Math 280B Winter 2010

6. Recursion on Well-Founded Relations

We work in ZF without foundation for the following:

6.1 Recall: For a binary relation R (may be a proper class):

(i) $\mathrm{pred}_R(a) = \{z | \langle z, a\rangle \in R\}$

(ii) R is set-like iff for each $a \in V$: $\mathrm{pred}_R(a)$ is a set, i.e. $\mathrm{pred}_R(a) \in V$.

(iii) If R is set-like then for any $A \in V$ we let

$$T_0 = A$$
$$T_{n+1} = \bigcup_{a \in T_n} \mathrm{pred}_R(a)$$

and then we let the transitive closure of A with respect to A:

(End of Chapter 2)

$$\mathrm{trcl}_R(A) = \bigcup_{n \in \omega} T_n$$

Note that $\mathrm{trcl}_\in(A) = \mathrm{trcl}(A)$.

(iv) R (now need R to bet set-like) is well-founded iff every nonempty A has an R-minimal element, i.e. some element $a \in A$ such that:

$$\langle z, a\rangle \notin R \text{ for all } z \in A.$$

Equivalently: $A \cap \mathrm{pred}_R(a) = \varnothing$.

6.2 Theorem: Construction by Recursion on Well-Founded Relations; Bar Induction/Recursion

Assume R is a binary relation that is well-founded and set-like. Let

$$G : V \to V$$

be a class function. Then there is a unique class function $F : V \to V$ such that

$$F(x) = G(F \restriction \mathrm{pred}_R(x)) \text{ for all } x \in V.$$

Proof. Uniqueness: Assume F, F' are two such functions and $F \neq F'$.

So $X = \{a \in V | F(a) \neq F(a')\}$ is a nonempty class. We proved before that if R is well-founded + set-like and $X \neq \varnothing$ then X has an R-minimal element. So let $b \in X$ be R-minimal. Then $\operatorname{pred}_R(b) \cap X = \varnothing$. Hence, $F(y) = F'(y)$ for all $y \in \operatorname{pred}_R(b)$. So $F \upharpoonright \operatorname{pred}_R(b) = F' \upharpoonright \operatorname{pred}_R(b)$. Hence

$$F(b) = G(F \upharpoonright \operatorname{pred}_R(b)) = G(F' \upharpoonright \operatorname{pred}_R(b)) = F'(b)$$

Existence: We show: to each $x \in V$ there is a unique function

$$f_x : \operatorname{trcl}_R(\{x\}) \to V$$

such that

$$(1)\ f_x(z) = G(f_x \upharpoonright \operatorname{pred}_R(z)) \text{ for all } z \in \operatorname{trcl}_R(\{x\}).$$

This implies that if $x, x' \in V$ then:

$$(2)\ f_x = f_{x'} \text{ whenever } z \in \operatorname{dom}(f_x) \cap \operatorname{dom}(f_{x'}) = \operatorname{trcl}_R(\{x\}) \cap \operatorname{trcl}_R(\{x'\})$$

This is because if $z \in \operatorname{trcl}_R(\{x\}) \cap \operatorname{trcl}_R(\{x'\})$ then $\operatorname{trcl}_R(\{x\}) \subseteq \operatorname{trcl}_R(\{x\}) \cap \operatorname{trcl}_R(\{x'\})$.

So $f_x \upharpoonright \operatorname{trcl}_R(\{z\}) = f_z = f_{x'} \upharpoonright \operatorname{trcl}_R(\{z\})$ because $f_x, f_{x'}$ and f_z follow (1).

Hence $f_x(z) = f_z(z) = f_{x'}(z)$.

So if we let

$$\mathcal{F} = \text{the class of all functions } f_x \text{ satisfying (1)}$$

we can let

$$F = \bigcup \mathcal{F}.$$

Then check that this is as required.

Now show that we have these functions:

Uniqueness: Pick x and show that there is at most one f_x satisfying (1). This is like the proof of uniqueness above.

Existence: Show that f_x exists for each $x \in V$. If not:

$$Y = \text{the class for all } x \in V \text{ such that there is no } f_x \text{ as in (1).}$$

Then $Y \neq \varnothing$. Since R is well-founded and set-like: Y has an R-minimal element c.

This means that f_z exists and is unique for each $z \in \operatorname{pred}_R(c)$.

By the fact that $\operatorname{pred}_R(c)$ is a set + Replacement:

$$\bigcup_{z \in pred_R(c)} f_z \text{ is a set function.}$$

In fact: $f' = F \upharpoonright \operatorname{pred}_R(c)$, then $f_x = f' \cup \{\langle x, G(f') \rangle\}$.

So $x \notin Y$ after all, a contradiction. This proves existence. $\square$

6.3 Corollary:

Let R be a binary relation that is well-founded and set-like. Then there is a unique class function $\sigma : R \to O_n$ such that:

$$\sigma(x) = \sup\{\sigma(y) + 1 | y \in x\}$$

Notice: $\langle x, x'\rangle \in R \Rightarrow \sigma(x) < \sigma(x')$.

Moreover: If σ' is any map such that $\langle x, x'\rangle \in R \Rightarrow \sigma'(x) < \sigma'(x')$ then $\sigma(x) \leq \sigma'(x)$ for all $x \in \text{Field}(R) = \text{dom}(R) \cup \text{rng}(R)$.

6.4 Definition: Rank

Let R be a binary relation that is well-founded and set-like. The unique map from 6.3 is called the *rank*. We write $\text{rank}_R(x)$ instead of $\sigma(x)$.

6.5 Example (Foundation):

We write $\text{rank}(x)$ instead of $\text{rank}_{\in}(x)$.

In fact:

$$\text{rank}(x) = \text{the unique } \alpha \in O_n \text{ such that } x \in V_{\alpha+1} - V_\alpha.$$

6.6 Definition: Extensional Binary Relation

A binary relation R is *extensional* iff for all x, y we have

$$\text{pred}_R(x) = \text{pred}_R(y) \Rightarrow x = y$$

Equivalently,

$$(\forall z)(zRx \leftrightarrow zRy) \Rightarrow x = y$$

6.7 Example:

By the Axiom of Extensionality, $\in$ is extensional.

6.8 Theorem: (Mostowski Collapsing Theorem)

Let A be a class and let $R \subseteq A \times A$ be a binary relation that is

- set-like
- extensional
- well-founded

Then there is a unique pair (U, σ) such that

- U is a transitive class

- $\sigma : (A, R) \to (U, \in)$ is an isomorphism

Proof. U is transitive: (U is a class by T.6.2). Let $a \in U$. Then $a = \sigma(x)$ for some $x \in A$. Now if $b \in a$ then $b = \sigma(z)$ for some b such that bRa since $a = \sigma(x) = \{\sigma(z) | zRx\}$. So $b \in \mathrm{rng}(\sigma) = U$.

Uniqueness: Since σ is an isomorphism for each $x \in A$ we have

$$(\forall z)\ zRx \leftrightarrow \sigma(z) \in \sigma(x) \text{ which means that } \sigma(x) = \{\sigma(z) | zRx\}$$

By the theorem on construction by recursion T.6.2 there is a unique σ that satisfies this recursion formula. Since $U = \sigma[A]$, U is also unique.

Existence: By T.6.2 there is some $\sigma : V \to V$ that satisfies $\sigma(x) = \{\sigma(z) | zRx\}$. Let U be $\sigma[A]$. From now on write σ instead of $\sigma \restriction A$. Then

(i) $\sigma : A \to U$ is surjective

(ii) $zRx \Rightarrow \sigma(z) \in \sigma(x)$ for all $x, z \in A$.

We show that $\sigma : (A, R) \to (U, \in)$ is an isomorphism.

σ is injective: Suppose not. This means that we have some $x, y \in A$ such that $x \neq y$ and $\sigma(x) = \sigma(y)$. Because R is set-like and well-founded, we can minimize x, i.e. we can find an $x \in A$ such that

(a) $\sigma(x) = \sigma(y)$ for some $y \neq x$

(b) If zRx then $\sigma(z) \neq \sigma(z')$ whenever $z \neq z'$.

We show: $\mathrm{pred}_R(x) \subseteq \mathrm{pred}_R(y)$.

Pick zRx. Then $\sigma(z) \in \sigma(x)$ by the definition of σ. Since $\sigma(x) = \sigma(y)$ we have $\sigma(z) \in \sigma(y) = \{\sigma(z') | z'Ry\}$. Hence there is some z' such that $z'Ry$ and $\sigma(z') = \sigma(z)$. But by (b) above: $z' = z$. Hence zRy.

A symmetric argument shows: $\mathrm{pred}_R(y) \subseteq \mathrm{pred}_R(x)$. (Check this!)
So we have: $\mathrm{pred}_R(x) = \mathrm{pred}_R(y)$.

Because R is extensional: we have that $x = y$. Contradiction, as we assumed $x \neq y$.
This proves injectivity of σ.

$\sigma(z) \in \sigma(x) \Rightarrow zRx$ for all $z, x \in A$:

If $\sigma(z) \in \sigma(x)$ then $\sigma(z) = \sigma(z')$ for some z' such that $z'Rx$ because $\sigma(x) = \{\sigma(z')|z'Rx\}$ by definition. Since σ is injective: $z = z'$. So zRx.

This has finished the proof of the theorem. □

6.9 Example:

Let $\langle A, <\rangle$ be a well-ordered set. Notice that $<$ is extensional on A. So by Mostowski there is exactly one transitive set α and exactly one map σ such that $\sigma : \langle A, <\rangle \to \langle \alpha, \in\rangle$ is an isomorphism. Notice: $\alpha = \text{otp}(A, <)$. Similarly, if A is a proper class and $<$ is a set-like well-ordering on A then $(A, <)$ is isomorphic to $(O_n, \in)$ and the isomorphism is unique.

6.10 Proposition: (ZF)

Assume U, U' are transitive classes and

$$\sigma : (U, \in) \to (U', \in)$$

is an isomorphism. Then $U = U'$ and $\sigma = id$.

Proof: Immediate from Mostowski.

6.11 Corollary: (ZF)

If M is a proper class and $\sigma : (M, \in) \to (V, \in)$ is an isomorphism, then $M = V$ and $\sigma = id$.

7. Elements of Cardinal Arithmetic

For the moment we work without Foundation in ZF.

Recall that if $\alpha \in O_n$ then a set $A \subseteq \alpha$ is *cofinal* in α or *unbounded* in α iff

$$(\forall \xi < \alpha)(\exists \zeta \in A)(\xi \leq \zeta).$$

If α is a limit ordinal then we can replace "$\xi \leq \zeta$" by "$\xi < \zeta$".

7.1 Definition: Cofinality of an Ordinal

Let $\alpha \in O_n$. The *cofinality of* α is the least ordinal γ such that there is a set $A \subseteq \alpha$ cofinal in α such that $\text{otp}(A) = \gamma$. We denote it by $\text{cf}(\alpha)$. So:

$$\text{cf}(\alpha) = \min\{\ \text{otp}(A) | A \subseteq \alpha \text{ is cofinal in } \alpha\}$$

7.2 Proposition:

If α is a successor ordinal then $\text{cf}(\alpha) = 1$.

Proof. Say $\alpha = \bar{\alpha} + 1$. Then $\{\bar{\alpha}\}$ is cofinal in α. □

7.3 Proposition:

For every ordinal α we have: $\mathrm{cf}(\alpha) \leq \alpha$.

Proof. This is because α is a cofinal subset of α. $\square$

7.4 Examples:

(a) $\mathrm{cf}(\omega) = \omega$.

(b) $\mathrm{cf}(\omega + \omega) = \omega$; $A = \{\omega, \omega + 1, ..., \omega + n, ...\} = \{\omega + n | n \in \omega\}$.

