

## Arithmetic aspects of self-similar groups

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**Abstract.** We prove that an irreducible lattice in a semisimple algebraic group is virtually isomorphic to an arithmetic lattice if and only if it admits a faithful self-similar action on a rooted tree of finite valency.

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### 1. Introduction

Let  $\Gamma$  be a group. A *virtual endomorphism* of  $\Gamma$  is a homomorphism  $\varphi: \Lambda \rightarrow \Gamma$ , where  $\Lambda$  is a finite index subgroup in  $\Gamma$ . An *invariant normal subgroup* of  $\varphi$  is a normal subgroup  $N \triangleleft \Gamma$  contained in  $\Lambda$  such that  $\varphi(N) < N$ . A virtual endomorphism  $\varphi$  is *essential* (or the group  $\Gamma$  is  *$\varphi$ -simple*) if  $\Gamma$  contains no nontrivial normal  $\varphi$ -invariant subgroups. We will be interested in the question

**Question 1.** Which virtual endomorphisms are essential and when does  $\Gamma$  admit essential virtual endomorphisms?

We will only consider the case of lattices  $\Gamma$  in semisimple algebraic linear Lie groups  $G$ . (Note that the case of nilpotent groups was analyzed in [2].) Therefore, without loss of generality, the subgroups  $\Lambda$  can be taken torsion-free and  $G$  can be assumed to have no compact factors. In particular, existence of a nontrivial invariant normal subgroup is equivalent to the existence of an infinite invariant normal subgroup. Our main result is

**Theorem 2.** *Let  $\Gamma$  be an irreducible lattice in a linear semisimple algebraic Lie group  $G$ . Then the following are equivalent:*

- (a)  $\Gamma$  is virtually isomorphic to an arithmetic lattice in  $G$ , i.e., contains a finite index subgroup isomorphic to such arithmetic lattice.
- (b)  $\Gamma$  admits an essential virtual endomorphism  $\varphi: \Lambda \rightarrow \Gamma$ .

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Moreover, except for the case when the symmetric space of  $G$  is the hyperbolic plane  $\mathbb{H}^2$ , we will classify all essential virtual endomorphisms; see Corollary 13.

We note that conjugacy classes of irreducible faithful self-similar actions of a group  $\Gamma$  on a rooted tree (of finite valency) are in 1-1 correspondence with conjugacy classes of essential virtual endomorphisms of  $\Gamma$ , [20], [22], [21]. Therefore,

**Corollary 3.** *Let  $\Gamma$  be an irreducible lattice in a linear semisimple algebraic Lie group  $G$ . Then the following are equivalent:*

- (a)  $\Gamma$  is virtually isomorphic to an arithmetic lattice in  $G$ .
- (b)  $\Gamma$  admits an irreducible faithful self-similar action on a regular rooted tree (of finite valency).

An explicit example of a self-similar action on a rooted tree using the construction from the proof of Theorem 2 was worked out by Zoran Šunić and is presented in Section 4.

## 2. Preliminaries

**Rooted trees and group actions** (see e.g. [20]). A *rooted tree* is a simplicial tree  $T$  with a choice of a vertex  $r \in T$ , called the *root*. We will consider only rooted trees of finite valence which is the same (equal to  $n + 1$ ) for all vertices different from the root, while the root has valence  $n$ . Such rooted trees are called *regular*. A *descending edge* of a vertex  $u \in T$  is an edge  $e$  emanating from  $u$  which is separated from the root  $r$  by  $u$ . The *descending tree*  $T_u$  of a vertex  $u \in T$  is the subtree of  $T$  where all edges are separated from  $r$  by  $u$ . A group action on a rooted tree is a simplicial action of a group  $G$  on a tree  $T$  which fixes the root. Such action is called *irreducible* if it is transitive on the set of edges emanating from the root of the tree. Let  $X = \{x_1, \dots, x_n\}$  be an alphabet. Thus, for every vertex  $u$  of  $T$  we can (bijectively) label all edges descending from  $u$  by the elements of  $X$ . We can then identify vertices of  $T$  with words  $w \in X^*$  in the alphabet  $X$ . An action  $G \curvearrowright T$  on a rooted tree is called *self-similar* if the following property holds:

For every  $g \in G$ ,  $x \in X$  and a word  $w \in X^*$  there exist  $h \in G$ ,  $y \in X$  such that  $g(xw) = yh(w)$ .

More geometrically, one can define self-similarity as follows. For every vertex  $u$  of  $T$  connected to  $r$  by an edge, we fix an isomorphism  $\phi_u: T_u \rightarrow T$  sending  $u$  to  $r$ . Then, we require that for every vertex  $u$  as above, for every  $g \in G$ , there exists  $h \in G$  such that

$$\phi_v \circ g|_{T_u} = h \circ \phi_u$$

where  $v = g(u)$ . Note that the isomorphisms  $\phi_u$  define isomorphisms  $\phi_w: T_w \rightarrow T$  for every vertex  $w \in T$ . Self-similarity of  $G \curvearrowright T$  then means that for every  $g \in G$  and vertex  $w$  of  $T$  there exists  $h \in G$  such that

$$\phi_{g(w)} \circ g|_{T_w} = h \circ \phi_w.$$

Informally speaking, the restriction of  $g$  to  $T_w$  is a *copy* of the action of  $h$  on  $T$ , which explains the name *self-similar*.

**Lie groups and symmetric spaces** (see e.g. [14]). Let  $G$  be a *reductive* Lie group with finitely many connected components. (Reductivity of  $G$  means that the Cartan–Killing form of  $G$  is nondegenerate.) We endow  $G$  with a left-invariant Riemannian metric which is invariant under the action of the maximal compact subgroup  $K < G$  on the right. Then the quotient  $X = G/K$  with the ( $G$ -invariant) metric projected from  $G$  is a *symmetric space*. The action of  $G$  on  $X$  may have kernel, but this kernel is necessarily compact (and contained in  $K$ ). The above metric always has nonpositive (sectional) curvature. The symmetric space  $X$  is said to have *rank*  $r$  if  $r$  is maximal dimension of a Euclidean space that can be isometrically embedded in  $X$ . (The number  $r$  is also the real rank of the Lie group  $G$ .) The space  $X$  has rank 1 iff its sectional curvature  $K_X$  is strictly negative. Then (after rescaling the metric on  $X$ ), we can assume that  $-1 \leq K_X \leq -a^2 < 0$ .

The symmetric space  $X$  is said to be *irreducible* if it does not split nontrivially as a Riemannian direct product. Equivalently, the isometry group of  $X$  is *simple* as Lie group (i.e., has simple Lie algebra).

A Lie group  $G$  is *semisimple* if every irreducible component of the Lie algebra of  $G$  is *simple* (and has dimension at least 3). Every such group is reductive. Moreover, a reductive group (without compact factors) is semisimple iff the symmetric space  $X = G/K$  does not split off a Euclidean factor.

