

# Lecture 5: Natural Boundary Conditions

Note Title

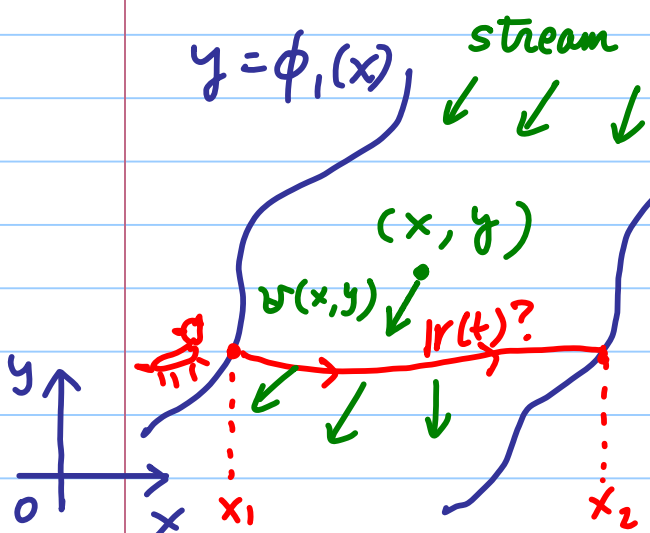
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## \* A Problem of **Zermelo** in Modified Form

Ernst Zermelo (1871-1953)

- Worked on C.V. for his Ph.D.
- Became more famous on the axiomatic set theory: "axiom of choice"
- Returned to the navigation problems in 1930's!

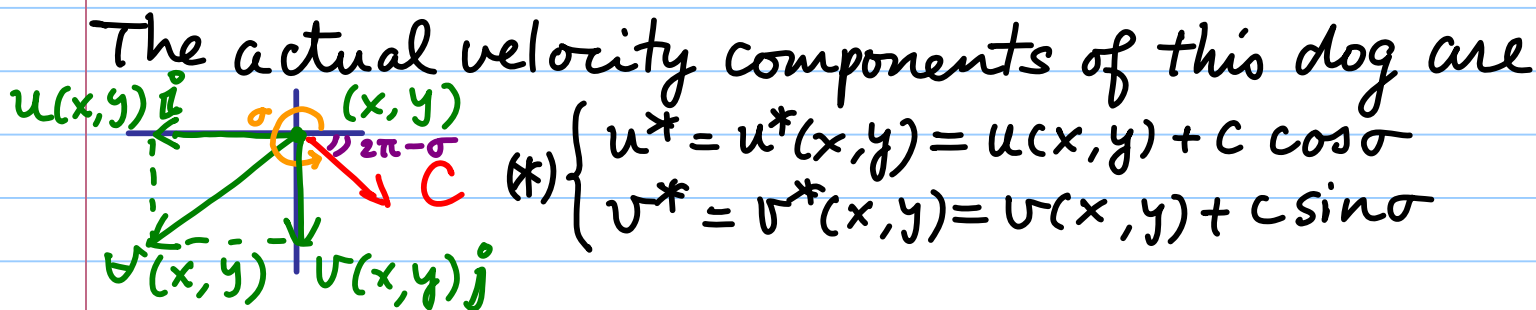
Given a stream, the left bank profile  $y = \phi_1(x)$ ,  
 " right " " "  $y = \phi_2(x)$



the velocity at  $(x,y)$  is specified as  
 $v(x,y) = u(x,y)\mathbf{i} + v(x,y)\mathbf{j}$   
 So, we consider a vector field sandwiched between left & right banks.

A dog can swim with the const. speed  $c$  in still water. He (or she) wants to cross the stream with a min. amount of time.

Question: Where should he embark his swimming and what path should he take?



$$(*) \begin{cases} u^* = u^*(x,y) = u(x,y) + c \cos \sigma \\ v^* = v^*(x,y) = v(x,y) + c \sin \sigma \end{cases}$$

Let  $r(t) = (x(t), y(t))^T$  be a path that dog takes.

Then,  $u^* = \dot{x}$ ,  $v^* = \dot{y}$

$$\text{So, } (*) \Rightarrow (\dot{x} - u)^2 + (\dot{y} - v)^2 = c^2$$

If  $r(t) = r(\tau(t))$ , then

$$\left(\frac{dx}{d\tau} \dot{\tau} - u\right)^2 + \left(\frac{dy}{d\tau} \dot{\tau} - v\right)^2 = c^2$$

Let's use  $\tau(t) = t$ . Then  $\dot{\tau} = \dot{x}$ ,  $\frac{dx}{d\tau} = 1$ .

$$\frac{dy}{d\tau} = \frac{dy}{dx} = y' \Rightarrow (\dot{x} - u)^2 + (y' \dot{x} - v)^2 = c^2$$

$$\Leftrightarrow (1 + y'^2) \dot{x}^2 - 2(u + v y') \dot{x} + u^2 + v^2 - c^2 = 0$$