(c) $\mathrm{cf}(\omega \cdot \omega) = \omega$; $A = \{\omega \cdot n | n \in \omega\}$.

(d) $\mathrm{cf}(\omega^\omega) = \omega$; $A = \{\omega^n | n \in \omega\}$.

(e) $\mathrm{cf}(\aleph_\omega) = \omega$; $A = \{\aleph_n | n \in \omega\}$.

7.5 Proposition:

Let α be a limit ordinal. Then $\gamma = \mathrm{cf}(\alpha)$ iff

(a) $\gamma =$ the least ordinal such that there is a strictly increasing cofinal map $f : \gamma \to \alpha$.

(b) $\gamma =$ the least ordinal such that there is a closed unbounded $C \subseteq \alpha$ in α with $\mathrm{otp}(C) = \gamma$.

(c) $\gamma =$ the least ordinal such that there is a normal cofinal map $f : \gamma \to \alpha$. (Recall: normal = strictly increasing and continuous).

Proof. (a) If $\gamma = \mathrm{cf}(\alpha)$, we have some unbounded $A \subseteq \alpha$ with $\mathrm{otp}(A) = \gamma$. Then if $f : \gamma \to A$ is the corresponding isomorphism, it is also a strictly increasing cofinal map in α. On the other hand: if $\bar{\gamma} < \gamma$ and $g : \bar{\gamma} \to \alpha$ is strictly increasing and cofinal then let $A = \mathrm{rng}(g)$. Then $A \subseteq \alpha$ is cofinal and $\mathrm{otp}(A) = \bar{\gamma} < \gamma = \mathrm{cf}(\alpha)$.

Contradiction.

(b) Follows from HW1: If $A \subseteq \alpha$ cofinal and $\bar{A} =$ the topological closure of A in α then $\mathrm{otp}(\bar{A}) = \mathrm{otp}(A)$.

(c) Then: if $A \subseteq \alpha$ is cofinal and $\mathrm{otp}(A) = \gamma = \mathrm{cf}(\alpha)$ then $\mathrm{otp}(\bar{A}) = \gamma = \mathrm{cf}(\alpha)$. Now if $g : \gamma \to \bar{A}$ is the isomorphism then $g : \gamma \to \alpha$ is normal and cofinal and by (a) there is no such map with domain $\bar{\gamma} < \gamma$. $\square$

7.6 Proposition:

Assume α, δ are limit ordinals and $f : \delta \to \alpha$ is cofinal (not necessarily increasing). Then $\mathrm{cf}(\alpha) \leq \delta$.

Proof. Let

$$D = \{\xi | \xi < \delta \text{ such that } f(\xi) > f(\eta) \text{ for all } \eta < \xi\}.$$

So $D \subseteq \delta$. This means: $\text{otp}(D) \leq \delta$.

We show:

(i) $f \upharpoonright D$ is strictly increasing.

(ii) $f \upharpoonright D$ is cofinal in α.

(i) is immediate from the definition: if $\bar{\xi} < \xi$ are in D then $f(\xi) > f(\eta)$, in particular, $f(\xi) > f(\bar{\xi})$.

(ii) Pick some $\bar{\alpha} < \alpha$. Assume for a contradiction that $f(\xi) < \bar{\alpha}$ for all $\xi \in D$. But we know f is cofinal, so there is some $\theta < \delta$ such that $f(\theta) \geq \bar{\alpha}$. Let

$$\theta^* = \text{the least } \theta \text{ such that } f(\theta) \geq \bar{\alpha}.$$

Then: If $\theta < \theta^*$ then $f(\theta) < \bar{\alpha} \leq f(\theta^*)$, so $\theta^* \in D$. Contradiction. □

7.7 Proposition:

Let α be a limit ordinal.

(a) $\text{cf}(\text{cf}(\alpha))=\text{cf}(\alpha)$

(b) $\text{cf}(\alpha)$ is a cardinal.

Proof. (a) Suppose not. Let $\gamma = \text{cf}(cf(\alpha))$. Since $\text{cf}(\text{cf}(\alpha)) \leq \text{cf}(\alpha)$ by P.7.3, we must have $\gamma < \text{cf}(\alpha)$. By P.7.5:

There is a cofinal strictly increasing map $f : \gamma \to \text{cf}(\alpha)$.

There is a cofinal strictly increasing map $g : \text{cf}(\alpha) \to \alpha$.

So $g \circ f : \gamma \to \alpha$ is a cofinal strictly increasing map into α (Exercise). This would mean that $\text{cf}(\alpha) \leq \gamma < \text{cf}(\alpha)$. Contradiction.

(b) Assume α is not a cardinal. So let $\kappa = \text{card}(\alpha)$. So there is a surjection $f : \kappa \xrightarrow{onto} \alpha$. Of course, f is cofinal by P.7.6: $\text{cf}(\alpha) \leq \kappa < \alpha$ (we assume that α is not a cardinal). Since $\text{cf}(\text{cf}(\alpha)) = \text{cf}(\alpha)$: $\text{cf}(\alpha)$ must be a cardinal.

□

7.8 Definition: Regular/Singular Cardinals

A cardinal κ is *regular* iff $\text{cf}(\kappa) = \kappa$.

Cardinals that are not regular are called *singular*.

Examples

(a) ω is regular — $\text{cf}(\omega) = \omega$.

(b) If $\alpha \in (\omega, \omega_1)$ $\mathrm{cf}(\alpha) = \omega$.

(c) $\mathrm{cf}(\aleph_\omega) = \omega$ so $\aleph_\omega$ is singular.

7.9 Proposition:

Let α, β be limit ordinals. Assume there is a cofinal strictly increasing map $f : \alpha \to \beta$. Then $\mathrm{cf}(\alpha) = \mathrm{cf}(\beta)$.

Proof. We know there is a cofinal strictly increasing map $g : \mathrm{cf}(\alpha) \to \alpha$. Then $f \circ g : \mathrm{cf}(\alpha) \to \beta$ is a cofinal strictly increasing map into β. So $\mathrm{cf}(\beta) \leq \mathrm{cf}(\alpha)$.

We also know there is a cofinal strictly increasing map $h : \mathrm{cf}(\beta) \to \beta$.

Notice: $f^{-1} \circ h$ need not be cofinal. So we need to define a new map: We define $k : \mathrm{cf}(\beta) \to \alpha$ by recursion:

$$k(\xi) = \text{the least } \eta < \alpha \text{ such that}$$

- $f(\eta) \geq h(\xi)$ (This will guarantee that f is cofinal).
- $f(\eta) > k(\xi')$ for all $\xi' < \xi$. This will guarantee k is strictly increasing.

If $k(\xi')$ is defined for all values $< \xi$, then k s strictly increasing. So we have $k \restriction \xi$. Now $\xi < \mathrm{cf}(\beta) \leq \mathrm{cf}(\beta)$, so $k \restriction \xi$ cannot be cofinal in α.

So there is some $\eta < \alpha$ such that $\eta > k(\xi')$ for all $\xi' < \xi$.

So we can find $\eta < \alpha$ such that $f(\eta) > h(\xi)$ and $\eta > k(\xi')$ for all $\xi' < \xi$.

This tells us that

(i) $k(\xi)$ is defined for all $\xi < \mathrm{cf}(\beta)$

(ii) k is strictly increasing.

(iii) k is cofinal. If $\eta < \alpha$: Since h is cofinal in β we can find $\xi < \mathrm{cf}(\beta)$ such that $h(\xi) > f(\eta)$. Then $k(\xi) \geq \eta$ by the definition of k.

Conclusion: $k : \mathrm{cf}(\beta) \to \alpha$ is a cofinal strictly increasing map. Hence, $cf(\alpha) \leq \mathrm{cf}(\beta)$. □

Remark: Alternatively, we could define $k' : \mathrm{cf}(\beta) \to \alpha$ by dropping the second clause in the definition of k. Then k' is still cofinal in α but not necessarily strictly increasing. By P.7.6 $\mathrm{cf}(\alpha) \leq \mathrm{cf}(\ \mathrm{cf}(\beta)) = \mathrm{cf}(\beta)$.

What we proved so far about cofinalities is practically all what can be done in ZF alone.

We have seen: examples where α was a limit ordinal and $\mathrm{cf}(\alpha) = \omega$. This is all one can prove in ZF alone, by the result of Gitik (1980):

$$\mathrm{Con(ZFC + (*))} \Rightarrow \mathrm{Con(ZF+ \text{ Every limit ordinal is } \omega\text{-cofinal})}$$

Here (*) is so-called *large cardinal axiom*. See below.

7.10 Definition: Weakly Inaccessible Cardinal

A cardinal that is both regular and limit is called *weakly inaccessible*.

The following holds:

(i) It cannot be proved in ZF that a weakly inaccessible cardinal exists.

(ii) We have seen in 280A examples like CH where one could prove:

$$\text{Con(ZFC)} \Rightarrow \text{Con(ZFC+CH)}$$
$$\text{Con(ZFC)} \Rightarrow \text{Con(ZFC+}\neg\text{CH)}$$

Let WI abbreviate the statement "there is a weakly inaccessible cardinal." Then

$$\text{Con(ZF)} \Rightarrow \text{Con(ZF+}\neg\text{WI)}$$

is provable. However, the implication

$$\text{Con(ZF)} \Rightarrow \text{Con(ZF+WI)}$$

cannot be proved in ZF. This resembles the situation where, letting

ZF^{fin} : ZF without the Axiom of Infinity

Inf : Axiom of Infinity

Then

$$\text{Con(ZF)} \Rightarrow \text{Con}(\text{ZF}^{fin} + \neg \text{ Inf})$$

But

$$\text{Con}(\text{ZF}^{fin}) \Rightarrow \text{Con}(\text{ZF}^{fin} + \text{Inf})$$

is not provable in ZF^{fin}.

For this reason, inaccessible cardinals are considered as "infinities of higher level."

As of today: there is a decent collection of known large cardinal axioms. Interestingly, they are all linearly ordered in terms of consistency. WI is the weakest of them.

From now on, work in ZFC.

7.11 Proposition: Let $\gamma \leq \kappa$ be cardinals and let $\langle A_\xi | \xi \leq \gamma \rangle$ by a sequence of sets such that $|A_\xi| \leq \kappa$ for all $\xi < \gamma$. Then

$$|\bigcup_{\xi<\gamma} A_\xi| \leq \kappa.$$

Proof. Since $|A_\xi| \leq \kappa$, there is a surjection $f_\xi : \kappa \to A_\xi$. So $F_\xi = \{f_\xi | f_\xi : \kappa \to A_\xi$ is a surjection $\} \neq \varnothing$.

Notice: $\langle F_\xi | \xi < \gamma \rangle$ is a set. By AC there is a sequence $\langle f_\xi | \xi < \gamma \rangle$ such that $f_\xi \in F_\xi$, i.e. $f_\xi : \kappa \to A_\xi$ is a surjection.

Now define a function $g : \gamma \times \kappa \to \bigcup_{\xi<\gamma} A_\xi$ by

$$g(\xi, \eta) = f_\xi(\eta).$$

Since each f_ξ is a surjection onto A_ξ, we have: $g : \gamma \times \kappa \to \bigcup_{\xi<\gamma} A_\xi$ is a surjection.

Since $\gamma \leq \kappa$, we have $|\gamma \times \kappa| = \kappa$. Hence there is a surjection of κ onto $\bigcup_{\xi<\gamma} A_\xi$. □

7.12 Remark:

The previous proof can be "localized"in terms of hypothesis.

Let A be a class, κ be an infinite cardinal. Then

$$\mathrm{AC}_\kappa(A)$$

is the following statement:

Whenever $\langle X_\xi | \xi < \kappa \rangle$ is a sequence of nonempty subsets of A, there is a sequence $\langle a_\xi | \xi < \kappa \rangle$ of elements a such that $a_\xi \in X_\xi$ for all $\xi < \kappa$.

