

Boundedness of moduli of varieties^① of general-type

Preliminaries II

X
↓ projective
normal
 U

D on X a \mathbb{R} -Cartier divisor
and C on X a prime
divisor.

If D is big on X define

$$\delta_C(D) := \inf \{ \text{mult}_C(D') : D' \sim_{\mathbb{R}} D, D' \geq 0 \}$$

* only depends on \cong -

* for D, D' big

$$\delta_C(D + D') \leq \delta_C(D) + \delta_C(D')$$

Ex: D^R is big and nef. There $\exists E \geq 0$
s.t. for $\forall \epsilon > 0 \exists A_\epsilon$ tuple s.t.

$$D \sim_{\mathbb{R}} A_\epsilon + \epsilon E \quad \text{hence} \quad \delta_C(D) \leq \delta_C(A_\epsilon) + \delta_C(\epsilon E)$$

$$\delta_C(D) = \inf_{\epsilon} \{ \epsilon \text{mult}_C(E) \} = 0 \quad \text{"0"}$$

Extend definition to pseudo-effective ⁽²⁾

$$D. \quad \delta_c'(D) := \lim_{\epsilon \rightarrow 0} \delta_c(D + \epsilon A) \quad \text{for } A \text{ ample}$$

Lemma: the two definitions agree for D big.

Proof: $\lim_{\epsilon \rightarrow 0} \delta_c(D + \epsilon A) \leq \lim_{\epsilon \rightarrow 0} (\delta_c(D) + \delta_c(\epsilon A))$
 \parallel
 0

$= \delta_c(D)$. For the converse,

since D is big there $\exists \Delta \geq 0$

$$\text{s.t. } B \underset{\mathbb{R}}{\sim} \delta A + \Delta \quad \text{for } \delta \in \mathbb{R}$$

$$\Rightarrow (1+\epsilon) B \underset{\mathbb{R}}{\sim} B + \epsilon \delta A + \epsilon \Delta$$

$$(1+\epsilon) \delta_c(B) \leq \delta_c(B + \epsilon \delta A) + \epsilon \delta_c(\epsilon \Delta)$$

$\lim_{\epsilon \rightarrow 0}$ then this gives

$$\delta_c(B) \leq \lim_{\epsilon \rightarrow 0} \delta_c(B + \epsilon \delta A)$$

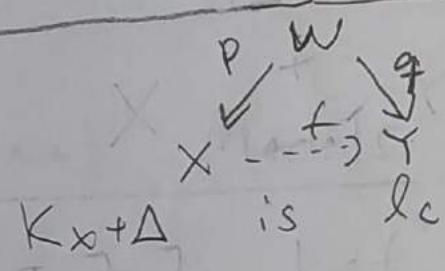
□

For D -pseudo-effective we define

$$N_S(X/\mu, D) := \sum_C \delta_C(D) C$$

Sections 2.6-2.9	prime in X	Minimal models and good minimal models
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Recall:



f -birational contraction over U
 $\Gamma = f_* \Delta$

- (1) (Y, Γ) is a weak log canonical model if $K_Y + \Gamma$ nef of non-positive i.e.,

$$p_*^*(K_X + \Delta) = q^*(K_Y + \Gamma) + E \quad E \geq 0 \text{ } q\text{-exc.}$$
- (2) A w.l.c.m is a semi-ample model if $K_Y + \Gamma$ is semi-ample.
- (3) (Y, Γ) is minimal model if (X, Δ) dlt, Y \mathbb{Q} -factorial w.l.c.m and f is negative (i.e.) $p_*^1 \text{Supp}(E) \subseteq \text{Supp}(E)$
- (4) (Y, Γ) is a good minimal model if minimal model + semi-ample model.

