

## MAT 145: PROBLEM SET 4

DUE TO FRIDAY FEB 22

ABSTRACT. This problem set corresponds to the sixth week of the Combinatorics Course in the Winter Quarter 2019. It was posted online on Friday Feb 15 and is due Friday Feb 22 at the beginning of the class at 9:00am.

**Information.** These are the solutions for the Problem Set 3 corresponding to the Winter Quarter 2019 class of MAT 145 Combinatorics, with Prof. Casals and T.A. A. Aguirre. The Problems are written in black and the solutions in blue.

**Problem 1.** Show that there exists no graph  $G = (V, E)$  with  $|V| = 48$  vertices such that the degrees of 30 of the vertices are 16, the degree of 15 vertices is 9 and the degree of the remaining 3 vertices is 12.

**Solution.** The number of odd-degree vertices is even, and thus no such graph can exist, since it should have 15 vertices of degree 9. Alternatively, the sum of the degrees of the vertices is twice the number of edges and therefore even. However  $30 \times 16 + 15 \times 9 + 3 \times 12$  is odd.

**Problem 2.** Let  $G = (V, E)$  be a connected graph, an edge  $e \in E$  is a *cut-edge* if  $G \setminus \{e\}$  is disconnected. Show that if  $G$  admits an Euler circuit, then there exist **no** cut-edge  $e \in E$ .

**Solution.** By the results in class, a connected graph has an Eulerian circuit if and only if the degree of each vertex is a nonzero even number. Suppose connects the vertices  $v$  and  $v'$  if we remove  $e$  we now have a graph with exactly 2 vertices with odd degrees. Recall that a graph has an Eulerian path (not circuit) if and only if it has exactly two vertices with odd degree. Thus the existence of such Eulerian path proves  $G - \{e\}$  is still connected so there are no cut edges.

**Problem 3.** (20 pts) For each of the three graphs in Figure 1, determine whether they have an Euler walk and/or an Euler circuit. Justify your answer, i.e. if an Euler walk or circuit exists, construct it explicitly, and if not give a proof of its non-existence.

**Solution.** The vertices of  $K_5$  all have even degree so an Eulerian circuit exists, namely the sequence of edges 1, 5, 8, 10, 4, 2, 9, 7, 6, 3. The 6 vertices on the right side of this bipartite  $K_{3,6}$  graph have odd degree. Recall that an Eulerian walk exists if and only if the number of vertices with odd degree is at most 2 hence there are no Eulerian walks (nor Eulerian circuits since that is an even stronger condition). Finally the wheel  $W_{10}$  has no Eulerian walks either since all its vertices have odd degree.

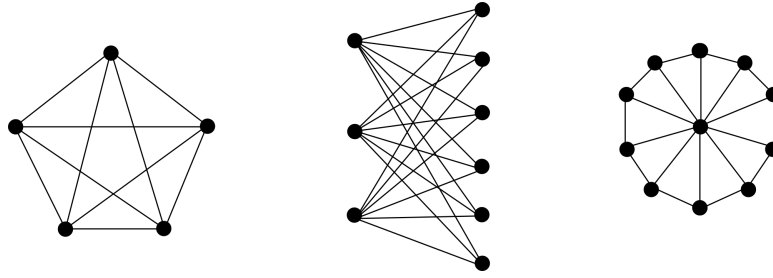


FIGURE 1. The three graphs for Problem 3  $K_5$ ,  $K_{3,6}$  and the Wheel  $W_{10}$ .

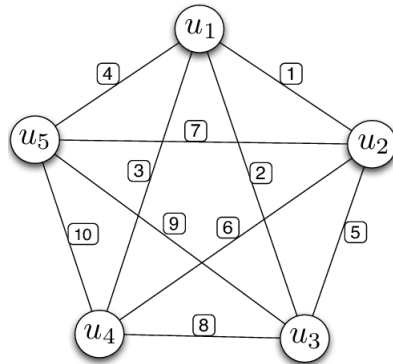


FIGURE 2. The Eulerian circuit for  $K_5$ .

**Problem 4.** (20 pts) Let  $n, m \in \mathbb{N}$  be two natural numbers. Let  $K_n$  be the complete graph in  $n$  vertices, and  $K_{n,m}$  the complete bipartite graph in  $n$  and  $m$  vertices<sup>1</sup>. See Figure 3 for two Examples of such graphs.

