows that $\|f - \phi\|_u \leq \varepsilon/2 < \varepsilon$. ■

LEMMA 1.3. *Fix $n \in \mathbb{N}$ and $\varepsilon > 0$. If $\phi \in C([0,1])$ is piecewise linear, then there exists a piecewise linear $\psi \in C([0,1])$ whose linear pieces have slopes greater than or equal to $2n$ in absolute value, and such that $\|\phi - \psi\|_u < \varepsilon$.*

Proof. We claim that if $\phi(x) = mx + c$ is affine linear on $[a,b]$ and $|y_0 - \phi(a)| \leq \frac{\varepsilon}{2}$, then there exists ψ piecewise linear on $[a,b]$ such that $\psi(a) = y_0$, each linear piece of ψ has slope greater than or equal to $2n$ in absolute value, and

$$|\psi(x) - \phi(x)| \leq \frac{\varepsilon}{2}, \qquad \forall x \in [a,b].$$

Indeed, if $|m| \geq 2n$, then $\psi(x) = m(x-a) + y_0$ works. Otherwise, approximate $mx + c$ with a "see-saw" function whose slopes have absolute value $2n$ (Figure 1.1).
From this, the lemma follows easily: let $0 = x_0 < x_1 < \ldots < x_N = 1$ be such that each $\phi|_{[x_{k-1}, x_k]}$ is affine linear. Apply the claim on $[0, x_1]$ with $y_0 = \phi(0)$ to define ψ on $[0, x_1]$. Then apply the claim on $[x_1, x_2]$ with $y_0 = \psi(x_1)$ (which satisfies $|y_0 - \phi(x_1)| \leq \frac{\varepsilon}{2}$ by construction) to define ψ on $[x_1, x_2]$. Continuing this iterative process defines ψ on $[0, 1]$. By construction, ψ is piecewise linear and each linear piece has slope greater than or equal to $2n$ in absolute value, and

$$\|\phi - \psi\|_u \leq \frac{\varepsilon}{2} < \varepsilon.$$

■

Proof of Proposition 1.1(2). By Lemma 1.2, there exists a piecewise linear $\phi \in C([0, 1])$ such that $\|f - \phi\|_u < \varepsilon/2$. By Lemma 1.3, there exists a piecewise linear $\psi \in C([0, 1])$ whose linear pieces have slopes greater than or equal to $2n$ in absolute value, and such that $\|\phi - \psi\|_u < \varepsilon/2$. By the triangle inequality,

$$\|f - \psi\|_u \leq \|f - \phi\|_u + \|\phi - \psi\|_u < \varepsilon.$$

■

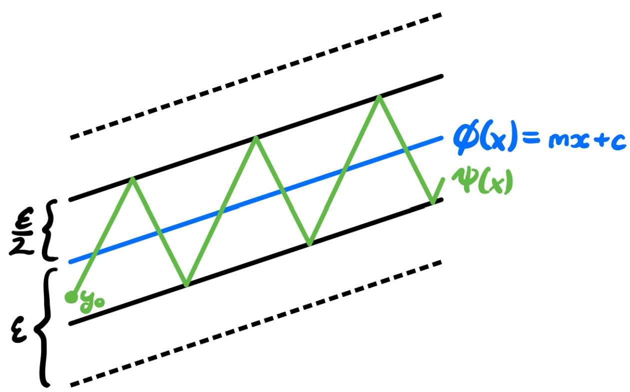

Figure 1.1: Approximating ϕ when $|m| < 2n$.

Question 2

Problem 2(a)

Show that every second-countable, locally compact Hausdorff space is metrizable. Conclude that every n-manifold is metrizable.

Solution

Let X be a second-countable, locally compact Hausdorff space. In view of Urysohn's metrization theorem, it suffices to show that X is regular.

Case 1. If X is compact, then recall that compact + Hausdorff $\implies$ normal, hence in particular X is regular.
Case 2. If X is not compact, then consider its one-point compactification X^* (see Problem Set 4 Q4). Since X^* is compact and Hausdorff, it is normal, hence in particular regular. Since the property of being regular is hereditary and the original topology on X coincides with the subspace topology induced by the inclusion $X \hookrightarrow X^*$, it follows that X is regular.

In particular, every n-manifold is by definition second-countable, Hausdorff, and locally homeomorphic to $\mathbb{R}^n$. Since $\mathbb{R}^n$ is locally compact, so too is every n-manifold; thus manifolds are metrizable.

Problem 2(b)

For a topological space, a sequence $\{K_n\}_{n\in\mathbb{N}}$ of compact sets is called an exhaustion of X by compact sets if the $\bigcup_n K_n = X$ and $K_n \subseteq \operatorname{Int} K_{n+1}$ for every $n \in \mathbb{N}$. Show that every second countable, locally compact Hausdorff space, and in particular an n-manifold, admits an exhaustion by compact sets.

