Boundedness of moduli of varieties (1)

of General-type / Preliminaries II

$X \xrightarrow{\text{projective normal}} U$ D on X a $\mathbb{R}$-Cartier divisor and C on X a prime divisor.

If D is big on X define

$$\delta_C(D) := \inf\{ \operatorname{mult}_C(D') : D' \sim_{\mathbb{R}} D \ D' \geq 0\}$$

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* only depends on $\equiv_p$.
* for D, D' big
$$\delta_C(D+D') \leq \delta_C(D) + \delta_C(D')$$

<u>Ex</u>: D is big and nef. There $\exists E \geq 0$ s.t for $\forall \varepsilon > 0$ $\exists A_\varepsilon$ ample s.t

$D \sim_{\mathbb{R}} A_\varepsilon + \varepsilon E$ hence $\delta_C(D) \leq \underbrace{\delta_C(A_\varepsilon)}_{= 0} + \delta_C(\varepsilon E)$

$$\delta_C(D) = \inf_\varepsilon \{\varepsilon \operatorname{mult}_C(E)\} = 0$$

Extend definition to pseudo-effective

D. $\delta_c'(D) := \lim_{\varepsilon \to 0} \delta_c(D + \varepsilon A)$ for A ample

<u>Lemma:</u> the two definitions agree for

D big.

<u>Proof:</u> $\lim_{\varepsilon \to 0} \delta_C(D+\varepsilon A) \leq \lim_{\varepsilon \to 0} \left(\delta_C(D) + \underset{0}{\underset{\|}{\delta_C(\varepsilon A)}}\right)$

$= \delta_c(D)$. For the converse,

since D is big there $\exists$ $\Delta \geq 0$

s.t $B \underset{\mathbb{R}}{\sim} \delta A + \Delta$ for $\delta \in \mathbb{R}$

$\Rightarrow$ $(1+\varepsilon) B \underset{\mathbb{R}}{\sim} B + \varepsilon \delta A + \varepsilon \Delta$

$(1+\varepsilon)\delta_c(B) \leq \delta_c(B + \varepsilon \delta A) + \varepsilon \delta_c(\varepsilon \Delta)$

$\lim_{\varepsilon \to 0}$ then this gives

$$\delta_c(B) \leq \lim_{\varepsilon \to 0} \delta_c(B + \varepsilon \delta A)$$

$\square$

For D-pseudo-effective we define

$$N_\sigma(X/U, D) := \sum_{\substack{C \\ \text{prime} \\ \text{in } X}} \sigma_C(D)\, C$$

Sections 2.6-2.9 | Minimal models and good minimal models

Recall:

$$\begin{array}{ccc} & W & \\ {}^{p}\swarrow & & \searrow^{q} \\ X & \overset{f}{\dashrightarrow} & Y \end{array}$$

f - birational contraction over U

$K_X+\Delta$ is l_c $\quad \Gamma = f_*\Delta$

(1) (Y,Γ) is a weak log canonical model if $K_Y+\Gamma$ nef of non-positive i.e.,

$$p^*(K_X+\Delta) = q^*(K_Y+\Gamma) + E \qquad E \geq 0 \ q\text{-exc.}$$

(2) A w.l.c.m is a semi-ample model if $K_Y+\Gamma$ is semi-ample.

(3) (Y,Γ) is minimal model if (X,Δ) dlt, Y $\mathbb{Q}$-factorial w.l.c.m and f is negative (i.e., $p_*^{-1}\mathrm{Supp}(\mathrm{Exc}(f)) \subseteq \mathrm{Supp}(E)$)

(4) (Y,Γ) is a good minimal model if minimal model + semi-ample model.

- Minimal Models

Lemma 2.7.1 Let (X,Δ) be a lc pair, where X is projective and let $X \overset{f}{\dashrightarrow} Y$ be a weak log-canonical model. Suppose that the rational map ϕ associated to $|r(K_X+\Delta)|$ is birational. Then

(1) Every component of $N_\sigma(K_X+\Delta)$ is ~~f~~ exceptional

Skip

2) If P is a prime divisor s.t P is not a component of the base-locus of $|r(K_X+\Delta)|$ and s.t $\phi_{|P}$ is birational then P is not f-exceptional.

