q-Holonomic Systems and Quantum Invariants

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Abstract

The topics of this dissertation fall under the purview of quantum topology, which seeks to build connections between the insights and constructions of quantum physics and classical topology. A pivotal theme will be the appearance of topologically interesting *q-holonomic systems* in quantum invariants. These manifest in the quasiperiodic behavior of Witten-Reshetikhin-Turaev (WRT) invariants, and as certain modules associated to lagrangians in quantized character varieties. This work was motivated by the AJ conjecture [**Gar04a, Guk05a**], which predicts that these two manifestations are the two sides of a single coin.

The main result of this dissertation is that the ADO invariant is q-holonomic, meaning it exhibits strong recursive behavior. Some subtlety is involved in the definition of q-holonomicity in this setting, as the ADO invariant exhibits a topologically uninteresting quasi-periodicity because of the appearance of roots of unity. This invariant is closely related to the colored Jones polynomial of the AJ conjecture, and acts as its analytic continuation.

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CHAPTER 1

Background

1.1. Quantum Invariants

In the 1980s Vaughan Jones introduced a novel knot invariant built from representations of von Neumann algebras, as detailed in his survey article [**Jon85**]. His treatment was based on representing planar diagrams as closures of braids, but the question was soon posed: Is there an intrinsically 3d description of the Jones polynomial? Later, Edward Witten published the seminal paper [**Wit89a**] identifying the Jones polynomial as a partition function in a 3d quantum field theory with complex Chern-Simons action.

Shortly after this, Nicolai Reshetikhin and Vladimir Turaev announced an algebraic implementation of Jones' original invariant in terms of what are now called *modular tensor categories*. Their work [**RT90, RT91**] gave a concrete procedure for producing knot invariants from categories with extra structures mimicking the behavior of tangles.

Since their inception three decades ago, the Witten-Reshetikhin-Turaev (WRT) invariants of links and three manifolds have been the subject of intense study and have been generalized widely. Their construction still revolves around the existence of a unique functor

$$F_{\mathcal{C}} : \mathcal{D}_{\mathcal{C}} \longrightarrow \mathcal{C} \tag{1.1.1}$$

that sends a link diagram T whose ℓ components are colored by objects $\{V_1, \ldots, V_\ell\}$ of a category $\mathcal{C}$ to an endomorphism $\langle T \rangle \in \mathrm{End}(\mathbb{1}_{\mathcal{C}}) \simeq \mathbb{C}(q)$. The *colored Jones polynomial* is one of the best known WRT invariants and arises when $\mathcal{C} = \mathrm{Rep}_q \mathrm{SL}_2 \mathbb{C}$.

WRT invariants depend on the specific objects (typically representations) attached to the link components, with the exact formulas for the invariants varying widely between representations. There is a strong but non-obvious quasi-periodicity to the variation, which manifests as recursive relationships between invariants at different representations. The exact expressions of these recursion relations is the core of the AJ conjecture in its mathematical formulation [**Gar04a, GL05**].

A new class of quantum invariants of links and three-manifolds was introduced in [**ADO92, M$^+$08, GPMT09, CGPM15a**], based on representation categories of quantum groups with vanishing quantum dimensions. Such categories are often non-semisimple. These invariants generalize Witten-Reshetikhin-Turaev (WRT) invariants [**Tur88, Wit89b, RT91**], which are instead constructed from semisimple categories where quantum dimensions are all nonzero. This paper arose from studying the recursive properties of this new class of invariants with the goal of comparing their behavior to that of better understood quantum invariants.

This work is in part motivated by the search for a physical manifestation of this new class of invariants. Many WRT invariants have a physical origin in Chern-Simons theory with compact gauge group [**Wit89b**]. An analogous physical origin for the new class of invariants would be a 3d continuum quantum field theory whose partition functions compute the new invariants. The results of this dissertation support the existence of such a theory, by proving that the relevant invariants exhibit the prerequisite strong recursive behavior.

It was shown by Garoufalidis and Lê in [**GL05**] that the sequence of colored Jones polynomials $\left(J_N^K(q)\right)_{N\in\mathbb{N}}$ of a knot K always obey a finite-order recursion relation. More precisely, the function $J^K : \mathbb{N} \to \mathbb{C}[q, q^{-1}]$ generates a *q-holonomic module* for the q-Weyl algebra

$$\mathbb{E}_1 = \mathbb{C}(q)[x^{\pm}, y^{\pm}]/(yx - qxy)\,, \tag{1.1.2}$$

where x and y act on functions $f : \mathbb{N} \to \mathbb{C}(q)$ as multiplication by q^N and shifting $N \mapsto N+1$, respectively. The theory of q-holonomic modules, central to the work of [**GL05**], was developed by Sabbah [**Sab93**] and generalized classic work on D-modules by Bernstein, Sato, Kashiwara, and others.

It was conjectured in [**Gar04b**] that the classical A-polynomial of a knot K divides the $q \to 1$ limit of any element $A(x, y; q) \in \mathbb{E}_1$ that annihilates the colored Jones polynomial J^K. Since the A-polynomial is defined using the $SL(2,\mathbb{C})$ representation variety of the knot complement $S^3\backslash K$ [**CCG$^+$94b**], this *AJ conjecture* established a new connection between colored Jones invariants and classical geometry. The conjecture is open but has been confirmed in many examples, e.g. [**GS10, GK12**].

The fact that the colored Jones polynomials should be annihilated by a recursion operator related to the A-polynomial was independently predicted by Gukov [**Guk05b**], based on the physics of Chern-Simons theory. The approach of [**Guk05b**] was to analytically continue Chern-Simons theory with compact gauge group $SU(2)$ to a complex group $SL(2,\mathbb{C})$; then an operator $A(x,y;q)$ providing recursion relations for the colored Jones was identified with an effective Hamiltonian that must annihilate the analytically continued Chern-Simons wave function This operator had to be a quantization of the classical A-polynomial, which was the classical Hamiltonian of the system. (This insight was subsequently used in [**Guk05b**] to generalize the Volume Conjecture of [**Kas97**].)

From a physical perspective, the presence of an operator $A(x,y;q)$ that quantizes the classical A-polynomial and annihilates quantum wave functions is now known to be a robust feature of Chern-Simons theory with gauge group $SU(2)$ and many other versions of Chern-Simons theory with gauge group $SL(2,\mathbb{C})$, including its analytic continuation (*cf.* [**DGLZ09, Dim15, GM19**]). It is therefore natural to ask whether the new class of quantum invariants of [**ADO92, GPMT09, CGPM15a**] satisfy recursion relations related to A-polynomial.

The invariants considered in this paper are defined using the representation category of the *unrolled quantum group* $\mathcal{U}^H_{\zeta_{2r}}(\mathfrak{sl}_2)$ at the $2r$-th root of unity $\zeta_{2r} := e^{\frac{i\pi}{r}}$, $r \in \mathbb{N}_{\geq 2}$. (See Section 2 for details.) This quantum group admits a continuous family of 'typical' representations $\{V_\alpha\}_{\alpha\in(\mathbb{C}\setminus\mathbb{Z})\cup(-1+r\mathbb{Z})}$ that are irreducible but have vanishing quantum dimensions.

Let L be a framed, oriented link in S^3, with n components colored by typical representations $V_{\alpha_1},\dots,V_{\alpha_n}$. It was shown in [**ADO92, GPMT09**] how to overcome the problem of vanishing quantum dimensions to define a non-vanishing link invariant $N^r_L(\alpha_1,\dots,\alpha_n)$. After restricting to $\alpha_i \in \mathbb{C}\setminus\mathbb{Z}$, it is useful to view these invariants as a family of holomorphic functions

$$N^r_L : (\mathbb{C}\setminus\mathbb{Z})^n \to \mathbb{C}, \qquad (r \in \mathbb{N}_{\geq 2}) \tag{1.1.3}$$

that admit metamorphic continuations to $(\mathbb{C}/2r\mathbb{Z})^n$. Though this family of invariants can be defined using the methods of [**GPMT09**], they appeared in the earlier work of Akutsu, Deguchi, and Ohtsuki [**ADO92**]. Thus we call $N^r_L(\alpha)$ the ADO invariants.

We prove that the ADO invariants N_L^r of any framed, oriented link L are indeed q-holonomic. Moreover, in the case of a knot $L = K$, we prove that all recursion relations satisfied by the ADO invariants are also satisfied by the colored Jones function J^K.

A physical interpretation of the ADO invariant has appeared in the work [**GHN⁺20**] of Gukov, Hsin, Nakajima, Park, & Pei. Therein, ADO invariants are related physically to a number of other invariants, including the homological blocks of [**GPPV20**]. It is conjectured in [**GHN⁺20**, Sec 4] that ADO invariants obey the same recursion operations as Jones polynomials. The results of this chapter prove that this is indeed the case.

1.1.1. Roots of unity, q-holonomic families, and Hamiltonian reduction. It is not obvious what should be meant when considering whether the ADO invariants are q-holonomic. At each fixed r, the ADO invariant N_L^r of an n-component link L turns out to be quasi-periodic in each variable α_i, with period $2r$. (We review this property in Proposition 2.2.1 and Corollary 2.2.2.) The result is that the ADO invariant $N_L^r(\alpha_1, \dots, \alpha_n)$ at fixed r will satisfy n independent recursion relations, of the form

$$\Big(\prod_{j=1}^{n} x_j^{-2rC_{ij}} y_i^{2r} - 1\Big) N_L^r(\alpha) = 0\,, \qquad i = 1, \dots, n \tag{1.1.4}$$

where each x_i acts as multiplication by $\zeta_{2r}^{\alpha_i} := e^{\frac{i\pi}{r}\alpha_i}$ and each y_i acts as a shift $\alpha_i \mapsto \alpha_i + 1$, and C_{ij} is the integer linking matrix of L. These recursion relations, which depend only on the linking matrix, do not have a deep connection with the A-polynomial.

To obtain topologically rich recursion relations, we work independently of the choice of r. This leads us to introduce the notion of a q-holonomic family. Let

$$\mathbb{E}_n = \mathbb{C}(q)[x_1^{\pm}, y_1^{\pm}, \dots, x_n^{\pm}, y_n^{\pm}]/(y_i x_j - q^{\delta_{ij}} x_j y_i) \tag{1.1.5}$$

be a q-Weyl algebra in n pairs of variables. Given an n-component link L with ADO invariants $\{N_L^r(\alpha)\}_{r\geq 2}$, define an analog of the annihilation ideal $\mathcal{I}[N_L] \subseteq \mathbb{E}_n$ by

$$\mathcal{I}[N_L] = \{A(x, y; q) \in \mathbb{E}_n \;\; A(x, y; \zeta_{2r}) N_L^r(\alpha) = 0 \text{ for all but finitely many } r \geq 2\}\,, \tag{1.1.6}$$

with the action $x_i N_L^r(\alpha) = \zeta_{2r}^{\alpha_i} N_L^r(\alpha)$ and $y_i N_L^r(\alpha) = N_L^r(\alpha_1, \dots, \alpha_i + 1, \dots, \alpha_n)$. Note that the specialization of elements of $\mathbb{E}_n$ to $q = \zeta_{2r}$ may not be defined at some finite number of r's, which we discard in defining the ideal $\mathcal{I}[N_L]$. We prove

THEOREM 2.3.3 *For any framed, oriented link L, the left $\mathbb{E}_n$-module $\mathbb{E}_n/\mathcal{I}[N_L]$ is q-holonomic.*

In particular, this implies that each ADO function $N^r_L(\alpha)$ satisfies n independent recursion relations, which come from operators $A(x, y; q) \in \mathbb{E}_n$ that are *independent* of r.

Our method of proof is to first show that the ADO invariants $N^r_L(\alpha)$ may be lifted (or analytically continued) to functions $G_{\mathbb{D}}(r; x_1, \ldots, x_n; z_{11}, z_{12}, \ldots, z_{nn}; q)$ of $1 + n + \frac{1}{2}n(n+1)$ variables $r, x_i, z_{ij} = z_{ji}$, and q, in such a way that

$$N^r_L(\alpha) = G_{\mathbb{D}}\left(r; \zeta_{2r}^{\alpha_1}, \ldots, \zeta_{2r}^{\alpha_n}; \zeta_{2r}^{\alpha_1^2/2}, \zeta_{2r}^{\alpha_1\alpha_2/2}, \ldots, \zeta_{2r}^{\alpha_n^2/2}; \zeta_{2r}\right). \tag{1.1.7}$$

The *diagram invariant* $G_{\mathbb{D}}$ is neither canonical nor a link invariant, as it depends on the *choice* of a diagram $\mathbb{D}$ for a $(1, 1)$-tangle whose closure is the link L.

The virtue of $G_{\mathbb{D}}$ is that it generates a q-holonomic module for the q-Weyl algebra $\mathbb{E}_{n+1}$, in the same n pairs of generators x_i, y_i as (1.1.5) together with a final pair $\hat{x}, \hat{y}$ that act as multiplication by q^r and shift $r \mapsto r+1$. The proof that $G_{\mathbb{D}}$ is q-holonomic (Section 1.2) is a simple generalization of the original work of [**GL05**].

In Section 2.3 it's argued that the specialization (1.1.7), which in particular sets q to be a $2r$-th root of unity, may be understood as a version of *quantum Hamiltonian reduction*. The Hamiltonian reduction reduces $\mathbb{E}_{n+1}$ to $\mathbb{E}_n$ by eliminating the shift $\hat{y}$ in r and setting $\hat{x} = \zeta_{2r}^r = -1$. It takes the annihilation ideal of $G_{\mathbb{D}}$ in $\mathbb{E}_{n+1}$ and explicitly constructs elements of our desired ideal $\mathcal{I}[N_L]$. Appendix A has a self-contained proof that the relevant Hamiltonian reduction preserves q-holonomic modules.

Our result that the family of ADO invariants is q-holonomic (Theorem 2.3.3) does not on its own guarantee the existence of topologically significant recursion relations. We prove in Section 2.3.3 that the ideal $\mathcal{I}[N_L]$ is included in the annihilation ideal of the colored Jones function (up to a rescaling of variables.)

More concretely, suppose that $L = K$ is an oriented knot with framing $\phi \in \mathbb{Z}$, and $J^K_N(q)$ are its colored Jones polynomials, normalized so that $J^{unknot}_N(q) = (q^N - q^{-N})/(q - q^{-1})$. Then we have:

THEOREM 2.3.4 *Every element $A(x, y; q) \in \mathcal{I}[N_K]$ satisfies $A(q^{-1}x, (-1)^{\phi+1}y; q)J^K_N = 0$.*

This result follows from a relation between the representations involved in defining the ADO and colored Jones invariants [**CGPM15b**]. When the parameter α of a typical module V_α for the unrolled quantum group is an integer $N-1 \in \mathbb{Z}\backslash r\mathbb{Z}$, the module is reducible and its simple quotient is the $(N-1)$-dimensional module used in defining the Nth colored Jones polynomial.

It has been shown [**Wil20, BB**] that *both* ADO and colored Jones invariants of links may be obtain by specializations of more universal invariants valued in the Habiro ring [**Hab04, Hab07**]. One might expect that such relations lead to an independent proof that the family of ADO invariants is q-holonomic, with recursion relations equivalent to those satisfied by the colored Jones. Indeed, [**Wil20**, Thm 66] proves that every element in the annihilation ideal of the colored Jones of a knot will also annihilate the family of ADO invariants. This is a converse to our Theorem 2.3.4. Taken together, the two results establish that the annihilation ideals of the colored Jones and ADO invariants are equivalent.

1.1.2. Example: figure-eight knot. The Jones polynomials $\big(J^{\mathbf{4_1}}_N(q)\big)_{N\in\mathbb{N}}$ of the zero-framed figure-eight knot,

$$J^{\mathbf{4_1}}_1(q) = 1\,, \qquad J^{\mathbf{4_1}}_2(q) = q^5 + q^{-5}\,,$$

(1.1.8) $$J^{\mathbf{4_1}}_3(q) = q^{14} - q^{10} + q^2 + 1 + q^{-2} - q^{-10} + q^{-14}$$

$$J^{\mathbf{4_1}}_4(q) = q^{27} - q^{23} - q^{21} + q^{17} + q^{11} + q^{9} + q^{-9} + q^{-11} + q^{-17} - q^{-21} - q^{-23} + q^{27}\,, \text{ etc.}$$

normalized so that $J^{unknot}_N(q) = \frac{q^N - q^{-N}}{q-q^{-1}}$, satisfy the 2nd-order in-homogeneous recursion

(1.1.9) $$(q-q^{-1})A_{\mathbf{4_1}}(x,y;q)J^{\mathbf{4_1}}_N(q) = B_{\mathbf{4_1}}(q^N;q)\,,$$

where[1]

(1.1.10) $$A_{\mathbf{4_1}}(x,y;q) = (\tfrac{x^2}{q} - \tfrac{q}{x^2})y - (x^2 - \tfrac{1}{x^2})(x^4 - x^2 - (q^2 + \tfrac{1}{q^2}) - \tfrac{1}{x^2} + \tfrac{1}{x^4}) + (qx^2 - \tfrac{1}{qx^2})y^{-1}$$
$$B_{\mathbf{4_1}}(x;q) = (x + \tfrac{1}{x})(qx^2 - \tfrac{1}{qx^2})(\tfrac{x^2}{q} - \tfrac{q}{x^2})\,.$$

[1]This differs slightly from the recursion relation found in [**GL05**], only because of the normalization of the colored Jones polynomials we are using here. The recursions are completely equivalent.

and x and y act as multiplication by q^N and shift $N \mapsto N+1$, respectively. The in-homogeneous recursion above implies the existence of a homogeneous recursion of one order higher,

$$(1.1.11)\qquad \big[B_{\mathbf{4_1}}(x;q)y - B_{\mathbf{4_1}}(qx;q)\big]A_{\mathbf{4_1}}(x,y;q)J_N^{\mathbf{4_1}}(q) = 0\,.$$

The operator $\widetilde{A}_{\mathbf{4_1}}(x,y;q) := \big[B_{\mathbf{4_1}}(x;q)y - B_{\mathbf{4_1}}(qx;q)\big]A(x,y;q)$ generates the annihilation ideal of the colored Jones. At $q=1$, it is easy to see that

$$(1.1.12)\quad \widetilde{A}_{\mathbf{4_1}}(m,\ell;q=1) = (m+m^{-1})(m^2-m^{-2})^3(\ell-1)\big(\ell-(m^4-m^2-2-m^{-2}+m^{-4})+\ell^{-1}\big)$$

is divisible by the A-polynomial of the figure-eight knot, namely

$$(1.1.13)\qquad (\ell-1)(\ell-(m^4-m^2-2-m^{-2}+m^{-4})+\ell^{-1}).$$

A compact formula for the ADO invariants of the zero-framed figure-eight knot was given in [**Mur08**]; adjusted for our conventions in this paper, it reads

$$(1.1.14)\qquad N^r_{\mathbf{4_1}}(\alpha-1) = \frac{-i^{1-r}}{x^r-x^{-r}}\sum_{k=0}^{r-1} x^{2k+1}(q^{-2k}x^{-2};q^2)_{2k+1}\quad x=\zeta_{2r}^{\alpha}, q=\zeta_{2r}$$

Letting $\hat{N}^r_{\mathbf{4_1}}(\alpha) := i^{1-r}(x^r-x^{-r})N^r_{\mathbf{4_1}}(\alpha-1)$, the first few ADO invariants are

$$(1.1.15)\qquad \begin{aligned}
\hat{N}^2_{\mathbf{4_1}}(\alpha) &= (x+x^{-1})(x^2+3+x^{-2}) && (x=e^{\frac{i\pi}{2}\alpha})\\
\hat{N}^3_{\mathbf{4_1}}(\alpha) &= (x+x^{-1})(x^4+3x^2+5-3x^{-2}+x^{-4}) && (x=e^{\frac{i\pi}{3}\alpha})\\
\hat{N}^4_{\mathbf{4_1}}(\alpha) &= (x-x^{-1})(x^2+1+x^{-1})^3 && (x=e^{\frac{i\pi}{4}\alpha})\\
\hat{N}^5_{\mathbf{4_1}}(\alpha) &= (x-x^{-1})(x+x^{-1})^2\big[x^6+x^4\\
&\quad +(3+q^2-q^3)(x^2+x^{-2})+(2-q^2+q^3)+x^{-4}+x^{-6}\big] && (x=e^{\frac{i\pi}{5}\alpha}, q=e^{\frac{i\pi}{5}})
\end{aligned}$$

Further values appear in Appendix B. We verify for each $2\le r\le 20$ that

$$(1.1.16)\qquad A_{\mathbf{4_1}}(x,y;\zeta_{2r})\hat{N}^r_{\mathbf{4_1}}(\alpha) = -(\zeta_{2r}^{2r\alpha}-3+\zeta_{2r}^{-2r\alpha})B_{\mathbf{4_1}}(\zeta_{2r}^{\alpha};\zeta_{2r})\,,$$

for exactly the *same* $A_{\mathbf{4_1}}$ and $B_{\mathbf{4_1}}$ as in (1.1.10), with x and y now acting as multiplication by ζ_{2r}^{α} and shift $\alpha\mapsto\alpha+1$, respectively. These in-homogeneous recursions imply that for each r the ADO invariant satisfies a homogeneous recursion

$$(1.1.17)\qquad \widetilde{A}_{\mathbf{4_1}}(x,y;\zeta_{2r})\hat{N}^r_{\mathbf{4_1}}(\alpha) = 0\qquad r\in\mathbb{N}_{\ge 2}$$

for exactly the same $\widetilde{A}_{\mathbf{4_1}}(x, y; q) = [B_{\mathbf{4_1}}(x; q)y - B_{\mathbf{4_1}}(qx; q)] A_{\mathbf{4_1}}(x, y; q)$ that annihilated the colored Jones. Note that (1.1.17) is equivalent to $\widetilde{A}_{\mathbf{4_1}}(qx, -y; \zeta_{2r}) N^r_{\mathbf{4_1}}(\alpha) = 0$ in the 'un-hatted' normalization, in agreement with Theorem 2.3.4.