**Lattices in  $G$**  (see e.g. [17], [23] or [28]). Let  $G$  be a semisimple group above. A *lattice* in  $G$  is a discrete subgroup  $\Gamma < G$  such that  $\Gamma \backslash G$  has finite volume. Equivalently,  $X/\Gamma$  has finite volume. Let  $X = X_1 \times \cdots \times X_n$  be the de Rham decomposition of  $X$  in irreducible factors. Then a lattice  $\Gamma < G$  (and  $G$  itself) contains a finite-index subgroup  $\Gamma'$  which preserves each factor  $X_i$  in this decomposition. For a proper subset  $I \subset \{1, \dots, n\}$  let  $X_I$  denote the product

$$\prod_{i \in I} X_i.$$

Let  $\Gamma'_I$  denote the image of  $\Gamma'$  in the isometry group  $\text{Isom}(X_I)$  of  $X_I$ . The lattice  $\Gamma$  is said to be *irreducible* if every projection  $\Gamma'_I$  is a dense subgroup of  $\text{Isom}(X_I)$ . For instance, if  $n = 1$  then  $\Gamma$  is irreducible.

Every lattice  $\Gamma < G$  is finitely generated. If  $G$  is a linear algebraic group (more precisely, is the set of real or complex points of an algebraic group), then by a theorem of Selberg [24],  $\Gamma$  contains a torsion-free subgroup of finite index, i.e., is *virtually torsion-free*. (Discreteness of  $\Gamma$  is irrelevant here, what is important is that  $\Gamma$  is finitely generated.)

**Thick-thin decomposition.** Let  $M$  be a manifold of negative sectional curvature  $-1 \leq K_M \leq -a^2 < 0$  and  $\varepsilon$  be a positive number. The *thick-thin decomposition*

of  $M$  (with respect to  $\varepsilon$ ) is the partition of  $M$  in two subsets  $M_{\text{thick}} = M_{[\varepsilon, \infty)}$  and  $M_{\text{thin}} = M_{(0, \varepsilon)}$  where  $M_{(0, \varepsilon)}$  consists of points  $x \in M$  such that there exists a homotopically nontrivial loop  $\gamma$  in  $M$  based at  $x$ , whose length is  $< \varepsilon$ . The key result is that there exists a number  $\varepsilon = \mu(n, a)$  (which depends only on the dimension of  $M$  and the number  $a$ , and is called a *Margulis constant*) such that every component of the thin part of  $M$  has virtually nilpotent fundamental group. Moreover, if  $M$  has finite volume, then  $M_{\text{thin}}$  has only finitely many components. Furthermore, one can choose  $\varepsilon < \mu(n, a)$  such that  $M_{\text{thin}}$  is diffeomorphic to  $N \times \mathbb{R}_+$ , where  $N$  is the boundary of  $M_{\text{thin}}$ , while  $M_{\text{thick}}$  is diffeomorphic to  $M$ . In particular, one cannot embed  $\pi_1(M)$  in  $\pi_1(C)$  for any component  $C$  of  $M_{\text{thin}}$ . (See [1] or [26].) We will apply these results in the case when  $M$  is *locally symmetric*, i.e., is the quotient of a rank 1 symmetric space  $X$  by a discrete torsion-free group  $\Gamma$  of isometries. The group  $\Gamma$  then acts faithfully on  $X$ . Finiteness of the volume of  $M$  is equivalent to saying that  $\Gamma$  is a *lattice* in  $G$ .

**Discrete groups of isometries of rank 1 symmetric spaces.** We refer to [15] or [5] for this material. Let  $X$  be a rank 1 symmetric space as above; it has the *ideal boundary*  $\partial X$ . Let  $\Gamma < \text{Isom}(X)$  be a discrete group of isometries of  $X$ . Then we have the following dichotomy: Either  $\Gamma$  is *non-elementary*, or *elementary*. Elementary groups can be characterized by the property that they are *virtually nilpotent*, equivalently, every such group fixes a point in  $X \cup \partial X$  or has an invariant geodesic in  $X$ . Moreover, unless  $\Gamma$  is finite, the fixed-point set of  $\Gamma$  in  $\partial X$  contains at most two points; accordingly,  $\Gamma$  can have at most one invariant geodesic. In contrast, every nonelementary  $\Gamma$  contains a free nonabelian subgroup. If two elements of a discrete group  $\Gamma < \text{Isom}(X)$  have a common fixed point in  $\partial X$ , then they belong to an elementary subgroup of  $\Gamma$ . If two elements of  $\Gamma$  commute then they have a common fixed point in  $X \cup \partial X$ . Thus, if  $\alpha, \beta, \gamma \in \Gamma$  and

$$[\alpha, \beta] = 1, \quad [\beta, \gamma] = 1,$$

then  $\alpha, \beta, \gamma$  generate an elementary subgroup of  $\Gamma$ . In particular, a discrete group  $\Gamma$  cannot contain a subgroup isomorphic to  $F_2 \times F_2$ , where  $F_2$  is the rank 2 free group. If  $\Gamma$  is a lattice in  $\text{Isom}(X)$ , then  $\Gamma$  never acts on  $X$  as an elementary group (unless  $X$  is 1-dimensional). One way to see this is to note that  $\Gamma$  is Zariski dense in  $G$  (see [3]); thus, if  $\Gamma$  is virtually nilpotent, then  $G$  is virtually nilpotent as well. On the other hand, the isometry group of  $X$  is always a simple Lie group. Alternatively, one observes that elementary groups have finite *limit sets*, which, in turn, implies that  $X/\Gamma$  always has infinite volume (unless  $X$  is 1-dimensional).

Let  $N$  be a normal subgroup in a discrete group  $\Gamma < \text{Isom}(X)$ , where  $X$  is a rank 1 symmetric space. If  $N$  is infinite and elementary, then its fixed point set (or the invariant geodesic) is invariant under  $\Gamma$ . Thus, in this case,  $\Gamma$  is elementary itself. We, therefore, conclude that every normal subgroup  $N$  in a lattice  $\Gamma < G = \text{Isom}(X)$  is either elementary (in which case it is finite and, hence, trivial) or is nonelementary, in which case it contains a free nonabelian subgroup.

**Commensurators.** Let  $G$  be an Lie group and  $\Gamma < G$  be a subgroup. The *commensurator* of  $\Gamma$  in  $G$ , denoted  $\text{Comm}_G(\Gamma)$  is the subset of  $G$  consisting of elements  $g$  such that the groups  $\Gamma$  and  $g\Gamma g^{-1}$  are *commensurable*, i.e.,  $\Gamma \cap g\Gamma g^{-1}$  has finite index in both  $\Gamma$  and  $g\Gamma g^{-1}$ . Most of the time we will abbreviate  $\text{Comm}_G(\Gamma)$  to  $\text{Comm}(\Gamma)$  when the group  $G$  is clear. It is immediate that  $\text{Comm}(\Gamma)$  is a subgroup of  $G$  containing  $\Gamma$ . Moreover,  $\text{Comm}(\Gamma)$  is the same for all commensurable subgroups  $\Gamma < G$ . For an abstract group  $\Gamma$  one also defines the *abstract commensurator*  $\text{Comm}_a(\Gamma)$ , which is the group of (equivalence classes of) isomorphisms between finite-index subgroups of  $\Gamma$ . If  $\Gamma$  is an irreducible lattice in  $G$  such that  $X = G/K \neq \mathbb{H}^2$ , then by the Mostow Rigidity Theorem,  $\text{Comm}(\Gamma) \cong \text{Comm}_a(\Gamma)$ .