Pf: $\boxed{5}$

1)
$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed, "f"] & & Y \end{tikzcd}$$

$p^*(K_X+\Delta) = q^*(K_Y+\Gamma) + E$ ← nef

$E \geq 0$ q-exceptional

Notice p_*E is f-exceptional.

Claim: $N_\delta(K_X+\Delta) = p_*E$,

$$N_\delta(K_X+\Delta) = N_\delta\big(\underbrace{p_* q^*(K_Y+\Gamma)}_{\text{nef}} + p_*E\big) \overset{\text{(no cancellation)}}{=} N_\delta(p_*E)$$

$= p_*E$.

$\boxed{\text{Skip}}$

2) $p_*^{-1}P = Q \subset W$
$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ P \subset X \arrow[rr, dashed] & & Y \end{tikzcd}$$

$|r(K_X+\Delta)|_{|P}|$ birational $\Rightarrow$ $|r(q^*(K_Y+\Gamma))|_{|Q}|$ birational

$\Rightarrow$ Q is not q-exceptional

$\Rightarrow$ P is not f-exceptional

Lemma 2.7.2 Let (X,Δ) be dlt X $\mathbb{Q}$-factorial & projective. Assume that $K_X+\Delta$ is pseudo-effective.

Suppose that we run a $K_X+\Delta$-MMP with scaling of an ample A,

$X \xrightarrow{f} Y$ so that $(Y, \Gamma+tB)$ is nef where $\Gamma=f_*\Delta$ $B=f_*A$, for $t>0$

(1) If F is f-exceptional then $F \subset \mathrm{Supp}(N_\sigma(K_X+\Delta))$

(2) If t is small enough, then $\mathrm{Supp}(N_\sigma(K_X+\Delta)) \subseteq \mathrm{Supp}(\mathrm{Ex}(f))$

(3) If (X,Δ) has a minimal model then $N_\sigma(K_X+\Delta)$ is a $\mathbb{Q}$-divisor.

Proof:

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed, "f"] & & Y \end{tikzcd}$$

$*$ f is a minimal model' for $(X, \Delta + tA)$ for some $t \geq 0$

$$N_\sigma(K_X + \Delta + tA) = p_* E$$

$$\left(p^*(K_X + \Delta + tA) = q^*(K_Y + \Gamma + tB) + E \right)$$

where E is q-exceptional &

since f is negative

$$\operatorname{Supp}(\operatorname{Exc}(f)) \subset p_* E$$

$$* \quad \operatorname{Supp}(\operatorname{Exc}(f)) \subset N_\sigma(K_X + \Delta + tA) \leq N_\sigma(K_X + \Delta)$$

this shows (1).

(2) we have seen that $\operatorname{Supp}(N_\sigma(K_X + \Delta + tA))$ is f-exceptional so sufficient to show that for t small enough

$$\operatorname{Supp}(N_\sigma(K_X + \Delta + tA)) = \operatorname{Supp}(N_\sigma(K_X + \Delta))$$

which is ok.

3) If (X, Δ) has a minimal model then $t = 0$ and so $N_\sigma(K_X+\Delta) = p_* E$

$\mathbb{Q}$-divisor

Lemma 2.7.3 (X,Δ) dlt X $\mathbb{Q}$-factorial & projective. Assume that $K_X+\Delta$ is pseudo-effective. If $f: X \dashrightarrow Y$ is a bir. contraction s.t Y is $\mathbb{Q}$-factorial $K_Y+\Gamma = f_*(K_X+\Delta)$ is nef and f only contracts components of $N_\sigma(K_X+\Delta)$ then f is a minimal model of (X,Δ).

Pf: Sufficient to show that f is negative.