Further examples of in-homogeneous and homogeneous recursions for the $\mathbf{3_1}$ and $\mathbf{5_2}$ knots are collected in Appendix B.

1.2. q-Holonomicity

q-Holonomic systems are a q-difference analog of classical holonomic ones. They are useful for describing and measuring recursive behaviors of functions with discrete arguments. We give a brief introduction here, deferring most technical details to the following Section 1.2.1.

Let $\mathbb{W}_n$ be the nth q-Weyl algebra, generated by q-difference operators on n variables. Certain WRT invariants, including the colored Jones and ADO invariants, generate $\mathbb{W}_n$ modules. (Here n is the number of link components.) The 'strength" of the recursion of a $\mathbb{W}_n$-module N is measured by its *homological dimension*.

Let $\mathcal{F}$ be the *Bernstein filtration* on $\mathbb{W}_n$ given by total degree in its generators [**Sab93**, §1.5.1]. Suppose $\mathcal{G}$ is an ascending filtration on a finitely generated $\mathbb{W}$ module N whose filtered components $\mathcal{G}_j N$ are finite dimensional over $\mathbb{C}(q)$, and compatible with $\mathcal{F}$ in the sense that

$$\mathcal{F}_j \mathbb{W}_n \cdot \mathcal{G}_k N \subseteq \mathcal{G}_{j+k} N \tag{1.2.1}$$

Given any such *good filtration*,[2] there exists a unique *Hilbert polynomial* p such that $\dim_{\mathbb{C}(q)} \mathcal{G}_i N = p(i)$ for $i \gg 0$.

DEFINITION 1.2.1. (Homological Dimension.) *For a finitely generated $\mathbb{W}$ module N with good filtration $\mathcal{G}$, the degree of the Hilbert polynomial p is the homological dimension of N as a $\mathbb{W}_n$ module. It is denoted $d(N)$.*

Intuitively, $d(N)$ is the growth rate of the filtered components of N. It does not depend on the choice of filtration [**Sab93**, Thm 1.5.3]. The lower $d(N)$ is, the more linear relations there are between elements of N. There are bounds on the homological dimension.

[2]When $N = \mathbb{W}_n\langle f\rangle$ is cyclic, the filtration given by $\mathcal{F}_j N = (\mathcal{F}_j \mathbb{W})\langle f\rangle$ satisfies these requirements.

THEOREM 1.2.2. (Bernstein's Inequality) [**Sab93**] *Let $\mathbb{W}_n$ be the q-Weyl algebra of rank n and N a finitely generated $\mathbb{W}_n$ module. Then either $d(N) = 0$, in which case $N = 0$, or else $n \leq d(N) \leq 2n$.*

A $\mathbb{W}_n$ module N is called *q-holonomic* when $d(N) = n$ or 0. Any generator of a cyclic q-holonomic module is likewise called q-holonomic. The central result of this dissertation is that the ADO invariants are a q-holonomic family. This is a necessary condition of the AJ conjecture, which further claims that the recursion relations are a quantized version of the A-polynomial [**CCG$^+$94a**].

The closure properties of q-holonomic functions under addition, multiplication, multi-sums, etc. are consequences of universal algebraic features of q-holonomic modules. In particular the closure properties are independent of the actual functional spaces on which q-Weyl algebras are represented.

To maintain a reasonably self-contained and pedagogical exposition, we will review basic definitions and examples of q-holonomic modules in Section 1.2.1, following the classic work of Sabbah [**Sab93**], which in turn was based on work of Bernstein [**Ber71**], Sato, Kashiwara, and others on D-modules. In the process we will introduce the functional spaces $\mathcal{V}_{m,n}$ relevant for the diagram invariant $G_{\mathbb{D}}$ defined in Section 2.2.1.

Other good references include the classic [**Zei90, WZ92**], as well as the more recent survey [**GL16**]. There are powerful derived methods available to study generalizations of q-Weyl modules and functors among them, such as [**KS12**]. We will not require or discuss these methods.

In Section 1.2.2 we explain how standard closure properties of q-holonomic modules apply to $G_{\mathbb{D}}$. Then in Section 2.2.3 we emulate [**GL05**] to prove that $G_{\mathbb{D}}$ is q-holonomic — by verifying that all the building blocks of Section 2.2.1 are q-holonomic and that their composition to form $G_{\mathbb{D}}$ preserves this property.

1.2.1. q-holonomic modules and functions. Let $\mathbb{C}_q = \mathbb{C}(q)$ denote the field of fractions in a formal variable q. Recall the q-Weyl algebras in n pairs of variables

$$\begin{aligned} \mathbb{W}_n &= \mathbb{C}_q[x_1, \dots, x_n, y_1, \dots, y_n]/\mathrm{rel}_q \\ \mathbb{E}_n &= \mathbb{C}_q[x_1^{\pm 1}, \dots, x_n^{\pm 1}, y_1^{\pm 1}, \dots, y_n^{\pm 1}]/\mathrm{rel}_q^{\pm} \end{aligned} \tag{1.2.2}$$

Namely, these consist of polynomials (resp. Laurent polynomials) in $2n$ non-commutative formal variables x_i, y_i (resp. $x_i^{\pm}, y_i^{\pm}$), subject to the relations

$$(1.2.3) \qquad \mathrm{rel}_q: \quad \begin{aligned} y_i x_j &= q^{\delta_{ij}} x_j y_j \\ x_i x_j &= x_j x_i \\ y_i y_j &= y_j y_i \end{aligned} \qquad \mathrm{rel}_q^{\pm}: \quad \begin{aligned} y_i^{\varepsilon} x_j^{\varepsilon'} &= q^{\varepsilon\varepsilon'\delta_{ij}} x_j^{\varepsilon'} y_j^{\varepsilon} \\ x_i^{\varepsilon} x_j^{\varepsilon'} &= x_j^{\varepsilon'} x_i^{\varepsilon} \\ y_i^{\varepsilon} y_j^{\varepsilon'} &= y_j^{\varepsilon'} y_i^{\varepsilon} \end{aligned} \qquad (\varepsilon, \varepsilon' \in \{\pm 1\})$$

as well as the implicit relations $x_i x_i^{-1} = y_i y_i^{-1} = 1$.

Both algebras have a notion of a q-holonomic module, though their respective definitions differ some. The notion of q-holonomic $\mathbb{W}_n$ modules is based on homological dimension, which quantifies the quasi-periodicity of the module elements under the action of $\mathbb{W}_n$. Homological dimension is based on the *Bernstein Filtration* $\mathcal{F}_\bullet \mathbb{W}_n$ given by total degree in x and y,

$$(1.2.4) \qquad \mathcal{F}_k \mathbb{W}_n = \mathbb{C}_q \langle x^a y^b \text{ s.t. } |a| + |b| \leq k \rangle ,$$

where we write $|a| = \sum_j a_j$ for a multi-index $a = (a_1, \ldots, a_n)$. Given a left $\mathbb{W}_n$-module M, an ascending filtration $\mathcal{F}_\bullet M$ is called a 'good filtration" if the associated Rees module is a finitely generated module for the Rees algebra of $\mathbb{W}_n$. In particular, this implies that the filtrations on $\mathbb{W}_n$ and M are compatible (*i.e.* $\mathcal{F}_k \mathbb{W}_n \cdot \mathcal{F}_\ell M \subseteq \mathcal{F}_{k+\ell} M$), and that each $\mathcal{F}_k M$ is finite-dimensional.

Remarkably, for every good filtration there exists a Hilbert polynomial p such that $\dim_{\mathbb{C}_q} \mathcal{F}_k M = p(k)$ for $k \gg 0$. The degree of the Hilbert polynomial is denoted $d(M)$ and called the *homological dimension* of M. It is independent of the choice of good filtration. In other words, $d(M)$ is the polynomial order of growth of the filtered components of any good filtration.

The q-analogue of Bernstein's inequality guarantees that if M is a finitely generated $\mathbb{W}_n$ module and has no monomial torsion[3] then $d(M) \geq n$. We are interested in the case when the homological dimension is as small as possible.

Definition 1.2.3. *A left $\mathbb{W}_n$-module M is called* q-holonomic *if it is finitely generated, has no monomial torsion, and either $M = 0$ or $d(M) = n$.*

Since elements of $\mathbb{E}_n$ may have arbitrarily large negative degree, the components of the Bernstein filtration will be infinite dimensional. Being q-holonomic is instead defined in terms of homological

[3]Given a left $\mathbb{W}_n$-module M, its *monomial torsion* $\mathrm{mtor}(M) \subseteq M$ is the subspace consisting of $v \in M$ such that $x^a y^b v = 0$ for some monomial $x^a y^b := x_1^{a_1} \cdots x_n^{a_n} y_1^{a_1} \cdots y_n^{a_n} \in \mathbb{W}_n$.

codimension. Given a left $\mathbb{E}_n$-module M, its *homological codimension* $c(M)$ is the smallest integer k such that $\mathrm{Ext}^k_{\mathbb{E}_n}(M,\mathbb{E}_n)\neq 0$.

DEFINITION 1.2.4. *A left $\mathbb{E}_n$-module M is called q-holonomic if it is finitely generated and $M=0$ or $c(M)=n$.*

Results in [**Sab93**, Sec. 2] show that for any finitely-generated $\mathbb{E}_n$-module M one has $c(M)\leq n$ (an analogue of Bernstein's inequality), and that M is q-holonomic if and only if $\mathrm{Ext}^k_{\mathbb{E}_n}(M,\mathbb{E}_n)=0$ for all $k\neq n$.

There is a close relationship between $\mathbb{E}_n$ and $\mathbb{W}_n$ modules. First, $\mathbb{E}_n$ has a natural right $\mathbb{W}_n$-module structure, which provides a map from (left) $\mathbb{W}_n$ modules to $\mathbb{E}_n$ modules, *i.e.*

$$\begin{array}{ccc} \mathbb{W}_n\text{-mod} & \to & \mathbb{E}_n\text{-mod} \\ M & \mapsto & \mathbb{E}_n\otimes_{\mathbb{W}_n} M\,. \end{array} \tag{1.2.5}$$

Note that the kernel of this map consists precisely of $\mathbb{W}_n$-modules with monomial torsion. Conversely, any finitely-generated $\mathbb{E}_n$-module M can be written as $M=\mathbb{E}_n\otimes_{\mathbb{W}_n} N$ for some N (e.g. the $\mathbb{W}_n$-span of the generators of M). A simple result of [**Sab93**, Sec. 2] is

PROPOSITION 1.2.5. *A left $\mathbb{E}_n$-module M is q-holonomic if and only if there exists a q-holonomic left $\mathbb{W}_n$ module N with $M=\mathbb{E}_n\otimes_{W_n} N$.*

1.2.1.1. *Cyclic modules.* We will mainly be interested in cyclic modules, *i.e.* modules of the form $M=\mathbb{E}_n v$ or $N=\mathbb{W}_n v$ generated by a single element v. In the case of $\mathbb{W}_n$-modules, a useful observation is that every cyclic module has a canonical good filtration, given by

$$\mathcal{F}_k N := (\mathcal{F}_k\mathbb{W}_n)v\,. \tag{1.2.6}$$

In the case of $\mathbb{E}_n$-modules, another structural result of [**Sab93**, Sec. 2] shows that

PROPOSITION 1.2.6. *Every q-holonomic $\mathbb{E}_n$-module is cyclic.*

We recall that any cyclic module may be written in the form

$$M=\mathbb{E}_n\big/\mathrm{Ann}_{\mathbb{E}_n}(v)\qquad\text{or}\qquad N=\mathbb{W}_n\big/\mathrm{Ann}_{\mathbb{W}_n}(v)\,, \tag{1.2.7}$$

where the *annihilation ideal* $\text{Ann}_A(v) = \{a \in A \text{ s.t. } av = 0\}$ is the left ideal in the algebra $A = \mathbb{E}_n$ or $\mathbb{W}_n$ consisting of elements that kill the generator.

For a cyclic module M, being q-holonomic roughly implies that the annihilation ideal has at least n independent generators. This can be made precise by introducing the characteristic variety $\text{char}M \in (\mathbb{C}^*)^{2n}$; by (*e.g.*) Prop. 7.1.9 of [**KS12**], M is q-holonomic if and only if $\dim(\text{char}M) = n$. The corresponding statement for D-modules is a classic result in the theory, *cf.* [**Kas77**]. A weaker, specialized result, which is sufficient for all the examples we need to consider in this paper, is the following:

LEMMA 1.2.7. *Let $M = \mathbb{E}_n v$ be a cyclic $\mathbb{E}_n$-module whose annihilation ideal contains elements of the form $p_j(x){y_j}^{d_j} + q_j(x)$ for each $j = 1, ..., n$, with $p_j(x), q_j(x) \in \mathbb{C}_q[x_1, ..., x_n]$, $p_j, q_j \neq 0$. Then M is q-holonomic.*

PROOF. We will prove that the associated $\mathbb{W}_n$-module $N = \mathbb{W}_n v$ is q-holonomic, by showing that the dimensions of the filtered components $\mathcal{F}_k N = (\mathcal{F}_k \mathbb{W}_n)v$ obey $\dim \mathcal{F}_k N \leq C k^n$ for some fixed constant C. Then it follows from Prop. 1.2.5 that $M = \mathbb{E}_n v$ is q-holonomic.

Choose any $k \geq \max\{n, d_1, ..., d_n\}$. The filtered component $\mathcal{F}_k N$ is certainly spanned by all the monomials $x^a y^b v := x_1^{a_1} ... x_n^{a_n} y_1^{b_1} ... y_n^{b_n} v$ with $a + b \leq k$. However, the relations

$$(p_j(x){y_j}^{d_j} + q_i(x))v = 0\,, \qquad j = 1, ..., n \tag{1.2.8}$$

make some of these monomials redundant, and reduce the dimension. Let $c_j = \deg_{x_j} p_j(x)$. Then, for any j, we observe that if $x^a y^b v$ is divisible by $y_j^{d_j}$, the relations (1.2.8) imply that it is sufficient to consider x^a such that $\deg_{x_j} x^a < c_j$. In other words, $\mathcal{F}_k N$ is spanned by

$$\Big\{ x^a y^b v \quad a + b \leq k\,, \quad a_j, b_j \geq 0 \text{ for all } j\,, \quad \text{and for all } j,\ b_j \geq d_j \text{ only if } a_j < c_j \Big\} \tag{1.2.9}$$

We seek an upper bound for dimension of this space of monomials. Let $\mathbf{d} = \max\{d_1, ..., d_n\}$ and $\mathbf{c} = \max\{c_1, ..., c_n\}$. For each $0 \leq m \leq n$, let

$$S_m := \left\{ (a, b) \quad \begin{array}{l} a_j, b_j \leq k \quad \text{for all } j, \\ b_j \geq \mathbf{d} \text{ for exactly } m \text{ values of } j, \\ \text{and } b_j \geq \mathbf{d} \text{ only if } a_j < \mathbf{c} \end{array} \right\}$$

Then $\bigcup_{m=0}^{n} S_m$ contains the set of (a, b) such that $x^a y^b v$ is in the set (1.2.9), and we can count

$$S_m = \binom{n}{m} \mathbf{d}^{n-m}(k - \mathbf{d} + 1)^m (k+1)^{n-m} \mathbf{c}^m \leq C_m k^n, \qquad 0 \leq m \leq n \tag{1.2.10}$$

for some constants C_m (depending on $m, n, \mathbf{c}, \mathbf{d}$). Thus

$$\dim \mathcal{F}_k N \leq \sum_{m=0}^{n} S_m \leq (\sum_{m=0}^{n} C_m) k^n,$$

so that the homological dimension of N is at most n, and N is a q-holonomic $\mathbb{W}_n$ module. We conclude that $M \simeq \mathbb{E}_n \otimes_{\mathbb{W}_n} N$ is likewise q-holonomic. □

1.2.1.2. *The function spaces $\mathcal{V}_{m,n}$.* The cyclic modules relevant to this work arise from a particular representation of the $\mathbb{W}$ and $\mathbb{E}$ algebras. For any non-negative integers m and n, we define

$$\mathcal{V}_{m,n} = \{\text{functions} : \mathbb{Z}^m \to \mathbb{V}_n\}, \tag{1.2.11}$$

where $\mathbb{V}_n$ is the field of rational functions in $q^{\frac{1}{2}}, \{x_i^{\frac{1}{2}}\}_{i=1}^n, \{z_{ij}\}_{i,j=1}^n$ as in (2.2.7). (Recall that the diagram invariant $G_{\mathbb{D}}$ is valued in $\mathbb{V}_n$.) We will think of $\mathcal{V}_{m,n}$ as a vector space over the fraction field $\mathbb{C}_q = \mathbb{C}(q)$. The appearance of the continuous variables x_i, z_{ij} is a departure from the setting of the colored Jones polynomial.

The space $\mathcal{V}_{m,n}$ has a left action of $\mathbb{E}_{n+m}$ (and hence of its sub-algebra $\mathbb{W}_{n+m}$) defined as follows. Let us relabel the last m pairs of generators of of $\mathbb{E}_{n+m}$ as $x_i, y_i \rightsquigarrow \hat{x}_{i-n}, \hat{y}_{i-n}$ $(i > n)$, so that

$$\mathbb{E}_{n+m} = \mathbb{C}_q[x_1^{\pm}, y_1^{\pm}, \ldots, x_n^{\pm}, y_n^{\pm}, \hat{x}_1^{\pm}, \hat{y}_1^{\pm}, \ldots, \hat{x}_m^{\pm}, \hat{y}_m^{\pm}]/(\mathrm{rel}_q^{\pm}). \tag{1.2.12}$$

Where $\mathrm{rel}_q^{\pm}$ are as in (1.2.3). Let $f : \mathbb{Z}^m \to \mathbb{V}_n$ denote a function in $\mathcal{V}_{m,n}$. The m pairs of generators $\hat{x}_1, \hat{y}_1, \ldots, \hat{x}_m, \hat{y}_m$ have a familiar action

$$\begin{aligned} (\hat{x}_i^{\pm} \cdot f)(a_1, \ldots, a_m) &= q^{\pm a_i} f(a_1, \ldots, a_m) \\ (\hat{y}_i^{\pm} \cdot f)(a_1, \ldots, a_m) &= f(a_1, \ldots, a_i \pm 1, \ldots, a_m). \end{aligned} \tag{1.2.13}$$

The other n pairs of generators $x_1, y_1, \dots, x_n, y_n$ have an action on $\mathcal{V}_{m,n}$ induced from that of $\mathbb{E}_n$ on the codomain $\mathbb{V}_n$, which is given by

$$x_i^{\pm} : \text{multiplication by } x_i^{\pm}, \qquad y_i^{\pm} : \begin{cases} x_j & \mapsto q^{\pm\delta_{ij}} x_j \\ z_{j\ell} & \mapsto q^{\frac{1}{2}\delta_{ij}\delta_{i\ell}} x_j^{\pm\frac{1}{2}\delta_{i\ell}} x_\ell^{\pm\frac{1}{2}\delta_{ij}} z_{j\ell} \end{cases} \tag{1.2.14}$$

A more intuitive way to understand the action of $\mathbb{E}_{n+m}$ on $\mathcal{V}_{m,n}$ is to fix q to be a generic complex number and to set $x_i = q^{\alpha_i}$ and $z_{ij} = q^{\alpha_i\alpha_j/2}$ for $\alpha_i, \alpha_j \in \mathbb{C}$. Then the generators x_i (resp. $\hat{x}_i$) of $\mathbb{E}_{n+m}$ act as multiplication by q^{α_i} (resp. q^{a_i}) and the generators y_i (resp. $\hat{y}_i$) act by shifting $\alpha_i \mapsto \alpha_i + 1$ (resp. $a_i \mapsto a_i + 1$). In particular, the awkward transformation of $z_{j\ell}$ in (1.2.14) is just that induced from a shift in α_i.