**Valuations, symmetric spaces and buildings.** Let  $F$  be an (algebraic) number field, i.e., a finite extension of  $\mathbb{Q}$ . Each embedding  $\sigma : F \hookrightarrow \mathbb{C}$  determines an *archimedean* norm  $|\cdot|_\sigma$  on  $F$  (the restriction of the absolute value on  $\mathbb{C}$ ); we let  $F_\sigma$  denote the completion of  $F$  with respect to this norm. Let  $\mathcal{O}$  be the ring of integers of  $F$ . Given a prime ideal  $p$  in  $\mathcal{O}$  we let  $F_p$  denote the completion of  $F$  with respect to the nonarchimedean valuation  $v_p$  on  $F$  determined by  $p$ . Typical examples of nonarchimedean valuations are the *p-adic valuations* on  $F = \mathbb{Q}$ , where the valuation is given by the norm

$$\left| \frac{a}{bp^n} \right|_p = p^n,$$

where  $p$  is prime and  $a, b$  are integers coprime with  $p$ . The completion  $\mathbb{Q}_p$  is then the field of  $p$ -adic numbers. In general,  $F$  is a finite extension of  $\mathbb{Q}$  and  $v_p$  restricts to some  $q$ -adic valuation on  $\mathbb{Q}$  for some prime  $q \in \mathbb{Z}$ . The reader unfamiliar with valuations can think of this example as the model since the general case is very similar.

Let  $\underline{G}$  be a semisimple algebraic group defined over  $\mathcal{O}$ . We let  $G_\infty$  denote the finite product

$$\prod_{\sigma : F \hookrightarrow \mathbb{C}} \underline{G}(F_\sigma)$$

where the product is taken over all embeddings  $\sigma : F \hookrightarrow \mathbb{C}$  (there are only finitely many of these). The image of  $\underline{G}(F)$  under the diagonal embedding is dense in  $G_\infty$  (this is a special case of the *weak approximation theorem*). The group  $G_\infty$  is a real or complex semisimple Lie group. Let  $K_\infty < G_\infty$  be a maximal compact subgroup. We will be interested in the symmetric space  $X = G_\infty/K_\infty$ . Note that  $G_\infty$  need not act faithfully on  $X$ , the kernel  $C$  of the action is compact and comes from compact factors of the group  $G_\infty$  as well as from finite normal subgroups in the noncompact factors of  $G_\infty$ . We let  $G := G_\infty/C$ . We will regard  $G$  as the isometry group of the symmetric space  $X$ . Then  $G$  has no compact factors and nontrivial finite normal subgroups. Similarly, for each prime ideal  $p$ , the group  $G_p := \underline{G}(F_p)$  acts as a group of isometries of a *Euclidean building*  $X_p$ , which is a certain regular cell complex where each facet is isometric to a polytope in  $\mathbb{R}^n$ . The group of integer points  $\underline{G}(\mathcal{O}_p)$  is the stabilizer in  $G_p$  of a *special vertex*  $o_p \in X_p$ . See e.g. [6].

**Arithmetic groups** (see e.g. [17], [27], [28]). An *arithmetic subgroup* of the group  $G_\infty$  as above is a group  $\Gamma$  commensurable to the group  $\underline{G}(\mathcal{O})$ , embedded diagonally. (Note that the group  $\underline{G}(\mathcal{O})$  need not be discrete in the group  $\underline{G}(F)$ .) Projections of arithmetic groups to the group  $G$  are also called arithmetic. The difference between these two notions of arithmeticity comes from finite normal subgroups which  $\Gamma < \underline{G}(F)$  may contain and can be safely ignored for the purpose of this paper. Whenever convenient, we will consider arithmetic subgroups of  $G$  or their lifts to  $G_\infty$ .

The basic examples of arithmetic groups the reader should think about are:

1.  $\underline{G} = \mathrm{SL}(n)$ ,  $F = \mathbb{Q}$ ,  $G = \mathrm{SL}(n, \mathbb{R})$ ,  $\underline{G}(\mathcal{O}) = \mathrm{SL}(n, \mathbb{Z})$ .
2.  $F = \mathbb{Q}(\sqrt{2})$ ,  $\mathcal{O} = \mathbb{Z}[\sqrt{2}]$ ,  $\underline{G} < \mathrm{GL}(n)$  is the automorphism group  $O(q)$  of the quadratic form

$$q(x) = x_1^2 + \cdots + x_n^2 - \sqrt{2}x_{n+1}^2.$$

Then  $\underline{G}(\mathcal{O})$  is  $O(q, \mathcal{O})$ , the group of matrices with coefficients in  $\mathcal{O}$  that preserve  $q$ . The field  $F$  is totally real and has two embeddings to  $\mathbb{C}$ . Accordingly,  $G_\infty \cong O(n, 1) \times O(n+1)$ , since  $q$  has the signature  $(n, 1)$ , while its image  $q^\sigma$  under the embedding  $\sigma: F \rightarrow \mathbb{R}$  sending  $\sqrt{2}$  to  $-\sqrt{2}$ , is positive-definite. Then  $G = \mathrm{PO}(n, 1)$  is the group of isometries of the hyperbolic  $n$ -space (which can be realized as the projectivization of the hyperboloid  $\{x : q(x) = -1\}$ ). The group  $\underline{G}(\mathcal{O})$  acts as a lattice on  $\mathbb{H}^n$ . This lattice is irreducible even though  $G_\infty$  splits as a nontrivial direct product.

The following deep results of Margulis (see [17] as well as [28]) will be the key for this paper:

**Theorem 4** (Margulis Arithmeticity Theorem). *If  $G$  has rank  $\geq 2$  then every irreducible lattice in  $G$  is arithmetic.*

**Theorem 5** (Margulis Commensurator Theorem). *Let  $\Gamma < G$  be an irreducible lattice (where  $G$  may have rank 1). Then  $\Gamma$  is arithmetic iff  $\mathrm{Comm}(\Gamma)$  is dense in  $G$ .*

**Theorem 6** (Margulis Finiteness Theorem). *Let  $\Gamma < G$  be an irreducible lattice of rank  $\geq 2$ . Then for every normal subgroup  $\Lambda \triangleleft \Gamma$  either  $\Lambda$  is finite or  $\Gamma/\Lambda$  is finite.*

In the arithmetic case, the commensurator of  $\Gamma = \underline{G}(\mathcal{O})$  in  $G_\infty$  is described by A. Borel (see [3]):

$$\mathrm{Comm}(\Gamma) = \underline{G}(F),$$

embedded diagonally in  $G_\infty$ .