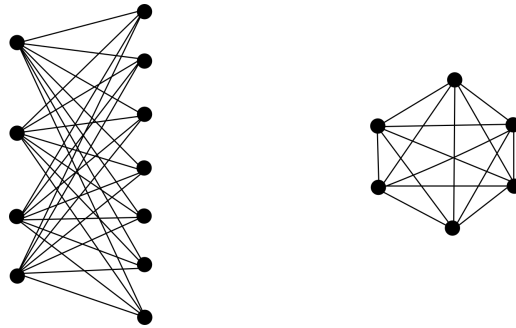


FIGURE 3. The  $K_{4,7}$  on the Left and  $K_6$  on the Right.

- Determine the number of edges of  $K_n$ , and the degree of each of its vertices. Given a necessary and sufficient condition on the number  $n \in \mathbb{N}$  for  $K_n$  to admit an Euler circuit.
- Determine the number of edges of  $K_{n,m}$ , and the degree of each of its vertices. Given a necessary and sufficient condition on the numbers  $n, m \in \mathbb{N}$  for  $K_{n,m}$  to admit an Euler circuit.

<sup>1</sup>In class, we also called  $K_{n,m}$  the *utility* graph.

- (c) Show that the complete bipartite graph  $K_{n,m}$  admits a Hamiltonian cycle if and only if  $n = m$ .

**Solution.**

- (a) Fix a given vertex  $v_1$ , then since  $K_n$  is the complete graph it is connected to the other  $(n - 1)$  vertices. Take any another vertex  $v_2$ , we have already counted the edge connecting it with  $v_1$  hence at this step we add  $n - 2$  edges connecting it the rest of the vertices and so on. Iterating this procedure we have  $(n - 1) + (n - 2) + \dots + 2 + 1 = \frac{(n-1)n}{2}$  vertices. With this construction it is clear that each vertex of the complete graph has degree  $(n - 1)$ . Thus  $K_n$  admits an Euler circuit if and only if  $n$  is odd.
- (b) Each of the  $n$  vertices on the left side of  $K_{n,m}$  is connected to the  $m$  vertices on the right. Therefore there are  $n \times m$  vertices, with  $n$  vertices have degree  $m$  and  $m$  vertices having degree  $n$ . Thus  $K_{n,m}$  has an Eulerian circuit if and only if both  $n$  and  $m$  are even.
- (c) If  $n = m$  then there are  $2n$  vertices each with degree  $n$  so by Dirac's theorem there is a Hamiltonian cycle. Conversely, let's suppose that  $K_{n,m}$  has partition  $\{v_1, v_2 \dots v_n\} \cup \{w_1, w_2 \dots w_m\}$ . Note that in a bipartite graph any Hamiltonian cycle must alternate between the two subsets of the partition. Now assume that we have a Hamiltonian cycle starting and ending at  $v_1$ . Since the graph is complete, let's make it  $v_1 w_1 v_2 w_2 \dots v_n w_n v_1$ . Now every vertex (except  $v_1$ ) has been reached exactly once so  $m = n$ . In other words if  $m > n$  some of the  $w_i$ 's would not have been reached and conversely if  $m < n$  some of the  $w_i$ 's would have been reached more than once.

**Problem 5.** (20 pts) Show that there are exactly three connected graphs with 4 vertices or less which admit an Euler circuit. In addition, list four different connected graphs with 5 vertices which admit Euler circuits, and find five different connected graphs with 6 vertices with an Euler circuits.

**Solution.** By convention we say the graph on one vertex admits an Euler circuit. There is only one connected graph on two vertices but for it to be a cycle it needs to use the only edge twice. On 3 vertices, we have exactly two connected graphs, a "straight line"  $v_1 e_1 v_2 e_2 v_3$  (here  $v_i, e_i$  represents the  $i$ -th vertex and edge respectively) and the triangle  $v_1 e_1 v_2 e_2 v_3 e_3 v_1$ . It's easy to check that the straight line has two vertices of odd degree while the triangle only has vertices of even degree hence it admits an Eulerian circuit. Thus we must only have one Eulerian connected graph on 4 vertices. Indeed, here are all the connected graphs on four vertices. By the parity criterion we can see that only the one on the top right is Eulerian.

Again, by the parity criterion, we can find 4 connected graphs on 5 vertices below are Eulerian.

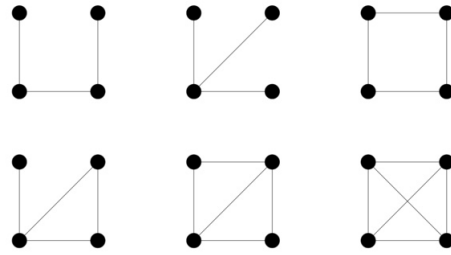


FIGURE 4. All connected graphs on 4 vertices.