We are interested in  $\frac{dt}{dx}$  (time behavior), and dividing the above by  $\dot{x}^2 = \left(\frac{dx}{dt}\right)^2$  to get

$$\begin{aligned} (c^2 - u^2 - v^2) \left(\frac{dt}{dx}\right)^2 + 2(u + v y') \frac{dt}{dx} - (1 + y'^2) &= 0 \\ \Rightarrow \frac{dt}{dx} &= \frac{-(u + v y') \pm \sqrt{(u + v y')^2 + (c^2 - u^2 - v^2)(1 + y'^2)}}{c^2 - u^2 - v^2} \\ &= \frac{-(u + v y') \pm \sqrt{c^2(1 + y'^2) - (u y' - v)^2}}{c^2 - u^2 - v^2} \end{aligned}$$

Integrating w.r.t.  $x$  from  $x_1$  to  $x_2$

$$t = \int_{x_1}^{x_2} \frac{-(u + v y') \pm \sqrt{c^2(1 + y'^2) - (u y' - v)^2}}{c^2 - u^2 - v^2} dx$$

$\rightarrow$  min! (Find a curve  $y = y^*(x)$  minimizing this  $t$ .) Note that

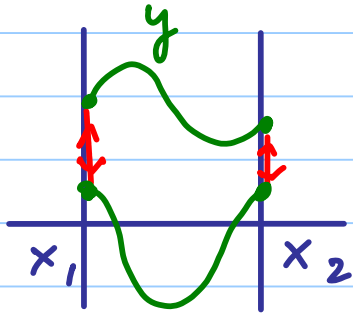
$x_1$  and  $x_2$  are not fixed!

With this generality, we cannot proceed too much from here. Note that a dog actually tries to do this kind of minimization. "dog curves"

## ★ Natural B.C.'s for the 1D Problem

Consider  $I = \int_{x_1}^{x_2} f(x, y, y') dx \rightarrow \min!$

But now  $y = y(x)$  slides freely at  $X = X_i, i = 1, 2$ .



Yet, let's consider the variation of the form:

$$y_\varepsilon = y(x, \varepsilon), \quad x \in [x_1, x_2].$$

$y^* = y_0 = y(x, 0)$  is the minimizer.

As before, set  $\frac{\partial y_\varepsilon}{\partial \varepsilon} = \frac{\partial y(x, \varepsilon)}{\partial \varepsilon} =: \eta(x, \varepsilon)$ .

But now,  $\eta(x_i, \varepsilon) \neq 0, i = 1, 2$ , unlike before. What's the consequence?

$\Rightarrow$  The first term via Int. by Parts does not vanish and needs more thinking!

$$\delta I = \frac{dI}{d\varepsilon}(0) = \underbrace{\eta f_{y'}}_{\neq 0 \text{ at } x_1 \& x_2} \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta \left( \underbrace{\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'}}_{=0} \right) dx$$

Yet, if we consider all the curves passing through fixed  $P_1$  &  $P_2$ , the min. of  $I(\varepsilon)$  should still lead to the E-L eqn.

$\Rightarrow \delta I = \eta f_{y'} \Big|_{x_1}^{x_2} = 0$  is a must.

$\Rightarrow \underline{f_{y'}(x_i) = 0}$   $i = 1, 2$  since  $\eta \in C^1[x_1, x_2]$  is arbitrary. This is called natural bdy cond's.

Example: Find a curve  $y=y(x)$  on  $0 \leq x \leq 1$   
 s.t.  $I = \int_0^1 \underbrace{\sqrt{1+y'^2}}_{=f} dx \rightarrow \min!$

Ans. The natural bdy cond's yield  
 $\frac{\partial f}{\partial y'}(x_i) = 0 \Leftrightarrow \frac{y'(x_i)}{\sqrt{1+y'(x_i)^2}} = 0$  Carl Neumann

i.e.,  $y'(0) = y'(1) = 0$ .  $\nearrow$  (1832-1925)

This is called the **Neumann** B.C.

Now, how about the E-L eqn.?

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0 - \frac{d}{dx} \left( \frac{y'}{\sqrt{1+y'^2}} \right) = 0$$

$$\Leftrightarrow \frac{y''}{\sqrt{1+y'^2}} - \frac{y'' \cdot y'}{(1+y'^2)^{3/2}} = 0$$

So, we have  $y'' = 0$  with  $y'(0) = y'(1) = 0$ .

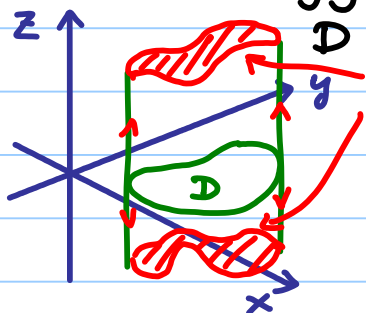
$\Leftrightarrow y = \text{const.}$  (a horizontal line).

and  $\min. I = 1$ . //

### ★ Natural Bdy. Cond. for the 2D Problem

Consider the minimization of

$$(*) \quad I = \iint_D f(x, y, z(x, y), z_x, z_y) dx dy.$$



$z(x, y)$  can slide along this cylinder wall!