In view of the Margulis Arithmeticity Theorem, the study of non-arithmetic lattices is reduced to the case of rank 1 symmetric spaces (and Lie groups). There are four classes of such symmetric spaces:  $\mathbb{H}^n$  (the real-hyperbolic  $n$ -space),  $\mathbb{C}\mathbb{H}^n$  (the complex-hyperbolic  $n$ -space,  $n \geq 2$ ),  $\mathbb{H}\mathbb{H}^n$  (the quaternionic-hyperbolic  $n$ -space,

$n \geq 2$ ) and  $\mathbf{O}\mathbb{H}^2$  (the octonionic hyperbolic plane). In the last two cases the Margulis Arithmeticity Theorem is supplemented by the following theorem which is a combination of work by Corlette [7] and by Gromov and Schoen [12]:

**Theorem 7** (Corlette, Gromov and Schoen). *Every lattice in the isometry group of  $\mathbf{H}\mathbb{H}^n$  ( $n \geq 2$ ) and  $\mathbf{O}\mathbb{H}^2$  is arithmetic.*

On the other hand, for  $n \geq 2$ , the isometry of  $\mathbb{H}^n$  contains non-arithmetic lattices (Gromov and Piatetsky-Shapiro [11]) while the isometry groups of  $\mathbb{C}\mathbb{H}^2$  and  $\mathbb{C}\mathbb{H}^3$  also contain non-arithmetic lattices (Deligne and Mostow [8]). Arithmeticity of lattices in the isometry groups  $\text{PU}(n, 1)$  of  $\mathbb{C}\mathbb{H}^n$  ( $n \geq 4$ ) is an open problem.

**Rigidity and non-rigidity.** Let  $\Gamma < G$  be an irreducible lattice in a semisimple Lie group  $G$  as above. Then, by the Mostow Rigidity Theorem [19], unless  $X = G/K$  is the hyperbolic plane, every discrete embedding  $\phi: \Gamma \rightarrow G$  is given by conjugating the identity embedding  $\Gamma \hookrightarrow G$  by an isometry of  $X$ . In particular, for every discrete embedding  $\phi: \Gamma \rightarrow G$ , there exists a finite-index subgroup  $\Gamma' < \Gamma$  such that  $\phi|_{\Gamma'}$  is induced by an automorphism of  $G$ . The group of automorphisms of  $G$  is a finite extension of the group of inner automorphisms of  $G$ .

Therefore, arithmeticity of  $\Gamma$  is the intrinsic group-theoretic property of  $\Gamma$ , independent of the embedding  $\Gamma \hookrightarrow G$ . In contrast, if  $\Gamma$  is a lattice in  $\text{PSL}(2, \mathbb{R})$ , the group of orientation-preserving isometries of the hyperbolic plane,  $\Gamma$  could be isomorphic to an arithmetic group without being an arithmetic group itself. For instance, all torsion-free lattices satisfy this property. Moreover, it is possible that  $\Gamma$  is not isomorphic to an arithmetic group but is commensurable to a group  $\Gamma'$  which is torsion-free and, hence, is isomorphic to an arithmetic group. For instance, consider groups  $T = T(p, q, r)$  with presentations

$$\langle x, y, z \mid xyz = 1, x^p = y^q = z^r = 1 \rangle, \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1.$$

Then every  $T$  embeds as a lattice (unique up to conjugation) in  $\text{PSL}(2, \mathbb{R})$ , but this lattice is arithmetic only for 85 different unordered triples  $(p, q, r)$ ; see [25].

By combining the Mostow Rigidity Theorem and the Margulis Finiteness Theorem, we get:

**Corollary 8.** *Let  $\widehat{G}(F)$  denote the finite extension of  $\underline{G}(F)$  by its outer automorphisms. Let  $\Gamma < \underline{G}(F)$  be an irreducible lattice of rank  $\geq 2$ . Then for every virtual endomorphism of  $\Gamma$ ,  $\varphi: \Lambda \rightarrow \Gamma$ , either  $\varphi$  has finite image or it is induced by some  $\alpha \in \widehat{G}(F)$ .*

*Proof.* Suppose that  $\varphi$  has finite kernel and, thus, infinite image. Thus,  $\varphi$  is induced by conjugation via some isometry  $\alpha$  of  $X$ . The orbifold  $M' := X/\varphi(\Gamma)$  has finite

volume, therefore, the orbifold-covering  $M' \rightarrow M = X/\Gamma$  (induced by  $\varphi$ ) has to have finite degree. This means that  $\varphi(\Gamma)$  has finite index in  $\Gamma$ . The isometry  $\alpha$  belongs to the finite extension of  $G$  by its outer automorphisms. We keep the notation  $\alpha$  for its lift to an automorphism of  $G_\infty$ . Since  $\alpha$  conjugates a lattice  $\Lambda < \underline{G}(F)$  to another lattice in  $\underline{G}(F)$ , it follows that  $\alpha$  belongs to  $\widehat{\underline{G}}(F)$ .  $\square$

We see, therefore, that every virtual automorphism of  $\Gamma$  with infinite image is induced by an element of the commensurator of  $\Gamma$ , which is an element of a finite extension of  $G_\infty$ . In order to avoid heavy notation, we will keep the notation  $\text{Comm}(\Gamma)$  for the above finite extension  $\widehat{\underline{G}}(F)$  of  $\underline{G}(F)$ . We note that  $\widehat{\underline{G}}(F)$  still acts by isometries of the Euclidean buildings  $X_p$  defined above, extending the actions of  $\underline{G}(F)$ .

### 3. Proof of Theorem 2

In what follows,  $\Gamma$  is a lattice (arithmetic or not) in a real or complex Lie group  $G_\infty$ . Given an element  $\alpha \in \text{Comm}(\Gamma)$  we let  $\varphi = \varphi_\alpha$  denote the automorphism of  $G_\infty$  induced by conjugation

$$\varphi(g) = \alpha g \alpha^{-1}.$$

Then  $\alpha$  induces *virtual endomorphisms* of  $\Gamma$ :  $\varphi(\Gamma) \cap \Gamma$  is a finite-index subgroup  $\Gamma_1 < \Gamma$ . Thus,  $\Gamma_2 := \varphi^{-1}(\Gamma_1)$  is also a finite-index subgroup in  $\Gamma$  and, hence  $\varphi: \Gamma_2 \rightarrow \Gamma_1$  is a virtual endomorphism of  $\Gamma$ .

**3.1. Arithmetic case.** Assume now that  $\Gamma$  is arithmetic. We will use the notation introduced in the previous section. We say that  $\alpha \in \text{Comm}(\Gamma)$  *virtually normalizes* the lattice  $\Gamma$  if there exists a finite-index subgroup  $\Gamma' < \Gamma$  which is normalized by  $\alpha$ .

**Lemma 9.** *One of the following mutually exclusive possibilities holds: Either  $\alpha$  virtually normalizes  $\Gamma$  or there exists a prime ideal  $p$  in  $\mathcal{O}$  such that  $\alpha$  acts on  $X_p$  with unbounded orbits (i.e., is a **hyperbolic isometry**).*

*Proof.* Suppose that  $\alpha$  acts on every  $X_p$  with bounded orbits. Then, as in the proof of the Margulis Arithmeticity Theorem (see [17] or [28], pp. 120–121), matrix entries of powers of  $\alpha$  have uniformly bounded denominators which, in turn, implies that the groups  $\alpha^N \underline{G}(\mathcal{O}) \alpha^{-N}$  intersect  $\underline{G}(\mathcal{O})$  along subgroups of finite index which are bounded by a constant independent of  $N$ . Taking the intersection of all these subgroups, we obtain a finite-index subgroup  $\Lambda < \underline{G}(\mathcal{O})$  which is normalized by  $\alpha$ . Therefore,  $\varphi$  preserves a finite-index subgroup of  $\Lambda \cap \Gamma$ .