Unfortunately, we will need to keep q a formal algebraic variable (we cannot set it to a generic complex number) in order to gain control over the specialization to the ADO invariant later on. Explicitly, the induced action on $f \in \mathcal{V}_{m,n}$ is

$$\begin{aligned} &(x_i^{\pm} \cdot f)(a_1, \dots, a_m) \mapsto x_i^{\pm} f(a_1, \dots, a_m)\,, \\ &(y_i^{\pm} \cdot f)(a_1, \dots, a_m) \mapsto f(a_1, \dots, a_m)\ {}_{x_i \to q^{\pm} x_i\,,\, z_{ij} \to x_j^{\pm\frac{1}{2}} z_{ij}\,(j \neq i),\, z_{ii} \to q^{\frac{1}{2}} x_i^{\pm} z_{ii}}\,. \end{aligned} \tag{1.2.15}$$

It is straightforward to check that the q-commutation relations of $\mathbb{E}_{n+m}$ are respected by these combined actions.

With the above action, any function $f \in \mathcal{V}_{m,n}$ generates a cyclic module for $\mathbb{E}_{n+m}$

$$M_f := \mathbb{E}_{n+m} f \simeq \mathbb{E}_{n+m} / \operatorname{Ann}_{\mathbb{E}_{n+m}}(f)\,. \tag{1.2.16}$$

Definition 1.2.8. *We say that* the function f is q-holonomic *if the corresponding* $\mathbb{E}_{n+m}$*-module* M_f *is* q*-holonomic.*

Similarly, f generates a cyclic $\mathbb{W}_{n+m}$-module $N_f = \mathbb{W}_{n+m} f = \mathbb{W}_{n+m} / \operatorname{Ann}_{\mathbb{W}_{n+m}}(f)$. Note that such a $\mathbb{W}_{n+m}$-module can never have monomial torsion, because the $\mathbb{W}_{n+m}$ action on $\mathcal{V}_{m,n}$ extends to an $\mathbb{E}_{n+m}$ action, for which the generators $x_i, y_i, \hat{x}_i, \hat{y}_i$ are invertible. By Prop. 1.2.5, if N_f is a q-holonomic $\mathbb{W}_{n+m}$-module, then M_f is a q-holonomic $\mathbb{E}_{n+m}$-module.

1.2.1.3. *Examples.* We list some classic examples of q-holonomic functions $f \in \mathcal{V}_{m,n}$ which will be useful in proving the q-holonomicity of the diagram invariant $G_{\mathbb{D}}$. In what follows we use a_i to denote a discrete variable and α_i for a continuous one. The actions of the x_i, y_i, $\hat{x}_i$, and $\hat{y}_i$ are as in (1.2.13) and (1.2.14).

Constant and delta functions.

The constant function $f(a_1,\dots,a_n) \equiv 1$ ($f \in \mathcal{V}_{m,n}$) has annihilation ideal

$$\mathrm{Ann}_{\mathbb{E}_{n+m}}(f) = \mathbb{E}_{n+m}(y_i - 1, \hat{y}_j - 1)_{i=1}^{n}{}_{j=1}^{m}, \tag{1.2.17}$$

and is q-holonomic by an application of Lemma 1.2.7.

The delta function in discrete variables $h(a_1,\dots,a_n) = \delta_{a_1,0}\cdots\delta_{a_n,0}$ is annihilated by the ideal

$$\mathrm{Ann}_{\mathbb{E}_{n+m}}(h) = \mathbb{E}_{n+m}(y_i - 1, \hat{x}_j - 1)_{i=1}^{n}{}_{j=1}^{m}\,. \tag{1.2.18}$$

It is q-holonomic by Lemma 1.2.7 applied with the $\hat{x}_j$ and $\hat{y}_j$ swapped. This swap, written more precisely as the map sending $(\hat{x}_j, \hat{y}_j) \mapsto (\hat{y}_j, \hat{x}_j^{-1})$, is an automorphism of $\mathbb{E}_{n+m}$ known as Mellin or Fourier transform, *cf.* [**Sab93**, Sec. 1.3].

We also consider a cyclic $\mathbb{E}_{n+m}$-module $M = \mathbb{E}_{n+m}v$ with annihilation ideal

$$\mathrm{Ann}_{\mathbb{E}_{n+m}}(v) = \mathbb{E}_{n+m}(x_i - 1, \hat{y}_j - 1)_{i=1}^{n}{}_{j=1}^{m}\,. \tag{1.2.19}$$

It is q-holonomic, by Lemma 1.2.7 now with the x_i and y_i swapped. This plays the role of the cyclic module generated by a delta-function in the *continuous* variables, namely $f = \delta(x_1-1)\cdots\delta(x_n-1)$. However, such a Dirac delta-function does not exist in our algebraic functional space $\mathbb{V}_n$, so the cyclic module $M = \mathbb{E}_{n+m}v$ is not embedded in $\mathcal{V}_{m,n}$.

Indicator functions.

Generalizing the delta-function example above, the indicator function

$$\vartheta_{[a_2,a_3]}(a_1) = \begin{cases} 1 & a_2 \le a_1 \le a_3 \\ 0 & \text{else} \end{cases} \quad \in \mathcal{V}_{3,0} \tag{1.2.20a}$$

is annihilated by the elements $(\hat{x}_1 - q\hat{x}_3)(\hat{y}_3 - 1)$, $(\hat{x}_1 - \hat{x}_2)(\hat{y}_2 - 1)$, and $(\hat{x}_1 - \hat{x}_3)(\hat{x}_1 - q^{-1}\hat{x}_2)(\hat{y}_1 - 1)$, and thus by Lemma 1.2.7 is q-holonomic for $\mathbb{E}_3$. Its half-infinite cousin

$$\vartheta_{a_1 \le a_2} = \vartheta_{(-\infty,a_2]}(a_1) = \vartheta_{[a_1,\infty)}(a_2) := \begin{cases} 1 & a_1 \le a_2 \\ 0 & \text{else} \end{cases} \quad \in \mathcal{V}_{2,0} \tag{1.2.20b}$$

has annihilation ideal containing $(\hat{x}_2 - q^{-1}\hat{x}_1)(\hat{y}_2 - 1)$ and $(\hat{x}_2 - \hat{x}_1)(\hat{y}_1 - 1)$, and thus is q-holonomic for $\mathbb{E}_2$. Specializations of these functions to constant a_1, a_2, and/or a_3 are similarly q-holonomic.

Linear exponentials.

The linear functions

$$f(a_1) = q^{a_1} \text{ in } \mathcal{V}_{1,0} \quad \text{and} \quad g = x_1 \text{ in } \mathcal{V}_{0,1} \tag{1.2.21}$$

are both q-holonomic, with annihilation ideals

$$\mathrm{Ann}(f) = \mathbb{E}_1(\hat{y}_1 - q) \quad \text{and} \quad \mathrm{Ann}(g) = \mathbb{E}_1(y_1 - q)\,. \tag{1.2.22}$$

More generally, given any non-zero integer vectors $\hat{A} = (\hat{A}_1, \ldots, \hat{A}_m)$ in $\mathbb{Z}^m$ and $A = (A_1, \ldots, A_n)$ in $\mathbb{Z}^n$, we consider the linear function in $\mathcal{V}_{m,n}$ given by

$$f(a_1, \ldots, a_m) = q^{\frac{1}{2}\hat{A}\cdot a} x^{\frac{1}{2}A} := q^{\frac{1}{2}\hat{A}_1 a_1 + \ldots + \frac{1}{2}\hat{A}_m a_m} x_1^{\frac{1}{2}A_1} \cdots x_n^{\frac{1}{2}A_n}\,. \tag{1.2.23}$$

Its annihilation ideal in $\mathbb{E}_{n+m}$ has generators

$$\begin{cases} \hat{y}_i - q^{\frac{1}{2}\hat{A}_i} & \hat{A}_i \text{ even} \\ \hat{y}_i^2 - q^{\hat{A}_i} & \hat{A}_i \text{ odd} \end{cases} \quad (i = 1, \ldots, m) \quad \text{and} \quad \begin{cases} y_i - q^{\frac{1}{2}A_i} & A_i \text{ even} \\ y_i^2 - q^{A_i} & A_i \text{ odd} \end{cases} \quad (i = 1, \ldots, n)\,, \tag{1.2.24}$$

and thus is q-holonomic by a direct application of Lemma 1.2.7.

Quadratic exponentials.

The quadratic functions

$$f(a_1) = q^{a_1^2} \text{ in } \mathcal{V}_{1,0} \quad \text{and} \quad g = z_{11}^2 \text{ in } \mathcal{V}_{0,1} \tag{1.2.25}$$

are q-holonomic with annihilation ideals $\mathbb{E}_1(\hat{y}_1 - q\hat{x}_1^2)$ and $\mathbb{E}_1(y_1 - qx_1^2)$, respectively. (Recall the specialization $z_{ii} \to \zeta_{2r}^{\alpha_i^2/2}$, under which z_{ii} depends quadratically on α_i.) Similarly, the quadratic function

$$f(a_1) = q^{\frac{1}{2}a_1^2} \text{ in } \mathcal{V}_{1,0} \quad \text{and} \quad g = z_{11} \text{ in } \mathcal{V}_{0,1} \tag{1.2.26}$$

are q-holonomic with annihilators $\mathbb{E}_1(\hat{y}_1^2 - q^2\hat{x}_1^2)$ and $\mathbb{E}_1(y_1^2 - q^2x_1^2)$. We may also consider mixed quadratic functions such as

$$f(a_1) = x_1^{a_1} \text{ in } \mathcal{V}_{1,1}\,, \tag{1.2.27}$$

which is q-holonomic with annihilation ideal $\mathrm{Ann}(f) = \mathbb{E}_2(y_1 - \hat{x}_1, \hat{y}_1 - x_1)$.

More generally, let $\hat{A} \in \mathbb{Z}^m$, $A \in \mathbb{Z}^n$ be non-zero integer vectors, let $\hat{B} : \mathbb{Z}^m \times \mathbb{Z}^m \to \mathbb{Z}$ and $B : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ be symmetric bilinear forms, and let $C : \mathbb{Z}^n \times \mathbb{Z}^m \to \mathbb{Z}$ be a bilinear map. Then the function f in $\mathcal{V}_{m,n}$ given by

$$f(a) = q^{\frac{1}{2}\hat{B}(a,a)+\frac{1}{2}\hat{A}\cdot a} x^{\frac{1}{2}C(-,a)+\frac{1}{2}A} z^{B} \tag{1.2.28}$$
$$:= q^{\frac{1}{2}\sum_{ij}\hat{B}_{ij}a_i a_j + \frac{1}{2}\sum_i \hat{A}_i a_i} \prod_{ij} x_i^{\frac{1}{2}C_{ij}a_j} \prod_i x_i^{\frac{1}{2}A_i} \prod_{ij} z_{ij}^{B_{ij}}$$

is q-holonomic. Its annihilation ideal is cumbersome to write down in general form because it depends on whether various parameters are even or odd, but possible to analyze. It is generated by expressions of the form $\hat{y}_j-$(monomial in $q^{\pm}, x^{\pm}, \hat{x}^{\pm}$) or $\hat{y}_j^2-$(monomial in $q^{\pm}, x^{\pm}, \hat{x}^{\pm}$) and by y_i-(monomial in $q^{\pm}, x^{\pm}, \hat{x}^{\pm}$) or y_i^2-(monomial in $q^{\pm}, x^{\pm}, \hat{x}^{\pm}$), for each $j = 1, \dots, m$ and $i = 1, \dots, n$. Thus being q-holonomic follows directly from Lemma 1.2.7.

Warning! The function $f(a_1) = q^{a_1^3}$ ($f \in \mathcal{V}_{1,0}$) is well known to *not* be q-holonomic, *cf.* [**GL16**, Ex. 2.2]. Similarly, the analogous "cubic" functions involving continuous variables, such as $g(a_1) = x_1^{a_1^2}$ ($g \in \mathcal{V}_{1,1}$) and $h(a_1) = z_{11}^{a_1}$ ($h \in \mathcal{V}_{1,1}$) are *not* q-holonomic.

***q*-Factorials.**

Many types of q-factorials (or quantum dilogarithms) are q-holonomic. For $a \in \mathbb{Z}$, we recall the q-Pochhammer symbol (2.1.10) given by

$$(x;q)_a := \begin{cases} (1-x)(1-qx)\cdots(1-q^{a-1}x) & a \geq 1 \\ 1 & a = 0 \\ 0 & a \leq -1 \end{cases} \tag{1.2.29}$$

This is an element of $\mathcal{V}_{1,1}$ and its annihilation ideal contains[4] $(\hat{x}-q^{-1})(\hat{y}+\hat{x}x-1)$ and $(1-x)y+\hat{x}x-1$, hence by Lemma 1.2.7 it generates a q-holonomic $\mathbb{E}_2$-module.

1.2.2. Closure properties. A notable feature of q-holonomic modules is that they are closed under many algebraic operations. These closure properties enabled Garoufalidis and Lê to efficiently

[4]Note that the $(\hat{x} - q^{-1})$ factor in the first equation accounts for setting $(x;q)_a = 0$ at negative values of a; at all positive a, the function $(x;q)_a = 0$ is simply annihilated by $\hat{y} + \hat{x}x - 1$.

prove that the colored Jones invariants of knots formed a q-holonomic family. They are of similar importance here.

We review some of the closure properties that will be used in the current work, as they apply to our functional spaces $\mathcal{V}_{m,n}$ containing both discrete and continuous variables. Even though the initial application to Jones polynomials [**GL05**] only involved acting on functions of discrete variables, the closure properties themselves are much more general. They all derive from purely *algebraic* properties of q-holonomic $\mathbb{E}_n$-modules, which make no reference to representations in a particular functional space. If one happens to be working with cyclic $\mathbb{E}_n$-modules generated by functions, the algebraic closure properties can simply be applied to that setting. This perspective was also espoused in the survey [**GL16**].

Thus, altogether, there is nothing mathematically novel in this section. Our goal is to illustrate how established closure properties apply in our setting.

PROPOSITION 1.2.9. **Closure properties** *Suppose that $f, g \in \mathcal{V}_{m,n}$ are q-holonomic, with arguments $a = a_1, \dots, a_m$.*

- **(a)** *(Addition and Multiplication) The functions $f + g \in \mathcal{V}_{m,n}$ and $fg \in \mathcal{V}_{m,n}$ are q-holonomic.*
- **(b)** *(Shifts) Choose vectors $c \in (\mathbb{C}_q \setminus 0)^n$ and $d \in \mathbb{Z}^m$. Then*

(1.2.30a) $$f(a_1 + d_1, \dots, a_m + d_m)\big|_{x_i \mapsto c_i x_i \text{ for } i = 1, \dots, n}$$

is q-holonomic.

- **(c)** *(Linear transformations) Let $A \in \mathrm{Mat}(n \times n', \mathbb{Z})$, $C \in \mathrm{Mat}(n \times m', \mathbb{Z})$ and $D \in \mathrm{Mat}(m \times m', \mathbb{Z})$. We define a q-holonomic function $h(a') \in \mathcal{V}_{m',n'}$, given by*

(1.2.30b) $$h(a') := f(Da')\big|_{x \mapsto q^{Ca} x^A}$$

Explicitly, the transformation of the x s here is $x_i \mapsto \prod_{j'=1}^{m'} q^{C_{ij'} a_{j'}} \prod_{i'=1}^{n'} x_{i'}^{A_{ii'}}$. Important special cases include specializations of discrete variables:

(1.2.30c) $$f(a_1, \dots, a_{m-1}, a_m) \in \mathcal{V}_{m,n} \text{ } q\text{-holonomic} \;\Rightarrow\; f(a_1, \dots, a_{m-1}, 0) \in \mathcal{V}_{m-1,n} \text{ } q\text{-holonomic};$$

specializations in continuous variables:

(1.2.30d) $$f(a) \in \mathcal{V}_{m,n} \text{ } q\text{-holonomic} \;\Rightarrow\; f(a)\big|_{x_n = 1} \in \mathcal{V}_{m,n-1} \text{ } q\text{-holonomic};$$

and extensions in both types of variables: when $m \leq m'$ and $n \leq n'$, we can view $f \in \mathcal{V}_{m,n}$ as an element of $\mathcal{V}_{m',n'}$ (a function independent of any extra a or x variables), and f being q-holonomic for $\mathbb{E}_{n+m}$ implies that f is q-holonomic for $\mathbb{E}_{n'+m'}$ as well.

(d) *The sum over a discrete variable*

$$h(a_1, \ldots, a_m, a_{m+1}) := \sum_{b=a_m}^{a_{m+1}} f(a_1, \ldots, a_{m-1}, b), \quad g \in \mathcal{V}_{m+1,n} \tag{1.2.30e}$$

is likewise q-holonomic. Similarly, when they converge, the half-infinite sums

$$\sum_{b=a_m}^{\infty} f(a_1, \cdots, a_{m-1}, b) \text{ and } \sum_{b=-\infty}^{a_m} f(a_1, \ldots, a_{m-1}, b)$$

are q-holonomic functions in $\mathcal{V}_{m,n}$; and $\sum_{b=-\infty}^{\infty} f(a_1, \ldots, a_{m-1}, b)$ is q-holonomic in $\mathcal{V}_{m-1,n}$.

Proof. The proofs of these statements are essentially identical to the arguments given in [**GL05, GL16**].

For (a), let $M = \mathbb{E}_{n+m} f$ and $N = \mathbb{E}_{n+m} g$ be the modules generated by f and g. The $\mathbb{E}_{n+m}$-module generated by the sum $f + g$ is a sub-quotient of the (algebraic) direct sum $M \oplus N$, and both sub-quotients and direct sums of q-holonomic modules are q-holonomic [**Sab93**]. Similarly, the $\mathbb{E}_{n+m}$-module generated by fg is a submodule of the algebraic tensor product $M \otimes_{\mathbb{C}_q[x^{\pm}]} N$, and tensor products of q-holonomic modules are q-holonomic [**Sab93**].

For (b), we may simply note that for any $c \in (\mathbb{C}_q^*)^n$ and $d \in \mathbb{Z}^m$, there is an automorphism of the algebra $\mathbb{E}_{n+m}$ given by $\gamma : (x_i, \hat{x}_j, y_i, \hat{y}_j) \mapsto (c_i x_i, q^{d_j} \hat{x}_j, y_i, \hat{y}_j)$, and a corresponding linear automorphism of $\mathcal{V}_{m,n}$ sending

$$h(a_1, \ldots, a_m) \quad \mapsto \quad h(a_1 + d_1, \ldots, a_m + d_m)\ {}_{x_i \mapsto c_i x_i \text{ for } i = 1, \ldots, n} \tag{1.2.31}$$

as in (1.2.30a) that intertwines the automorphism γ of the algebra. The property of being q-holonomic is preserved by any such automorphism.

For (c), we assemble A, C, D into an $(n + m) \times (n' + m')$ matrix

$$U = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}. \tag{1.2.32}$$

This linear transformation defines a function $F : (\mathbb{C}^*)^{n+m} \to (\mathbb{C}^*)^{n'+m'}$ under which the pullback of coordinates is $F^* x_i = \prod_{j=1}^{n+m} x_j^{U_{ij}}$. This in turn induces an inverse image functor $F^! : \mathbb{E}_{m+n}$-mod$\to \mathbb{E}_{m'+n'}$-mod, which is shown in [**Sab93**, Sec. 2.3] to preserve q-holonomic modules. Letting $M = \mathbb{E}_{n+m} f$, one finds that the module $N = \mathbb{E}_{n'+m'} h$ generated by the function in (1.2.30b) is a sub-quotient of $F^!(M)$, and so q-holonomic.

For (d), we use the result of [**Sab93**, Sec 2.4] that the algebraic convolution product of q-holonomic modules is q-holonomic. For any $h(a_1, \dots, a_m)$ and $h'(a_1, \dots, a_m)$, the function

$$h *_m h' := \sum_{b=-\infty}^{\infty} h(a_1, \dots, b + a_m) h'(a_1, \dots, -b) \quad \in \mathcal{V}_{m,n}, \tag{1.2.33}$$

when it exists, generates a submodule of the algebraic convolution product $(\mathbb{E}_{n+m} h) * (\mathbb{E}_{n+m} h')$, and thus is q-holonomic. Then we recall that indicator functions (1.2.20a) are q-holonomic. The summation given by (1.2.30e) is obtained by convolving f (extended to an element of $\mathcal{V}_{m+2,n}$) with an indicator function; similarly, the half-infinite and infinite sums below (1.2.30e) are obtained by convolving f with half-infinite indicator functions and with the constant function, respectively. □

CHAPTER 2

The ADO Invariants are a q-Holonomic Family

2.1. An Extension of the Drinfel'd-Jimbo Algebra

Here we consider a particular quantum group whose representation category is used to construct the ADO invariant. This object was first fully established in [**GPMT09**], though ideas of its formulation were already present in [**ADO92, Oht02**]. For more details about the unrolled quantum group and its representation theory see [**CGPM15c, GPM18**].

Let q be a formal variable. Fix a positive integer $r \geq 2$, and let $\zeta_{2r} = e^{\frac{\pi\sqrt{-1}}{r}}$ be a $2r^{th}$-root of unity. Let $\mathbb{K}_r$ be the subring of $\mathbb{C}(q)$ consisting of elements with no poles at $q = \zeta_{2r}$. A $\mathbb{K}_r$-module can be specialized at $q = \zeta_{2r}$ by tensoring with the module $\mathbb{K}_r/(q - \zeta_{2r})$.