Solution

We claim that X has a countable basis consisting of *precompact* open sets (i.e., open sets whose closures are compact). To show this, start with any countable basis $\mathcal{B}_0$ of X. Since X is locally compact, every $x \in X$ admits an open neighbourhood U and a compact set K such that $x \in U \subseteq K$. Since X is Hausdorff, K is closed. Consider $\mathcal{N}_x = \{B \in \mathcal{B}_0 \mid B \subseteq U\}$. Each $B \in \mathcal{N}_x$ satisfies $\overline{B} \subseteq \overline{U} \subseteq \overline{K} = K$, so $\overline{B}$ is compact (as a closed subset of the compact set K). Then

$$\mathcal{B} := \bigcup_{x\in X} \mathcal{N}_x$$

is the desired basis (note that $\mathcal{B} \subseteq \mathcal{B}_0$, so $\mathcal{B}$ is also countable).
Indexing $\mathcal{B} = \{U_n\}_{n=1}^{\infty}$, we define an exhaustion $\{K_n\}_{n=1}^{\infty}$ by compact sets inductively. First, set $K_1 = \overline{U_1}$. Suppose that K_n has been defined. Since K_n is compact and $\mathcal{B}$ is an open cover, there are finitely many subindices $n_1, \dots, n_j$ such that $K_n \subseteq U_{n_1} \cup \cdots \cup U_{n_j}$. We then define $K_{n+1} = \overline{U_{n+1}} \cup \overline{U_{n_1}} \cup \cdots \cup \overline{U_{n_j}}$. This is compact (as a finite union of compact sets) and satisfies

$$K_n \subseteq U_{n+1} \cup U_{n_1} \cup \cdots \cup U_{n_j} \subseteq \operatorname{Int}\left(\overline{U_{n+1} \cup U_{n_1} \cup \cdots \cup U_{n_j}}\right) = \operatorname{Int} K_{n+1}.$$

The collection $\{K_n\}_{n=1}^{\infty}$ covers X as each $x \in X$ belongs to some U_n (since $\mathcal{B}$ is a basis), which is a subset of K_n by construction.

Problem 2(c)

Let X be a second countable, locally compact Hausdorff space and let $\{U_\alpha\}_{\alpha\in J}$ be an open cover for X. Show that there exists an open cover $\{V_\alpha\}_{\alpha\in J}$ such that $\overline{V}_\alpha \subseteq U_\alpha$ and that for every $x \in X$, there exists a neighbourhood of x that intersects V_α for finitely many $\alpha \in J$.
Hint: Fix an exhaustion $\{K_n\}_{n\in\mathbb{N}}$ by compact sets and define $A_n := K_{n+1} \setminus \operatorname{Int} K_n$ and

$U_n := \operatorname{Int} K_{n+2} \setminus K_{n-1}$.

Solution

With K_n, A_n, and U_n defined as in the hint, we have that A_n is compact, U_n is open, and $A_n \subseteq U_n$. The collection $\mathcal{O} = \{U \subseteq X \text{ open} \mid \overline{U} \subseteq U_\alpha \text{ for some } \alpha \in J\}$ is an open cover of X; thus, for each $n \in \mathbb{N}$, there is a finite subcover $\mathcal{U}_n \subseteq \mathcal{O}$ of A_n. Then $\mathcal{U} = \bigcup_{n=1}^\infty \mathcal{U}_n$ is another open cover of X with the property that every $x \in X$ admits an open neighbourhood that intersects finitely many sets in $\mathcal{U}$. For each $W \in \mathcal{U}$, there exists some $\alpha \in J$ such that $\overline{W} \subseteq U_\alpha$. Choosing one such α for each $W \in \mathcal{U}$ defines a function $f\colon \mathcal{U} \to J$. Then setting

$$V_\alpha = \bigcup_{W \in f^{-1}(\alpha)} W$$

yields the desired open cover $\{V_\alpha\}_{\alpha \in J}$.

Problem 2(d) (bonus)

Generalize our proof for the existence of a partition of unity to second countable, locally compact Hausdorff spaces and, in particular, to n-manifolds.

Solution

See Theorem 4.85 in Lee's *Introduction to Topological Manifolds*.

Question 3

Problem 3(a)

Show that if X is Hausdorff and A is a retract of X, then A is closed.

Solution

Let $r\colon X \to A$ be a retraction. We will show that A is closed by showing that $X \setminus A$ is open. Given $x \in X \setminus A$, we have $r(x) \neq x$; since X is Hausdorff, there are disjoint open neighbourhoods U and V of x and $r(x)$, respectively. Then $V \cap A$ is open in A and contains $r(x)$, hence $r^{-1}(V \cap A)$ is open in X and contains x. Define $U_x = r^{-1}(V \cap A) \cap U$. Then U_x is an open neighbourhood of x which is disjoint from A, for if there existed $y \in U_x \cap A$, then $r(y) = y \in V \cap A$ and $y \in U$; this is impossible, as $U \cap V = \varnothing$. Since

$$X \setminus A = \bigcup_{x \in X \setminus A} U_x,$$

this shows that $X \setminus A$ is open.