Suppose that  $\alpha$  acts hyperbolically on some  $X_p$  and virtually normalizes  $\Gamma$ , i.e.,  $\alpha \Gamma' \alpha^{-1} = \Gamma'$  for some finite-index subgroup  $\Gamma' < \Gamma$ . Since  $\Gamma'$  is commensurable to  $\underline{G}(\mathcal{O})$ , it also fixes a point in  $X_p$  (although, not necessarily  $o_p$ ). Since  $\alpha$  acts hyperbolically on  $X_p$  and normalizes  $\Gamma'$ , the fixed-point set of  $\Gamma'$  in  $X_p$  is unbounded.



Therefore, this fixed-point set contains an ideal boundary point  $\xi$  of the building  $X_p$ . Algebraically, this means that  $\Gamma'$  normalizes a parabolic subgroup  $P < G_p$ . Since  $\Gamma'$  is commensurable to  $\underline{G}(\mathcal{O})$ , it is also Zariski dense in  $G_p$ . Therefore,  $G_p$  also would have to normalize  $P$  (and, thus, fix  $\xi$ ). However (since  $\underline{G}$  is semisimple),  $G_p$  cannot have proper normal parabolic subgroups.

One can give a more geometric argument as follows. The ideal boundary  $\partial X_p$  of  $X_p$  is a spherical building and, by the Cartan decomposition for  $G_p$ , the group  $\underline{G}(\mathcal{O})$  acts transitively on the chambers of  $\partial X_p$ . Since  $\Gamma'$  is commensurable to  $\underline{G}(\mathcal{O})$ , the orbit of  $\xi$  under  $\underline{G}(\mathcal{O})$  is finite. This contradicts the fact that  $\partial X_p$  is infinite.  $\square$

We note that the property that  $\alpha \in \text{Comm}(\Gamma)$  acts hyperbolically on some  $X_p$  amounts to saying that the matrix coefficients of powers of  $\alpha$  have arbitrarily high denominators. For instance, if  $\underline{G} = \text{SL}(n)$  then any diagonal matrix where some diagonal entries belong to  $F \setminus \mathcal{O}$ , satisfies this property.

**Theorem 10.** *Let  $\Gamma < \underline{G}(F)$  be an arithmetic group,  $\alpha \in \text{Comm}(\Gamma)$  be an element which acts as a hyperbolic isometry of some  $X_p$ , and  $\Lambda < \Gamma$  be a finite-index subgroup such that  $\varphi_\alpha(\Lambda) \subset \Gamma$ . If  $N < \Lambda$  is a normal subgroup of  $\Gamma$  such that  $\varphi_\alpha(N) \subset N$ , then  $N$  is finite.*

*Proof.* As before, let  $X$  be the symmetric space of  $G$  and let  $M := X/N$ . The manifold  $M$  isometrically covers  $X/\Gamma$ . We also obtain an isometric covering

$$q = \bar{\alpha}: M \rightarrow M$$

induced by the endomorphism  $\varphi = \varphi_\alpha$  of the fundamental group of  $M$ .

1. First, consider the case when  $G$  has rank  $\geq 2$ . Then, by Margulis Finiteness Theorem, every normal subgroup of  $\Gamma$  is either finite or has finite index in  $\Gamma$ . Assuming  $N$  has finite index in  $\Gamma$ , the manifold  $M$  has finite volume and, since  $q$  preserves the volume form, it has to be a diffeomorphism. Thus,  $\varphi(N) = N$ , which (by the previous lemma) contradicts our choice of  $\alpha$ .

2. We now consider the more interesting case when  $G$  has rank 1. We assume that  $N$  is an infinite group. Therefore, it is Zariski dense in  $G$ . In what follows, we will lift thick-thin decompositions from  $X/\Gamma$  to  $M$  and, by abuse of terminology, will refer to them as *lifted thick-thin decompositions* of  $M = M_{\text{thick}} \cup M_{\text{thin}}$ . By choosing  $\varepsilon$  in the definition of the thick-thin decomposition sufficiently small, we can assume that each component of  $M_{\text{thin}}$  is non-compact; in particular,  $M_{\text{thick}}$  is connected. (The thick part of  $M$  can be disconnected only if  $X \cong \mathbb{H}^2$ .)

Note that the group  $\Gamma$  acts on  $M$  isometrically, we use the notation  $\bar{\gamma}$  for the isometry of  $M$  induced by  $\gamma \in \Gamma$ . For any lifted thick-thin decomposition of  $M$ , the action  $\Gamma \curvearrowright M_{\text{thick}}$  is cocompact since  $(X/\Gamma)_{\text{thick}}$  is compact.

Consider the iterations  $q^k$  of the isometric endomorphism  $q: M \rightarrow M$ . Pick a connected compact subset  $C \subset M_{\text{thick}}$  whose fundamental group maps onto a

Zariski dense subgroup  $H$  of  $N$  and which contains a fundamental domain for the (cocompact) action  $\Gamma \curvearrowright M_{\text{thick}}$ .

Since each component of  $M_{\text{thin}}$  has virtually nilpotent fundamental group, while  $H$  is not virtually nilpotent, it follows that  $q^k(C)$  is never contained in  $M_{\text{thin}}$ . Therefore, for each  $k$  there exists  $x_k \in C$  such that  $q^k(x_k) \in M_{\text{thick}}$ .

Since the action  $\Gamma \curvearrowright M_{\text{thick}}$  is cocompact, for every  $k$  there exists  $\gamma_k \in \Gamma$  such that

$$\bar{\gamma}_k \circ q^k(x_k) \in C.$$

Set  $\bar{\beta}_k := \bar{\gamma}_k \circ q^k$  and let  $D$  be the diameter of  $C$ . Since  $q$  is isometric,  $q^k(C)$  also has diameter  $\leq D$ , which implies that  $\bar{\beta}_k(C)$  is contained in the  $D$ -neighborhood of  $C$ . Therefore, by Arzela–Ascoli theorem, the sequence of isometries  $\bar{\beta}_k$  is precompact.