• Minimal Models

(4)

Lemma 2.7.1 Let (X, Δ) be a lc pair, where X is projective and

let $X \xrightarrow{f} Y$ be a weak

log-canonical model. Suppose

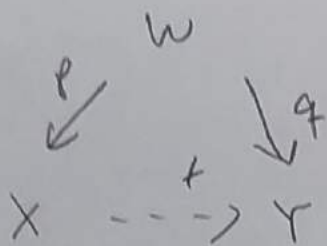
that the rational map ϕ associated to $|r(K_X + \Delta)|$ is birational. Then

(1) Every component of $N_S(K_X + \Delta)$ is f -exceptional. Step

2) If P is a prime divisor s.t. P is not a component of the base-locus of $|r(K_X + \Delta)|$ and s.t. $\phi|_P$ is birational then P is not f -exceptional.

Pf: (5)

1)



$$p^*(k_x + \Delta) = q^*(k_y + \Gamma) + E$$

$E \geq 0$ q -exceptional

← net

Notice $p_* E$ is t -exceptional.

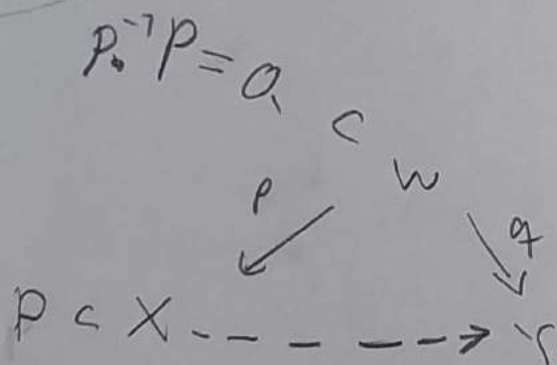
Claim: $N_S(k_x + \Delta) = p_* E$, (no cancellations)

$$N_S(k_x + \Delta) = N_S(\underbrace{q^*(k_y + \Gamma)}_{\text{net}} + p_* E) \stackrel{\text{no cancellations}}{=} N_S(p_* E)$$

$$= p_* E.$$

Skip

2)



$$|r(k_x + \Delta)|_{\mathbb{R}} \text{ birational} \Rightarrow |r(q^*(k_y + \Gamma))|_{\mathbb{R}} \text{ birational}$$

$\Rightarrow Q$ is not q -exceptional

$\Rightarrow P$ is not t -exceptional

Lemma 2.7.2 Let (X, Δ) be dlt $\textcircled{6}$
 X \mathbb{Q} -factorial & projective. Assume
that $\mathbb{Q}X + \Delta$ is pseudo-effective.

Suppose that we run a $\mathbb{Q}X + \Delta$ -MMP
with scaling of an ample A ,

$X \xrightarrow{f} Y$ so that $(Y, \Gamma + tB)$ is
nef where $\Gamma = f_*\Delta$ $B = f_*A$, for $t > 0$

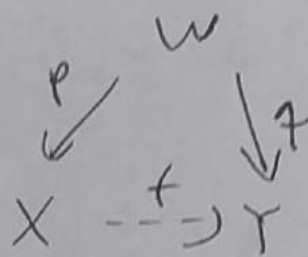
(1) If F is t -exceptional then
 $F \subset \text{Supp}(N_S(\mathbb{Q}X + \Delta))$

(2) If t is small enough, then
 $\text{Supp}(N_S(\mathbb{Q}X + \Delta)) \subseteq \text{Supp}(E_X(f))$

(3) If (X, Δ) has a minimal model
then $N_S(\mathbb{Q}X + \Delta)$ is a \mathbb{Q} -divisor.

Proof:

* f is a minimal model for



$(X, \Delta + tA)$
for some $t \geq 0$

$$N_S(K_X + \Delta + tA) = p_* E$$

$$\left(p^*(K_X + \Delta + tA) - q^*(K_Y + \Gamma + tB) + E \right)$$

where E is q -exceptional &

since f is negative

$$\text{Supp}(E_{\text{exc}}(f)) \subset p_* E$$

$$* \text{Supp}(E_{\text{exc}}(f)) \subset N_S(K_X + \Delta + tA) \subseteq N_S(K_X + \Delta)$$

this shows (1).