Consider the $\mathbb{C}(q)$-algebra $\mathcal{U}_q = \mathcal{U}_q(\mathfrak{sl}_2)$ generated by E, F, K, K^{-1} with relations

$$KF = q^{-2}FK, \quad KE = q^2EK, \quad KK^{-1} = K^{-1}K = 1, \text{ and } \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}. \tag{2.1.1}$$

This is a Hopf algebra with co-product, co-unit, and antipode defined on generators by:

$$\begin{aligned} \triangle(E) &= 1 \otimes E + E \otimes K, & \varepsilon(E) &= 0, & S(E) &= -EK^{-1}, \\ \triangle(F) &= K^{-1} \otimes F + F \otimes 1, & \varepsilon(F) &= 0, & S(F) &= -KF, \\ \triangle(K) &= K \otimes K, & \varepsilon(K) &= 1, & S(K) &= K^{-1}. \end{aligned} \tag{2.1.2}$$

The Hopf algebra $\mathcal{U}_q$ is usually called the Drinfel'd-Jimbo quantum group.

The *unrolled quantum group* $\mathcal{U}^H_{\zeta_{2r}} = \mathcal{U}^H_{\zeta_{2r}}(\mathfrak{sl}_2)$ is the $\mathbb{C}$-algebra generated by E, F, K, K^{-1}, H with relations (2.1.1) specialized to $q = \zeta_{2r}$, together with the relations

$$HK = KH, \qquad [H, E] = 2E, \qquad [H, F] = -2F. \tag{2.1.3}$$

The algebra $\mathcal{U}^H_{\zeta_{2r}}$ is a Hopf algebra with coproduct, counit and antipode defined as above on $K^{\pm}, E, F$ and defined on the element H as

$$\triangle(H) = H \otimes 1 + 1 \otimes H, \qquad \varepsilon(H) = 0, \qquad S(H) = -H. \tag{2.1.4}$$

To connect with the ADO invariant, we will further pass to the central quotient

$$\overline{\mathcal{U}}^H_{\zeta_{2r}} = \overline{\mathcal{U}}^H_{\zeta_{2r}}(\mathfrak{sl}_2) := \mathcal{U}^H_{\zeta_{2r}}/(E^r, F^r)\,. \tag{2.1.5}$$

2.1.1. Representations of $\overline{\mathcal{U}}^H_{\zeta_{2r}}$. Let V be a finite-dimensional $\overline{\mathcal{U}}^H_{\zeta_{2r}}$ module. An eigenvalue $\lambda \in \mathbb{C}$ of H is called a *weight* and the associated eigenspace is called the *weight space*. We say V is a *weight module* if it splits as a direct sum of weight spaces and $q^H = K$ as operators on V, i.e. $Kv = {\zeta_{2r}}^{\lambda} v$ for any weight vector v with $Hv = \lambda v$. Let $\mathrm{Rep}_{\mathrm{wt}}\,\overline{\mathcal{U}}^H_{\zeta_{2r}}$ denote the category of finite dimensional weight modules of $\overline{\mathcal{U}}^H_{\zeta_{2r}}$.

Consider the following two families of modules. For $\alpha \in \mathbb{C}$, let V_α be the representation in $\mathrm{Rep}_{\mathrm{wt}}\,\overline{\mathcal{U}}^H_{\zeta_{2r}}$ with a basis $\{v_0, \ldots, v_{r-1}\}$ on which the $\overline{\mathcal{U}}^H_{\zeta_{2r}}$-action is given by

$$Ev_i = \frac{{\zeta_{2r}}^{\alpha-i+1} - {\zeta_{2r}}^{-(\alpha-i+1)}}{\zeta_{2r} - {\zeta_{2r}}^{-1}} v_{i-1}, \qquad Fv_i = \frac{{\zeta_{2r}}^{i+1} - {\zeta_{2r}}^{-(i+1)}}{\zeta_{2r} - {\zeta_{2r}}^{-1}} v_{i+1}, \tag{2.1.6}$$

$$Hv_i = (\alpha - 2i)v_i, \qquad Kv_i = {\zeta_{2r}}^{\alpha-2i} v_i$$

where $v_{-1} = v_r = 0$. When $\alpha \in (\mathbb{C} \setminus \mathbb{Z}) \cup (-1 + r\mathbb{Z})$ the module V_α is simple and called *typical*. As we will now discuss, when $\alpha \in \mathbb{Z} \setminus (-1 + r\mathbb{Z})$ the module V_α is not decomposable — it has a simple submodule which is not a direct summand.

For each $n \in \mathbb{Z}_{\geq 0}$, let S^q_n be the usual $(n+1)$-dimensional irreducible highest weight $\mathcal{U}_q$-module with highest weight n. The module S^q_n has a basis $\{s_0, s_1, \ldots, s_n\}$ on which the $\mathcal{U}_q$-action is given by $Ks_i = q^{n-2i}s_i$ and

$$Es_i = \frac{q^{n-i+1} - q^{-(n-i+1)}}{q - q^{-1}} s_{i-1}, \qquad Fs_i = \frac{q^{i+1} - q^{-(i+1)}}{q - q^{-1}} s_{i+1} \tag{2.1.7}$$

where $s_{-1} = s_{n+1} = 0$. If $n \in \{0, \ldots, r-1\}$ then by setting $q = \zeta_{2r}$ and $Hs_i = (n-2i)s_i$, the $\mathcal{U}_q$-module S^q_n becomes a simple $\overline{\mathcal{U}}^H_{\zeta_{2r}}$-module S_n. In general, if $m \in \{0, \ldots, r-1\}$ and $k \in \mathbb{Z}$ then we can define a simple $(m+1)$-dimensional $\overline{\mathcal{U}}^H_{\zeta_{2r}}$-module S_{m+kr} with basis $\{s_0, \ldots, s_m\}$ on which the $\overline{\mathcal{U}}^H_{\zeta_{2r}}$-action is given by

$$Hs_i = (m + kr - 2i)s_i, \qquad Ks_i = q^{m+kr-2i}s_i$$

and (2.1.7) with $q = \zeta_{2r}$ and $n = m + rk$ (here we set $s_{-1} = s_{m+1} = 0$). Notice that the definitions of V_{kr-1} and S_{kr-1} coincide.

LEMMA 2.1.1. *Every irreducible representation of* $\mathrm{Rep}_{\mathrm{wt}}\overline{\mathcal{U}}^H_{\zeta_{2r}}$ *is isomorphic to exactly one of the modules in the list:*

- S_{n+kr}, *for* $n = 0, \cdots, r-2$ *and* $k \in \mathbb{Z}$,
- V_α *for* $\alpha \in (\mathbb{C} \setminus \mathbb{Z}) \cup (-1 + r\mathbb{Z})$.

PROOF. An argument analogous to that of finite dimensional $\mathfrak{sl}_2$-modules (see for example [**Kas95**, Section V.4]) shows the following: 1. Every non-zero finite dimensional weight $\overline{\mathcal{U}}^H_{\zeta_{2r}}$-module has a highest weight vector and 2. If W is furthermore a simple module then it is uniquely determined up to isomorphism by its highest weight $\lambda \in \mathbb{C}$. The lemma then follows from the fact that the highest weights of modules in the above list are in bijection with the elements of $\mathbb{C}$. □

When $\alpha = n + kr$, $n = 0, \ldots, r-2$, the module V_α is no longer irreducible. Instead, there is a non-split short exact sequence

$$0 \to S_{n+kr-2(n+1)} \to V_{n+kr} \to S_{n+kr} \to 0$$

where the first morphism is determined by sending the highest weight vector of $S^{\zeta_{2r}}_{n+kr-2(n+1)}$ to v_{n+1} and the second morphism is given by sending the highest weight vector V_{n+kr} to the highest weight vector of $S^{\zeta_{2r}}_{n+kr}$. The families

$$\{V_\alpha\}_{\alpha \in (\mathbb{C}\setminus\mathbb{Z})\cup(-1+r\mathbb{Z})} \quad \text{and} \quad \{S^q_n\}_{n \in \mathbb{Z}_{\geq 0}}$$

are used to define the ADO invariant and colored Jones polynomial, respectively.

2.1.2. The ribbon structure on $\mathrm{Rep}_{\mathrm{wt}}\overline{\mathcal{U}}^H_{\zeta_{2r}}$**.** Here we recall that $\mathrm{Rep}_{\mathrm{wt}}\overline{\mathcal{U}}^H_{\zeta_{2r}}$ is a ribbon category, for details see for example [**GPMT09, CGPM15c, GPM18**]. We will describe the ribbon structure in terms of dualities and a braiding. This formulation follows [**GPM18**], where it is shown that a ribbon category can be defined as a pivotal braided category satisfying certain compatibility constraints on the natural twist morphism defined from the braiding and dualities. This structure will be used while defining link invariants.

Since $\overline{\mathcal{U}}^H_{\zeta_{2r}}$ is a Hopf algebra, $\mathrm{Rep}_{\mathrm{wt}}\overline{\mathcal{U}}^H_{\zeta_{2r}}$ is a monoidal category where the unit $\mathbb{1}$ is the 1-dimensional trivial module $\mathbb{C}$. Moreover, $\mathrm{Rep}_{\mathrm{wt}}\overline{\mathcal{U}}^H_{\zeta_{2r}}$ is $\mathbb{C}$-linear: hom-sets are $\mathbb{C}$-modules, the composition and tensor product of morphisms are $\mathbb{C}$-bilinear, and $\mathrm{End}_{\mathrm{Rep}_{\mathrm{wt}}\overline{\mathcal{U}}^H_{\zeta_{2r}}}(\mathbb{1}) = \mathbb{C}\,\mathrm{Id}_{\mathbb{1}}$. We will often denote the unit $\mathbb{1}$ by $\mathbb{C}$.

Duality. Let V and W be representations in $\mathrm{Rep}_{\mathrm{wt}}\overline{\mathcal{U}}^H_{\zeta_{2r}}$. Let $v_0, \dots, v_{r-1}$ be a basis of V and $v_0^*, \dots, v_{r-1}^*$ the dual basis of the dual space $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$. The duality morphisms

$$\begin{aligned} \overrightarrow{\mathrm{coev}}_V &\colon \mathbb{C} \to V \otimes V^*, & \overrightarrow{\mathrm{ev}}_V &\colon V^* \otimes V \to \mathbb{C} \\ 1 &\mapsto \sum_{i=0}^{r-1} v_i \otimes v_i^*, & f \otimes w &\mapsto f(w), \\ \overleftarrow{\mathrm{coev}}_V &\colon \mathbb{C} \to V^* \otimes V & \overleftarrow{\mathrm{ev}}_V &\colon V \otimes V^* \to \mathbb{C} \\ 1 &\mapsto \sum_{i=0}^{r-1} v_i^* \otimes K^{r-1} v_i, & w \otimes f &\mapsto f(K^{1-r} w) \end{aligned}$$

define a pivotal structure on $\mathrm{Rep}_{\mathrm{wt}}\overline{\mathcal{U}}^H_{\zeta_{2r}}$ [**GPM18**]. Taking $V = V_\alpha$, the cup and cap morphisms can be written

$$\overleftarrow{\mathrm{coev}}\colon 1 \mapsto \sum_{i=0}^{r-1} {\zeta_{2r}}^{(r-1)(\alpha-2i)} v_i^* \otimes v_i, \qquad \overleftarrow{\mathrm{ev}}\colon v_i \otimes v_j^* \mapsto {\zeta_{2r}}^{(1-r)(\alpha-2i)} \delta_{ij}. \tag{2.1.8}$$

Braiding. In [**Oht02**], Ohtsuki truncates the usual formula of the h-adic quantum R-matrix to define an operator on $V \otimes W$ by

$$R = {\zeta_{2r}}^{H \otimes H/2} \sum_{k=0}^{r-1} \frac{(\zeta_{2r} - {\zeta_{2r}}^{-1})^{2k}}{{\zeta_{2r}}^k ({\zeta_{2r}}^{-2}; {\zeta_{2r}}^{-2})_k} E^k \otimes F^k, \tag{2.1.9}$$

where the q-factorial (*a.k.a.* q-Pochhammer symbol or quantum dilogarithm [**FK94**]) is given by

$$(x; p)_n := \begin{cases} \prod_{k=0}^{n-1} (1 - x p^k) & \text{if } n > 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.1.10}$$

and ${\zeta_{2r}}^{H \otimes H/2}$ is the operator given by

$${\zeta_{2r}}^{H \otimes H/2}(v \otimes v') = {\zeta_{2r}}^{\lambda\lambda'/2} v \otimes v'$$

for weight vectors v and v' of weights of λ and λ'. We call R the *truncated R-matrix*. It is not an element in $\overline{\mathcal{U}}^H_{\zeta_{2r}} \otimes \overline{\mathcal{U}}^H_{\zeta_{2r}}$, but its action on the tensor product of two objects of $\mathrm{Rep}_{\mathrm{wt}}\overline{\mathcal{U}}^H_{\zeta_{2r}}$ is a well-defined linear map. Moreover, R gives rise to a braiding $c_{V,W} : V \otimes W \to W \otimes V$ on $\mathrm{Rep}_{\mathrm{wt}}\overline{\mathcal{U}}^H_{\zeta_{2r}}$ defined by $v \otimes w \mapsto \tau(R(v \otimes w))$ where τ is the permutation $x \otimes y \mapsto y \otimes x$. The inverse of the operator R is

$$R^{-1} = \left(\sum_{k=0}^{r-1} (-1)^k \frac{({\zeta_{2r}} - {\zeta_{2r}}^{-1})^{2k}}{{\zeta_{2r}}^{k^2} ({\zeta_{2r}}^{-2}; {\zeta_{2r}}^{-2})_k} E^k \otimes F^k \right) {\zeta_{2r}}^{-H \otimes H/2}. \tag{2.1.11}$$

For later reference, we compute the coefficients of R acting on $v_a \otimes w_b \in V_\alpha \otimes V_\beta$:

$$\begin{aligned} R(v_a \otimes w_b) &= q^{\frac{1}{2} H \otimes H} \sum_{k=0}^{r-1} \frac{(q - q^{-1})^{2k}}{q^k (q^{-2}; q^{-2})_k} E^k v_a \otimes F^k w_b \\ &= q^{\frac{1}{2} H \otimes H} \sum_{k=0}^{r-1} (-1)^k q^{k(\alpha - a - b - 1)} \frac{(q^{-2(\alpha - a + 1)}; q^{-2})_k (q^{2(b+1)}; q^2)_k}{(q^{-2}; q^{-2})_k} v_{a-k} \otimes w_{b+k} \\ &= \sum_{k=0}^{r-1} (-1)^k q^{k(\alpha - a - b - 1)} q^{\frac{1}{2} \lambda^\alpha_{a-k} \lambda^\beta_{b+k}} \frac{(q^{-2(\alpha - a + 1)}; q^{-2})_k (q^{2(b+1)}; q^2)_k}{(q^{-2}; q^{-2})_k} v_{a-k} \otimes w_{b+k} \qquad (2.1.12) \end{aligned}$$

where $\lambda^\beta_{b+k} = \beta - 2(b+k)$ and $\lambda^\alpha_{a-k} = \alpha - 2(a-k)$ are the weights of $w_{b+k} \in V_\beta$ and $v_{a-k} \in V_\alpha$, respectively and $q = \zeta_{2r}$. A similar calculation reveals the following coefficients for the inverse:

$$R^{-1}(v_a \otimes w_b) = \sum_{k=0}^{r-1} (-1)^k q^{-\frac{1}{2} \lambda^\alpha_a \lambda^\beta_b} q^{k(\alpha - a - b + 1)} \frac{(q^{-2(\alpha - a + 1)}; q^{-2})_k (q^{2(b+1)}; q^2)_k}{(q^2; q^2)_k} v_{a-k} \otimes w_{b+k}, \tag{2.1.13}$$

again with $q = \zeta_{2r}$.

2.2. The ADO Invariant

A cousin of the WRT family of invariants [**RT90**], the ADO invariant is based on a functor from a category formalizing link diagrams to the category of representations detailed in Section 2.1.1. This section is meant to provide a concise review of this invariant, along the way establishing notation.

We consider framed oriented tangles whose components are colored by objects of $\mathrm{Rep}_{\mathrm{wt}} \overline{\mathcal{U}}^H_{\zeta_{2r}}$. Such tangles are called $\mathrm{Rep}_{\mathrm{wt}} \overline{\mathcal{U}}^H_{\zeta_{2r}}$-*colored ribbons*. Let $\mathcal{R}_{\mathrm{Rep}_{\mathrm{wt}} \overline{\mathcal{U}}^H_{\zeta_{2r}}}$ be the category of $\mathrm{Rep}_{\mathrm{wt}} \overline{\mathcal{U}}^H_{\zeta_{2r}}$-colored ribbons [**Kas95**, XIV.5.1]. The well-known Reshetikhin-Turaev construction defines a $\mathbb{C}$-linear functor

$$F : \mathcal{R}_{\mathrm{Rep}_{\mathrm{wt}} \overline{\mathcal{U}}^H_{\zeta_{2r}}} \to \mathrm{Rep}_{\mathrm{wt}} \overline{\mathcal{U}}^H_{\zeta_{2r}}$$

(for details see *e.g.* [**Kas95, Tur16**]). The value of any $\mathrm{Rep}_{\mathrm{wt}} \overline{\mathcal{U}}^H_{\zeta_{2r}}$-colored ribbon under F can be computed using the six *building blocks*, which are the morphisms ⤪, ⤩, ∪, ∪, ↶, ↷ in $\mathcal{R}_{\mathrm{Rep}_{\mathrm{wt}} \overline{\mathcal{U}}^H_{\zeta_{2r}}}$.

The functor F transforms these building blocks as follows:

$$F(\times) = \tau \circ R, \qquad F(\curvearrowleft) = \overrightarrow{\mathrm{coev}}_V, \quad F(\smile) = \overleftarrow{\mathrm{coev}}_V,$$
$$F(\times) = \tau \circ R^{-1}, \quad F(\frown) = \overrightarrow{\mathrm{ev}}_V, \qquad F(\frown) = \overleftarrow{\mathrm{ev}}_V. \tag{2.2.1}$$

where $\tau(v \otimes w) = w \otimes v$ permutes the factors. Vertical lines are sent to the identity morphism and reversing the direction of an arrow is equivalent to coloring instead by the dual module.

Suppose T is a $(1,1)$ tangle whose endpoints are both colored with the irreducible representation $V \in \mathrm{Rep}_{\mathrm{wt}} \overline{\mathcal{U}}^H_{\zeta_{2r}}$. By definition $F(T) \in \mathrm{End}_{\mathrm{Rep}_{\mathrm{wt}} \overline{\mathcal{U}}^H_{\zeta_{2r}}}(V)$. Since V is simple, this endomorphism is the product of the identity $\mathrm{Id}_V : V \to V$ with an element $\langle T \rangle$ of the ground ring of $\mathrm{Rep}_{\mathrm{wt}} \overline{\mathcal{U}}^H_{\zeta_{2r}}$, i.e. $F(T) = \langle T \rangle \, \mathrm{Id}_V$.

Let L be the closed link obtained by joining the loose ends of T. Then

$$\begin{aligned} F(L) &= F\left(\boxed{T}\, V \right) = \langle T \rangle \, F\left(\boxed{\mathrm{Id}_V}\, V \right) \\ &= \langle T \rangle \, F\left(\bigcirc\, V \right) = \langle T \rangle (\overleftarrow{\mathrm{ev}}_V \circ \overrightarrow{\mathrm{coev}}_V) = \langle T \rangle \, \mathrm{qdim}_{\mathrm{Rep}_{\mathrm{wt}} \overline{\mathcal{U}}^H_{\zeta_{2r}}}(V). \end{aligned} \tag{2.2.2}$$

When $V = V_\alpha$ is typical, direct calculation shows that quantum dimension vanishes:

$$\mathrm{qdim}_{\mathrm{Rep}_{\mathrm{wt}} \overline{\mathcal{U}}^H_{\zeta_{2r}}}(V_\alpha) := (\overleftarrow{\mathrm{ev}}_{V_\alpha} \circ \overrightarrow{\mathrm{coev}}_{V_\alpha}) = 0. \tag{2.2.3}$$

Details may be found in [**GPMT09**]. Thus, from equation (2.2.2) we have that $F(L) = 0$ if any component of L is colored by a typical module V_α.

In [**ADO92**], Akutsu, Deguchi, and Ohtsuki showed that one can replace such a vanishing quantum dimension in equation (2.2.2) with a *modified dimension* $\mathsf{d}(V)$ and obtain an invariant which is now known as the ADO invariant. This process was extended to a general theory in [**GPMT09**]. We will briefly recall this construction.