**Lemma 11.** *The set  $\bar{I} := \{\bar{\beta}_k : k \in \mathbb{N}\}$  is finite.*

*Proof.* If not, then there will be arbitrarily large  $k, m$  such that  $\bar{\beta}_k \neq \bar{\beta}_m$  and the restrictions  $\bar{\beta}_k|_C, \bar{\beta}_m|_C$  are arbitrarily close in the sup-metric. In particular, for large  $k, m$ , they are homotopic and, hence, induce the same (up to conjugation in  $N$ ) map  $H \rightarrow N$  given by

$$h \mapsto \beta_k h \beta_k^{-1}, \quad h \mapsto \beta_m h \beta_m^{-1}.$$

Since  $H$  is Zariski dense in  $G$ , its centralizer in  $G$  is trivial and, thus, we have the equality of the cosets

$$\beta_k \cdot N = \beta_m \cdot N.$$

Hence,  $\bar{\beta}_k = \bar{\beta}_m$ . Contradiction. □

We continue with the proof of Theorem 10. Let  $I \subset \widehat{G}(F)$  denote the finite set of representatives of lifts of the isometries  $\bar{\beta}_k \in \bar{I}$ . For each  $k \in \mathbb{N}$ ,

$$\gamma_k \cdot \alpha^k \cdot N \subset I \cdot N,$$

and, thus,

$$\alpha^k \in \gamma_k^{-1} \cdot I \cdot N,$$

where  $\gamma_k \in \Gamma$ .

We now consider the action of the isometries in the above equation on the building  $X_p$ , where  $p$  is such that  $\alpha : X_p \rightarrow X_p$  is hyperbolic. The group  $\Gamma$  fixes a point  $x_p \in X_p$ , the images  $I \cdot x_p$  form a finite set. Therefore, the set

$$\gamma_k^{-1} \cdot I \cdot N \cdot x_p = \gamma_k^{-1} \cdot I \cdot x_p$$

is bounded in  $X_p$ . However, by our assumption, orbits of  $\langle \alpha \rangle$  on  $X_p$  are unbounded. Contradiction. □

**Corollary 12.** *Suppose that  $\Gamma$  has no finite normal subgroups. Then every  $\alpha$  as in Theorem 10 induces an essential virtual endomorphism  $\varphi_\alpha$  of  $\Gamma$ .*

The following corollary then classifies all virtual endomorphisms of arithmetic groups induced by conjugation (which is the case unless  $X \cong \mathbb{H}^2$ ):

**Corollary 13.** *Let  $\Gamma < \underline{G}(F)$  be arithmetic with no finite normal subgroups,  $\Lambda < \Gamma$  be a finite-index subgroup and  $\varphi: \Lambda \rightarrow \Gamma$  be an essential virtual endomorphism of  $\Gamma$  such that  $\varphi = \varphi_\alpha$ , where  $\alpha \in \text{Comm}(\Gamma)$ . Then  $\alpha$  does not normalize finite-index subgroups of  $\Gamma$ , equivalently,  $\alpha$  acts hyperbolically on some  $X_p$ . Conversely, for every  $\alpha \in \text{Comm}(\Gamma)$  that acts hyperbolically on some  $X_p$ , the virtual endomorphism  $\varphi_\alpha$  of  $\Gamma$  is essential.*

**3.2. Non-arithmetic case.** We now consider virtual endomorphisms of non-arithmetic lattices. Then  $\Gamma$  is a lattice in a rank 1 Lie group  $G$ . Without loss of generality, we may assume that  $G$  acts faithfully on the associated symmetric space  $X = G/K$ . In this section we will also assume that the Lie algebra of  $G$  is not isomorphic to  $sl(2, \mathbb{R})$ , i.e., the symmetric space  $X = G/K$  of  $G$  is not isometric to the hyperbolic plane. We assume that  $\Gamma < G$  is non-arithmetic. Our assumptions imply that  $\Gamma$  contains no nontrivial normal finite subgroups. The same, of course, applies to finite-index subgroups  $\Lambda < \Gamma$ .

**Proposition 14.** *Under the above assumptions, every virtual endomorphism  $\varphi$  of  $\Gamma$  is not essential.*

*Proof.* 1. Suppose that  $\varphi: \Lambda \rightarrow \Gamma$  is not injective. Let  $K$  denote the kernel of  $\varphi$ . Note that this subgroup is not necessarily normal in  $\Gamma$ . Since  $\Lambda$  contains no nontrivial normal subgroups, it follows that the group  $K$  is infinite.

We first consider the case when  $\Lambda$  is normal in  $\Gamma$ . Let  $\gamma_1, \dots, \gamma_n$  be the generators of  $\Gamma$ . Consider the conjugates

$$K_i := \gamma_i^{-1} K \gamma_i \subset \Gamma, \quad K_0 := K.$$

Define homomorphisms

$$\varphi_i: \Lambda \rightarrow \Gamma, \quad \varphi_i(g) = \varphi(\gamma_i g \gamma_i^{-1}), \quad i = 1, \dots, n; \quad \varphi_0 := \varphi.$$

For every nonempty subset  $I \subset \{0, \dots, n\}$  consider the normal subgroup  $K_I \triangleleft \Gamma$  given by the intersection

$$K_I := \bigcap_{i \in I} K_i$$

This subgroup is the kernel of the homomorphism

$$\Phi_I: \Lambda \rightarrow \prod_{i \in I} \Gamma,$$

where the  $i$ -th component of  $\Phi_I$  is  $\varphi_i$ ,  $i \in I$ . We let  $\Gamma_i$  denote the  $i$ -th factor of the product group  $\prod_{i \in I} \Gamma$ .

Suppose that for  $J = \{0, \dots, n\}$ ,  $K_J$  is infinite. Let  $I$  be a smallest (with respect to the inclusion) subset of  $J$  such that  $K_I$  is finite (i.e., trivial, since it is a normal subgroup in  $\Lambda$ ). Since each  $\varphi_i$  has infinite kernel,  $I$  contains at least two elements. By the choice of  $I$ , for every  $i \in I$ ,

$$H_i := \Phi_I(\Lambda) \cap \Gamma_i$$

is infinite. Since  $H_i$  is normal in  $\Phi_I(\Lambda)$ , the subgroup  $H_i$  contains a free nonabelian subgroup  $F_i$ . Hence, the group  $\Lambda$  contains a direct product of free nonabelian subgroups. This is impossible since  $\Lambda$  is isomorphic to a discrete subgroup of isometries of the negatively curved symmetric space  $X$  defined above. Contradiction. Thus,  $K_J$  is infinite and we obtain an infinite normal subgroup  $N = K_J \triangleleft \Lambda$  of the group  $\Gamma$  such that  $\varphi(N) = 1 \subset N$ .

We now consider the case when  $\Lambda$  is not necessarily normal in  $\Gamma$ . If  $\varphi: \Lambda \rightarrow \Gamma$  is a virtual endomorphism with infinite kernel, we find a finite index subgroup  $\Lambda' \subset \Lambda$  which is normal in  $\Gamma$ . Then the restriction  $\varphi' = \varphi|_{\Lambda'}$  still has infinite kernel and we obtain a contradiction as above.