Problem 3(b)

Suppose that A is a retract of X, Show that for any $x_0 \in A$, the homomorphism $\iota_* : \pi(A, x_0) \to \pi(X, x_0)$ induced by the inclusion map $\iota : A \to X$ is injective and the homomorphism $r_* : \pi(X, x_0) \to \pi(A, x_0)$ induced by the retraction r is surjective. Conclude that a retract of a simply connected space is simply connected.

Solution

To say that $r\colon X \to A$ is a retraction is equivalent to saying that $r \circ \iota = \mathrm{id}_A$, where $\iota\colon A \hookrightarrow X$ is the inclusion. Thus, for any $x_0 \in A$, the induced maps on fundamental groups $\iota_*\colon \pi_1(A, x_0) \to \pi_1(X, x_0)$ and $r_*\colon \pi_1(X, x_0) \to \pi_1(A, x_0)$ satisfy

$$r_* \circ \iota_* = (r \circ \iota)_* = (\mathrm{id}_A)_* = \mathrm{id}_{\pi_1(A,x_0)} .$$

Whenever a composition $g \circ f$ of functions is bijective, the function f must be injective and the function g must be surjective. Thus ι_* is injective and r_* is surjective.
In particular, if X is simply connected, then so is A: it is the image of the path-connected space X under r, hence path-connected, and since $\pi_1(X, x_0) = 0$, the homomorphism r_* is a surjective map from the trivial group to $\pi_1(A, x_0)$, which shows that $\pi_1(A, x_0) = 0$.

Problem 3(c)

Show that for any $n \in \mathbb{N}$, S^{n-1} is a retract of $\mathbb{R}^n \setminus \{0\}$. Conclude that $\mathbb{R}^2 \setminus \{0\}$ is not simply connected.

Solution

A retract $r\colon \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ is given by $r(x) = x/\|x\|$, where $\|\cdot\|$ denotes the Euclidean norm. By part (b), it follows that $r_*\colon \pi_1(\mathbb{R}^n \setminus \{0\}) \to \pi_1(S^{n-1})$ is a surjective homomorphism (suppressing the choice of basepoint from our notation). Specializing to $n = 2$, we see that there is a surjection $\pi_1(\mathbb{R}^2 \setminus \{0\}) \to \pi_1(S^1) \cong \mathbb{Z}$. Thus $\pi_1(\mathbb{R}^2 \setminus \{0\}) \neq 0$.

Problem 3(d)

Show that the torus $T = S^1 \times S^1$ is not simply connected by finding a retract that is homeomorphic to S^1.

Solution

Many choices of retraction are possible. For instance, fix any $p \in S^1$ and define a map $r\colon S^1 \times S^1 \to S^1 \times \{p\}$ by $r(x, y) = (x, p)$. This is a retraction, and $S^1 \times \{p\}$ is homeomorphic to S^1 via the projection $S^1 \times \{p\} \to S^1$.

Problem 3(e)

Let $n \geq 2$ and let N be the north pole in S^n. Let $p \in S^n \setminus \{N\}$. Show that any loop based at p is path-homotopic to one that doesn't contain N. Conclude that S^n is simply connected.

Hint: Recall that $S^n \setminus \{N\}$ is homeomorphic to $\mathbb{R}^n$.

Solution

Let $U \subset S^n$ be an open neighbourhood of N which is homeomorphic to a disk. Let V be an open neighbourhood of $S^n \setminus U$ which does not contain N. For any loop $\gamma\colon [0,1] \to S^n$ based at p, the preimages $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$ are open subsets of $[0,1]$, hence together form a collection of intervals which are open in $[0,1]$. By compactness, there exist finitely many numbers $0 = t_0 < t_1 < \ldots < t_n = 1$ such that $\gamma_i := \gamma|_{[t_i,t_{i+1}]}$ has image contained in either U or V. For those intervals $[t_i, t_{i+1}]$ with image contained in U, observe that $U \setminus \{N\}$ is path-connected (this step uses that $n \geq 2$), hence there exists a path γ_i' parametrized by the same interval which has the same start and endpoints as γ_i, but avoids N. For those intervals $[t_i, t_{i+1}]$ with image contained in V, we simply set $\gamma_i' = \gamma_i$. Then γ is path-homotopic to the concatenation of the γ_i's, which is path-homotopic to the concatenation of the γ_i''s; this loop avoids N.

From this, it follows that S^n is simply connected for $n \geq 2$: any loop based at $p \in S^n \setminus \{N\}$ is path-homotopic to one with image contained in $S^n \setminus \{N\}$. Since $S^n \setminus \{N\}$ is homeomorphic to $\mathbb{R}^n$ and the latter space is simply connected, it follows that S^n is also simply connected.

Problem 3(f) (bonus)

Define the figure eight $Y \subset \mathbb{R}^2$ to be the union of the circles of radius one and centres $(0,1)$ and $(0,-1)$. Show that Y is not simply connected.

Solution

Let $A = \{(x,y) \mid x^2 + (y-1)^2 = 1\} \subset Y$ be the top circle in Y. Then there is a retraction $r\colon Y \to A$ given by $r(x,y) = (x,|y|)$. Since A is homeomorphic to S^1, the same reasoning as in parts (c) and (d) shows that Y is not simply connected.