Also, let AC_κ stand for $\mathrm{AC}_\kappa(V)$.

It is easy to see that the conclusion in T.7.11 follows from AC_κ.

7.13 Proposition:

For every infinite cardinal κ: we have κ^+ is regular.

Proof. Suppose not then $\gamma \overset{def}{=} \mathrm{cf}(\kappa^+) < \kappa^+$. Since there are no cardinals in the interval (κ, κ^+) and $\mathrm{cf}(\kappa^+)$ is a cardinal: $\gamma \leq \kappa$. By P.7.5 there is a cofinal function $f : \gamma \to \kappa^+$. Now each $f(\xi) < \kappa^+$, so $|f(\xi)| \leq \kappa$. But then

$$\kappa^+ = \bigcup_{\xi<\gamma} f(\xi)$$

By P.7.11, $\bigcup_{\xi<\gamma} f(\xi)$ is of size $\leq \kappa$. This gives $|\kappa^+| \leq \kappa$. Contradiction. □

7.14 Remark:

The conclusion in P.7.13 can be proved from $\mathrm{AC}_\kappa(\mathcal{P}(\kappa))$.

7.15 Example:

(i) By the above: $\omega_1, \omega_2, ..., \omega_n$ are all regular.

(ii) There is no sequence $\langle \alpha_n | n \in \omega \rangle$ that would converge to ω_1.

(iii) There are unboundedly many $\alpha < \omega_2$ such that $\operatorname{cf}(\alpha) = \omega_1$.

Given $\beta < \omega_2$, we find $\alpha < \omega_2$ such that $\alpha \geq \beta$ and $\operatorname{cf}(\alpha) = \omega_1$.

By recursion define a function $f : \omega_1 \to \omega_2$ as follows:

$f(0) = \beta$

$f(\xi + 1) =$ some ordinal $< \omega_2$ that is larger than $f(\xi)$.

$f(\xi) = \sup\{f(\bar{\xi} | \bar{\xi} < \xi\}$ if ξ is a limit.

Note: $f(\xi)$ is defined for all $\xi < \omega_1$:

- If $f(\xi)$ is defined then $f(\xi) < \omega_2$ and ω_2 is a limit ordinal. So this is an ordinal in the interval $(f(\xi), \omega_2)$.
- If ξ is limit and $f(\bar{\xi})$ is defined for all $\bar{\xi} < \xi$, i.e. $f \restriction \xi$ is defined. Now $f \restriction \xi$ cannot be cofinal in ω_2 because $\xi < \omega_1 < \omega_2$ and $\operatorname{cf}(\omega_2) = \omega_2$. Hence $\sup\{f(\bar{\xi} | \bar{\xi} < \xi\} < \omega_2$. This sup is the value of $f(\xi)$.

  Now let $\alpha = \sup_{\xi < \omega_1} f(\xi)$. Because $\operatorname{cf}(\omega_2) < \omega_2$: $\alpha < \omega_2$.

  Moreover: $f : \omega_1 \to \alpha$ is a strictly increasing cofinal map. So by 7.9, $\operatorname{cf}(\alpha) = \operatorname{cf}(\omega_1) = \omega_1$.

(iv) Also notice: If $\beta < \omega_2$ then $\beta + \omega_1 < \omega_2$ and $(\beta + \omega_1) = \omega_1$. I.e. the map $\xi \mapsto \beta + \xi$. (Or, in (iii) take "the least" instead of "some").

7.16 Proposition:

From the above we have the following formulae:

(a) If $\kappa, \lambda \in \omega$ then ordinal arithmetic = cardinal arithmetic.

From now on, let at least one of κ, λ be infinite.

(b) $\kappa + 0 = \kappa$

$\kappa \cdot 0 = 0$

If both $\kappa, \lambda > 0$ then

$\kappa + \lambda = \kappa \cdot \lambda = \max(\kappa, \lambda)$

(c) All cardinals:

$\kappa^{\lambda + \mu} = \kappa^\lambda \cdot \kappa^\mu$

$(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$

$(\kappa^\lambda)^\mu = \kappa^{\lambda \cdot \mu}$

(d) If $\kappa \geq 2$ then $\kappa^\lambda > \lambda$ (Cantor Theorem).

(e) $\kappa \leq \kappa'$ and $\lambda \leq \lambda' \Rightarrow \kappa^\lambda \leq {\kappa'}^{\lambda'}$ and similarly for $+$ and $\cdot$.

7.17 Definition:
Let $\lambda \leq \kappa$ be cardinals.

$$[\kappa]^\lambda = \{\eta \subseteq \kappa : |\eta| = \lambda\} \text{ (= the set of all subsets of } \kappa \text{ that have size } \lambda).$$

The notation $[\kappa]^{<\lambda}$, $[\kappa]^{\leq\lambda}$ is then self-explanatory, as well as $[A]^\lambda, [A]^{<\lambda}, [A]^{\leq\lambda}$.

7.18 Proposition:
Let $\lambda \leq \kappa$ be cardinals and κ be infinite. Then

$$\kappa^\lambda = |[\kappa]^\lambda|$$

Proof: application of Schroëder-Bernstein (HW).

7.19 Definition:
Let $I \neq \varnothing$ and $\langle \kappa_i | i \in I\rangle$ be an indexed system of cardinals.

$$\text{(a) } \sum_{i\in I} \kappa_i = \big|\bigcup_{i\in I}(\{i\} \times \kappa_i)\big|$$

$$\text{(b) } \prod_{i\in I} \kappa_i = \big|\bigtimes_{i\in I} \kappa_i\big|$$

7.20 Remark:

(a) These definitions are consistent with the definitions of $+$ and $\cdot$ if we let $I = \{0, 1\}$.

(b) Assume all κ_i agree, say $\kappa_i = \kappa$ for all $i \in I$. Then

$$\sum_{i\in I} \kappa_i = \big|\bigcup_{i\in I}(\{i\} \times \kappa_i)\big| = \big|\bigcup_{i\in I}\{i\} \times \kappa\big| = |I \times \kappa| = |I| \cdot \kappa.$$
$$\prod_{i\in I} \kappa_i = \big|\bigtimes_{i\in I} \kappa_i\big| = |\kappa^I| = \kappa^{|I|}.$$

7.21 Proposition: Let $\langle \kappa_i | i \in I\rangle$ be an indexed system of cardinals, at least one κ_i be infinite. Then

$$\sum_{i\in I} \kappa_i = |I| \cdot \sup_{i\in I} \kappa_i.$$

Proof. Let $\kappa = \sup_{i\in I} \kappa_i$. So $\kappa_i \leq \kappa$ for all $i \in I$. Let $f_i : \kappa_i \to \kappa$. Then the map

$$f : \bigcup_{i \in I} \{i\} \times \kappa_i \to I \times \kappa$$

defined by

$$f(i, \xi) = (i, \xi)$$

is an injection. So $\sum_{i \in I} \kappa_i \leq |I| \cdot \kappa$. $\square$

For the converse, notice WLOG we may assume $\kappa_i > 0$ for all $i \in I$. (Check this!).

Then: $\kappa_j \leq \sum_{i \in I} \kappa_i$ for all $j \in I$. Hence

$$(*) \ \sup_{j \in I} \kappa_j \leq \sum_{i \in I} \kappa_i$$

Since we are assuming that $\kappa_i > 0$ for all $i \in I$:

$$(**) \ |I| = \Big| \bigcup_{i \in I} (\{i\} \times \{0\}) \Big| \leq \Big| \bigcup_{i \in I} \{i\} \times \kappa_i \Big| = \sum_{i \in I} \kappa_i$$

$1 = \{0\} \leq \kappa_i$.

From (*) and (**) we get

$$\sup_{j \in I} \kappa_j \cdot |I| \leq \Big(\sum_{i \in I} \kappa_i \Big) \cdot \Big(\sum_{i \in I} \kappa_i \Big) = \sum_{i \in I} \kappa_i.$$

Where the last step is justified because $(\sum_{i \in I} \kappa_i)$ is infinite, because we are assuming that at least one of κ_i is infinite.

7.22 Proposition:

Let λ be an infinite cardinal and $\langle \kappa_\xi | \xi < \lambda \rangle$ be an increasing sequence of cardinals such that $\kappa_0 > 0$. Then the product

$$\prod_{\xi < \lambda} \kappa_\xi = (\sup_{\xi < \lambda} \kappa_\xi)^\lambda$$

Proof. Let $\kappa = \sup_{\xi < \lambda} \kappa_\xi$. Assume WLOG that $\langle \kappa_\xi | \xi < \lambda \rangle$ is strictly increasing; for non-strictly increasing , it then easily follows. (Either pick a strictly increasing subsequence of else consider the largest element).

$\leq$ is easy: $\prod_{\xi < \lambda} \kappa_\xi \leq \prod_{\xi < \lambda} \kappa = \kappa^\lambda$, where the $\leq$ follows from the argument similar to the proof of $\leq$ in P.7.21.

To see $\geq$: Since λ is infinite: $\lambda \sim \lambda \times \lambda$. So we can re-index the system $\langle \kappa_\xi | \xi < \lambda \rangle$ and get $\langle \kappa'_{\xi,\eta} | \xi, \eta < \lambda \rangle$ so that $\{\kappa_\xi | \xi < \lambda\} = \{\kappa'_{\xi,\eta} | \xi, \eta < \lambda\}$.

Notice: If we fix $\xi = \xi_0$ then the set

$$\{\kappa'_{\xi_0,\eta} | \eta < \lambda\}$$

is cofinal in $\kappa = \sup_{\xi<\lambda} \kappa_\xi$. Why: Because A_{ξ_0} has size λ but for every $\alpha < \kappa$ the set $\{\kappa_\xi | \kappa_\xi < \alpha\}$ has size $< \lambda$: because let $\xi^* =$ the least ξ such that $\kappa_\xi \geq \alpha$. Then

$$\{\kappa_\xi | \kappa_\xi < \alpha\} = \{\kappa_\xi | \xi < \xi^*\} \text{ and } \xi^* < \lambda.$$

Point: if B is a well-ordered set whose order type is a cardinal, and B' is a proper initial segment of B then $|B'| < |B|$.

So: $\prod_{\xi<\lambda} \kappa_\xi = \prod_{\xi,\eta<\lambda} \kappa'_{\xi,\eta} = \prod_{\xi<\lambda} \left(\prod_{\eta<\lambda} \kappa'_{\xi,\eta}\right)$ which is $\geq \kappa$ by the "Notice"above. Then, that is $\geq \prod_{\xi<\lambda} \kappa = \kappa^\lambda$. □

7.23 Proposition:

Let λ be a cardinal and $\langle \kappa_\xi | \xi < \gamma \rangle$ be a sequence of cardinals. Then

$$\Big(\prod_{\eta<\gamma} \kappa_\eta\Big)^\lambda = \prod_{\eta<\gamma} \kappa_\eta^\lambda$$

Proof. We construct a bijection directly. Let $f \in^\lambda \left(\bigtimes_{\eta<\gamma} \kappa_\eta\right)$, i.e. $f : \lambda \to \bigtimes_{\eta<\gamma} \kappa_\eta$.

We define

$$F_f : \lambda \times \gamma \to O_n \text{ by } F_f(\xi, \eta) = f(\xi)(\eta).$$

Now let

$$g_{f,\eta} : \lambda \to O_n \text{ defined by } g_{f,\eta}(\xi) = F_f(\xi, \eta) = f(\xi)(\eta) \in \kappa_\eta$$

(We are fixing the first argument, then we are fixing the second argument.)