$D$ : a simply-connected domain  $\subset \mathbb{R}^2$   
 $C = \partial D$ : a rectifiable curve

As before, consider a variation  $z_\varepsilon = z(x, y, \varepsilon)$  with  $z^* = z_0 = z(x, y, 0)$  being the minimizer of  $I$ . Also  $\frac{\partial z_\varepsilon}{\partial \varepsilon} =: \zeta(x, y, \varepsilon)$

$\zeta$  does not necessarily vanish along  $C$ .

Recall the derivative of  $I$  using the Green's Thm:  $\delta I = \frac{dI}{d\varepsilon}(0) = \iint_D \zeta \left( \frac{\partial f}{\partial z} - \frac{\partial}{\partial x} \frac{\partial f}{\partial z_x} - \frac{\partial}{\partial y} \frac{\partial f}{\partial z_y} \right) dx dy - \oint_C \zeta \left( \frac{\partial f}{\partial z_y} dx - \frac{\partial f}{\partial z_x} dy \right) = 0$

$= 0$  on  $C$  may not hold.

However, the E-L eqn.  $\frac{\partial f}{\partial z} - \frac{d}{dx} \frac{\partial f}{\partial z_x} - \frac{d}{dy} \frac{\partial f}{\partial z_y} = 0$  is a must since  $z = z^*$  must yield a min. for each fixed bdy. So we must have

$$\oint_C \zeta \left( \frac{\partial f}{\partial z_y} dx - \frac{\partial f}{\partial z_x} dy \right) = 0 \quad \forall \zeta \in C^1(D)$$

$$\Rightarrow \frac{\partial f}{\partial z_y} - \frac{\partial f}{\partial z_x} \frac{dy}{dx} = 0 \quad \text{on } C$$

F.L.C.V.  $\Rightarrow$  2D version of a nat. bdy. cond.

If  $z = z^*$  minimizes  $I$  in (\*) without specifying bdy values on  $C$ , then it must satisfy both the E-L eqn. & the natural bdy. cond.

Example: Let  $f = z_x^2 + z_y^2 = \|\nabla z\|^2$  and the min. of  $I = \iint_D f dx dy$ , i.e.,

the total gradient is minimized.

Such a surface  $z = z(x, y)$  must be smooth!

$$\frac{\partial f}{\partial z_x} = 2z_x, \quad \frac{\partial f}{\partial z_y} = 2z_y$$

So, the E-L. eqn  $\frac{\partial f}{\partial z} - \frac{d}{dx} \frac{\partial f}{\partial z_x} - \frac{d}{dy} \frac{\partial f}{\partial z_y} = 0$

$$\Leftrightarrow z_{xx} + z_{yy} = 0 \text{ in } D, \text{ i.e.,}$$

$$\Delta z = 0 \text{ in } D \text{ Laplace's eqn!}$$

Now how about the nat. bdy. cond.?

$$\frac{\partial f}{\partial z_y} - \frac{\partial f}{\partial z_x} \frac{dy}{dx} = 0 \text{ on } C$$

$$\Leftrightarrow 2z_y - 2z_x \cdot \frac{dy}{dx} = 0 \quad (**)$$

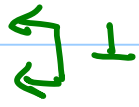
Let's use a parametric form to describe  $C$ , i.e.,  
 $(x, y) = (x(s), y(s))$ ,  $s \in J$ : some 1D interval.

Then  $\frac{dy}{dx} = \frac{dy/ds}{dx/ds} = \frac{\dot{y}}{\dot{x}}$ ,  $\frac{d}{ds}(\cdot) = (\cdot)'$

Now  $(**)$  becomes

$$z_y \dot{x} - z_x \dot{y} = 0 \quad (***)$$

Tangent vector to  $C = \mathbf{t} := \dot{x} \mathbf{i} + \dot{y} \mathbf{j}$

Normal " " " =  $\mathbf{v} := -\dot{y} \mathbf{i} + \dot{x} \mathbf{j}$  

$$\text{So, } \frac{\partial z}{\partial \nu} := \mathbf{v} \cdot \nabla z = (-\dot{y} \mathbf{i} + \dot{x} \mathbf{j}) \cdot (z_x \mathbf{i} + z_y \mathbf{j})$$

$$= -\dot{y} z_x + \dot{x} z_y = 0 \text{ by } (***)!!$$

i.e.,  $\frac{\partial z}{\partial \nu} \Big|_C = 0$  the Neumann B.C.

On  $C$ ,  $z = z^*$  must be flat in the normal directions.