(2) we have seen that

$\text{Supp}(N_S(K_X + \Delta + tA))$ is f -exceptional

so sufficient to show that for t small

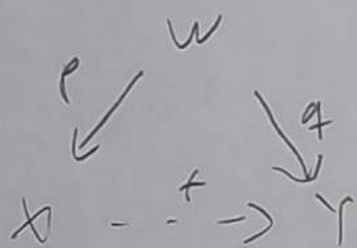
$$\text{enough} \quad \text{Supp}(N_S(K_X + \Delta + tA)) = \text{Supp}(N_S(K_X + \Delta))$$

which is ok.

3) If (X, Δ) has a normal model
then $t=0$ and so $N_S(K_X + \Delta) = p_X \mathbb{E}$
 \uparrow
Q-divisor

Lemma 2.7.3, (X, Δ) dH X \mathbb{Q} -factorial & projective. Assume that $K_X + \Delta$ is pseudo-effective. If $f: X \dashrightarrow Y$ is a bir. contraction s.t Y is \mathbb{Q} -factorial $K_Y + \Gamma = f_*(K_X + \Delta)$ is nef and f only contracts components of $N_S(K_X + \Delta)$ then f is a minimal model of (X, Δ) .

Pf: Sufficient to show that f is nef.



$$p^*(K_X + \Delta) + E = q^*(K_Y + \Gamma) + F \quad \begin{array}{l} \text{disjoint} \\ E, F \geq 0 \\ q\text{-exceptional} \end{array}$$

$$\bullet E=0 \quad N_S(q^*(K_Y + \Gamma) + F) = N_S(F)$$

therefore $N_S(p^*(K_X + \Delta) + E)$ is supported on

F , every component of E is in support

$$\rightarrow E=0 \quad (K_Y + \Gamma \text{ nef} \Rightarrow f \text{ non-positive})$$

next page
→

$$N_{\delta}(K_x + \Delta) = p_{\ast} E$$

\cup

$\rightarrow f$ is negative

$\text{Exc}(f)$

□

Good

Minimal

Models

9] (X, Δ) dlt \mathbb{Q} -factorial projective Lemmas 2.9.1

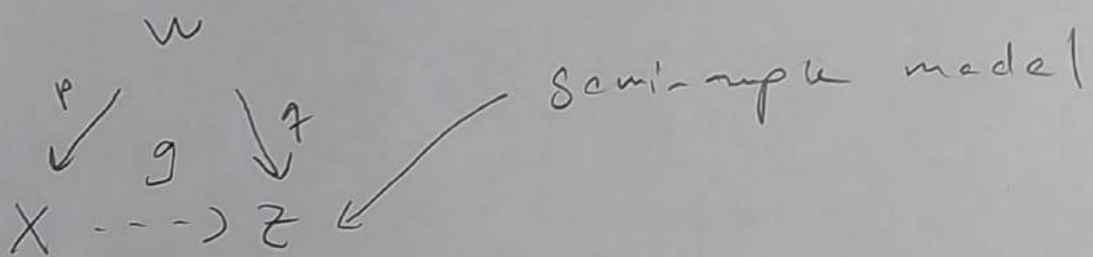
If (X, Δ) has w.l.c.m. then

(X, Δ) has semi-ample model



(X, Δ) has good minimal model.

Proof:



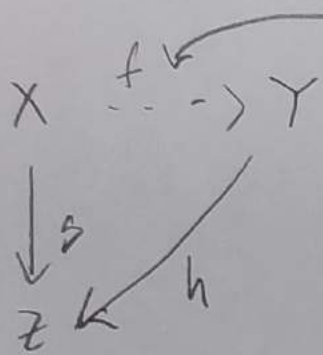
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$$p^*(K_X + \Delta) + E \cong_w K_w + F \quad E \text{ } p\text{-exc.}$$

then by 2.10 in "Hacon - Xu, Existence of log-canonical closures" (X, Δ) has good minimal model \Leftrightarrow
 (w, \emptyset) has good minimal model.