So $g_{f,\eta} \in ({}^\lambda\kappa_\eta)$. Now let

$$g_f : \gamma \to V \text{ defined by } g_f(\eta) = g_{f,\eta} \in ({}^\lambda\kappa_\eta)$$

So $g_f \in \bigtimes_{\eta<\gamma} \kappa_\eta^\lambda$

We defined a function

$$\text{from } {}^\lambda\Big(\bigtimes_{\eta<\gamma} \kappa_\eta\Big) \text{ into } \bigtimes_{\eta<\gamma} \kappa_\eta^\lambda \text{ by } f \mapsto g_f.$$

Injectivity: Assume $f \neq f'$. Then there is some $\xi < \lambda$ such that $f(\xi) \neq f'(\xi)$. Hence there is some $\eta < \gamma$ such that $f(\xi)(\eta) \neq f'(\xi)(\eta) \Rightarrow g_f(\eta)(\xi) \neq g'_f(\eta)(\xi)$. Hence, $g_f(\eta) \neq g'_f(\eta)$, i.e. $g_f \neq g'_f$.

Surjectivity: Switch the coordinates back. Given $g \in \bigtimes_{\eta<\gamma} ({}^\lambda\kappa_\eta)$. Let $f : \lambda \to \bigtimes_{\eta<\gamma} \kappa_\xi$ be defined by $f(\xi)(\eta) = g(\eta)(\xi)$. Then $g_f = g$.

□

7.24 Proposition:

Let $\langle \kappa_\xi | \xi < \lambda \rangle$ be an increasing sequence of cardinals.

Then, letting $\kappa = \sup_{\xi<\lambda} \kappa_\xi$:

$$\kappa^\lambda = \prod_{\xi<\lambda} \kappa_\xi^\lambda$$

Proof. $\prod_{\xi<\lambda} \kappa_\xi^\lambda \overset{7.23}{=} \big(\prod_{\xi<\lambda} \kappa_\xi\big)^\lambda \overset{7.22}{=} (\kappa^\lambda)^\lambda = \kappa^\lambda$

□

7.25 Definition: ℷ

The function ℷ (third hebrew letter of alphabet) is defined by:

$$\kappa \mapsto \kappa^{cf(\kappa)}$$

The function ℷ is fundamental in cardinal arithmetic, because it completely determines the value of $\aleph_\alpha^{\aleph_\beta}$.

Before we prove the theorem, one more proposition:

7.26 Proposition: Let λ be a limit cardinal, say $\lambda = \sup_{\xi<\gamma} \lambda_\xi$ where $\langle \lambda_\xi | \xi < \lambda \rangle$ is strictly increasing and $\langle 2^{\lambda_\xi} | \xi < \gamma \rangle$ is increasing, but not eventually constant.

$$2^\lambda = \gimel\big(\sup_{\xi<\gamma} 2^{\lambda_\xi}\big)$$

Proof. $\geq$: $\sup_{\xi<\gamma} 2^{\lambda_\xi} \leq 2^\lambda$

$\mathrm{cf}(\sup_{\xi<\gamma} 2^{\lambda_\xi}) \leq \gamma$. This is true if $\langle 2^{\lambda_\xi} | \xi < \gamma \rangle$ is not eventually constant because $\xi \mapsto 2^{\lambda_\xi}$ is a cofinal map from γ into $\sup_{\xi<\gamma} 2^{\lambda_\xi}$ by 7.6.

So $\gimel(\sup_{\xi<\gamma} 2^{\lambda_\xi}) \leq (2^\lambda)^\gamma = 2^\lambda$

$\leq$: We inject $\mathcal{P}(\lambda)$ into $\bigtimes_{\xi<\gamma} \mathcal{P}(\lambda_\xi)$ by

$$A \mapsto \langle A \cap \lambda_\xi | \xi < \gamma \rangle$$

Easy to see this is an injection.

But $\big|\bigtimes_{\xi<\gamma} \mathcal{P}(\lambda_\xi)\big| = \prod_{\xi<\gamma} 2^{\lambda_\xi}$.

Let $K \leq \gamma$ be such that $\xi \mapsto 2^{\lambda_\xi}$ is strictly increasing on K. Also, choose such a K with smallest possible order-type. Then the above argument shows that the assignment

$$\langle A \cap \lambda_\xi | \xi \in K \rangle$$

is injective as well. But then

$$\big|\bigtimes_{\xi\in K} \mathcal{P}(\lambda_\xi)\big| = \prod_{\xi\in K} 2^{\lambda_\xi} \overset{7.22}{=} \big(\sup_{\xi\in K} 2^{\lambda_\xi}\big)^{|K|} \overset{(1)}{=} \gimel\big(\sup_{\xi\in K} 2^{\lambda_\xi}\big)$$

(1) is because $|K| = \mathrm{otp}(K) = \mathrm{cf}(\sup_{\xi\in K} 2^{\lambda_\xi})$. □

7.27 Theorem:

The value κ^λ are completely determined by function $\beth$.

Proof. By induction on κ, we show that the values κ^λ for all $\lambda \leq \kappa$ are determined by $\beth$. This sufices, as if $\lambda > \kappa$ then $\kappa^\lambda = \lambda^\lambda$.

For $\kappa \geq 2$:

$$2^\lambda \leq \kappa^\lambda \leq \lambda^\lambda \leq (2^\lambda)^\lambda = 2^{\lambda\cdot\lambda} = 2^\lambda$$

Induction: Assume $\lambda \leq \kappa$ and $\kappa^{\lambda'}$ is determined by $\beth$ for all $\lambda' < \lambda$. Want to compute κ^λ.

Claim 1: $\lambda < \mathrm{cf}(\kappa)$.

$$\kappa^\lambda = |{}^\lambda\kappa| \overset{(1)}{=} \Big|\bigcup_{\lambda<\alpha<\kappa}({}^\lambda\alpha)\Big| \overset{(2)}{=} \sup_{\alpha<\kappa}|\alpha|^\lambda$$

(1) Since $\lambda < \mathrm{cf}(\kappa)$: Every $f : \lambda \to \kappa$ is bounded.

(2) Note: $\alpha < \beta$ so $({}^\lambda\alpha) \subsetneq ({}^\lambda\beta)$.

The rest follows from the fact: If $\langle X_\alpha|\alpha < \kappa\rangle$ $(X_0 \neq \varnothing)$ is an increasing sequence with respect to $\subseteq$ sequence of sets, then $|\bigcup_{\alpha<\kappa}(X_\alpha) = \kappa \cdot \sup_{\alpha<\kappa}|X_\alpha|$ since

$$|X_\beta| \leq \Big|\bigcup_{\alpha} < \kappa\Big| \qquad\qquad \kappa \leq \Big|\bigcup_{\alpha<\kappa} X_\alpha\Big| \tag{1}$$

$\Rightarrow \sup_{\beta<\kappa}|X_\beta| \leq \big|\bigcup_{\alpha<\kappa} X_\alpha\big|$.

Converse: $\bigcup_{\xi<\kappa} X_\xi = \bigcup_{\xi<\kappa}(X_{\xi+1} - X_\xi) = \ldots$

Where the above is a disjoint union.

Or else: notice

$$\Big|\bigcup_{\xi<\kappa} X_\xi\Big| \leq \Big|\bigcup_{\xi<\kappa}\{\xi\} \times X_\xi\Big| = \kappa \cdot \sup_{\xi<\kappa}|X_\xi|.$$

Now since $\lambda < \alpha$: $\lambda \leq |\alpha|$ so we already know that $|\alpha|^\lambda$ is determined by $\beth$ by the induction hypothesis, as $|\alpha| < \kappa$. This means that

$$\kappa^\lambda = \sup_{\lambda<\alpha<\kappa}|\alpha|^\lambda$$

Is determined by $\beth$.

Case 2: $\mathrm{cf}(\kappa) \leq \lambda < \kappa$.

Hence $\mathrm{cf}(\kappa) < \kappa$ so κ is a singular cardinal. Let $\langle \kappa_\xi | \xi < \mathrm{cf}(\kappa) \rangle$ be a strictly increasing sequence of cardinals converging to κ.

Case 2A: The values κ_ξ^λ for $\xi < \mathrm{cf}(\kappa)$ are eventually constant. Then

$$\kappa^\lambda \overset{7.24}{=} \prod_{\xi < cf\kappa} \kappa_\xi^\lambda \overset{(1)}{=} \prod_{\xi_0 < \xi < cf\kappa} \kappa_\xi^\lambda = (\kappa_{\xi_0}^\lambda)^{cf(\kappa)} = \kappa_{\xi_0}^{\lambda \cdot cf(\kappa)} = \kappa_{\xi_0}^\lambda$$

(1) ξ_0 is such that $\kappa_\xi^\lambda = \kappa_{\xi_0}^\lambda$ for all $\xi > \xi_0$. Use Schroëder-Bernstein.
Now pick ξ_0 so that $\lambda < \kappa_{\xi_0}$. This is possible, as $\lambda < \kappa$.
So we have $\kappa^\lambda = \kappa_{\xi_0}^\lambda$ and this value is determined by $\gimel$ by the induction hypothesis.

Case 2B: Otherwise. In this case we can pick the sequence $\langle \kappa_\xi | \xi < \mathrm{cf}(\kappa) \rangle$ so that the values of κ_ξ^λ are strictly increasing. So we get

$$\begin{aligned} \kappa^\lambda &\overset{7.24}{=} \prod_{\xi < cf(\kappa)} \kappa_\xi^\lambda \\ &\overset{7.22}{=} \big(\sup_{\xi < cf(\kappa)} \kappa_\xi^\lambda \big)^{cf(\kappa)} \\ &= \big(\sup_{\xi < cf(\kappa)} \kappa_\xi^\lambda \big)^{cf(\sup_{\xi < cf(\kappa)} \kappa_\xi^\lambda} = \gimel(\sup_{\xi < cf(\kappa)} \kappa_\xi^\lambda) \end{aligned}$$

Notice the function $\xi \mapsto \kappa_\xi^\lambda$ is a strictly increasing function from $\mathrm{cf}(\kappa)$ into $\sup_{\xi < cf(\kappa)} \kappa_\xi^\lambda$. By 7.9: $\mathrm{cf}(\sup_{\xi < cf(\kappa)} \kappa_\xi^\lambda) = \mathrm{cf}(\mathrm{cf}(\kappa))$ which is the same as $\mathrm{cf}(\kappa)$.

Case 3: $\lambda = \kappa$

Case 3A: κ is regular. Then $\mathrm{cf}(\kappa) = \kappa$. So $\kappa^\kappa = \kappa^{cf(\kappa)} = \gimel(\kappa)$.

Case 3B: κ is singular.

$$\kappa^\kappa = 2^\kappa = \gimel\big(\sup_{\mu < \kappa} 2^\mu\big)$$

Where μ^μ is known by the induction hypothesis.

□

7.28 Definition: Strong Limit Cardinal

A cardinal κ is *strong limit* iff

$$2^\mu < \kappa \text{ for all } \mu < \kappa.$$

7.29 Proposition:

If κ is strong limit singular cardinal then

$$2^\kappa = \beth(\kappa)$$

Proof. $2^\kappa \overset{7.26}{=} \beth(\sup_{\mu<\kappa} 2^\mu) = \beth(\kappa)$

Where the middle supremum is equal to κ since $\mu < 2^\mu\kappa$ (κ is strong limit). $\square$

7.30 Definition: Strongly Inaccessible Cardinal

A cardinal that is that is both strong limit and regular is called *strongly inaccessible*. So in particular:

$$\kappa \text{ strongly inaccessible} \Rightarrow \kappa \text{ weakly inaccessible.}$$

Hence the existence of strongly inaccessible cardinals cannot be proved in ZFC, and actually the statement

$$\mathrm{Con(ZF+SI)}$$

cannot be proved in ZFC (where SI="There is a strongly inaccessible cardinal").