FIGURE 5. All Eulerian graphs on 5 vertices.

And likewise for these 5 connected graphs on 6 vertices.

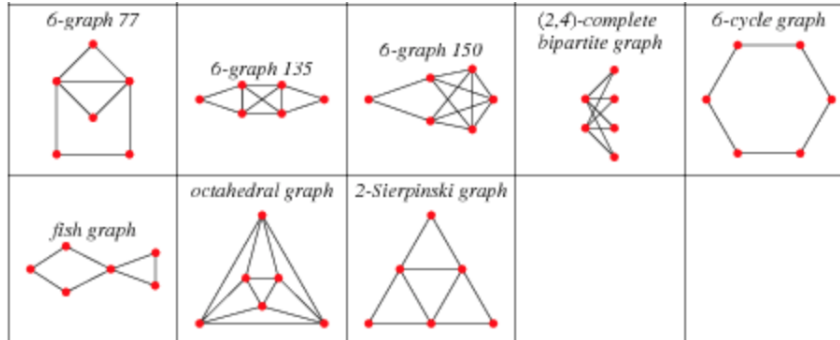


FIGURE 6. All Eulerian graphs on 6 vertices.

Although not needed for this problem, this is in fact the full classification of connected Eulerian graphs of 5 and 6 nodes respectively. See the Wolfram MathWorld entry for Eulerian Graph.

**Problem 6.** (20 pts) Decide whether the following statements are true or false. In case the statement is true, provide a proof, and if it is false, provide a counter-example.

- (a) The Petersen Graph does admit a Hamiltonian cycle.  
See Figure 7 (Left) for a depiction of the Petersen graph.

- (b) The Herschel Graph does *not* admit a Hamiltonian cycle.  
See Figure 7 (Right) for a depiction of this graph.

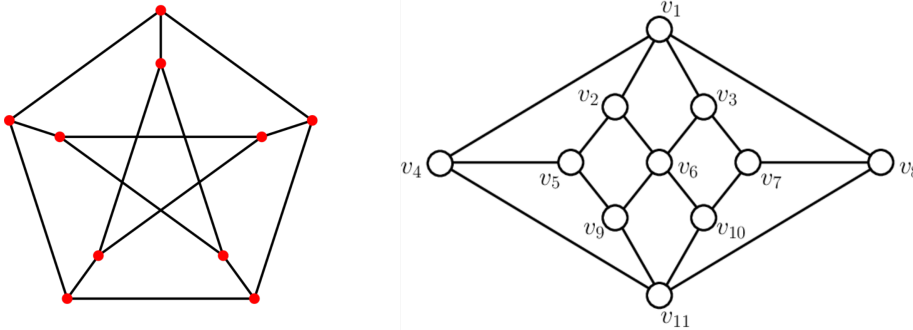


FIGURE 7. The Petersen Graph (Left) and the Herschel Graph (Right).

- (c) Every connected graph in 7 vertices admits a Hamiltonian cycle.
- (d) Let  $G = (E, V)$  be a graph such that for all non-adjacent vertices  $x, y \in V$

$$\deg(x) + \deg(y) \geq |V| - 1.$$

Then  $G$  is connected.

**Solution.**

- (a) **False.** Notice that there are 2 subgraphs; the inner star and the outer pentagon and that they are connected by 5 edges. Any Hamiltonian cycle must then travel along these edges an even number of times. We thus have that we use these edges either twice or four times. Without loss of generality let's suppose we wish to start and end our cycle in some point in the outside pentagon (if we start in the star the argument is different). By rotational symmetry all points are the same so say we start at the bottom left vertex of the pentagon. If we visit the outer pentagon first we enter the inner star at the bottom right corner and then notice that once we only have one path to visit the vertices of the star. If we follow it we either stop before visiting each vertex or visit a vertex (other than the starting one) twice before visiting them all. Likewise if we enter the star right by following this unique (Hamiltonian) path we see that we exit it at the rightmost vertex of the pentagon. Since this is not adjacent to the base vertex (bottom left), in order to preserve the Hamiltonian condition we have to choose either the clockwise or counterclockwise direction to get back to the base vertex. Either way we will miss at least one vertex. We can argue similarly to show that there is no way to have a Hamiltonian cycle by using four of the vertices connecting the star and the pentagon.
- (b) **True.** First observe that this is a bipartite graph with partition

$$\{v_1, v_5, v_6, v_7, v_{11}\} \cup \{v_2, v_3, v_4, v_8, v_9, v_{10}\} := A \cup B.$$

Therefore any Hamiltonian cycle must alternate between vertices from each set. Suppose we wish to start our Hamiltonian cycle at  $v_1$ . After having visited each vertex of  $A$  and 5 of  $B$  we have an acceptable cycle but it has missed one vertex  $B$  so it is not Hamiltonian. If we insist on having visited all vertices of  $B$  we must have visited either  $v_1$  3 times or some other vertex of  $A$  twice; at any rate this would not be a Hamiltonian cycle either.