The aforementioned *modified dimension* d is given on the set of typical modules by

$$\mathsf{d}(V_\alpha) = \prod_{j=0}^{r-2} \frac{1}{{\zeta_{2r}}^{\alpha+r-j} - {\zeta_{2r}}^{-(\alpha+r-j)}} = -{\zeta_{2r}}^{\frac{1}{2}r(1-r)} \frac{{\zeta_{2r}}^{\alpha+1} - {\zeta_{2r}}^{-(\alpha+1)}}{{\zeta_{2r}}^{r\alpha} - {\zeta_{2r}}^{-r\alpha}}. \tag{2.2.4}$$

Let L be a $\mathrm{Rep}_{\mathrm{wt}} \overline{\mathcal{U}}^H_{\zeta_{2r}}$-colored framed link with at least one component colored by a typical module V_α. Cutting this component, we obtain a $(1,1)$ tangle T_α. Then [**GPMT09**, Prop. 35] implies that the assignment

$$L \mapsto F'(L) := \mathsf{d}(V_\alpha)\langle T_\alpha \rangle$$

is independent of the choice of diagram and cut component and yields a well-defined isotopy invariant of L. This is the ADO invariant.

In the remainder of this paper we will assume that all colors are typical modules. Given a link with colors $V_{\alpha_1}, \ldots, V_{\alpha_n}$, we will choose without loss of generality to cut the component colored V_{α_1}. The corresponding ADO invariant defines a function

$$N_L^r : (\mathbb{C} \backslash \mathbb{Z})^n \to \mathbb{C}, \qquad N_L^r(\alpha_1, \ldots, \alpha_n) = \mathsf{d}(V_{\alpha_1})\langle T_{\alpha_1} \rangle . \tag{2.2.5}$$

Establishing q-holonomic properties of this family of functions for $r \geq 2$ is the central focus of this paper.

The diagrammatic calculus summarized here computes the ADO invariant in a blackboard framing. One may use the ribbon element in the category (or add extra loops to a diagram) to change to an arbitrary framing. Changing the framing of the i-th strand by ϕ units multiples the ADO invariant by a prefactor

$${\zeta_{2r}}^{\frac{1}{2}\phi[\alpha^2+2(1-r)\alpha]}. \tag{2.2.6}$$

2.2.1. A two-step reconstruction of the ADO invariant. For analyzing the q-holonomic properties of the ADO invariant N_L^r, it will be useful to split its construction into two steps:

(1) Cut an n-strand link L to get a $(1,1)$ tangle T with a particular choice of diagram $\mathbb{D}$, arranged so that all crossings are of the form $\nearrow\!\!\!\!\searrow$ or $\nwarrow\!\!\!\!\swarrow$. To this diagram we will associate a function $G_{\mathbb{D}} : \mathbb{Z} \to \mathbb{V}_n$, where

$$\mathbb{V}_n := \mathbb{C}(q^{\frac{1}{2}}, x_1^{\frac{1}{2}}, \ldots, x_n^{\frac{1}{2}}, z_{11}, z_{12}, \ldots, z_{nn}) \tag{2.2.7}$$

is the field of rational functions in the $1+n+\frac{1}{2}n(n+1) = \frac{1}{2}(n+1)(n+2)$ formal variables $q^{\frac{1}{2}}$, $\{x_i^{\frac{1}{2}}\}_{i=1}^n$, and $\{z_{ij}\}_{i,j=1}^n$, with $z_{ij} = z_{ji}$. We call the function $G_{\mathbb{D}}$ the *diagram invariant*.

(2) For each $r \in \mathbb{Z}_{\geq 2}$ we specialize the variables in $G_{\mathbb{D}}(r)$ as

$$q^{1/2} = {\zeta_{2r}}^{1/2}, \quad x_i^{1/2} = {\zeta_{2r}}^{\alpha_i/2}, \quad z_{ij} = {\zeta_{2r}}^{\alpha_i\alpha_j/2}. \tag{2.2.8}$$

to get the ADO invariant $N_L^r(\alpha)$. It will follow from the construction of $G_{\mathbb{D}}$ that this specialization is well defined. More compactly: if we write $G_{\mathbb{D}}(r; x^{\frac{1}{2}}, z; q^{\frac{1}{2}})$ to explicitly emphasize the dependence on x, z, q for each value of r, then

$$N_L^r(\alpha) = G_{\mathbb{D}}(r; {\zeta_{2r}}^{\alpha}/2, {\zeta_{2r}}^{\alpha\otimes\alpha/2}; {\zeta_{2r}}^{1/2}) . \tag{2.2.9}$$

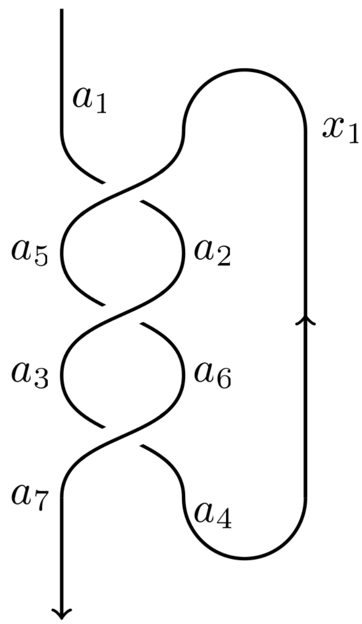
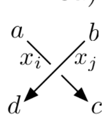


FIGURE 2.1. Labeled tangle diagram whose closure is a $\mathbf{3_1}$ knot.

We assign each component of the tangle a distinct variable $x_1, \ldots, x_n$. These parameterize the typical modules which color the diagram. By convention the unique open component will be given x_1. We also label each arc with a distinct parameter $a_1, \ldots, a_m$, which parameterize basis elements of the relevant typical module. Arcs end at crossings *regardless* of whether they are the upper or lower strand. A general $(1,1)$ tangle diagram with C crossings and U disjoint flat unknot components has $2C + U + 1$ arcs. See Figure 2.1.

We decompose the diagram $\mathbb{D}$ into crossings, cups, and caps. To each of these building blocks we associate a function of $m+1$ variables $a_1, \ldots, a_m, r$, which is valued in $\mathbb{V}_n$, as follows:

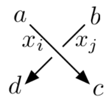
(2.2.10a)



$\rightsquigarrow R^{a,b}_{c,d}[x_i, x_j]$

$$:= \delta^{d-b}_{a-c} \vartheta_{c\leq a} \vartheta_{d\geq b} \, (-x_i)^{a-c} q^{(c-a)(a+b+1)+2cd} z_{ij} x_i^{-d} x_j^{-c} \frac{(q^{2(a-1)} x_i^{-2}; q^{-2})_{a-c} (q^{2(b+1)}; q^2)_{a-c}}{(q^{-2}; q^{-2})_{a-c}}$$

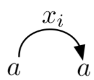
(2.2.10b)



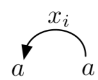
$\rightsquigarrow (R^{-1})^{a,b}_{c,d}[x_i, x_j]$

$$:= \delta^{d-b}_{a-c}\vartheta_{c\le a}\vartheta_{d\ge b}\,(-x_i)^k q^{(c-a)(a+b-1)-2ab} z_{ij}^{-1} x_i^b x_j^a \frac{(q^{2(a-1)}x_i^{-2}; q^{-2})_{a-c}(q^{2(b+1)}; q^2)_{a-c}}{(q^2; q^2)_{a-c}}$$

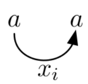
(2.2.10c)



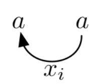
$\rightsquigarrow \epsilon_a[x_i] = 1$



$\rightsquigarrow \epsilon^*_a[x_i] = q^{2a(r-1)}x_i^{1-r}$



$\rightsquigarrow \eta_a[x_i] = 1$



$\rightsquigarrow \eta^*_a[x_i] = q^{2a(1-r)}x_i^{r-1}$

Here we have used a, b, c, d to denote the subset of arc variables $a_1, \ldots, a_m$ present at a particular crossing. We have also used

(2.2.10d)
$$\delta_{a,b} := \begin{cases} 1 & a = b \\ 0 & \text{otherwise} \end{cases}, \qquad \vartheta_{a\le b} := \begin{cases} 1 & a \le b \\ 0 & \text{otherwise} \end{cases}.$$

We are thinking of each of the maps $R[x_i, x_j], R^{-1}[x_i, x_j], \epsilon[x_i], \epsilon^*[x_i], \eta[x_i], \eta^*[x_i]$ as functions of the full set of arc variables $a_1, \ldots, a_m$ together with r — though they are independent of the arc variables that do not appear in the building block under consideration.

We similarly rewrite[1] the modified quantum dimension (2.2.4) associated to component labeled x_i as

(2.2.10e)
$$\mathsf{d}[x_i] = \prod_{j=2}^{r} \frac{1}{q^j x_i - q^{-j}x_i^{-1}} = (-x_i)^{r-1} q^{\frac{1}{2}r(r+1)-1} \frac{1}{(q^4 x_i^2; q^2)_{r-1}},$$

thought of as a function of all $m+1$ integer variables, which depends non-trivially only on r.Each function in (2.2.10a)–(2.2.10e) has domain $\mathbb{Z}^{m+1}$ and is valued in $\mathbb{V}_n$.

We define a function $G^{\times}_{\mathbb{D}} : \mathbb{Z}^{m+1} \to \mathbb{V}_n$ by multiplying together together the functions associated to every crossing, cup, and cap in the diagram; a function $\mathsf{d}[x_1]$ for the open link component (labeled

[1]One might wonder why we did not "analytically continue" the simpler formula on the right hand side of (2.2.4) to obtain $\mathsf{d}[x_i] = -q^{\frac{1}{2}r(1-r)} \frac{qx_i-(qx_i)^{-1}}{x_i^r - x_i^{-r}}$. The answer is that (2.2.10e) turns out to be q-holonomic, whereas this latter expression is not!

x_1 by convention), and delta-functions $\delta_{a_1,0}$, $\delta_{a_m,0}$ for the two arcs at the open ends of the (1,1) tangle (labeled, say, a_1 and a_m). Schematically,

$$G_{\mathbb{D}}^{\times}(a_1,\ldots,a_m;r) = \mathsf{d}[x_1]\delta_{a_1,0}\delta_{a_m,0}\prod_{\nwarrow\!\!\!\!\nearrow} R \prod_{\nearrow\!\!\!\!\nwarrow} R^{-1}\prod_{\curvearrowright}\epsilon\prod_{\curvearrowleft}\epsilon^*\prod_{\circlearrowleft}\eta\prod_{\circlearrowright}\eta^* . \tag{2.2.11}$$

From this we define the *diagram invariant* $G_{\mathbb{D}} : \mathbb{Z} \to \mathbb{V}_n$ as the multi-sum of $G_{\mathbb{D}}^{\times}$ over the arc variables

$$G_{\mathbb{D}}(r) := \sum_{a_1,\ldots,a_m\in[0,r-1]^m} G_{\mathbb{D}}^{\times}(a_1,\ldots,a_m;r) \tag{2.2.12}$$

The multi-sum in (2.2.12) reproduces the composition of building blocks (2.2.1) by summing over the basis elements of the typical representations. Once we fix $r \geq 2$ and specialize $q = \zeta_{2r}$, $x_i = {\zeta_{2r}}^{\alpha_i}$, and $z_{ij} = {\zeta_{2r}}^{\alpha_i\alpha_j/2}$, each of the functions $R, R^{-1}, \epsilon, \epsilon^*, \eta, \eta^*, \mathsf{d}$ above simply becomes a matrix element of the building blocks from (2.2.1). This can be seen by comparing with the formulas (2.1.12), (2.1.13), (2.1.8), (2.2.4). Finally, the specialization of $G_{\mathbb{D}}(r)$ as in (2.2.8) reproduces the ADO invariant $N_L^r(\alpha)$.

Example: The labeled diagram of a (1,1) tangle whose closure is a $\mathbf{3_1}$ knot is shown in Figure 2.1. There are seven arcs with associated variables $a_1,\ldots,a_7$ and one component with associated variable x_1. The corresponding diagram invariant is

$$G_{\mathbb{D}}(r) = \sum_{a_1,\ldots,a_7=0}^{r-1} \mathsf{d}[x_1]\delta_{a_1,0}\delta_{a_7,0}R^{a_1,a_4}_{a_2,a_5}[x_1,x_1]R^{a_5,a_2}_{a_6,a_3}[x_1,x_1]R^{a_3,a_6}_{a_4,a_7}[x_1,x_1]\epsilon^*_{a_4}[x_1]\eta_{a_4}[x_1]$$

2.2.2. Poles and relation to the colored Jones. It follows from its construction in Section 2.2.1 that the ADO invariant is a meromorphic function.

The only non-monomial denominators that appear in the functions $R^{\pm}, \epsilon^{(*)}, \eta^{(*)}, \mathsf{d}$ of Section 2.2.1 are $(q^4x_i^2;q^2)_{r-1}$ in the modified dimension $\mathsf{d}[x_i]$ and $(q^2;q^2)_{a-c}$, $(q^{-2};q^{-2})_{a-c}$ in the R-matrices. A short exercise shows that the denominators in the R-matrices divide the numerators, as both $\frac{(q^{2(b+1)};q^2)_k}{(q^2;q^2)_k}$ and $\frac{(q^{2(b+1)};q^2)_k}{(q^{-2};q^{-2})_k}$ belong to $\mathbb{C}[q,q^{-1}]$ for all $k,b \in \mathbb{Z}_{\geq 0}$. Thus after simplification the only *possible* denominator in $G_{\mathbb{D}}(r)$ is $x_1^r - x_1^{-r}$.

Moreover, only integral powers of x, z, q appear, and the only place that z_{ij} (resp. z_{ij}^{-1}) appears is as a prefactor in the R function (resp. R^{-1} function) for positive (resp. negative) crossing of components i and j. Altogether this implies that

Proposition 2.2.1. *For each $r \geq 1$,*

$$G_{\mathbb{D}}(r) \in \frac{1}{(q^4x_1^2;q^2)_{r-1}} \prod_{i,j=1}^{n} z_{ij}^{C_{ij}} \cdot \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}, q^{\pm}], \tag{2.2.13}$$

where C_{ij} is the linking matrix of the original framed link L (C_{ii} being the framing of the i-th component).

Recall that the ADO invariant $N_L^r(\alpha)$ is obtained from $G_{\mathbb{D}}$ by the specialization $q \to \zeta_{2r}$, $x_i \to \zeta_{2r}^{\alpha_i}$, and $z_{ij} \to \zeta_{2r}^{\alpha_i\alpha_j/2}$ in (2.2.8). Proposition 2.2.1 then translates into the following functional properties of the ADO invariant.

Corollary 2.2.2.

(i) *If L is a knot ($n = 1$ strands), the ADO invariant $N_L^r : \mathbb{C}\backslash\mathbb{Z} \to \mathbb{C}$ may be extended to a meromorphic function of $\alpha = \alpha_1 \in \mathbb{C}$ with at most simple poles at each integer.*

(ii) *If L is a link with $n > 1$ strands, the ADO invariant $N_L^r : (\mathbb{C}\backslash\mathbb{Z})^n \to \mathbb{C}$ may be extended to a holomorphic function of $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$.*

(iii) *For any n, N_L^r is quasi-periodic, satisfying*

$$N_L^r(\alpha_1, \dots, \alpha_i + 2r, \dots, \alpha_n) = \Big(\prod_{j=1}^{n} {\zeta_{2r}}^{2rC_{ij}\alpha_j}\Big) N_L^r(\alpha_1, \dots, \alpha_i, \dots, \alpha_n) \tag{2.2.14}$$

In other words, N_L^r is a section (holomorphic if $n > 1$, meromorphic if $n = 1$) of a complex line bundle on $(\mathbb{C}/2r\mathbb{Z})^n$ determined by the linking matrix C_{ij}.

The relations in (2.2.14) are those mentioned in Section 1.1.1. They are our motivation for excluding r-dependent recursion relations from our q-holonomic systems.

Proof. For (i) we observe that after specializing $q = \zeta_{2r}$ and $x_1 = {\zeta_{2r}}^{\alpha}$, the modified quantum dimension may be rewritten as on the right hand side of (2.2.4), so the only denominator that could give rise to poles is ${\zeta_{2r}}^{r\alpha} - {\zeta_{2r}}^{-r\alpha} = 2i\sin(\pi\alpha)$.

For part (ii), recall that the ADO invariant does not depend on which strand is cut to represent the initial link diagram as a (1,1) tangle. From (2.2.13) and the reasoning of Part (i), it follows that there are no poles in α_i for $i \neq 1$. Since ADO can be constructed with a different choice of cut strand it follows that there can be no poles in α_1 either.

Part (iii) follows from observing that, with the exception of the $z_{ij}^{C_{ij}}$ prefactors, the expression (2.2.13) is a function of the $x_i = {\zeta_{2r}}^{\alpha_i}$, which satisfy $\zeta_{2r}^{\alpha_i+2r} = \zeta_{2r}^{\alpha_i}$. The prefactors $z_{ij}^{C_{ij}} = {\zeta_{2r}}^{\frac{1}{2}C_{ij}\alpha_i\alpha_j}$ lead precisely to the quasi-periodicity (2.2.14). □

In the case of a knot, the residues of the poles at integer values of α are related to colored Jones polynomials. This is because for $\alpha = N-1$ (with $N \in \mathbb{Z}\backslash r\mathbb{Z}$), the typical module V_α becomes reducible and contains as a simple quotient the module S_{N-1} used to define the N-th colored Jones polynomial. This expectation was made precise in [**CGPM15b**, Cor. 15], which we restate here:

Proposition 2.2.3. *([**CGPM15b**, Cor. 15]) Let K be an oriented knot with framing ϕ. Let $r \geq 2$, and let $N \in \mathbb{Z} \setminus r\mathbb{Z}$. Let $J_N^K(q) \in \mathbb{C}[q^{\pm}]$ denote the N-th colored Jones polynomial of K, normalized so that $J_N^{unknot}(q) = (q^N - q^{-N})/(q-q^{-1})$. Then*

$$\operatorname{Res}_{\alpha=N-1} N_K^r(\alpha) = \frac{i^{1-r}}{\pi} \sin\left(\frac{\pi}{r}\right)(-1)^{N+(N-1)\phi} J_N(\zeta_{2r})\,. \tag{2.2.15}$$

Note that the right hand side differs slightly from that in [**CGPM15b**]. The r-dependent prefactors differ due to a different normalization for the modified dimension $\mathsf{d}[V_\alpha]$. The extra $(-1)^{N+(N-1)\phi}$ appears because the pivotal structure and ribbon element in the category $\overline{\mathcal{U}}_{\zeta_{2r}}^H(\mathfrak{sl}_2)$-mod discussed above — the *only* pivotal structure that exists for generic $\alpha \in \mathbb{C}$ — differs from the pivotal structure and ribbon element used in the standard definitions of the colored Jones polynomial.

We remark that at $\alpha = N-1$ with $N \in r\mathbb{Z}$, the ADO invariant does not have a pole, and may simply be evaluated. In particular, it was shown some time ago by J. Murakami and H. Murakami [**MM01**] that $N_K^r(r-1)$ coincides with the re-normalized Jones polynomial $\hat{J}_r(\zeta_{2r})$, where $\hat{J}_N(q) = \frac{q-q^{-1}}{q^N-q^{-N}} J_N(q)$, as well as with the Kashaev invariant [**Kas97**]. This observation allowed Kashaev's famous volume conjecture to be reformulated in terms of colored Jones polynomials.

2.2.3. $G_{\mathbb{D}}$ is q-holonomic. Our next goal is to prove that the diagram invariant $G_{\mathbb{D}}$ defined in Section 2.2.1 is q-holonomic. Specifically, for an n-component tangle, we show that $G_{\mathbb{D}}$ generates a q-holonomic module for a q-Weyl algebra with $n+1$ pairs of generators: $x_1, y_1, \ldots, x_n, y_n$ acting by multiplication and q-shifts of the variables x_i in $G_{\mathbb{D}}$, together with $\hat{x}, \hat{y}$ acting by multiplication by q^r and shift $r \mapsto r+1$.

Proving that $G_{\mathbb{D}}$ is q-holonomic in this sense is a straightforward generalization of the classic results of Garoufalidis and Lê [**GL05**] on the Jones polynomial. We adapt the methods there to the function spaces to which $G_{\mathbb{D}}$ belongs.

With the machinery of q-holonomic modules in place, we obtain

PROPOSITION 2.2.4. *The diagram invariant $G_{\mathbb{D}}(r)$ defined in Section 2.2.1, which is an element of $\mathcal{V}_{1,n}$, generates a q-holonomic module for $\mathbb{E}_{n+1}$.*

PROOF. All the individual functions (2.2.10) associated to crossings, cups, and caps that get multiplied to define $G_{\mathbb{D}}^{\times}$ in (2.2.11) are q-holonomic in $\mathcal{V}_{m+1,n}$. Specifically:

- The discrete delta-functions $\delta_{a,0} \in \mathcal{V}_{1,0}$ that enter the final product $G_{\mathbb{D}}^{\times}$ are q-holonomic (Example (1.2.18)).
- The modified quantum dimension $\mathsf{d}[x_i] = (-x_i)^{r-1} q^{\frac{1}{2}r(r+1)-1} \frac{1}{(q^4 x_i^2; q^2)_{r-1}} \in \mathcal{V}_{1,1}$ can be assembled as a product of
  (1) A q-factorial $\frac{1}{(x_i;q^2)_r}$, which was explained to be q-holonomic below (1.2.29), and in which we use Prop. 1.2.9b to shift $x_i \to q^2 x_i$ and $r \to r-1$.
  (2) A general quadratic exponential $x_i^r q^{\frac{1}{2}r^2}$ as in Example 1.2.28, in which we shift $x_i \to -x_i$.
  (3) A linear exponential $x_i^{-1} q^{\frac{1}{2}r}$ as in Example (1.2.23), in which we shift $x_i \to -x_i$.
  (4) An overall constant q^{-1}.