On the other hand, the following is provable:

$$\mathrm{Con(ZFC+WI)} \Leftrightarrow \mathrm{Con(ZFC+SI)}$$

7.31 Proposition:

Let $I \neq \varnothing$ be a nonempty index set and $\kappa_i \leq \lambda_i$, $i \in I$ be cardinals such that $\lambda_i > 1$. Then

$$\sum_{i\in I} \kappa_i \leq \prod_{i\in I} \lambda_i$$

7.32 Theorem: (König's Inequality)

Let I be a nonempty index set $\kappa_i \leq \lambda_i$, $i \in I$ be cardinals. Then

$$\sum_{i\in I} \kappa_i < \prod_{i\in I} \lambda_i.$$

Proof. Notice: WLOG $\kappa_i > 0$ for all $i \in I$. Then $\lambda_i > \kappa_i \geq 1$. So 7.31 applies and we get $\leq$. We want to see that $<$ holds, i.e.:
There is no surjection $F : \bigcup_{i\in I}(\{i\} \times \kappa_i) \to \bigtimes_{i\in I} \lambda_i$. So let

$$F : \bigcup_{i\in I}(\{i\} \times \kappa_i) \to \bigtimes_{i\in I} \lambda_i.$$

We find a function $f \in \bigtimes_{i\in I} \lambda_i$ that is not in $\mathrm{rng}(F)$. This is a diagonal argument. To each $(i, \xi) \in \bigcup_{i\in I}(\{i\} \times \kappa_i)$, $F(i, \xi)$ is a function in $\bigtimes_{i\in I} \lambda_i$. If we fix $i \in I$ then

$$A_i = \{F(i, \xi)(i) | \xi \in \kappa_i\} \text{ is of size} \leq \kappa_i.$$

(Because $F(i,\xi)(i)$ depends only on ξ and there are κ_i many ξs).

By the assumption that $\kappa_i < \lambda_i$: $\lambda_i - A_i \neq \varnothing$. So if $f \in \bigtimes_{i\in I} \lambda_i$ such that $f(i) \in \lambda_i - A_i$ then $f \neq F(i,\xi)$ for all $\xi \in \kappa_i$. So let $f \in \bigtimes_{i\in I} \lambda_i$ be defined by

$$f(i) = \min(\lambda_i - A_i).$$

Then $f(i) \neq F(i,\xi)(i)$ for all $i \in I$ and all $\xi \in \kappa_i$, so $f \neq F(i,\xi)$ for all $i \in I$ and all $\xi \in \kappa_i$.

This means: $f \notin \mathrm{rng}(F)$. $\square$

7.33 Proposition:

(a) Let κ be a cardinal. Notice:

$$1 < 2$$
$$\underbrace{1+1+...+1}_{\kappa} < \underbrace{2\cdot 2...\cdot 2}_{\kappa}$$
$$\text{So } \kappa < 2^\kappa$$

(Cantor)

(b) $\kappa < \mathrm{cf}(2^\kappa)$ (tells us that you cannot split the continuum into countably many sets).

If κ is regular, this follows from (a). Now assume that κ is singular. Let $\gamma = \mathrm{cf}(2^\kappa)$. Let $\langle \kappa_\xi | \xi < \gamma\rangle$ be a strictly ascending sequence of cardinals converging to 2^κ. Assume $\gamma \leq \kappa$.

$$2^\kappa (\overset{7.21}{=} \gamma \cdot \sup_{\xi<\gamma} \kappa_\xi) = \textstyle\sum_{\xi<\gamma} \kappa_x i < \prod_{\xi<\gamma} 2^\kappa = (2^\kappa)^\gamma = 2^{\kappa\cdot\gamma} = 2^\kappa.$$

So $2^\kappa < 2^\kappa$, a contradiction.

(c) $\kappa < \beth(\kappa)$, κ infinite.

Again, if κ is regular, this follows from (a). If κ is singular, then pick a strictly increasing sequence $\langle \kappa_\xi | \xi < \mathrm{cf}(\kappa)\rangle$ converging to κ. Then

$$\kappa \overset{7.21}{=} \sum_{\xi<cf(\kappa)} \kappa_\xi \overset{Konig}{<} \prod_{\xi<cf(\kappa)} \kappa = \kappa^{cf(\kappa)} = \beth(\kappa).$$

Since (b)$\Rightarrow$ (a) in 7.33, we can summarize the known facts about the function $\kappa \mapsto 2^\kappa$ as follows:

(i) $\kappa \leq \lambda \Rightarrow 2^\kappa \leq 2^\lambda$

(ii) $\kappa < \mathrm{cf}(2^\kappa)$.

For regular cardinals, this is all one can prove in ZFC, by a theorem of Easton.

Easton (1970): Assume E is a class of ordinals and

$$f : E \to O_n$$

is a function satisfying for each $\alpha, \beta \in E$

(i) $\alpha \leq \beta \to \aleph_{f(\alpha)} \leq \aleph_{f(\beta)}$

(ii) $\mathrm{cf}(\aleph_{f(\alpha)}) > \aleph_\alpha$

If $\aleph_\alpha$ is regular for all $\alpha \in E$ then the statement

$$(\forall \alpha \in E)(\aleph_{f(\alpha)} = 2^{\aleph_\alpha})$$

is consistent with ZFC.

However, this is not the case for singular cardinals.

Silver (1974): If κ is a singular cardinal with uncountable cofinality and $2^\mu = \mu^+$ for all (acually only "many") $\mu < \kappa$ then $2^\kappa = \kappa^+$.

Jensen (1974): If κ is a singular strong limit cardinal and $2^\kappa > 2^{\kappa^+}$ then Con(ZFC+SI).

(He actually proved: there is an inner model L and a nontrivial elementary embedding $J : L \to L$).

If $\varphi(v_1, ..., v_l)$ is a formula, then for any $a_1, ..., a_l$

$$L| = \varphi(a_1, ..., a_l) \Leftrightarrow L| = \varphi(j(a_1), ..., j(a_l)).$$

7.34 Definition: Singular Cardinal Hypothesis (SCH)

SCH is the following statement: for a singular cardinal κ:

$$\mathrm{SCH}_\kappa\text{: } \daleth(\kappa) = \kappa^+.$$

We saw that if κ is singular strong limit then $\daleth(\kappa) = 2^\kappa$ so for singular strong limit κ:

$$\mathrm{SCH}_\kappa \Leftrightarrow 2^\kappa = \kappa^+ \equiv \mathrm{GCH}_\kappa$$

SCH is one of the major topics in set theory with many questions open.

Milestones:

κ singular strong limit and $2^\kappa > \kappa^+ \Rightarrow \mathrm{Con}(\mathrm{ZFC}+\ o(\kappa) = \kappa^{++})$

(Mitchell 1980s)

And $\Leftarrow$ by Magidor; early 1970s.

Where $o(\kappa) = \kappa^{++}$ is a very strong large cardinal axiom; much larger than SI.

Another milestone: Galvin-Hajval-Shelah?

$$2^{\aleph_0} < \aleph_\omega \Rightarrow \beth(\aleph_\omega) < \aleph_{\omega_4} \text{ in ZFC}$$

Major open question:

$$2^{\aleph_0} < \aleph_\omega \overset{?}{\Rightarrow} \beth(\aleph_\omega) < \aleph_{\omega_1}$$

8. Boolean Algebras, Filters and Ideals

8.1 Definition: Boolean Algebra

A *Boolean Algebra* is a structure $\mathbb{B} = (B, \wedge, \vee, ', 0, 1)$ where $\vee, \wedge$ are binary operations, $'$ is a unary operation and $0, 1 \in B$.

And the following equalities hold:

Commutativity:

$$x \vee y = y \vee x; \; x \wedge y = y \wedge x$$

Associativity:

$$(x \vee y) \vee z = x \vee (y \vee z); \; (x \wedge y) \wedge z = x \wedge (y \wedge z)$$

Absorbtion:

$$x \wedge (x \vee y) = x; \; x \vee (x \wedge y) = x$$
$$x \wedge x = x; \; x \vee x = x$$

Distributivity:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z); \; x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

Complementation:

$$x \wedge x' = 0; \; x \vee x' = 1$$

8.2 Example:

Given any set A we have the power set algebra $(\mathcal{P}(A), \cap, \cup, ', \varnothing, A)$.

8.3 Proposition:

(i) $x \wedge 0 = 0$, $x \vee 0 = x$, $x \wedge 1 = x$, $x \vee 1 = 1$.

(ii) x' is unique.

(iii) $(x \wedge y)' = x' \vee y'$, $(x \vee y)' = x' \wedge y'$.

Proof: Exercise. Regarding (iii): Show $(x \wedge y) \wedge (x' \vee y') = 0$ and $(x \wedge y) \vee (x' \vee y') = 1$.

8.4 Proposition:
If $\mathbb{B}$ is a Boolean Algebra we define a binary relation $\leq$ on it by

$$x \leq y \text{ iff } x \wedge y = x.$$

Then

(i) $\leq$ is a partial ordering on $\mathbb{B}$.

(ii) 0 is the least element and 1 is the largest element.

(iii) $x \leq y$ iff $x \vee y = y$.

Proof. To see (i): $x \wedge x = x$, so $x \leq x$.
If $x \leq y$ and $y \leq x$ then $x \wedge y = x$, $x \wedge y = y$. This implies $x = y$.
$x \leq y$ and $y \leq z$ then $x \wedge z = (x \wedge y) \wedge z = x \wedge (y \wedge z) = x \wedge y = x$. □

In the above proposition, $x \vee y$ is the sup of x, y, i.e. the least upper bound. That is,

$$z = x \vee y \text{ iff } x \leq z, y \leq z \text{ and if } z' \text{ is such that } x, y \leq z' \text{ then } z \leq z'.$$

Dually: $x \wedge y$ is the infimum of x, y, i.e. the greatest lower bound.

8.5 Definition:
Let $\mathbb{B}$ be a Boolean Algebra and $X \subseteq B$.
$\bigvee X$ is the supremum (join) of X, if it exists. In general, $\bigvee X$ need not exist.
Similarly, $\bigwedge X$ is the infimum (meet) of X if exists.

So:

$$\bigvee X = \text{the least upper bound on } X$$
$$\bigwedge X = \text{the greatest lower bound on } X.$$

8.6 Exercise:
Let $\mathbb{B}$ be a Boolean Algebra and $X \subseteq B$. If $\bigvee X$ exists and $b \in B$ then also

$$\bigvee_{x \in X} (b \wedge x) \text{ holds and}$$
$$b \wedge \left(\bigvee X\right) = \bigvee_{x \in X} (b \wedge x).$$

Similarly with $\bigwedge X$.

8.7 Definition: κ-Complete

Let $\mathbb{B}$ be a Boolean Algebra and κ be a cardinal. $\mathbb{B}$ is *κ-complete* iff $\bigwedge X$, $\bigvee X$ exist for any $X \subseteq B$ such that $|X| < \kappa$.

(So by definition, every $\mathbb{B}$ is ω-complete). (By exercise 8.6 + DeMorgan it would suffice to postulate just the existence of $\bigwedge X$ or just $\bigvee X$).

8.8 Definition:

Let $\mathbb{B}$ be a Boolean Algebra.

(a) A *filter* on $\mathbb{B}$ is a set $F \subseteq B$ such that

  (i) $1 \in F$ and $0 \notin F$.

  (ii) $x, y \in F \Rightarrow x \wedge y \in F$.

  (iii) $(x \in F$ and $x \leq y) \Rightarrow y \in F$

(b) An *ideal* on $\mathbb{B}$ is a set $I \subseteq B$ such that:

  (i) $0 \in I$ and $1 \notin I$.

  (ii) $x, y \in I \Rightarrow x \vee y \in I$.