2. Suppose now that  $\varphi$  is injective. Then, by the Mostow Rigidity Theorem, the homomorphism  $\varphi$  is induced by conjugation via some  $\alpha \in \text{Comm}(\Gamma)$ . Recall that, by the Margulis Commensurator Theorem,  $\text{Comm}(\Gamma)$  is not dense in  $G$ . Since  $\Gamma$  is Zariski dense in  $G$ , it follows that  $\text{Comm}(\Gamma)$  is discrete. Therefore, since  $\Gamma$  is a lattice,  $\text{Comm}(\Gamma)$  is a finite extension of  $\Gamma$ . Thus,  $\Lambda$  has finite index in  $\text{Comm}(\Gamma)$  and, hence, contains a finite index subgroup  $N < \Lambda$  which is normal in  $\text{Comm}(\Gamma)$ . In particular,

$$\alpha N \alpha^{-1} = N$$

and  $N \triangleleft \Gamma$  is a normal subgroup. Clearly,  $N$  is an infinite  $\varphi$ -invariant subgroup of  $\Gamma$  and, thus,  $\varphi$  is not essential. □

**3.3. Subgroups of  $\text{SL}(2, \mathbb{R})$ .** Suppose now that  $\Gamma$  is a subgroup in a Lie group  $G$  which is locally isomorphic to  $\text{SL}(2, \mathbb{R})$ , i.e., the symmetric space of  $G$  is the hyperbolic plane  $\mathbb{H}^2$ . Then the abstract commensurator of a lattice  $\Gamma < G$  is not isomorphic to the commensurator of  $\Gamma$  in  $G$ .

**Lemma 15.** *Every lattice  $\Gamma < G$  contains a finite index torsion-free subgroup  $\Lambda$  which is isomorphic to an arithmetic subgroup of  $G$ . (This isomorphism, of course, need not be induced by conjugation, thus  $\Lambda$  itself need not be arithmetic.)*

*Proof.* It suffices to consider the case of  $G = \text{SL}(2, \mathbb{R})$ . Then  $G$  contains both cocompact and non-cocompact arithmetic lattices  $\Gamma', \Gamma''$ . (This is a special case of Borel’s theorem on existence of cocompact and non-cocompact arithmetic lattices in the given algebraic group, [3].) Specifically, one can take  $\Gamma'$  such that the quotient orbifold  $\mathbb{H}^2/\Gamma'$  is  $S^2(2, 4, 8)$ , the sphere with three cone points of the orders 2, 4, 8 (see [25]), while for  $\Gamma''$  one can take the modular group  $\text{SL}(2, \mathbb{Z})$ , whose quotient

is the orbifold  $S^2(2, 3, \infty)$ , the sphere with two cone points of orders 2, 3 and one puncture.

We leave it to the reader to construct an 8-fold orbifold cover  $S_2 \rightarrow S^2(2, 4, 8)$  where  $S_2$  is the surface of genus 2, and a 6-fold orbifold cover  $S^2(\infty, \infty, \infty) \rightarrow S^2(2, 3, \infty)$ , where  $S^2(\infty, \infty, \infty)$  is the sphere with 3 punctures. Thus,  $\Gamma'$  contains an index 8 subgroup isomorphic to  $\pi_1(S_2)$ , while  $\Gamma''$  contains an index 6 subgroup isomorphic to the free group of rank 2,  $F_2$ . Let  $\Gamma < G$  be a lattice. Then  $\Gamma$  contains a torsion-free subgroup of finite index  $\Lambda$ ; the quotient  $\mathbb{H}^2/\Lambda$  is a surface which is compact if and only if  $\mathbb{H}^2/\Gamma$  is. Suppose that  $\mathbb{H}^2/\Gamma$  is compact. Then  $\mathbb{H}^2/\Lambda$  is a surface  $S_g$  of genus  $g \geq 2$ . Every such surface is a  $(g - 2)$ -fold cover over  $S_2$ . Hence,  $\Lambda$  is isomorphic to an index  $8(g - 2)$  subgroup  $\Lambda' < \Gamma'$ , which is, of course, arithmetic. If  $\mathbb{H}^2/\Gamma$  is not compact, then  $\Lambda$  is isomorphic to a free group  $F_r$  of rank  $r \geq 2$ . Then  $F_2$  contains an index  $r - 2$  subgroup isomorphic to  $F_r$ . Hence,  $\Lambda$  is isomorphic to an index  $6(r - 2)$  subgroup  $\Lambda'' < \Gamma''$ , which is again arithmetic.

The reader uncomfortable with orbifold covers may instead use the following observation: Every two compact surfaces  $S_g, S_h$  of genus  $\geq 2$  are finite covers over  $S_2$ , while every two finite rank nonabelian free groups  $F_r, F_s$  are finite index subgroups in  $F_2$ . Therefore,  $S_g, S_h$  admit a common finite cover, while  $F_r, F_s$  admit isomorphic finite index subgroups. Therefore, for every cocompact lattice  $\Gamma < G$  there exists a finite index subgroup isomorphic to a finite index subgroup of  $\Gamma'$ , while for every non-cocompact lattice  $\Gamma < G$  there exists a finite index subgroup isomorphic to a finite index subgroup of  $\Gamma''$ . □

Now, in view of the above lemma, by Theorem 10,  $\Lambda$ , and, hence,  $\Gamma$ , admits essential virtual endomorphisms  $\varphi$ .

By combining this lemma with Theorem 10 and Proposition 14, we obtain Theorem 2.

#### 4. An example of a faithful, self-similar action of $\text{PSL}_2(\mathbb{Z})$ on the ternary rooted tree

The following explicit example of a faithful self-similar action on a tree associated with an element of  $\text{Comm}(\Gamma)$  was computed by Zoran Šunić.

Let  $X = \{0, \dots, k - 1\}$ . Every automorphism  $g$  of the tree  $X^*$  decomposes as

$$g = \pi_g (g|_0, g|_1, \dots, g|_{k-1}),$$

where  $\pi_g$  is a permutation of the alphabet  $X$ , called root permutation of  $g$ , describing the action of  $g$  on the first level of the tree, and  $g|_0, \dots, g|_{k-1}$  are tree automorphisms, called sections of  $g$ , describing the action of  $g$  on the subtrees below the first level. For every letter  $x$  in  $X$  and word  $w$  over  $X$ ,

$$g(xw) = \pi_g(x)g|_x(w).$$

We provide a brief description of the self-similar action on a rooted  $k$ -ary tree by automorphisms associated to a virtual endomorphism  $\phi: \Lambda \rightarrow \Gamma$  (for the original definition see [20], Proposition 4.9, or [21], Proposition 2.5.10).

Choose a left transversal  $T = \{t_0, t_1, \dots, t_{k-1}\}$  for  $\Lambda$  in  $\Gamma$ , with  $t_0 = 1$ , i.e., representatives of the quotient  $\Gamma/\Lambda$ . For  $g \in \Gamma$ , let  $\bar{g}$  denote the representative of the left coset  $g\Lambda$ .

A self-similar action of  $\Gamma$  on the rooted  $k$ -ary tree induced by  $\phi$  is defined as follows. For  $g \in \Gamma$ , define the root permutation  $\pi_g$  of  $X = \{0, 1, \dots, k-1\}$  by

$$\pi_g(x) = y \quad \text{if and only if} \quad \overline{gt_x} = t_y$$

and the section of  $g$  at  $x \in X$  by

$$g|_x = \phi(\overline{gt_x}^{-1}gt_x).$$

The action induced by the virtual endomorphism  $\phi$  may be not faithful. It is faithful precisely when the virtual endomorphism is essential. In the following example we will abuse the notation and use matrix notation for the elements of  $\text{PSL}(2, \mathbb{Z})$ .