  All these pieces are q-holonomic functions in $\mathcal{V}_{1,1}$, so Prop. 1.2.9a guarantees their product will be q-holonomic as well.
- The cup and cap functions $\eta_a[x_i] = 1$, $\epsilon_a[x_i] = 1$, are constant and therefore q-holonomic by Example 1.2.17.

  The cup and cap functions $\eta_a^*[x_i] = q^{2a(1-r)} x_i^{r-1}$ and $\epsilon_a^*[x_i] = q^{-2a} x_i$, both in $\mathcal{V}_{2,1}$ (the discrete variables are a and r), are products of general linear and quadratic exponentials, as in Examples 1.2.23, 1.2.28.
- The R-matrices $R_{c,d}^{a,b}[x_i, x_j]$, $(R^{-1})_{c,d}^{a,b}[x_i, x_j] \in \mathcal{V}_{5,2}$ are products of discrete delta-functions (Example (1.2.18)), indicator functions (Example (1.2.20)), linear and quadratic exponentials (Examples 1.2.23, 1.2.28), and q-factorials (Example (1.2.29)), all with various shifts

(Prop. 1.2.9b) and linear transformations (Prop. 1.2.9c). Some of the q-factorials involve q^{-2} rather than q^2, but can be put into the same form as Example (1.2.29) by observing that

$$(y;q^{-2})_a = (-y)^a q^{-a(a-1)}(y^{-1};q^2)_a\,, \tag{2.2.16}$$

which is a 'standard" q-factorial multiplied by linear and quadratic exponentials. Thus $R, R^{-1} \in \mathcal{V}_{5,2}$ are q-holonomic.

By Prop. 1.2.9c, the above functions remain q-holonomic when extended to $\mathcal{V}_{m+1,n}$. The product of $G^{\times}_{\mathbb{D}} \in \mathcal{V}_{m+1,n}$ of these q-holonomic functions is q-holonomic by Prop. 1.2.9a. The final diagram invariant $G_{\mathbb{D}} \in \mathcal{V}_{1,n}$ is obtained from $G^{\times}_{\mathbb{D}}$ by summing over every discrete variable, and then specializing the bounds of each summation to be 0 and $r-1$. It is therefore q-holonomic by Prop. 1.2.9d (for the summations) and Prop. 1.2.9c (for the specializations). □

2.3. Specializing to a Root of Unity

We proved in Proposition 2.2.4 that $G_{\mathbb{D}}$ generates a q-holonomic $\mathbb{E}_{n+1}$-module for any (1, 1)-tangle diagram $\mathbb{D}$. The relevant action of $\mathbb{E}_{n+1} = \mathbb{C}_q[x_1^{\pm}, y_1^{\pm}, \dots, x_n^{\pm}, y_n^{\pm}, \hat{x}^{\pm}, \hat{y}^{\pm}]$ on functions $f \in \mathcal{V}_{1,n}$ (including $G_{\mathbb{D}}$) is given by

$$\begin{aligned} x_i &: f \mapsto x_i f & \hat{x} &: f \mapsto q^r f \\ y_i &: f \mapsto f\,\big|_{x_i \mapsto q x_i,\, z_{ii} \mapsto q^{\frac12} x_i z_{ii},\, z_{ij} \mapsto x_j^{\frac12} z_{ij}} & \hat{y} &: f \mapsto f\,\big|_{r \mapsto r+1}\,. \end{aligned} \tag{2.3.1}$$

We would now like to prove that the ADO invariant N^r_L is q-holonomic in the sense detailed below. The ADO invariant is a topological invariant of the framed, oriented link L obtained by closing the tangle with diagram $\mathbb{D}$.

We recall from Section 2.2.1 that the ADO invariant is obtained from $G_{\mathbb{D}}$ by setting $q^{\frac12} = {\zeta_{2r}}^{\frac12}$, $x_i^{\frac12} = {\zeta_{2r}}^{\alpha_i/2}$, and $z_{ij} = {\zeta_{2r}}^{\alpha_i \alpha_j/2}$. More succinctly, if we make explicit the dependence on x, z, q in $G_{\mathbb{D}}(r, x^{\frac12}, z; q)$, then

$$N^r_L(\alpha) = G_{\mathbb{D}}(r; {\zeta_{2r}}^{\alpha/2}, {\zeta_{2r}}^{\alpha^2/2}; {\zeta_{2r}}^{1/2})\,. \tag{2.3.2}$$

As prefaced in the introduction, explaining what it means for functions defined at roots unity $q = \zeta_{2r}$ to be holonomic is a subtle matter. By Corollary 2.2.2, we may think of the ADO invariant at each fixed r as an element of the functional space

$$N_L^r \in \mathcal{V}_n^{(r)} := \{\text{quasi-periodic, meromorphic functions} : (\mathbb{C}/2r\mathbb{Z})^n \to \mathbb{C}\}, \tag{2.3.3}$$

with periodicity of the form $f(\alpha_1, \ldots, \alpha_i + 2r, \ldots, \alpha_n) = \zeta_{2r}^{\sum_j 2rC_{ij}\alpha_j} f(\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n)$ for some (unspecified) C_{ij}. Each space $\mathcal{V}_n^{(r)}$ has an action of the q-Weyl algebra at a $2r$-th root of unity

$$\mathcal{E}_n^{(r)} := \mathbb{C}[x_1^{\pm}, y_1^{\pm}, \ldots, x_n^{\pm}, y_n^{\pm}]/(y_i x_j - \zeta_{2r}^{\delta_{ij}} x_j y_i) \tag{2.3.4}$$

given by

$$x_i \cdot f(\alpha) = \zeta_{2r}^{\alpha_i} f(\alpha), \qquad y_i \cdot f(\alpha) = f(\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_n) \qquad (f \in \mathcal{V}_n^{(r)}). \tag{2.3.5}$$

However, due to the quasi-periodicity in Part (iii) of Corollary 2.2.2, it is also clear that at each fixed r the ADO invariant of an n-strand link will trivially satisfy n independent recursion relations $\big(\prod_j x_j^{-2rC_{ij}} y_i^{2r} - 1\big) N_L^r = 0$ $(i = 1, \ldots, n)$, where C_{ij} is the linking matrix of L. In order to obtain a topologically interesting statement, we work in a *family*, considering all $r \in \mathbb{N}_{\geq 2}$ at once.

Consider the evaluation maps

$$\mathrm{ev}_r : \begin{array}{ccc} \mathbb{E}_n & \dashrightarrow & \mathcal{E}_n^{(r)} \\ A(x, y; q) & \mapsto & A(x, y; \zeta_{2r}). \end{array} \tag{2.3.6}$$

$$\mathcal{I}[f] := \{A \in \mathbb{E}_n \;\; \mathrm{ev}_r(A) f_r = 0 \text{ for all but finitely many } r \in \mathbb{N}\}, \tag{2.3.7}$$

throwing out any r's for which $\mathrm{ev}_r(A)$ is not defined. Then we say:

Definition 2.3.1. *The family of functions $\{f_r \in \mathcal{V}_n^{(r)}\}_{r \in \mathbb{N}}$ is q-holonomic if the associated cyclic module $\mathbb{E}_n/\mathcal{I}[f]$ is q-holonomic.*

We will prove in this section that the *family* of ADO invariants $\{N_L^r \in \mathcal{V}_n^{(r)}\}_{r \geq 2}$ of any framed, oriented link L is q-holonomic. We will also prove that the associated ideal $\mathcal{I}[N_L]$ is contained in the annihilation ideal of the colored Jones polynomial of L.

2.3.1. Quantum Hamiltonian reduction. We introduce a preliminary result that will help us relate the annihilation ideal of $G_{\mathbb{D}}$ and the family of ADO invariants. The result is purely algebraic in nature, independent of particular functional spaces.

Suppose we have a left ideal $\mathcal{I}_n \subseteq \mathbb{E}_n$ and a nonzero element c in $\mathbb{C}_q$. Then we can construct a left ideal $\mathcal{I}_{n-1}^{(x_n \to c)} \subseteq \mathbb{E}_{n-1}$ by first taking the intersection of $\mathcal{I}_n$ with the subalgebra

$$\widetilde{\mathbb{E}}_{n-1} := \mathbb{C}_q[x_1^{\pm}, y_1^{\pm}, \ldots, x_{n-1}^{\pm}, y_{n-1}^{\pm}, x_n^{\pm}]/(y_i x_j - q^{\delta_{ij}} x_j y_i)_{i,j=1}^{n-1} \quad \subset \ \mathbb{E}_n, \tag{2.3.8}$$

in which x_n is central (because y_n is no longer present), and then specializing $x_n = c$, noting that $\mathbb{E}_{n-1} \simeq \widetilde{\mathbb{E}}_{n-1}/(x_n - c)$. All together,

$$\mathcal{I}_{n-1}^{(x_n \to c)} = \left(\mathcal{I}_n \cap \widetilde{\mathbb{E}}_{n-1}\right)\Big|_{x_n = c}. \tag{2.3.9}$$

Explicitly, the elements of $\mathcal{I}_n$ and $\mathcal{I}_{n-1}^{(x_n \to c)}$ are related as follows: $A(x_1, y_1, \ldots, x_{n-1}, y_{n-1})$ is in $\mathcal{I}_{n-1}^{(x_n \to c)}$ if and only if it has a lift $\widetilde{A}$ in $\mathcal{I}_n$ independent of y_n such that $A = \widetilde{A}\big|_{x_n = c}$.

Passing from the associated module $\mathbb{E}_n/\mathcal{I}_n$ to $\mathbb{E}_{n-1}/\mathcal{I}_{n-1}^{(x_n \to c)}$ is a version of quantum Hamiltonian reduction. In this case, the reduction is with respect to a multiplicative moment map x_n, and central character c.[2] Quantum Hamiltonian reduction is a familiar operation in the study of D-modules and representation theory, *cf.* [**EG02**, **CBEG07**, **Los12**, **Jor14**], which is generally expected to preserve holonomic modules (since it is the quantization of a Lagrangian correspondence). We will use the following result, whose proof is the subject of Appendix A:

PROPOSITION 2.3.2. *For $n \geq 2$, let $\mathcal{I}_n \subseteq \mathbb{E}_n$ be a left ideal, and let $\mathcal{I}_{n-1}^{(x_n \to c)} = (\mathcal{I} \cap \widetilde{\mathbb{E}}_{n-1})\big|_{x_n = c} \subseteq \mathbb{E}_{n-1}$ as above. If $\mathbb{E}_n/\mathcal{I}_n$ is a q-holonomic $\mathbb{E}_n$-module then $\mathbb{E}_{n-1}/\mathcal{I}_{n-1}^{(x_n \to c)}$ is a q-holonomic $\mathbb{E}_{n-1}$-module.*

A useful way to relate Hamiltonian reduction to more elementary operations on q-holonomic modules is the following. Let v be the generator of $\mathbb{E}_n/\mathcal{I}_n$ and let $\delta_c^{(n)}$ denote the generator of the module $\mathbb{E}_n/(y_1 - 1, \ldots, y_{n-1} - 1, x_n - c)$, a 'delta-function" module in the final variable x_n. Just like Example (1.2.19), this delta-function module is q-holonomic. We denote by $\mathbb{E}_n(v \otimes \delta_c^{(n)})$ the submodule of the tensor-product-module $(\mathbb{E}_n v) \otimes (\mathbb{E}_n \delta_c^{(n)})$ generated by $v \otimes \delta_c^{(n)}$.

Let us also consider the map of rings $f^* : \mathbb{C}_q[x_1^{\pm}, \ldots, x_n^{\pm}] \mapsto \mathbb{C}_q[x_1^{\pm}, \ldots, x_{n-1}^{\pm}]$ given by $f^*(x_i) = x_i$ for $1 \leq i \leq n-1$ and $f^*(x_n) = c$. There is a corresponding inverse-image functor $f^! : \mathbb{E}_n\text{-mod} \to \mathbb{E}_{n-1}\text{-mod}$ defined in [**Sab93**, Sec. 2.3]. It is explained in the proof of Prop. 2.3.2 in Appendix A

[2]Very similar reductions were used in [**Dim13**] to construct quantum A-polynomials from ideal triangulations of knot complements. The construction there was not yet rigorous, but could hopefully be made so using Prop. 2.3.2.

that

$$\mathbb{E}_{n-1}/\mathcal{I}_{n-1}^{(x_n\to c)} \simeq f^{!}\left(\mathbb{E}_n(v\otimes\delta_c^{(n)})\right). \tag{2.3.10}$$

With this in hand, it follows from the fact that taking tensor products, passing to submodules, and inverse images all preserve q-holonomic modules that $\mathbb{E}_{n-1}/\mathcal{I}_{n-1}^{(x_n\to c)}$ must be q-holonomic as well.

We note that the quantum Hamiltonian reduction discussed above is closely related to specialization of variables, in the case of cyclic $\mathbb{E}_n$-modules generated by functions. For example, if $\mathbb{E}_n$ acts on some space of functions of $(x_1,\dots,x_n)$, and the function $f(x_1,\dots,x_n)$ generates a cyclic module $\mathbb{E}_n f = \mathbb{E}_n/\mathcal{I}_n$, $\mathcal{I}_n = \mathrm{Ann}_{\mathbb{E}_n}(f)$, then the specialization $f_c(x_1,\dots,x_{n-1}) := f(x_1,\dots,x_{n-1},c)$ generates a module $\mathbb{E}_{n-1}f_c = \mathbb{E}_{n-1}/\mathcal{I}_{n-1}$, such that the Hamiltonian-reduction ideal $\mathcal{I}_{n-1}^{(x_n\to c)}$ above satisfies $\mathcal{I}_{n-1}^{(x_n\to c)} \subseteq \mathcal{I}_{n-1}$. In other words, the specialized module $\mathbb{E}_{n-1}f_c$ is a quotient of $\mathbb{E}_{n-1}/\mathcal{I}_{n-1}^{(x_n\to c)}$. Thus, a corollary of Proposition 2.3.2 is that when f is q-holonomic its specialization f_c must be q-holonomic as well. This was already established in Proposition 1.2.9c. The virtue of the algebraic formulation of quantum Hamiltonian reduction above is that it applies even when considering modules that are not generated by functions; that is how we will use it in the next section.

2.3.2. The ADO invariants are a q-holonomic family. We are now ready to prove one of our main results, by using quantum Hamiltonian reduction to implement the specializations $q^r = -1$ in the ADO invariants.

THEOREM 2.3.3. *Let L be a framed, oriented link with n components. Then the family of ADO invariants $\{N_L^r\}_{r\geq 2}$ is q-holonomic for $\mathbb{E}_n$. In other words, the associated ideal*

$$\mathcal{I}[N_L] := \{A\in\mathbb{E}_n \mid ev_r(A)N_L^r = 0 \text{ for all but finitely many } r\} \tag{2.3.11}$$

as in (2.3.7) *defines a q-holonomic $\mathbb{E}_n$-module $\mathbb{E}_n/\mathcal{I}[N_L]$.*

PROOF. Choose a diagram $\mathbb{D}$ of a $(1,1)$ tangle whose closure is L, as in Section 1.2, and let $G_{\mathbb{D}}(r; x^{\frac{1}{2}}, x; q^{\frac{1}{2}}) \in \mathcal{V}_{1,n}$ be the associated diagram invariant. From Proposition 2.2.4, we know that $G_{\mathbb{D}}$ generates a q-holonomic left $\mathbb{E}_{n+1}$-module via the action given in (2.3.1). Denote its annihilation ideal by

$$\mathcal{I}_{n+1} = \mathrm{Ann}_{\mathbb{E}_{n+1}} G_{\mathbb{D}}. \tag{2.3.12}$$

and construct the reduced ideal

$$\mathcal{I}_n^{(\hat{x}\to -1)} := \left(\mathcal{I}_{n+1} \cap \widetilde{\mathbb{E}}_n\right)_{\hat{x}=-1} \quad \subseteq \mathbb{E}_n \tag{2.3.13}$$

as in (2.3.9). This is quantum Hamiltonian reduction at $c = -1$. It eliminates the $\hat{y}$ variable (which shifted $r \mapsto r+1$) and sets the $\hat{x}$ variable (which acted as multiplication by q^r) to -1.

We claim that $\mathcal{I}_n^{(\hat{x}\to -1)} \subseteq \mathcal{I}[N_L]$. To see this, choose any $A(x, y; q) = A(x_1, y_1, \dots, x_n, y_n; q) \in \mathcal{I}_n^{(\hat{x}\to -1)}$. By the definition of $\mathcal{I}_n^{(\hat{x}\to -1)}$, there exists $\tilde{A}(x, \hat{x}, y; q) \in \widetilde{\mathbb{E}}_n = \mathbb{C}_q[x_1, y_1, \dots, x_n, y_n, \hat{x}] \subset \mathbb{E}_{n+1}$ such that

$$\tilde{A}(x, \hat{x} = -1, y; q) = A(x, y; q) \qquad \text{and} \qquad \tilde{A}(x, \hat{x}, y; q) G_{\mathbb{D}}(r; x^{\frac{1}{2}}, z; q)\,. \tag{2.3.14}$$

Choose any nonzero polynomial $f(q) \in \mathbb{C}[q]$ such that $f(q)A(x, y; q)$ and $f(q)A(x, \hat{x}, y; q)$ both have evaluations at $q = \zeta_{2r}$ for all $r \in \mathbb{N}$. (Recall that all singularities came from poles.) From the first equality in (2.3.14), we have $f(\zeta_{2r})\tilde{A}(x, \hat{x}, y; \zeta_{2r}) = f(\zeta_{2r})A(x, y; \zeta_{2r})$ for all r. Combining this with the second equality in (2.3.14), evaluated at $q = \zeta_{2r}$, we have

$$f(\zeta_{2r})A(x, y; \zeta_{2r}) G_{\mathbb{D}}(r; x^{\frac{1}{2}}, z; \zeta_{2r}) = 0\,. \tag{2.3.15}$$

We may further specialize $x^{\frac{1}{2}} = e^{\frac{i\pi}{2r}\alpha}$ and $z = e^{\frac{i\pi}{2r}\alpha^2}$ as in (2.3.2), leading to

$$f(\zeta_{2r})A(x, y; \zeta_{2r}) N_L^r(\alpha) = 0 \tag{2.3.16}$$

for all r, with action (2.3.5). Since $f(\zeta_{2r})$ can only vanish at (at most) finitely many values of $r \in \mathbb{N}$, we find that $A(x, y; \zeta_{2r}) \in \mathcal{I}[N_L]$.

From Proposition 2.3.2 we know that the module $\mathbb{E}_n/\mathcal{I}_n^{(\hat{x}\to -1)}$ is q-holonomic. Moreover, since $\mathcal{I}_n^{(\hat{x}\to -1)} \subseteq \mathcal{I}[N_L]$, we find that $\mathbb{E}_n/\mathcal{I}[N_L] \simeq (\mathbb{E}_n/\mathcal{I}_n^{(\hat{x}\to -1)})/(\mathcal{I}[N_L]/\mathcal{I}_n^{(\hat{x}\to -1)})$ is a quotient of $\mathbb{E}_n/\mathcal{I}_n^{(\hat{x}\to -1)}$. Since quotients of q-holonomic modules are q-holonomic by [**Sab93**, Cor. 2.1.6], it follows that $\mathbb{E}_n/\mathcal{I}[N_L]$ is q-holonomic. □

2.3.3. Relation to the AJ conjecture. Finally, we can relate the recursion relations satisfied by the ADO family to those satisfied by the colored Jones polynomials. Let $L = K$ be an oriented knot with framing ϕ.

THEOREM 2.3.4. *Let $\mathcal{I}[N_K] \in \mathbb{E}_1$ be the ideal in Theorem 2.3.3 that annihilates the ADO family. Let $\left(J_N(q)\right)_{N\in\mathbb{N}}$ be the sequence of colored Jones polynomials of K. Then for every element*

$A(x, y; q) \in \mathcal{I}[N_K]$ *we have*

$$A(q^{-1}x, (-1)^{\phi+1}y; q)\, J_N(q) = 0\,, \tag{2.3.17}$$

where x *acts as multiplication by* q^N *and* y *acts by shifting* $N \mapsto N+1$.