  (iii) $(x \in I$ and $y \leq x) \Rightarrow y \in I$.

A filter F on $\mathbb{B}$ is

(1) *Principal* iff there is some $a \in B$ such that

$$F = \{x \in B | a \leq x\}$$

(2) *Maximal* iff there is no filter F' on $\mathbb{B}$ such that $F' \supsetneq F$.

For ideals, we define these notions dually.

Intuition: Members of a filter are “large”while members of ideals are “small”.

Given a filter F on $\mathbb{B}$, we let

$$\breve{F} = \{x' | x \in F\}$$

Easy to check: $\breve{F}$ is an ideal on $\mathbb{B}$. This ideal is called the *ideal dual to F*.

Dually, if I is an ideal on $\mathbb{B}$ then

$$\breve{I} = \{x' | x \in I\}$$

is the *filter dual to I*.

In particular, we have

$$\breve{\breve{F}} = F \text{ and } \breve{\breve{I}} = I$$

If I is an ideal on $\mathbb{B}$ then

$$I^+ = B - I$$

The elements of I^+ are called *I-positive*.

If F is a filter on $\mathbb{B}$ then $F^+ \stackrel{def}{=} (\breve{F})^+$.

Notice: $a \in F^+$ iff $a \wedge x \neq 0$ for all $x \in F$. (Check this).

8.9 Definition: Ultrafilter

A filter F on $\mathbb{B}$ is an *ultrafilter* iff for every $x \in B$ we have

$$x \in F \text{ or } x' \in F.$$

8.10 Definition: Filter Base

Let $\mathbb{B}$ be a Boolean Algebra. A set $X \subseteq B$ is a *filter base* or a *centered system* iff $\bigwedge X' \neq 0$ for all Finite $X' \subseteq X$.

A base for an ideal is defined dually.

8.11 Proposition:

If $\mathbb{B}$ is a Boolean Algebra and $X \subseteq B$ is a filter base, then

$$F = \{x \in B | \exists \text{ finite } X' \subseteq X \text{ with } \bigwedge X' \leq x\}$$

is a filter; this filter is called the filter generated by X. Dually for ideals.

8.12 Proposition:

Let $\mathbb{B}$ be a Boolean Algebra and F be a filter on $\mathbb{B}$. Then

$$\text{either } F \cup \{a\} \text{ or } F \cup \{a'\} \text{ is a filter base.}$$

Proof is as in Fall quarter with sets.

8.13 Proposition:

Let $\mathbb{B}$ be a Boolean Algebra and F be a filter on $\mathbb{B}$

(i) F is an ultrafilter iff F is maximal.

(ii) F can be extended to an ultrafilter.

Proof: As in Fall.

8.14 Definition: Prime Ideal

An ideal I on a Boolean Algebra $\mathbb{B}$ is a *prime ideal* iff $\breve{I}$ is an ultrafilter. So an ideal is a prime ideal iff it is a maximal ideal.

Notice: if I is a prime ideal, then $I^+ = \breve{I}$.

8.15 Definition + Proposition:

An *atom* in a Boolean Algebra $\mathbb{B}$ is an element $a > 0$ such that for every $x \in B$: $x \leq a \Rightarrow x = a$ or $x = 0$.

An ultrafilter F is principal iff there is an atom a such that

$$F = \{x \in B | a \leq x\}$$

Next come a few remarks to exercise 8.6:

8.16 Proposition:

Let $\mathbb{B}$ be a Boolean Algebra. Then

(i) $(a \leq a^*$ and $b \leq b^*) \Rightarrow a \wedge b \leq a^* \wedge b^*$; $a \vee b \leq a^* \vee b^*$

(ii) $a \leq b \Leftrightarrow a \wedge b' = 0$.

Proof. Toward the proof of 8.6. We assume that $A \subseteq B$ and $\bigvee A$ exists.

Want to show: for every $b \in B$: $\bigvee_{a \in A}(b \wedge a)$ exists and

$$b \wedge \bigvee A = \bigvee_{a \in A}(b \wedge a)$$

$\geq$ is easy, as $b \wedge \bigvee A \geq b \wedge a$ for each $a \in A$ using P.8.16.

$\leq$: Assume x is an upper bound on all $b \wedge a$ where $a \in A$. So

$$b \wedge a \leq x \Rightarrow (x' \wedge b) \wedge a = x' \wedge (b \wedge a) \overset{P.8.16(ii)}{=} 0$$

Since $(x' \wedge b) \wedge a = 0$, by 8.16(ii) we have $a \leq (x' \wedge b)'$ for all $a \in A$.

So $\bigvee A \leq (x' \wedge b)' \overset{8.16}{\Rightarrow} (x' \wedge b) \wedge \bigvee A = 0$ and this tells us that $b \wedge \bigvee A \leq x$.

Summary: $(\forall a \in A)(b \wedge a \leq x) \Rightarrow b \wedge \bigvee A \leq x$.

So: $b \wedge \bigvee A$ is an upper bound on all $b \wedge a$, $a \in A$ and if x is any upper bound on all $b \wedge a$ then $b \wedge \bigvee A \leq x$.

This says:

$$b \wedge \bigvee A = \bigvee_{a \in A}(b \wedge a).$$

This proves 8.6. $\square$

We defined κ-complete Boolean Algebra. Now:

8.17 Definition: Complete Boolean Algebra

A Boolean Algebra is *complete* iff it is κ-complete for all κ.

8.18 Definition:

Let $\mathbb{B}$ be a κ-complete Boolean Algebra and F be a filter on $\mathbb{B}$. We say that F is κ-complete iff for every X we have:

$$(X \subseteq F \text{ and } |X| < \kappa) \Rightarrow \bigwedge X \in F.$$

κ-complete ideal is defined dually.

8.19 Definition:

Let $\mathbb{B}_1$, $\mathbb{B}_2$ be Boolean Algebras and $h : \mathbb{B}_1 \to \mathbb{B}_2$.

(a) The map h is a *homomorphism* iff h preserves the operations (including mapping 0 to 0 and 1 to 1).

(b) An injective homomorphism is called an *embedding*.

(c) If both $\mathbb{B}_1$ and $\mathbb{B}_2$ are κ-complete then a homomorphism $h : \mathbb{B}_1 \to \mathbb{B}_2$ is κ-complete iff h preserves joins and meets of size $< \kappa$.

(d) If $h : \mathbb{B}_1 \to \mathbb{B}_2$ is a homomorphism of complete Boolean Algebras then h is *complete* iff it preserves all joins (and meets).

8.20 Definition:

Let $\mathbb{B}_1$, $\mathbb{B}_2$ be Boolean Algebras with domains B_1, B_2.

(a) We say that $\mathbb{B}_1$ is a subalgebra of $\mathbb{B}_2$ iff $B_1 \subseteq B_2$ and the operations of $\mathbb{B}_1$, $\mathbb{B}_2$ agree on B_1. Equivalently, $\mathbb{B}_1$ is a subalgebra of $\mathbb{B}_2$ iff $id : \mathbb{B}_1 \to \mathbb{B}_2$ is an embedding.

(b) $\mathbb{B}_1$ is a κ-complete (complete) subalgebra of $\mathbb{B}_2$ iff $\mathbb{B}_1$ is a subalgebra of $\mathbb{B}_2$ and the meet and join of size $< \kappa$ (all meets and joins) agree in $\mathbb{B}_1, \mathbb{B}_2$.

Equivalently: iff $id : \mathbb{B}_1 \to \mathbb{B}_2$ is a κ-complete (complete) embedding.

Definition for isomorphism is obvious.

8.21 Notation:

We will write

$$a - b \text{ for } a \wedge b' \text{ (difference)}$$
$$a\Delta b = (a-b) \vee (b-a) \text{ (symmetric difference)}$$

8.22 Definition + Proposition:

Let $\mathbb{B}$ be a Boolean Algebra and I be an ideal on $\mathbb{B}$. We define a binary relation $\sim$ by

$$a \sim b \text{ iff } a\Delta b \in I.$$

Then $\sim$ is an equivalence relation on $\mathbb{B}$. We define the quotient algebra $\mathbb{B}/I$ by

$$\mathbb{B}/I = \text{the set of all equivalence classes } [a] \text{ where } a \in B.$$

The operations are defined in the obvious way:

$$[0]_{\mathbb{B}/I} = [0_B],\ [1]_{\mathbb{B}/I} = [1_B],\ [x] \wedge_{\mathbb{B}/I} [y] = [x \wedge y] ... \text{ etc.}$$

These operations are well-defined and

$$\mathbb{B}/I = (B/I, \vee_{B/I}, \wedge_{B/I}, '_{B/I}, 0_{B/I}, 1_{B/I}) \text{ is a Boolean algebra}$$

(check this). From this we get the quotient map

$$k : \mathbb{B} \to \mathbb{B}/I$$

Defined by $k(a) = [a]$. By what we said above, k is a homomorphism. Moreover

$$\ker(k) = k^{-1}[\{0\}] = I$$

because $a\Delta 0 = (a-0) \vee (0-a) = a$. We also have the usual factor theorem:

If $h : \mathbb{B} \to \mathbb{C}$ is a homomorphism of Boolean Algebras then $\ker(h)$ is an ideal on $\mathbb{B}$ and we have a unique isomorphism i that makes the following diagram commutate:

$$\begin{tikzcd} \mathbb{B} \arrow[rr, "h"] \arrow[dr, "k"'] & & \mathbb{C} \\ & \mathbb{B}/\ker(h) \arrow[ur, "i"'] & \end{tikzcd}$$

8.23 Proposition:

Assume $\mathbb{B}$ is a κ-complete BA and I is a κ-complete ideal on $\mathbb{B}$. Then $\mathbb{B}/I$ is κ-complete.

Proof. If $\gamma < \kappa$ and $\langle x_\xi | \xi < \gamma \rangle$, $\langle y_\xi | \xi < \gamma \rangle$ are such that $[x_\xi] = [y_\xi]$ for all $\xi < \gamma$, then

$$\begin{aligned}
&(\bigvee_{\xi<\gamma} x_\xi) \Delta (\bigvee_{\xi<\gamma} y_\xi) \\
&= (\bigvee_{\xi<\gamma} x_\xi - \bigvee_{\xi<\gamma} y_\xi) \vee (\bigvee_{\xi<\gamma} y_\xi - \bigvee_{\xi<\gamma} x_\xi) \\
&\overset{8.6}{=} \bigvee_{\xi<\gamma} (x_\xi - \bigvee_{\mu<\gamma} y_\mu) \\
&\leq \bigvee_{\xi<\gamma} (x_\xi - y_\mu) \qquad (2) \\
&\in I \qquad (3)
\end{aligned}$$

where (1) comes from $y_\xi \leq \bigvee_{\xi<\gamma} y_\xi$, and (2) since I is κ-complete.

This shows: the definition

$$\bigvee_{\xi<\gamma} [x_\xi] = [\bigvee_{\xi<\gamma} x_\xi]$$

is meaningful.

□

8.24 Definitions:

Let $\mathbb{B}$ be a BA.

(a) Two elements $a, b \in \mathbb{B}$ are *incompatible* iff $a \wedge b = 0$ for $a, b \neq 0$.

(b) $X \subseteq B$ is an *antichain* on $\mathbb{B}$ iff X consists of pairwise incompatible elements.

(c) The algebra $\mathbb{B}$ is κ*-saturated* (or κ-c.c.) iff every antichain in $\mathbb{B}$ is of size $< \kappa$.

The smallest cardinal κ such that $\mathbb{B}$ is κ-saturated is called the *saturation* of $\mathbb{B}$, $\mathrm{sat}(\mathbb{B})$.

It can be shown that $\mathrm{sat}(\mathbb{B})$ is always a regular cardinal.