**Example 16** ( $\Gamma = \text{PSL}_2(\mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3$ ). It is well known that the two free factors of  $\Gamma$  may be generated by the elements represented by the matrices

$$a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

of orders 2 and 3, respectively.

Let  $\Lambda$  be the subgroup

$$\Lambda = \left\{ \begin{pmatrix} x & 2y \\ z & w \end{pmatrix} \mid x, y, z, w \in \mathbb{Z}, \quad xw - 2yz = 1 \right\} \leq \Gamma.$$

It is easy to establish that, for any two matrices

$$M_1 = \begin{pmatrix} x_1 & y_1 \\ z_1 & w_1 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} x_2 & y_2 \\ z_2 & w_2 \end{pmatrix}$$

in  $\Gamma$ ,  $M_2^{-1}M_1 \in \Lambda$  if and only if

$$y_1 \equiv y_2 \pmod{2} \quad \text{and} \quad w_2 \equiv w_1 \pmod{2}.$$

Therefore  $|\Gamma : \Lambda| = 3$  and left coset representatives of  $\Lambda$  in  $\Gamma$  are given by

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad b^2 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Define  $\phi: \Lambda \rightarrow \Gamma$  by

$$\phi \begin{pmatrix} x & 2y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y \\ 2z & w \end{pmatrix}.$$

Since  $\phi$  is just conjugation by the diagonal matrix  $D = \text{Diag}(2, 1)$  with entries 2 and 1 along the main diagonal, it is indeed a homomorphism.

Any  $\phi$ -invariant subset of  $\Lambda$  must be a subset of the subgroup of lower triangular matrices. Indeed, if

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

belongs to a  $\phi$  invariant subset of  $\Lambda$ , then iterations of the endomorphism  $\phi$  can be applied to it arbitrarily many times. Thus  $y$  is divisible by arbitrarily high powers of 2, implying that  $y = 0$ . However, since the conjugation by  $a$  turns lower triangular matrices into upper triangular matrices, every normal,  $\phi$ -invariant subgroup of  $\Gamma$  is a subgroup of the group of diagonal matrices in  $\Gamma$ , which is trivial. Thus  $\phi$  is an essential endomorphism.

We calculate the root permutation of  $a$  and  $b$ ,

$$\begin{aligned} \overline{a \cdot 1} &= b^2, & \overline{a \cdot b} &= b, & \overline{a \cdot b^2} &= 1, \\ \overline{b \cdot 1} &= b, & \overline{b \cdot b} &= b^2, & \overline{b \cdot b^2} &= 1, \end{aligned}$$

as well as the sections,

$$\begin{aligned} a|_0 &= \phi(\overline{a \cdot 1}^{-1} \cdot a \cdot 1) = \phi(ba) = baba, \\ a|_1 &= \phi(\overline{a \cdot b}^{-1} \cdot a \cdot b) = \phi(b^{-1}ab) = bab^{-1}, \\ a|_2 &= \phi(\overline{a \cdot b^2}^{-1} \cdot a \cdot b^2) = \phi(ab^2) = ab^{-1}ab^{-1}, \end{aligned}$$

and, for  $i = 0, 1, 2$ ,

$$b|_i = \phi(\overline{b \cdot b^i}^{-1} \cdot b \cdot b^i) = 1.$$

Therefore, a faithful, self-similar action of  $\Gamma = \text{PSL}_2(\mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3$  on the ternary rooted tree is given by

$$a = (02) (baba, bab^{-1}, ab^{-1}ab^{-1}), \quad b = (012) (1, 1, 1).$$

Note that

$$ba = (12)(baba, bab^{-1}, ab^{-1}ab^{-1}).$$

It is easy to see that  $c = ba$  has infinitely many distinct sections. Indeed, since

$$c^{2n} = (c^{4n}, *, *),$$

we see that some of the sections of  $c$  are  $c, c^2, c^4, c^8, c^{16}, \dots$ . Since  $c = ba$  has infinite order, all these sections are distinct.

## 5. Concluding remarks

Lattices  $\Gamma$  in groups virtually isomorphic to  $SL(2, \mathbb{R})$  are the only case where we do not get a complete classification of essential virtual endomorphisms  $\varphi$  of  $\Gamma$ . By appealing to the results of the previous section, one sees that an essential virtual endomorphism  $\varphi$  of  $\Gamma$  has to be injective and cannot preserve a finite-index subgroup of  $\Gamma$ . We conjecture that, provided  $\Gamma$  is not virtually free, for every essential virtual endomorphism  $\varphi$  of such  $\Gamma$ , there exists a finite-index subgroup  $\Gamma' < \Gamma$  and a discrete embedding  $\iota: \Gamma' \hookrightarrow G$  such that  $\varphi|_{\Gamma'}$  is induced by some  $\alpha \in \text{Comm}(\iota(\Gamma'))$ . If this were the case, then the classification of essential virtual endomorphisms of  $\Gamma$  would reduce to the arithmetic case as in Corollary 13. On the other hand, if  $\Gamma$  is virtually free, its finite-index subgroups can be mapped isomorphically to infinite-index subgroups of  $\Gamma$  and we do not have even conjectural classification of essential virtual endomorphisms.

We observe that our results should, in principle, generalize to Gromov-hyperbolic groups which are not lattices. The problem, however, is that:

1. Among hyperbolic groups  $\Gamma$ , very few are known to be *weakly cohopfian*, in the sense that if  $\Lambda < \Gamma$  is a finite index subgroup and  $\varphi: \Lambda \rightarrow \Gamma$  is an injective homomorphism, then the image of  $\varphi$  is a finite index subgroup of  $\Gamma$ . Examples of weakly cohopfian hyperbolic groups are Poincaré duality groups (e.g., fundamental groups of closed aspherical manifolds). Hyperbolic groups which act geometrically on rank 2 hyperbolic buildings provide good candidates for weakly cohopfian groups in view of [4]. (One can show that such groups are weakly cohopfian provided that they are locally quasiconvex, such examples are given by [18].) On the other hand, there are no known examples of 1-ended hyperbolic groups which are not weakly cohopfian.

2. Among hyperbolic groups  $\Gamma$  which are Poincaré duality groups, the only known examples where the abstract commensurator  $\text{Comm}(\Gamma)$  is a finite extension of  $\Gamma$ , are the non-arithmetic lattices. There are few more classes of hyperbolic groups with small abstract commensurators: Fundamental groups of compact hyperbolic  $n$ -manifolds with totally-geodesic boundary,  $n \geq 3$ , surface-by-free groups [9], as well as some rigid examples constructed in [16]. (In all these examples, the entire quasi-isometry group of  $\Gamma$  is a finite extension of  $\Gamma$ .) Conjecturally, the fundamental groups of Gromov–Thurston manifolds [13] should also have small abstract commensurators.

**Question 17.** Does the faithful action of an arithmetic group  $\Gamma$  on a rooted tree constructed in Theorem 2 ever correspond to a finite-state automaton? In the example computed by Zoran Šunić in Section 4, the number of states is infinite. See however the examples of self-similar S-arithmetic groups constructed by Glasner and Mozes in [10].

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