Proof. From Corollary 2.2.2 we see that the poles in $N_K^r(\alpha)$ come entirely from the denominator ${\zeta_{2r}}^{r\alpha} - {\zeta_{2r}}^{-r\alpha}$ in the modified quantum dimensions. Then, noting that

$$\mathrm{Res}_{\alpha=N-1} \frac{1}{{\zeta_{2r}}^{r\alpha} - {\zeta_{2r}}^{-r\alpha}} = \mathrm{Res}_{\alpha=N-1} \frac{1}{2i \sin(\pi\alpha)} = \frac{(-1)^{N-1}}{2\pi i}\,, \tag{2.3.18}$$

we may rewrite Proposition 2.2.3 to say that

$$(\zeta_{2r}^{\alpha} - \zeta_{2r}^{-\alpha}) N_K^r(\alpha)\,\Big|_{\alpha=N-1} = \left(2i^{-r} \sin\frac{\pi}{r}\right) (-1)^{(N-1)\phi} J_N(\zeta_{2r})\,. \tag{2.3.19}$$

Also note that with y acting as a shift $\alpha \mapsto \alpha+1$ we have $y(\zeta_{2r}^{\alpha} - \zeta_{2r}^{-\alpha}) = (\zeta_{2r}^{\alpha} - \zeta_{2r}^{-\alpha})(-y)$, and so

$$A(x, y; \zeta_{2r}) N_K^r(\alpha) = 0 \quad \Leftrightarrow \quad A(x, -y; \zeta_{2r})(\zeta_{2r}^{\alpha} - \zeta_{2r}^{-\alpha}) N_K^r(\alpha) = 0\,. \tag{2.3.20}$$

Similarly, with y acting as a shift $N \mapsto N+1$ we have $y(-1)^N = (-1)^N(-y)$, so

$$A(x, y; \zeta_{2r})(-1)^{\phi(N-1)} J_N(\zeta_{2r}) = 0 \quad \Leftrightarrow \quad A(x, (-1)^{\phi}y; \zeta_{2r}) J_N(\zeta_{2r}) = 0\,. \tag{2.3.21}$$

Let $A(x, y; q)$ be any element of the ideal $\mathcal{I}[N_K]$. For every value of r such that $A(x, y; q)$ is non-singular at $q = \zeta_{2r}$ we have $A(x, y; \zeta_{2r}) N_K(\alpha) = 0$; and then from (2.3.19)–(2.3.21) we obtain

$$A(q^{-1}x, (-1)^{\phi+1}y; \zeta_{2r}) J_N(\zeta^{2r}) = 0\,. \tag{2.3.22}$$

(The extra shift $x \to q^{-1}x$ is made to ensure that x acting as q^{α} on the ADO is compatible with x acting as q^N (rather than q^{N-1}) on the colored Jones.) Now consider the functions

$$B_N(q) := A(q^{-1}x, (-1)^{\phi+1}y; q) J_N(q) \in \mathbb{C}_q\,, \qquad N \in \mathbb{N} \tag{2.3.23}$$

Due to (2.3.22), each rational function $B_N(q)$ has zeroes at an infinite set of distinct points $q = \zeta_{2r}$. (Note: there are at most finitely many poles in $B_N(q)$, and if they occur at roots of unity, the corresponding values of r may be thrown out without affecting this argument.) Each function $B_N(q)$ must therefore be identically zero. □

We have shown that, up to an algebra automorphism that rescales $(x, y) \mapsto (q^{-1}x, (-1)^{\phi+1}y)$, the annihilation ideal $\mathcal{I}[N_K]$ of the ADO family is included in the annihilation ideal of the colored Jones function. If we further assume the AJ Conjecture of [**Gar04b, Guk05b**], it follows that:

COROLLARY 2.3.5. *(Assuming the AJ Conjecture of* [**Gar04b**]*.) Let K be a knot with framing ϕ and let $A(x, y; q)$ be any element of the ADO ideal $\mathcal{I}[N_K]$ that admits evaluation at $q = 1$. Then $A(m, (-1)^{\phi+1}\ell; 1)$ is divisible by the A-polynomial $\mathbf{A}(m, \ell)$ of K.*

The converse of Theorem 2.3.4 is proven in [**Wil20**, Thm 66]: that the colored Jones annihilation ideal is included in the ADO annihilation ideal. Taken together, these results imply that the two annihilation ideals can be identified. Computations in Appendix B confirm this identification.

CHAPTER 3

Future Directions

It has been conjectured [**Guk05a, Gar04a**] that the q-difference operators which annihilate the colored Jones polynomial are defined by modules of quantized character varieties coming from knot complements. We have shown that such q-holonomic systems also appear for quantum invariants constructed using modified traces, which implies that the original conjecture holds beyond the semisimple case. This section proposes some lines of inquiry which would explore this relationship between recursive properties of quantum invariants and quantized character varieties.

3.1. The Quantum A-Polynomial

A pivotal next goal is to give a rigorous quantization of the A-polynomial [**CS83**] in terms of the quantum character stacks of [**JLSS21**]. These character stacks enjoy an excision property, which means the skein modules can be built from a 3d triangulation of the knot complement, while resolving the issues facing the quantization procedure detailed in [**Dim11**].

Quantum character stacks are non-commutative moduli spaces defined for surfaces with marked points on the boundary. The relevant surfaces consist of T and G regions, separated by one dimensional B-interfaces. The moduli spaces of [**JLSS21**] track the G- and T- local systems over the appropriate regions, along with B-reductions along interfaces and framing data along specified segments of the boundary.

This would be a major step towards resolving the AJ conjecture [**Gar04a, Guk05a**], which predicts that a (yet-undefined) quantization of the A-polynomial annihilates the colored Jones polynomial [**RT90**]. This work would place the quantum A-polynomial in the context of the Kapustin-Witten twist of $\mathcal{N} = 4$ 4d Yang-Mills theory [**KW07, BZBJ18**], which is also expected to manifest the colored Jones polynomial as a partition function. Significant incremental progress has been made towards solving the AJ conjecture, but this proposal represents a distinct approach centered on its physical formulation [**Guk05a**].

The A-polynomial of a knot $\mathcal{K}$ is defined by a lagrangian in the $\mathrm{SL}_2\,\mathbb{C}$ character variety of the torus. It consists of those characters which extend to the interior of the knot complement $S^3 \setminus \mathcal{K}$ [**CCG⁺94a**]. I will translate this classical construction into the quantum setting by first considering the 4-punctured sphere, thought of as the boundary of an ideal tetrahedron Δ. The triangulated surface has a quantum coordinate ring which acts on the *bulk module*. Defining and constructing this module is a central part of the project.

Bulk modules are then constructed for arbitrary triangulated 3-manifolds by gluing together the bulk modules of individual tetrahedra. Working with triangulated knot complements, we arrive at an algebraic and quantum version of the lagrangian which defines the A-polynomial. Recovering an expression for the quantum A-polynomial involves identifying the meridian and longitude of the boundary torus as elements in quantum coordinate ring of the boundary torus. The quantum A-polynomial is expected to be a two-variable non-commutative polynomial in terms of these elements.

As this construction depends on the choice of an ideal triangulation, proving invariance under the 2-3 Pachner move will be crucial. Furthermore, this invariance was the fundamental problem faced by the construction in [**Dim11**], where triangulations with self-folded tetrahedra could lead to trivial results. A major benefit of working in the framework provided by [**JLSS21**] is that the extra information tracked by decorated character stacks will produce non-trivial bulk modules even in the presence of self folded tetrahedra.

Goals:

- Formulate the decorated skein module of a 3-manifold using quantum character stacks.
- Give a triangulation-independent reformulation of the quantum A-polynomial [**Dim11**] based on the bulk module of knot complements.
- Compute the quantum A-polynomial of the $\mathbf{3_1}, \mathbf{4_1}, \mathbf{5_2}$ knots, using fixed triangulations.

3.1.1. Beyond SL_2**.** The classical A-polynomial is a knot invariant defined by a lagrangian in the $\mathrm{SL}_2\,\mathbb{C}$-character variety of a torus [**CS83**]. The proposed construction of the quantum A-polynomial from cluster coordinates on decorated character stacks likewise focuses on the $\mathrm{SL}_2\,\mathbb{C}$ case. A natural next step would be to leverage the generality of decorated character stacks to construct quantum A-polynomials based on $\mathrm{SL}_n\,\mathbb{C}$-local systems for $n > 2$.

In [**Sik05**], SU(n)-quantum invariants were introduced which coincide with those of the Kauffman bracket for $n = 2$ and Kuperburg bracket for $n = 3$. For any three manifold M, the SU(n)-skein module is isomorphic to the coordinate ring of the SL(n)-character variety of $\pi_1(M)$ [**Sik05**]. Furthermore, for any simple complex Lie algebra $\mathfrak{g}$, there is a quantum torus $W_{\mathfrak{g}}$ whose action on an appropriate function space captures the recursive properties of G-quantum invariants [**Sik08**].

An SL_n-bulk module built from the coordinates of [**JLSS21**] should have a natural interpretation as a module of $W_{\mathfrak{sl}_n}$, giving more weight to SL_n versions of the AJ conjecture.

3.2. Representation Theory and Recursion

This dissertation adds to the growing collection of quantum invariants known to be q-holonomic [**GL05, Gar12, GLL18, BDGG20**]. The relevant proofs, including the central result of this work, have focused on the expression of these invariants as functions, which are obtained from some category of representations $\mathcal{C}$ through a standard procedure. The idea behind the proposed project is simple: the q-holonomicity of a quantum invariant depends on the category used to construct it. Motivated by that, I hope to formulate q-holonomicity as a categorical notion.

My approach would be to first categorify the action of the quantum tori discussed in Section 1.2, and then the definition of homological dimension. The relevant quantum tori are generated by q-difference operators, which have natural candidates for categorification to $\mathrm{Rep}_q(G)$. Namely, the multiplication M of (1.2.13) becomes a grading shift and the shift L becomes tensoring with a distinguished object. An early check on these candidates is whether they q-commute after decategorification. Following [**BGHL14**], I believe that decategorification is given by zeroth Hochschild homology. This homology acts as a categorical trace and is used in the construction of WRT invariants, where it provides the notion of quantum dimension.

Goals:

- Categorify the action of q-difference operators, to the category $\mathrm{Rep}_q\, G$.
- Reformulate homological dimension as a property of morphisms in $\mathrm{Rep}_q\, G$.
- Prove the equivalent of Bernstein's Inequality for categories.
- (Re)-prove that the colored Jones polynomial is q-holonomic, using the categorical perspective.

APPENDIX A

Proof of Proposition 2.3.2

We give here an elementary proof of Proposition 2.3.2, on quantum Hamiltonian reduction. We use the same notation as in Section 2.3.1.

PROPOSITION 2.3.2. *For $n \geq 2$, let $\mathcal{I}_n \subseteq \mathbb{E}_n$ be a left ideal, and let $\mathcal{I}_{n-1}^{(x_n \to c)} = \left(\mathcal{I} \cap \widetilde{\mathbb{E}}_{n-1}\right)_{x_n=c} \subseteq \mathbb{E}_{n-1}$ as in (2.3.9). If $\mathbb{E}_n/\mathcal{I}_n$ is a q-holonomic $\mathbb{E}_n$-module then $\mathbb{E}_{n-1}/\mathcal{I}_{n-1}^{(x_n \to c)}$ is a q-holonomic $\mathbb{E}_{n-1}$-module.*

Without loss of generality, we may assume $c = 1$. Otherwise we may use the automorphism of $\mathbb{E}_n$ given by

$$(x_i, y_i) \mapsto \begin{cases} (x_i, y_i) & i \leq n-1 \\ (cx_i, y_i) & i = n \end{cases} \tag{A.1}$$

to intertwine the reduction at $x_n = c$ with reduction at $x_n = 1$.

Let $\mathcal{I}_n$ be the annihilation ideal of v, so that v generates the cyclic $\mathbb{E}_n$-module $\mathbb{E}_n/\mathcal{I}_n$. Let us denote by

$$\mathbb{E}_{n-1} = \mathbb{C}_q[x_1^{\pm}, y_1^{\pm}, \ldots, x_{n-1}^{\pm}, y_{n-1}^{\pm}]/(y_i x_j - q^{\delta_{ij}} x_j y_i), \quad \text{and} \quad \mathbb{E}_1 = \mathbb{C}_q[x_n^{\pm}, y_n^{\pm}]/(y_n x_n - q x_n y_n) \tag{A.2}$$

the standard q-Weyl algebra in the first $n-1$ pairs of variables and the last pair, respectively. We introduce the ‘delta-function” module

$$M_1^{\delta} = \mathbb{E}_1/(x_n - 1) = \mathbb{E}_1 \delta^{(1)}, \tag{A.3}$$

with formal[1] generator $\delta^{(1)}$ satisfying $(x_n - 1)\delta^{(1)} = 0$, and its extension to an $\mathbb{E}_n$-module

$$M_n^{\delta} = \mathbb{E}_n/(y_1 - 1, \ldots, y_{n-1} - 1, x_n - 1) = \mathbb{E}_n \delta^{(n)} \tag{A.4}$$

[1]Recall the delta functions in continuous variables are not in the function spaces $\mathcal{V}_{m,n}$, so we must describe them formally, in terms of their annihilation ideals.

with formal generator $\delta^{(n)}$ satisfying $(y_i - 1)\delta^{(n)} = 0$ for $i = 1, \ldots, n-1$ and $(x_n - 1)\delta^{(n)} = 0$. Both M_1^δ and M_n^δ are q-holonomic (for $\mathbb{E}_1$ and $\mathbb{E}_n$, respectively), as in Example (1.2.19).

It is also useful to recall that the *tensor product* of $\mathbb{E}_n$-modules $U \otimes W$ has underlying vector space $U \otimes_{\mathbb{C}_q[x^\pm]} W$ and action $x_i^\pm(u \otimes w) := (x_i^\pm u) \otimes w = u \otimes (x_i^\pm w)$, $y_i^\pm(u \otimes w) := (y_i^\pm u) \otimes (y_i^\pm w)$. In contrast, the *exterior product* of an $\mathbb{E}_{n-1}$-module U and an $\mathbb{E}_1$-module W is defined to have underlying vector space $U \otimes_{\mathbb{C}_q} W$ and action $x_i^\pm(u \otimes w) = (x_i^\pm u) \otimes w$, $y_i^\pm(u \otimes w) = (y_i^\pm u) \otimes w$ for $i \leq n-1$ and $x_n^\pm(u \otimes w) = u \otimes (x_n^\pm w)$, $y_n^\pm(u \otimes w) = u \otimes (y_n^\pm w)$. A special case is the exterior product of the algebras themselves, $\mathbb{E}_{n-1} \boxtimes \mathbb{E}_1 \simeq \mathbb{E}_n$.

Now let $\widetilde{M}$ denote the submodule of the tensor product $(\mathbb{E}_n/\mathcal{I}_n) \otimes M_n^\delta$ generated by $v \otimes \delta^{(n)}$,

$$\widetilde{M} = \mathbb{E}_n(v \otimes \delta^{(n)})\,. \tag{A.5}$$

$\widetilde{M}$ is q-holonomic because q-holonomic modules are closed under taking tensor products and passing to submodules (Section 1.2.2, [**Sab93**, Cor. 2.1.6, Prop 2.4.1]). We will show that

LEMMA A.1. *$\widetilde{M}$ decomposes as an exterior product of $\mathbb{E}_{n-1}$ and $\mathbb{E}_1$ modules*

$$\widetilde{M} \simeq (\mathbb{E}_{n-1}/\mathcal{I}_{n-1}^{(x_n\to 1)}) \boxtimes M_1^\delta\,, \tag{A.6}$$

whose first factor is precisely the module $\mathbb{E}_{n-1}/\mathcal{I}_{n-1}^{(x_n\to 1)}$ in the statement of Prop. 2.3.2.

PROOF OF PROP. 2.3.2. Assuming Lemma A.1, the most efficient way to prove the proposition is to consider the map

$$f : (\mathbb{C}^*)^{n-1} \to (\mathbb{C}^*)^n\,, \qquad f(x_1, \ldots, x_{n-1}) = (x_1, \ldots, x_{n-1}, 1) \tag{A.7}$$

and to apply the associated inverse image functor $f^!$ to $\widetilde{M}$. Explicitly, the inverse image functor $f^! : \mathbb{E}_n\text{-mod} \to \mathbb{E}_{n-1}\text{-mod}$ acts on an $\mathbb{E}_n$-module U by tensoring it over $\mathbb{E}_n$ with the $(\mathbb{E}_{n-1}, \mathbb{E}_n)$ bimodule

$$\mathcal{E} := \mathbb{C}_q[x_1^\pm, \ldots, x_n^\pm]/(x_n - 1) \underset{\mathbb{C}_q[x_1^\pm, \ldots, x_n^\pm]}{\otimes} \mathbb{E}_n \simeq (x_n - 1)\mathbb{E}_n \backslash \mathbb{E}_n\,. \tag{A.8}$$

Thus in general $f^! U := \mathcal{E} \otimes_{\mathbb{E}_n} U \simeq (x_n - 1)U \backslash U$. In the case of the product $\widetilde{M} = (\mathbb{E}_{n-1}/\mathcal{I}_{n-1}^{(x_n\to 1)}) \boxtimes M_1^\delta$, the inverse image functor just removes the M_1^δ factor, giving

$$f^! \widetilde{M} = \mathbb{E}_{n-1}/\mathcal{I}_{n-1}^{(x_n\to 1)}\,. \tag{A.9}$$

Since inverse image (the zeroth cohomology of the derived inverse image of [**Sab93**, Prop. 2.3.2]) preserves q-holonomic modules, $\mathbb{E}_{n-1}/\mathcal{I}_{n-1}^{(x_n\to 1)}$ is q-holonomic. $\square$

Comparing to $\mathbb{W}_n$-modules gives an alternative proof that $\widetilde{M} = (\mathbb{E}_{n-1}/\mathcal{I}_{n-1}^{(x_n\to 1)}) \boxtimes M_1^\delta$ being q-holonomic implies that $\mathbb{E}_{n-1}/\mathcal{I}_{n-1}^{(x_n\to 1)}$ is q-holonomic. We include this for completeness.

Denote by w the generator of $\mathbb{E}_{n-1}/\mathcal{I}_{n-1}^{(x_n\to 1)}$ and let $N_{n-1} = \mathbb{W}_{n-1}w = \mathbb{W}_{n-1}/(\mathcal{I}_{n-1}^{(x_n\to 1)} \cap \mathbb{W}_{n-1})$. The canonical good filtration on this module is given by

$$\mathcal{F}_k N_{n-1} = \{\beta w \quad \deg_{x,y}\beta \leq k\}, \tag{A.10}$$

where $\deg_{x,y}$ denotes total degree in $x_1, \dots, x_{n-1}$ and $y_1, \dots, y_{n-1}$. Let $d_k = \dim_{\mathbb{C}_q} \mathcal{F}_k N_{n-1}$.

Similarly, let $\widetilde{N} = \mathbb{W}_n(w \boxtimes \delta^{(1)})$. By [**GL16**, Prop. 3.4], $\widetilde{N}$ is a q-holonomic $\mathbb{W}_n$ module. The filtration (A.10) can likewise be defined on $\widetilde{N}$, tracking total degree in the $2n$ variables $x_1, \dots, x_n, y_1, \dots, y_n$. Let $\tilde{d}_k = \dim_{\mathbb{C}_q} \mathcal{F}_k \widetilde{N}$. As a vector space, $\widetilde{N}$ decomposes as

$$\widetilde{N} = \mathbb{W}_n(w \boxtimes \delta^{(1)}) = (\mathbb{W}_{n-1}w) \otimes_{\mathbb{C}_q} (\mathbb{W}_1\delta^{(1)}) \simeq (\mathbb{W}_{n-1}w) \otimes_{\mathbb{C}_q} \mathbb{C}_q[y_n] \tag{A.11}$$

we find that $\mathcal{F}_k\widetilde{N} \simeq \bigoplus_{\ell=0}^k \mathcal{F}_{k-\ell}N_{n-1} \otimes y_n^\ell$, so $\tilde{d}_k = \sum_{\ell=0}^k d_\ell$, or equivalently $d_k = \tilde{d}_k - \tilde{d}_{k-1}$. Since $\widetilde{N}$ is q-holonomic, there is a polynomial $p(k)$ of degree n such that $\tilde{d}_k = p(k)$ for all sufficiently large k. Therefore, $d_k = s(k) - s(k-1)$ is a polynomial of degree $n-1$ for all sufficiently large k, whence N_{n-1} is also q-holonomic. Then by Prop. 1.2.5, $\mathbb{E}_{n-1}/\mathcal{I}_{n-1}^{(x_n\to 1)} = \mathbb{E}_{n-1} \otimes_{\mathbb{W}_{n-1}} N_{n-1}$ is q-holonomic.