(d) Let I be an ideal. Elements $a, b \in \mathbb{B}$ are *incompatible* $\mod I$ iff $a \wedge b \in I$ for $a, b \in I^+$.

(e) $X \subseteq \mathbb{B}$ is an *antichain* $\mod I$ iff every $a, b \in X$ are incompatible $\mod I$.

(f) I is κ*-saturated* iff every antichain has size $< \kappa$. The *saturation* of I is the least κ such that $\mod I$ is κ-saturated.

Remark: (a)–(c) are just special cases of $I = \{0\}$.

8.25 Proposition: Let $\mathbb{B}$ be a BA and I be an ideal on $\mathbb{B}$. Then I is κ-saturated iff $\mathbb{B}/I$ is κ-saturated.

Proof (sketch): $a \wedge b \in I$ iff $[a] \wedge [b] = [0]$. Exercise.

8.26 Proposition:

Let $\mathbb{B}$ be a BA and $X \subseteq B$. Let X^* be the downward closure of X, i.e.

$$X^* = \{z \in B | (\exists x \in X)(z \leq x)\}$$

and let $A \subseteq X^*$ be an antichain that is maximal among all antichains contained in X^*.

If the join of one of these sets exists then the two other joins exist and are equal. In particular, if $\bigvee A$ exists, then also $\bigvee X^*$ and $\bigvee X$ exist and

$$\bigvee X = \bigvee X^* = \bigvee A.$$

Proof. Do the nontrivial one: $\bigvee X = \bigvee A$. Notice: Enough to prove $\bigvee A$ is an upper bound on X.

If not: we have some $x \in X$ such that $x \not\leq \bigvee A$. Hence $y = x - \bigvee A \neq 0$. But if $a \in A$ then $a \wedge y \leq (\bigcup A) \wedge y = 0$.

Also, $y \in X^*$, as $y \leq x$. Hence, $A \cup \{y\} \subseteq X^*$ is an antichain. This contradicts the maximality of A. □

8.27 Proposition:

Let $\mathbb{B}$ be a BA that is κ-complete and κ-saturated. Then $\mathbb{B}$ is complete.

Proof. It is enough to check that $\bigvee A$ exists whenever A is an antichain in $\mathbb{B}$. By saturation, if A is an antichain in $\mathbb{B}$ then $|A| < \kappa$. By κ-completeness of $\mathbb{B}$, $\bigvee A$ exists. □

8.28 Proposition:

Let $\mathbb{B}$ be a κ-complete BA and I be a κ-saturated κ-complete ideal on $\mathbb{B}$. Then $\mathbb{B}/I$ is complete.

Proof. From 8.23, 8.25 and 8.27. □

The obvious Boolean Algebra is $(\mathcal{P}(A), \cap, \cup, ', \varnothing, A)$. We show that any BA can be represented this way.

8.29 Definition:

Let $\mathbb{B}$ be a BA. We let

$$S(\mathbb{B}) = \{U \in \mathcal{P}(\mathbb{B}) | U \text{ is an ultrafilter on } \mathbb{B}\}$$

To each $a \in B$ let

$$N_a = \{U \in S(\mathbb{B}) | a \in U\}$$

8.30 Proposition:

Let $\mathbb{B}$ be a BA,

$$B^* = \{N_a | a \in B\}$$

and

$$\mathbb{B}^* = (B^*, \cap, \cup, ', \varnothing, S(\mathbb{B}))$$

Then $\mathbb{B}^*$ is a Boolean subalgebra of $\mathcal{P}(S(\mathbb{B}))$ and the map

$$a \mapsto N_a$$

is an isomorphism.

Proof. $N_0 = \varnothing$ because no ultrafilter contains 0.

$N_1 = S(\mathbb{B})$ because every ultrafilter contains 1.

$N_{a \wedge b} = N_a \cap N_b$ because any filter contains a, b iff it contains $a \wedge b$.

$N_{a \vee b} = N_a \cup N_b$ because any ultrafilter contains $a \vee b$ iff it contains at least one of the elements a, b.

$N_{a'} = S(\mathbb{B}) - N_a$ because any ultrafilter contains a' iff it does not contain a. □

8.31 Remark:

P.8.30 is called Stone representation theorem and $S(\mathbb{B})$ is called the Stone space of $\mathbb{B}$ (named by Marshall Stone).

8.32 Proposition:

$\mathbb{B}^*$ is a base for topology on $S(\mathbb{B})$.

Proof. (i) $\bigcup_{a \in \mathbb{B}^*} N_a = S(\mathbb{B})$ since $S(\mathbb{B}) = N_1$.

(ii) If $G_1, ..., G_n \in \mathbb{B}^*$ then $G_1 \cap ... \cap G_n$ can be expressed as a union of elements of $\mathbb{B}^*$. In any case:

$$N_{a_1} \cap ... \cap N_{a_k} = N_{a_1 \wedge ... \wedge a_k}$$

□

8.33 Proposition:

The topological space $S(\mathbb{B})$ is

(a) 0−dimensional; this means: it has a base for topology consisting of clopen sets.

(b) Hausdorff (a topological space in which distinct points have disjoint neighborhoods).

(c) Compact.

Proof. —

(a) $N_a = S(\mathbb{B}) - N_{a'}$.

(b) If $U_1, U_2 \in S(\mathbb{B})$ and $U_1 \neq U_2$ then we have some $a \in \mathbb{B}$ such that e.g. $a \in U_1$ and $a \notin U_2$. But then $a' \in U_2$. So: $U_1 \in N_a$ and $U_2 \in N_{a'}$. Now $N_a \cap N_{a'} = N_{a \wedge a'} = N_0 = \varnothing$.

(c) Enough to prove: If $\mathcal{C}$ is a centered system of sets in $\mathbb{B}^*$ then $\bigcap \mathcal{C} \neq \varnothing$. (Notice: $\{S(\mathbb{B}) - A | A \in \mathbb{B}^*\} = \mathbb{B}^*$. So every closed set can be expressed as $\bigcap X$ for some $X \subseteq \mathbb{B}^*$.)

So let $\mathcal{C} \subseteq \mathbb{B}^*$ be a centered system. Then we have some $F \subseteq \mathbb{B}$ such that $\mathcal{C} = \{N_a | a \in F\}$. Now if $a_1, ..., a_k \in F$ then

$$N_{a_1 \wedge ... \wedge a_k} = N_{a_1} \cap ... \cap N_{a_k} \neq \varnothing, \text{ as we are assuming the system is centered.}$$

Hence, $a_1 \wedge ... \wedge a_k \neq 0$. So we have:

$$a_1, ..., a_k \in F \Rightarrow a_1 \wedge ... \wedge a_k \neq 0$$

This means that F is a filter base. So we can extend F to some ultrafilter $U \supseteq F$, $U \in S(\mathbb{B})$. But then $a \in U$ for all $a \in F$, so $U \in N_a$ for all $N_a \in \mathcal{C}$. So $\bigcap \mathcal{C} \neq \varnothing$.

□

8.34 Proposition:

Let $(X, \mathcal{T})$ be a 0−dimensional compact Hausdorff space. Then there is a BA $\mathbb{B}$ such that $X = S(\mathbb{B})$.

Proof. Let $B =$ the collection of all clopen sets in $(X, \mathcal{T})$ and $\mathbb{B} = (B, \cap, \cup, ', \varnothing, X)$. Now define a map $f : S(\mathbb{B}) \to X$ by

$$f(U) = \text{the unique element in } \bigcap U.$$

This works, because if U is an ultrafilter on $\mathbb{B}$, then $\bigcap U$ is a singleton.

Why: $\bigcap U \neq \varnothing$, as U is a centered system of closed subsets of X.

$\bigcap U$ cannot have more than one element: if $x, y \in X$, then we can separate x, y by open sets since the space is Hausdorff. So e.g. we can find a clopen set A from the base of clopen sets such that $x \in A$ but $y \notin A$. So $y \in X - A$. But then $A \in U$ or $X - A \in U$. But only one of $A, X - A$ is in U.

So f is well-defined. Also f is injective, since if $U_1 \neq U_2$, then we can find clopen A such that $A \in U_1$ and $X - A \in U_2$, so $(\bigcap U_1) \cap (\bigcap U_2) = \varnothing$.

Also, f is surjective: if $x \in X$ then

$$U = \{A \in B | x \in A\}$$

is easily seen to be an ultrafilter on $\mathbb{B}$ such that $\bigcap U = \{x\}$.

So f is a bijection. To see that f is a homeomorphism (a continuous function between two topological spaces that has a continuous inverse function) it suffices to show that f is continuous, as both X, $S(\mathbb{B})$ are compact.

But notice: $f(U) \in A$ iff $A \in U$ iff $U \in N_A$, i.e. $f^{-1}[A] = N_A$. □

Conclusion: there is a 1-1 correspondence between Boolean Algebras and Compact Hausdorff 0-dimensional spaces, called *Stone duality*. So everything about Boolean Algebras can be expressed in the language of these spaces and vice versa, but the translation is a "mirror image": e.g.

If $f : \mathbb{B}_1 \to \mathbb{B}_2$ is an injective homomorphism, then $f' : S(\mathbb{B}_2) \to S(\mathbb{B}_1)$ is a surjective continuous map. Etc...

Constructing Complete Boolean Algebras

8.35 Definition:

Let $(X, \mathcal{T})$ be a topological space. Let

- $\bar{A}$= the topological closure of A.
- A^o= the topological interior of A.

Both of these operations are monotonic. Moreover:

$$\overline{A \cup B} = \bar{A} \cup \bar{B} \text{ and } (A \cap B)^o = A^o \cap B^o.$$

A set $A \subseteq X$ is *regular open* iff $\bar{A}^\circ = A$.

8.36 Proposition:

Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$.

(a) If $A \subseteq X$ then $\bar{A}^\circ$ is a regular open set.

(b) If A is open then $\bar{A}^\circ$ is the smallest regular open set that contains A. (A open $\Rightarrow A \subseteq \bar{A}^\circ$).

Proof. First: If $A \subseteq X$, then $\bar{A}^\circ$ is regular open:

By the monotonicity of the operations, we have that $\bar{A}^\circ \subseteq (\overline{\bar{A}^\circ})^\circ$.

On the other hand: $(\overline{\bar{A}^\circ}) \subseteq \bar{A}$ because $\bar{A}^\circ \subseteq \bar{A}$. Hence $\overline{\bar{A}^\circ}^\circ \subseteq \bar{A}^\circ$.

Now assume that R is regular open and $A \subseteq R$. Then $\bar{A}^\circ \subseteq \bar{R}^\circ = R$. □

8.37 Proposition:

Let $(X, \mathcal{T})$ be a topological space and $A, B \subseteq X$ be open sets. Then

$$\overline{A \cap B}^{\circ} \stackrel{1}{=} (\bar{A} \cap \bar{B})^{\circ} = \bar{A}^{\circ} \cap \bar{B}^{\circ}$$

Proof. Of (1) (the essential part), $\subseteq$ is easy, as $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$, and in fact holds for any $A, B \subseteq X$.

We prove $\supseteq$:

Let $G = (\bar{A} \cap \bar{B})^{\circ} - \overline{A \cap B}$. It is enough to prove that $G \neq \varnothing$.

To see that G is empty, notice that G is open. $G \subseteq \bar{A} \cap \bar{B}$. Therefore, if $G \neq \varnothing$ then $G \cap A \neq \varnothing$ and $G \cap B \neq \varnothing$.

Claim: $(G \cap A) \cap B = \varnothing$, because $G \cap (\bar{A} \cap \bar{B}) = \varnothing$. (Need $G \cap A$ open here).