w.l.o.g

$X \xrightarrow{g} Z$ is a morphism



run $K_X + \Delta$ MMP / Z
with scaling of
ample

absolute \Uparrow
 $K_X + \Delta + H$ - MMP
where $H = s^*(\text{ample})$

$N_S(K_X + \Delta)$ has same support as

$$N_S(K_X + \Delta + H)$$

w.w.s $K_Y + \Gamma$ is semi-ample

$h_*(K_Y + \Gamma)$ is semi-ample.

for t small enough $\text{Supp}(E_X(t)) = N_S(K_X + \Delta)$

$(E_X(t)) \subset N_S(K_X + \Delta) \Rightarrow h$ does

not contract any divisor

$\Rightarrow K_Y + \Gamma = h^*(h_*(K_Y + \Gamma))$ is semi-ample

2.9 Good minimal models

70

Lemma 2.9.3. Let k be any field of characteristic 0 and let (X, Δ) be a log pair over k . Let $(\bar{X}, \bar{\Delta})$ be the base change to \bar{k} .

Assume $(\bar{X}, \bar{\Delta})$ is dlt and \mathbb{Q} -factorial.

Then (X, Δ) has a good minimal model $\Leftrightarrow (\bar{X}, \bar{\Delta})$ has a good minimal model.

Remark: Assume (X, Δ) dlt, because this is a part of definition of (X, Δ) having minimal model.

$(\bar{X}, \bar{\Delta})$ dlt $\not\Rightarrow (X, \Delta)$ dlt
 \Leftarrow

Eg. $(\mathbb{A}^2, x^2 + y^2 = 0)$ this is dlt / \mathbb{C}

but not over \mathbb{R} . Since not
of the
form

Proof: Suppose (X, Δ) has a (1)
good minimal model

$$X \xrightarrow{f} Y, \quad \bar{X} \xrightarrow{\bar{f}} \bar{Y}$$

then $(\bar{Y}, \bar{\Delta})$ is a semi-ample
model, $(K_{\bar{Y}} + \bar{\Delta})$ is semi-ample, f is non-positive
We \Rightarrow use (Lemma 2.9.1) that (X, Δ) has
a good minimal model

Suppose $(\bar{X}, \bar{\Delta})$ has a good

minimal model $(\bar{Y}, \bar{\Delta})$ run

$$\text{a } K_X + \Delta + tA \text{ MMP}$$

$$f: X \dashrightarrow Y$$

then f is a minimal model for

$(X, \Delta + tA) \Rightarrow f$ is a weak lc
model for $(\bar{X}, \bar{\Delta} + t\bar{A}) \xrightarrow{\text{Lemma 2.9.2}}$

\Rightarrow if $(\bar{X}, \bar{\Delta})$ has a good minimal model
then $\exists \varepsilon > 0$ s.t. $\bar{X} \dashrightarrow \bar{Y}$ weak lc model of
 $(\bar{X}, \bar{\Delta} + t\bar{A})$ for $t \in [0, \varepsilon)$ then if
 $(\bar{X}, \bar{\Delta} + t\bar{A})$ is a semi-ample
model

i.e., $K_{\bar{Y}} + \bar{\Gamma} + t\bar{B}$ is semiample
 for $t \in (0, \varepsilon) \Rightarrow K_{\bar{Y}} + \bar{\Gamma}$ is semi-ample
 $\Rightarrow K_Y + \Gamma$ is semiample and so
 (Y, Γ) is a good minimal model

Left to show (Lemma 2.9.1)

(X, Δ) dlt \mathbb{Q} -factorial projective

If (X, Δ) has a weak lc-model
 then (X, Δ) has a semi-ample model
 $\Leftrightarrow (X, \Delta)$ has a good minimal model.

See previous lemma (page 9) for the proof of the part of
 Lemma 2.9.1 which is used here.