Proof of Lemma A.1. We introduce a $\mathbb{Z}$-grading on $\mathbb{E}_n$ given by degree with respect to y_n, with graded components $\mathbb{E}_n^{(k)} = \widetilde{\mathbb{E}}_{n-1}y_n^k$, where $\widetilde{\mathbb{E}}_{n-1} = \mathbb{C}_q[x_1^\pm, y_1^\pm, \dots, x_{n-1}^\pm, y_{n-1}^\pm, x_n^\pm]/(y_ix_j - q^{\delta_{ij}}x_jy_i)$ as in (2.3.8). With respect to this grading, M_n^δ of (A.4) may be given the structure of a graded module. Indeed,

$$M_n^\delta \simeq \mathbb{C}_q[x_1^\pm, \dots, x_{n-1}^\pm, y_n^\pm], \tag{A.12}$$

and we take the graded components to be $M_n^{\delta(k)} = \mathbb{C}_q[x_1^\pm, \dots, x_{n-1}^\pm]y_n^k$. The tensor product $(\mathbb{E}_n/\mathcal{I}_n)\otimes M_n^\delta$ and its submodule $\widetilde{M} = \mathbb{E}_n(v \otimes \delta^{(n)})$ inherit the $\mathbb{Z}$-grading from M_n^δ. Explicitly, the graded components are

$$\widetilde{M}^{(k)} = \widetilde{\mathbb{E}}_{n-1}(y_n^k v \otimes y_n^k \delta^{(n)}). \tag{A.13}$$

It follows that the annihilation ideal $\mathrm{Ann}_{\mathbb{E}_n}(v\otimes\delta^{(n)})$ must be generated by elements that are homogeneous in y_n. Combined with the fact that y_n is invertible, we find that $\mathrm{Ann}_{\mathbb{E}_n}(v\otimes\delta^{(n)})$ can be generated entirely in degree zero, *i.e.* its generators can be chosen to be elements of $\widetilde{\mathbb{E}}_{n-1}$. Moreover, we have $x_n-1\in\mathrm{Ann}_{\mathbb{E}_n}(v\otimes\delta^{(n)})$, since $(x_n-1)\cdot(v\otimes\delta^{(n)})=v\otimes(x_n-1)\delta^{(n)}=0$. All together, the annihilation ideal takes the form

$$\mathrm{Ann}_{\mathbb{E}_n}(v\otimes\delta^{(n)})=\mathbb{E}_n(p_1,\dots,p_\ell,x_n-1)\simeq\mathbb{E}_n(p_1\,{}_{x_n=1},\dots,p_\ell\,{}_{x_n=1},x_n-1) \tag{A.14}$$

for some $p_1,\dots,p_\ell\in\widetilde{\mathbb{E}}_{n-1}$. We have used the fact that x_n is central in $\widetilde{\mathbb{E}}_{n-1}$ to simply set $x_1=1$ in the p_i's, as indicated. This establishes a product decomposition

$$\widetilde{M}\simeq\mathbb{E}_{n-1}/(p_1\,{}_{x_n=1},\dots,p_\ell\,{}_{x_n=1})\boxtimes\mathbb{E}_1/(x_n-1)=\mathbb{E}_{n-1}/(p_1\,{}_{x_n=1},\dots,p_\ell\,{}_{x_n=1})\boxtimes M_1^\delta\,. \tag{A.15}$$

It remains to show that the ideal $\mathbb{E}_{n-1}(p_1\,{}_{x_n=1},\dots,p_\ell\,{}_{x_n=1})$ appearing on the left hand side of this product is equivalent to $\mathcal{I}_{n-1}^{(x_n\to1)}=(\mathcal{I}_n\cap\widetilde{\mathbb{E}}_{n-1})\,{}_{x_n=1}$. The following observation is key: for any $\beta\in\widetilde{\mathbb{E}}_{n-1}$, we can use the q-commutation relations to order variables in each monomial in β such that x's are placed to the left and y's are placed to the right. Then, using $y_i(v\otimes\delta_{x_n,c})=(y_iv)\otimes(y_i\delta_{x_n,c})=(y_iv)\otimes\delta_{x_n,c}$ for $i<n$ and $x_i(v\otimes\delta_{x_n,c})=(x_iv)\otimes\delta_{x_n,c}$ for all i, we find that $\beta\cdot(v\otimes\delta^{(n)})=(\beta v)\otimes\delta^{(n)}$ for all $\beta\in\widetilde{E}_{n-1}$. More so, using $(x_n-1)\delta^{(n)}=0$ we can extend this to

$$\beta\cdot(v\otimes\delta^{(n)})=(\beta v)\otimes\delta^{(n)}=(\beta\,{}_{x_n=1}v)\otimes\delta^{(n)}=\beta\,{}_{x_n=1}\cdot(v\otimes\delta^{(n)})\,. \tag{A.16}$$

Now, if $\beta\in\mathcal{I}_n\cap\widetilde{\mathbb{E}}_{n-1}=\mathrm{Ann}_{\widetilde{\mathbb{E}}_{n-1}}(v)$ then $\beta v=0$, so (A.16) implies $\beta\,{}_{x_n=1}\in\mathrm{Ann}_{\mathbb{E}_n}(v\otimes\delta^{(n)})$. From the form of the annihilation ideal (A.14), we therefore have

$$\beta\,{}_{x_n-1}\in\mathbb{E}_{n-1}(p_1\,{}_{x_n=1},\dots,p_\ell\,{}_{x_n=1}).$$

Conversely, suppose that $\gamma\in\mathbb{E}_{n-1}(p_1\,{}_{x_n=1},\dots,p_\ell\,{}_{x_n=1})$, so that $(\gamma v)\otimes\delta^{(n)}=0$. We now observe[2] that the map $\widetilde{\mathbb{E}}_{n-1}v\to(\widetilde{\mathbb{E}}_{n-1}v)\otimes\delta^{(n)}$ of left $\widetilde{E}_{n-1}$-modules has kernel $(x_n-1)\widetilde{E}_{n-1}v$. Therefore, $(\gamma v)\otimes\delta^{(n)}=0$ implies that there exists $\tilde\gamma\in\widetilde{\mathbb{E}}_{n-1}$ such that $\gamma v=(x_n-1)\tilde\gamma v$; or equivalently

[2]Explicitly: $(\widetilde{\mathbb{E}}_{n-1}v)\otimes\delta^{(n)}$ is a submodule of the tensor product of modules $(\widetilde{\mathbb{E}}_{n-1}v)\otimes(\widetilde{\mathbb{E}}_{n-1}\delta^{(n)})$, which by definition has underlying vector space $(\widetilde{\mathbb{E}}_{n-1}v)\underset{\mathbb{C}_q[x_1^\pm,\dots,x_n^\pm]}{\otimes}(\widetilde{\mathbb{E}}_{n-1}\delta^{(n)})$. But $\widetilde{\mathbb{E}}_{n-1}\delta^{(n)}\simeq\mathbb{C}_q[x_1,\dots,x_n]/(x_n-1)$. Thus, noting that x_n-1 is central in $\widetilde{\mathbb{E}}_{n-1}$, the full tensor product becomes $(\widetilde{\mathbb{E}}_{n-1}v)\underset{\mathbb{C}_q[x_1^\pm,\dots,x_n^\pm]}{\otimes}(\widetilde{\mathbb{E}}_{n-1}\delta^{(n)})\simeq(\widetilde{\mathbb{E}}_{n-1}v)\underset{\mathbb{C}_q[x_1^\pm,\dots,x_n^\pm]}{\otimes}\mathbb{C}_q[x_1,\dots,x_n]/(x_n-1)\simeq(\widetilde{\mathbb{E}}_{n-1}v)\big/\big((x_n-1)\widetilde{\mathbb{E}}_{n-1}v\big)$. Therefore, the map $(\widetilde{\mathbb{E}}_{n-1}v)\to(\widetilde{\mathbb{E}}_{n-1}v)\otimes\delta^{(n)}$ has kernel contained in $(x_n-1)\widetilde{E}_{n-1}v$; and one can check that the kernel also contains $(x_n-1)\widetilde{\mathbb{E}}_{n-1}v$.

that there exists $\hat{\gamma} \in \widetilde{\mathbb{E}}_{n-1}$ such that $\hat{\gamma} v = 0$ and $\hat{\gamma}\,_{x_n=1} = \gamma$ (just set $\hat{\gamma} = \gamma - (x_n - 1)\tilde{\gamma}$). Since $\hat{\gamma} \in \mathrm{Ann}_{\widetilde{\mathbb{E}}_{n-1}}(v) = \mathcal{I}_n \cap \widetilde{\mathbb{E}}_{n-1}$, it follows that $\gamma \in (\mathcal{I}_n \cap \widetilde{\mathbb{E}}_{n-1})\,_{x_n=1} = \mathcal{I}_{n-1}^{(x_n \to 1)}$. $\square$

APPENDIX B

Further Examples and Computations

The Jones polynomials for the zero-framed $\mathbf{3_1}$ and $\mathbf{5_2}$ knots are readily computed using a general formula for p-twist knots [**Mas03, Hab00**] (see also [**GS10**]):

$$J_n^p(q) := \sum_{k=0}^{n}\sum_{j=0}^{k}(-1)^{j+1}q^{k+pj(j+1)+\frac{1}{2}j(j-1)}\frac{(q^{2j+1}-1)(q;q)_k(q^{1-n};q)_k(q^{1+n};q)_k}{(q;q)_{k+j+1}(q;q)_{k-j}}\,. \tag{B.1}$$

In our normalization and choices of chirality, we have

$$J_N^{\mathbf{3_1}}(q) = \frac{q^N - q^{-N}}{q-q^{-1}}J_N^{p=1}(q^{-2})\,, \qquad J_N^{\mathbf{5_2}}(q) = \frac{q^N - q^{-N}}{q-q^{-1}}J_N^{p=2}(q^{-2})\,. \tag{B.2}$$

We computed ADO invariants directly, using the (1, 1)-tangle diagrams in Figure B.1, and then changing the framing from blackboard to zero framing. We performed computations for $2 \le r \le 11$. For convenience, we introduce the normalization

$$\hat{N}_K^r(\alpha) := (i\zeta_{2r}^{\alpha})^{1-r}\frac{\zeta_{2r}^{\alpha}-\zeta_{2r}^{-\alpha}}{\zeta_{2r}^{r\alpha}-\zeta_{2r}^{-r\alpha}}N_K^r(\alpha-1)\,. \tag{B.3}$$

Let $X^{(n)} := x^n - x^{-n}$, $q = \zeta_{2r}$, $x = \zeta_{2r}^{\alpha}$, and $X^{(n)} := x^n - x^{-n}$. The ADO invariants for $\mathbf{3_1}$ and $\mathbf{5_2}$ are shown in Figures B.2 and B.3, respectively.

Inhomogeneous recursion relations for the colored Jones polynomials of $\mathbf{3_1}$ and $\mathbf{5_2}$ were found in [**GL05, GS10**]. In the current normalization, the recursions take the form

$$(q-q^{-1})A_{\mathbf{K}}(x,y;q)J_N^{\mathbf{K}}(q) = B_{\mathbf{K}}(q^N;q) \tag{B.4}$$

Where for $\mathbf{3_1}$ the operators in (B.4) are

$$A_{\mathbf{3_1}}(x,y;q) = q^3x^6y - 1$$

$$B_{\mathbf{3_1}}(x;q) = q^2x(q^2x^4-1)$$

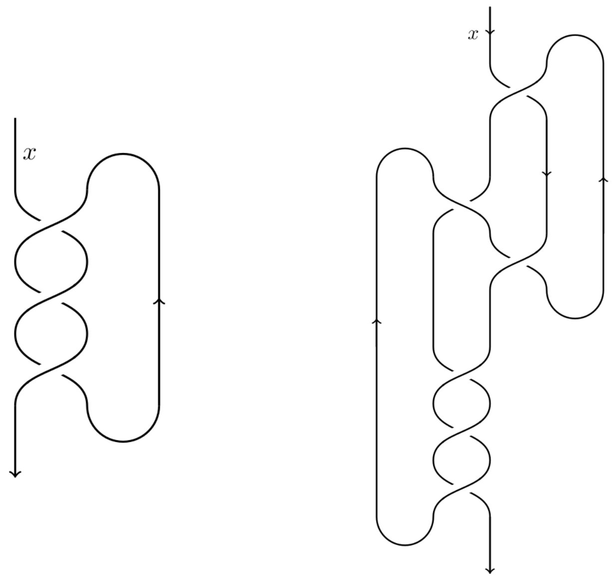


FIGURE B.1. Tangle diagrams whose closures are the $\mathbf{3_1}$ (left) and $\mathbf{5_2}$ knots (right).

r	$\hat{N}^r_{\mathbf{3_1}}(\alpha)$
2	$-X^{(3)}$
3	$q^2X^{(5)} + qX^{(1)}$
4	$q^2X^{(7)} + X^{(3)} + q^2X^{(1)}$
5	$q^2X^{(9)} - q^4X^{(5)} + qX^{(3)}$
6	$q^2X^{(11)} - q^4X^{(7)} + X^{(5)} + X^{(1)}$
7	$q^2X^{(13)} - q^4X^{(9)} - q^6X^{(7)} - q^5X^{(3)} + q^2X^{(1)}$
8	$q^2X^{(15)} - q^4X^{(11)} - q^6X^{(9)} - q^4X^{(5)} + X^{(3)}$
9	$q^2X^{(17)} - q^4X^{(13)} - q^6X^{(11)} - q^3X^{(7)} - q^7X^{(5)} - q^8X^{(1)}$
10	$q^2X^{(19)} - q^4X^{(15)} - q^6X^{(13)} - q^2X^{(9)} - q^6X^{(7)} - q^6X^{(3)} + q^2X^{(1)}$
11	$q^2X^{(21)} - q^4X^{(17)} - q^6X^{(15)} - qX^{(11)} - q^5X^{(9)} - q^4X^{(5)} - q^{10}X^{(3)}$

FIGURE B.2. The ADO invariant for the knot $\mathbf{3_1}$.

For $\mathbf{3_1}$ they are

$$A_{\mathbf{5_2}}(x; y; q) = -q^{28}(1 - q^2x^4)(1 - q^4x^4)x^{14}y^3 - q(1 - q^8x^4)(1 - q^{10}x^4)$$

$$- q^5(1 - q^2x^4)(1 - q^8x^4)x^4(1 - q^4x^2 - q^4(1 - q^2)(1 - q^4)x^4 + q^8(1 + q^6)x^6 + 2q^{14}x^8 - q^{18}x^{10})y^2$$

$$+ (1 - q^4x^4)(1 - q^{10}x^4)(1 - 2q^2x^2 - q^2(1 + q^6)x^4 + q^4(1 - q^2)(1 - q^4)x^6 + q^{10}x^8 - q^{12}x^{10})y$$

r	$\hat{N}^r_{\mathbf{5_2}}(\alpha)$
2	$-2X^{(3)} - X^{(1)}$
3	$(2q^2-1)X^{(5)} + 2q^2X^{(3)} + 2q^2X^{(1)}$
4	$(2q^2-2)X^{(7)} + (3q^2-q)X^{(5)} + (3q^2-1)X^{(3)} + (2q^2-1)X^{(1)}$
5	$(2q^2-q-2)X^{(9)} + (2q^3+2q^2-2)X^{(7)} + (2q^3+2q^2+q-3)X^{(5)}$ $+(2q^3+q^2+q-2)X^{(3)} + (q^3+q^2-2)X^{(1)}$
6	$-(4q^4+2)X^{(9)} - (6q^4+2)X^{(7)} - (6q^4+1)X^{(5)} - (4q^4+2)X^{(3)} - 2X^{(1)}$
7	$-(q^4+2q^3-2q^2+1)X^{(13)} + (4q^5-2q^4-4)X^{(11)} + (5q^5-2q^4+2q^3-7)X^{(9)}$ $+(6q^5-q^4+3q^3-2q^2+2q-7)X^{(7)} + (5q^5-2q^4+3q^3-q^2+q-7)X^{(5)}$ $+(3q^5-2q^4+q^3-q^2-4)X^{(3)} - (q^4+q^3-q^2+2)X^{(1)}$
8	$-(2q^6+2q^4-2q^2+2)X^{(15)} + (q^6-3q^4-q^2-5)X^{(13)} + (3q^6-q^4-3q^2-9)X^{(11)}$ $+(7q^6-3q^2-10)X^{(9)} + (7q^6-3q^2-10)X^{(7)} + (4q^6-q^4-3q^2-8)X^{(5)}$ $+(q^6-3q^4-2q^2-4)X^{(3)} - (q^6+2q^4-q^2+1)X^{(1)}$
9	$-(4q^5-4q^2+1)X^{(17)} - (4q^5+4q^3-2q^1+4q)X^{(15)} - (2q^5+q^4+5q^3+7q+5)X^{(13)}$ $+(q^5+2q^4-3q^3-6q^2-11q-8)X^{(11)} + (3q^5-q^3-7q^2-10q-12)X^{(9)}$ $+(2q^5+q^4-2q^3-7q^2-10q-8)X^{(7)} - (q^5+2q^4+4q^3+2q^2+6q+5)X^{(5)}$ $-(3q^5+2q^4+3q^3-q^2+2q)X^{(3)} - (3q^5+q^4-3q^2+1)X^{(1)}$
10	$-(q^6-4q^2)X^{(19)} - (6q^6+4q^4-2)X^{(17)} - (8q^6+10q^4+4q^2+4)X^{(15)}$ $-(6q^6+14q^4+10q^2+14)X^{(13)} - (22q^4+8q^2+22)X^{(11)} + (q^6-22q^4-8q^2-22)X^{(9)}$ $-(6q^6+14q^4+10q^2+14)X^{(7)} - (9q^6+8q^4+6q^2+3)X^{(5)} - (7q^6+2q^4+q^2-2)X^{(3)}$ $-(3q^6-q^4-2q^2-1)X^{(1)}$
11	$-(2q^7-q^5-2q^4-2q^2-2q+2)X^{(21)} - (2q^9+6q^7+4q^5-4q^4+4q^3-6q^2-2)X^{(19)}$ $-(3q^9+5q^8+7q^7+3q^6+7q^5+7q^3-4q^2+1)X^{(17)}$ $-(2q^9+5q^8+10q^7+6q^6+9q^5+5q^4+12q^3+2q^2+5q+3)X^{(15)}$ $+(3q^9-9q^8-6q^7-11q^6-8q^5-14q^4-10q^3-9q^2-3q-12)X^{(13)}$ $+(6q^9-8q^8-4q^7-15q^6-6q^5-18q^4-9q^3-15q^2-2q-13)X^{(11)}$ $+(3q^9-9q^8-5q^7-12q^6-7q^5-15q^4-9q^3-10q^2-3q-12)X^{(9)}$ $-(2q^9+5q^8+9q^7+7q^6+8q^5+7q^4+10q^3+3q^2+4q+3)X^{(7)}$ $-(4q^9+4q^8+7q^7+3q^6+6q^5+2q^4+6q^3-3q^2)X^{(5)}$ $-(2q^9+5q^7+3q^5-2q^4+3q^3-5q^2-3)X^{(3)} - (2q^7-q^5-q^4-2q^2-2q+1)X^{(1)}$

FIGURE B.3. The ADO invariant for the knot $\mathbf{5_2}$.

$$B_{\mathbf{5_2}}(x;q) = q^5x^3 + q^7(1+q^2)x^5 - q^7(1+q^8)x^7 + \frac{q^6-q^{-6}}{q-q^{-1}}(-q^{14}x^9 + q^{20}x^{13})$$
$$- q^{19}(1+q^8)x^{15} - q^{25}(1+q^2)x^{17} - q^{29}x^{19}.$$

These imply homogeneous recursions

$$\text{(B.5)} \qquad \widetilde{A}_K(x,y;q)J_N^K(q) := \left[B_K(x;q)y - B_K(qx;q)\right]A_K(x,y;q)J_N^K(q) = 0 \qquad (K = \mathbf{3_1}, \mathbf{5_2}),$$

just as in the figure-eight example (1.1.11) in the Introduction.

We checked explicitly for each $2 \le r \le 11$ that the ADO invariants satisfy inhomogeneous recursions

$$\text{(B.6)} \qquad \begin{aligned} A_{\mathbf{3_1}}(x,y;\zeta_{2r})\hat{N}^r_{\mathbf{3_1}}(\alpha) &= (\zeta_{2r}^{2r\alpha} - 1 + \zeta_{2r}^{-2r\alpha})B_{\mathbf{3_1}}(\zeta_{2r}^{\alpha};\zeta_{2r}) \\ A_{\mathbf{5_2}}(x,y;\zeta_{2r})\hat{N}^r_{\mathbf{5_2}}(\alpha) &= (2\zeta_{2r}^{2r\alpha} - 3 + 2\zeta_{2r}^{-2r\alpha})B_{\mathbf{5_2}}(\zeta_{2r}^{\alpha};\zeta_{2r}) \end{aligned}$$

with exactly the same A and B polynomials. Again, these imply homogeneous recursions

$$\text{(B.7)} \qquad \widetilde{A}_K(x,y;\zeta_{2r})\hat{N}^r_K(\alpha) = 0 \qquad r \in \mathbb{N}_{\ge 2} \qquad (K = \mathbf{3_1}, \mathbf{5_2}),$$

with the same $\widetilde{A}_K(x,y;q) = \left[B_K(x;q)y - B_K(qx;q)\right]A_K(x,y;q)$. Note that the prefactors $(x^{2r} - 1 + x^{-2r})$ and $(2x^{2r} - 3 + 2x^{-2r})$ appearing in (B.6) may be factored out from the homogeneous recursion (B.7), since they commute with y and just behave like overall constants.

In terms of the standard normalization of the ADO invariant used in the main body of the paper (N_K rather than $\hat{N}_K$), the homogeneous recursions take the form

$$\text{(B.8)} \qquad \widetilde{A}_K(qx,-y;\zeta_{2r})N^r_K(\alpha) = 0 \qquad r \in \mathbb{N}_{\ge 2} \qquad (K = \mathbf{3_1}, \mathbf{5_2}),$$

in perfect agreement with Theorem 2.3.4.

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