So: $G \cap A$ is open and nonempty, $(G \cap A) \cap B = \varnothing$ and $G \cap A \subseteq \bar{B}$. $\square$

8.38 Theorem:

Let $(X, \mathcal{T})$ be a topological space. Let

$$\text{RO}(X) = \text{ the collection of all regular open subsets of } X.$$

Then the Boolean Algebra ($\text{RO}(X), \subseteq$) is a complete Boolean Algebra. We denote this Boolean Algebra again by $\text{RO}(X)$.

Proof. If $a \subseteq \text{RO}(X)$, we let

$$\bigvee a = \overline{\bigcup a}^{\circ}$$

By 8.36, $\bigvee a$ is the smallest regular open set that contains all sets from a. We also let

$$\bigwedge a = (\bigcap a)^{\circ}.$$

Clearly $(\bigcap a)^{\circ}$ is the largest open set contained in all sets from a.

$\overline{(\bigcap a)^{\circ}}^{\circ} \subseteq \bar{A}^{\circ} = A$ for all $A \in a$.

So: $\overline{(\bigcap a)^{\circ}}^{\circ} \subseteq \bigcap a$. Since $\overline{(\bigcap a)^{\circ}}^{\circ}$ is open, we have $\overline{(\bigcap a)^{\circ}}^{\circ} \subseteq (\bigcap a)^{\circ}$.

This shows that $\overline{(\bigcap a)^{\circ}}^{\circ} \subseteq (\bigcap a)^{\circ}$; the converse is 8.36 (b).

In particular:

$$A_1 \vee ... \vee A_n = \overline{A_1 \cup ... \cup A_n}^{\circ}$$
$$A_1 \wedge ... \wedge A_n = A_1 \cap ... \cap A_n$$

So $\mathrm{RO}(X)$ is a complete lattice. From this, we abstractly get commutativity and associativity for $\vee$ and $\wedge$. The absorbtion laws can be easily verified.

Distributivity: $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$. $(A \cap \overline{B \cap C}^\circ = \overline{(A \cap B) \cup (A \cap C)})$. Equality:

$$\overline{(A \cap B) \cup (A \cap C)}^\circ = \overline{A \cap (B \cup C)}^\circ \overset{8.36}{=} \bar{A}^\circ \cap \overline{B \cup C}^\circ = A \cap \overline{B \cup C}^\circ.$$

Verification of the other distributive law is similar.

Complementation: If $A \in \mathrm{RO}(X)$ we let $A' = (X - A)^\circ$ $(= \overline{X - A}^\circ)$.

Obviously: $A \wedge A' = A \cap A' = \varnothing$. We must verify $A \vee A' = \overline{A \cup A'}^\circ = X$.

Enough to prove: $\overline{A \cup A'} = X$. However, $G = X - \overline{A \cup A'}$ is open and is disjoint with A. So $G \subseteq (X - A)^\circ = A'$. But G is also disjoint with A', so $G \subseteq A$. This implies $G = \varnothing$. $\square$

Remark: So $\mathrm{RO}(X) \subseteq \mathcal{P}(X)$. The orderings are the same on both sets. But $\mathrm{RO}(X)$ is not a subalgebra of $\mathcal{P}(X)$ because the operation $\vee$ is distinct in $\mathrm{RO}(X)$ and $\mathcal{P}(X)$.

Example 1: $\mathcal{P}(\omega)$ is a complete BA. Notice:

$$\mathrm{sat}(\mathcal{P}(\omega)) = \omega_1$$

Example 2: Let

$$\mathcal{I} = \text{the collection of all finite subsets of } \omega.$$

Then $\mathcal{I}$ is an ideal on $\mathcal{P}(\omega)$.

$$\mathrm{sat}(\mathcal{I}) = (2^\omega)^+.$$

Why: $\mathrm{sat}(\mathcal{I}) \leq (2^\omega)^+$ because $|\mathcal{P}(\omega)| = 2^\omega$.

However, there is an antichain a modulo $\mathcal{I}$ such that $|a| = 2^\omega$.

This means: if $A, B \in a$, then $A \cap B$ is finite and $a \subseteq \mathcal{I}^+$.

The corresponding quotient algebra $\mathcal{P}(\omega)/\mathcal{I}$, also referred to as $\mathcal{P}(\omega)/\mathrm{fin}$, is a BA such that $\mathrm{sat}(\mathcal{P}(\omega)/\mathcal{I}) = (2^\omega)^+$ by 8.? (saturation of ideal same as saturation of quotient).

Notice we have the quotient homomorphism

$$h : \mathcal{P}(\omega) \to \mathcal{P}(\omega)/\mathrm{fin}.$$

8.39 Proposition:

Let $\mathbb{B}$ be a complete BA. Let κ be a cardinal and $A \subseteq B$. Then there is a smallest κ-complete subalgebra $\mathbb{B}' \subseteq \mathbb{B}$ that contains A.

$\mathbb{B}$' is called the κ-complete subalgebra generated by A.

Proof. (1) Let

$$\mathcal{B}\text{= the family of all } \kappa\text{-complete subalgebras of } \mathbb{B} \text{ that contain } A.$$

Then $\mathbb{B}' = \bigcap \mathcal{B}$.

(2) By recursion define collections Σ_α, Π_α:

$$\Sigma_0 = A$$
$$\Pi_\alpha = \{a' | a \in \Sigma_\alpha\}$$
$$\Sigma_{\alpha+1} = \{\bigvee x | x \in \left(\bigcup_{\bar{\alpha}<\alpha} \Pi_{\bar{\alpha}} \cup \bigcup_{\bar{\alpha}<\alpha} \Sigma_\alpha\right)^{<\kappa}$$
$$\Sigma_\alpha = \bigcup_{\bar{\alpha}<\alpha} \Sigma_{\bar{\alpha}} (= \bigcup_{\bar{\alpha}<\alpha} \Sigma_{\bar{\alpha}} \cup \Pi_{\bar{\alpha}})$$

if α is limit.

Next, we get a diagram with sigmas on top where sigma 0 points to pi 0 on bottom nw arrow sigma 1 etc...

Easy to see: $\alpha < \beta \Rightarrow \Sigma_\alpha \subseteq \Sigma_\beta$, $\Pi_\alpha \subseteq \Sigma_\beta$.

Then $B' = \Sigma_{\kappa^+}$. In fact: if κ is regular then $B' = \Sigma_\kappa$.

To see the latter (the former then directly follows): Show $\Sigma_{\kappa+1} = \Sigma_\kappa$.

Assume κ is regular and $a \in \Sigma_{\kappa+1}$. This means we have some sequence $\langle a_\xi | \xi < \gamma \rangle$ where $a_\xi \in \Sigma_{\alpha_\xi}$, $\gamma < \kappa$ and $a = \bigvee_{\xi<\gamma} a_\xi$.

Each $a_\xi \in \bigcup_{\xi<\kappa} \Sigma_\xi$, i.e. to each ξ there is some $\alpha_\xi < \kappa$ such that $a_\xi \in \Sigma_{a_\xi}$.

Because κ is regular: there is some α such that $\alpha_\xi < \alpha$ for $\xi < \gamma$. This means: $a \in \Sigma_{\alpha+1} \subseteq \Sigma_\kappa$. □

Example 3: Borel sets. Consider topological space $(X, \mathcal{T})$. Borel sets is the ω_1-complete subalgebra of $\mathcal{P}(X)$ generated by $(X, \mathcal{T})$.

8.40 Proposition:

Let $\mathbb{B}$ be a complete BA, $\mathbb{B}'$ be a κ-complete subalgebra and $\mathcal{I}$ be a κ-complete ideal on $\mathbb{B}$. We can extend $\mathbb{B}'$ to a κ-complete subalgebra $\mathbb{B}'_{\mathcal{I}}$ modulo $\mathcal{I}$ as follows:

$$a \in \mathbb{B}'_{\mathcal{I}} \Leftrightarrow (\exists b \in \mathbb{B}')(a \Delta b \in \mathcal{I}).$$

Proof. Exercise. Similar to the proof that if an algebra is κ-complete, then its quotient modulo any κ-complete ideal is again κ-complete. □

Example 4: Baire property. If $(X, \mathcal{T})$ is a topological space then a set K is nowhere dense iff $\bar{K}^\circ = \varnothing$. A set is *meager*, or "of the first category"iff there is a countable sequence of closed nowhere dense sets $\langle F_i | i \in \omega \rangle$ such that $A \subseteq \bigcup_{i \in \omega} F_i$. Letting

$$\mathcal{I}_M = \text{the family of all meager subsets of } X$$

we have: $\mathcal{I}_M$ is an ω_1-complete ideal on $\mathcal{P}(X)$.

Note: ω_1-complete is the same as what is in mathematics called σ-complete. Let

$$\text{Baire}(X) = \text{Borel}(X)_{\mathcal{I}_M}.$$

Then Baire(X) is the so-called family of all Baire sets, the sets with Baire property. By 8.40, Baire(X) is a σ-complete subalgebra of $\mathcal{P}(X)$ that contains all Borel sets.

Also notice: if F is closed then $F - F^\circ$ is meager, and is even closed nowhere dense. So $F \Delta F^\circ \in \mathcal{I}_M$. This gives an alternative characterization of Baire(X):

$$A \in \text{Baire}(X) \Leftrightarrow \text{there is some open set } G \text{ such that } A \Delta G \in \mathcal{I}_M.$$

Most interesting space $(X, \mathcal{T})$ for this example are complete separable metric spaces $\stackrel{def}{=}$ Polish spaces. E.g. $\mathbb{R}^n$.

Another important example is the Baire space $\mathcal{N} = ({}^\omega\omega)$ = the topological product of ω copies of ω with discrete topology.

So: Elements of $\mathcal{N}$ are infinite sequences $x \in ({}^\omega\omega)$. Basic open sets have the form:

$$B_S = \{x \in ({}^\omega\omega) | x \text{ extends } s \ (s \subseteq x)\}$$

for $s \in ({}^{<\omega}\omega)$.

We can define the metric:

$$d(x, y) = \tfrac{1}{k+1} \text{ when } k = \text{the least such that } x(k) \neq y(k).$$

Note: $\mathcal{N}$ is homeomorphic to the space of irrational numbers.

Example 5: Let $(X, \mathcal{T})$ be a topological space. A set $A \subseteq X$ is G_δ iff A is an intersection of countably many open sets.

A measure μ on X is *regular from above* iff for any μ-measurable set K there is some G_δ set $A \supseteq K$ such that $\mu(K) = \mu(A)$.

Notice:

$$\mathcal{I}_N = \text{the family of all subsets } K \text{ of } X \text{ such that } \mu(K) = 0.$$

Then $\mathcal{I}_N$ is an ω_1-complete ideal, called the ideal of null sets. By 8.40, for regular measures μ we can define

$$\text{Meas}_\mu(X) = \text{Borel}(X)_{\mathcal{I}_N}.$$

$\mathrm{Meas}_\mu(X)$ is the family of all μ-measurable sets.

This gives an alternative definition of $\mathrm{Meas}_\mu(X)$. We can define measure on Borel(X) first. Then we can let

$$\mathcal{I}_N = \{Y \in \mathcal{P}(X) | \text{ There is some } A \in \text{Borel}(X) \text{ with } A \supseteq Y \text{ and } \mu(A) = 0\}$$

Hence: A is μ-measurable iff $A \Delta B \in \mathcal{I}_N$ for some Borel B.

8.41 Proposition:

Let $(X, \mathcal{T})$ be a Polish space and μ be a regular measure on $(X, \mathcal{T})$. Then both $\mathcal{I}_M, \mathcal{I}_N$ are ω_1-saturated. I.e. if $\mathcal{A}$ is a family of Baire μ-Measurable sets such that for all $A, B \in \mathcal{A}$ $A, B \in \mathcal{I}_M^+$ and $A \cap B \in \mathcal{I}_M$, $A, B \in \mathcal{I}_N^+$ and $A \cap B \in \mathcal{I}_N$.