- (c) **False.** Consider the "straight line" graph that is  $v_1e_1v_2e_2v_3e_3v_4e_4v_5e_5v_6e_6v_7$ . Clearly any loop that visits more than two vertices is visiting more than one vertex twice.
- (d) **True.** We assume that  $|V| > 1$  or else the problem is trivial. Now let's create another graph  $G' = (E', V')$  by adding another vertex  $v'$  and connecting it to every vertex in  $G$ . Now clearly this raises the degree of each vertex of  $G$  by one. We thus have that  $|E'| = |E| + |V|$  and  $|V'| = |V| + 1$  and moreover by this observation and our assumption,  $\deg(x) + \deg(y) \geq |V| + 1 = |V'|$  for every pair of non-adjacent vertices in  $V'$  (Note that  $v'$  is adjacent to everyone else so we don't need to worry about pairs containing it). Ore's theorem now implies that  $G'$  contains a Hamiltonian cycle. In particular we have can pick the following cycle  $\{v'x_1x_2\dots x_nv'\}$  where  $x_1, x_2, \dots, x_n$  are all the vertices of  $G$  which proves  $G$  is connected.

**Problem 7.** (20 pts) Let  $n \in \mathbb{N}$  be a natural number and  $K_n$  the complete graph in  $n$  vertices. Show that  $K_n$  admits  $(n - 1)!$  different<sup>2</sup> Hamiltonian cycles.

**Solution** Let's pick a starting vertex  $v_1$ . Since  $K_n$  is complete at any given step we can travel to any of the other  $(n - 1)$  vertices. However to keep it Hamiltonian we have to avoid the vertices we have already visited. This means that at the  $k$ -th step we have  $(n - k - 1)$  possibilities. Thus the number of Hamiltonian walk should be  $(n - 1) \times (n - 2) \dots (n - k) \dots 2 \times 1 = n!$ . Note that by the remark below we are considering lists up to cyclic ordering so we don't need to count the  $n$  ways to choose  $v_1$ .

**Problem 8.** (De Bruijn Graphs) Consider the set  $S(n)$  of binary sequences of length  $n$ , which is given by

$$S(n) := \{(s_1, \dots, s_n) : s_i \in \{0, 1\}, 1 \leq i \leq n\}.$$

Construct the directed<sup>3</sup> graph  $B_n$  whose vertex set is  $S(n)$ , and such that each vertex  $v = (s_1, \dots, s_n)$  has the following two edges going *out* of it, going to the vertices  $(s_2, s_3, \dots, s_n, 0)$  and  $(s_2, s_3, \dots, s_n, 1)$ . Show that  $B_n$  admits a (directed) Euler circuit for all  $n \in \mathbb{N}$ .

**Solution.** We use the following lemma: a directed graph admits an Eulerian circuit if and only if it is connected and the in degree of each vertex is the same as the out degree. Now note that although the construction is inductive this is *not* a graph with growing word length (such as one resembling a genalogical tree) since all the words are of fixed length. However the construction lets us identify exactly where the in edges are coming from and where the out edges are going to. Fix a word (vertex). For the out edges; delete the first letter and attach either a 0 or 1 at the end and those are the words (vertices) you are travelling to. For the in edges we do the opposite. Fix a word, delete the last letter and the in vertices are coming from the words with either a 1 or 0 concatenated in front of this  $n - 1$  substring. This shows that the in and out degrees for each vertex are the same. For connectedness we need only note that by applying the procedure of deleting the last letter and attaching a 0 or 1 at the beginning of it, we eventually visit every possible binary string.

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<sup>2</sup>A Hamiltonian cycle is considered to be an ordered list of all vertices, where only *adjacent* vertices are allowed to be consecutive. Such a list is only considered up to cyclic ordering.

<sup>3</sup>A graph is directed if the edges are oriented, i.e. each edge goes from a vertex to another vertex, that is, all the edges are directed from one vertex to another.