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed, "f"] & & Y \end{tikzcd}$$

$$p^*(K_X+\Delta) + E = q^*(K_Y+\Gamma) + F \qquad E, F \geq 0$$ disjoint, q-exceptional

• $E=0$ $\quad N_\sigma(q^*(K_Y+\Gamma)+F) = N_\sigma(F)$

therefore $N_\sigma(p^*(K_X+\Delta)+E)$ is supported on F, every component of E is in support

$\Rightarrow E=0 \qquad (K_Y+\Gamma \text{ nef} \Rightarrow f \text{ non-positive})$

next page

$$N_\delta(\ell_x+\Delta) = p_* E$$

$$\cup$$

$$\operatorname{Exc}(f)$$

$\Rightarrow$ f is negative

$\square$

Good

Minimal

Models

9] (X, Δ) dlt $\mathbb{Q}$-factorial projective **Lemma 2.9.1**

If (X, Δ) has w.l.c.m then

(X, Δ) has semi-ample model

$$\Updownarrow$$

(X, Δ) has good minimal model.

__Proof:__

$$\begin{tikzcd} & W \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[rr, dashed, "g"] & & Z \end{tikzcd}$$

semi-ample model

write $\quad p^*(K_X+\Delta) + E \underset{\mathbb{R}}{=} K_W + \Phi \qquad E$ p-exc.

then by 2.10 in "Hacon-Xu, Existence of log-canonical closures" (X, Δ) good min model $\Leftrightarrow$ (W, Φ) has good minimal model.

w.l.o.g

$$X \xrightarrow{g} Z \text{ is a morphism}$$

$$\begin{tikzcd} X \arrow[r, dashed, "f"] \arrow[d, "g"] & Y \arrow[dl, "h"] \\ Z & \end{tikzcd}$$

run $K_X+\Delta$ MMP/Z with scaling of ample

$$\Updownarrow$$

absolute $K_X+\Delta+H$ - MMP where $H = g^*(\text{ample})$

$N_\delta(K_X+\Delta)$ has same support as $N_\delta(K_X+\Delta+H)$

w.w.s $K_Y+\Gamma$ is semi-ample

$h_*(K_Y+\Gamma)$ is semi-ample.

for t small enough $\mathrm{Supp}(\mathrm{Ex}(f)) = N_\delta(K_X+\Delta)$

$(\mathrm{Exc}(g)) \subset N_\delta(K_X+\Delta) \Longrightarrow h$ does not contract any divisor

$\Longrightarrow K_Y+\Gamma = h^*(h_*(K_Y+\Gamma))$ is semi-ample

2.9 Good minimal models

Lemma 2.9.3. Let k be any field of characteristic 0 and let (X,Δ) be a log pair over k. Let $(\bar{X},\bar{\Delta})$ be the base change to $\bar{k}$.
Assume $(\bar{X},\bar{\Delta})$ is dlt and $\mathbb{Q}$-factorial.
Then (X,Δ) has a good minimal model $\Leftrightarrow$ $(\bar{X},\bar{\Delta})$ has a good minimal model.

Remark: Assume (X,Δ) dlt, because this is a part of definition of (X,Δ) having minimal model.

$$(\bar{X},\bar{\Delta}) \text{ dlt } \not\Rightarrow (X,\Delta) \text{ dlt}$$
$$\Longleftarrow$$

E.g. $(\mathbb{A}^2, x^2+y^2=0)$ this is dlt$/\mathbb{C}$ but not over $\mathbb{R}$. [snc not étale local]

Proof: Suppose (X, Δ) has a good minimal model 11

$$X \overset{f}{\dashrightarrow} Y, \quad \bar{X} \overset{\bar{f}}{\dashrightarrow} \bar{Y}$$

then $(\bar{Y}, \bar{\Gamma})$ is a semi-ample model, $(K_{\bar{Y}} + \bar{\Gamma})$ is semi-ample, f is non-positive $\implies$ (Lemma 2.9.1) that (X, Δ) has a good minimal model

Suppose $(\bar{X}, \bar{\Delta})$ has a good minimal model $(\bar{Y}, \bar{\Gamma})$ run a $K_X + \Delta + tA$ MMP

$$f : X \dashrightarrow Y$$

then f is a minimal model for $(X, \Delta + tA) \Rightarrow \bar{f}$ is a weak lc model for $(\bar{X}, \bar{\Delta} + t\bar{A}) \overset{\text{Lemma 2.9.2}}{\Longrightarrow}$

If $(\bar{X}, \bar{\Delta})$ has a good minimal model then $\exists\, \varepsilon > 0$ s.t. $\bar{X} \dashrightarrow \bar{Y}$ weak lc model of $(\bar{X}, \bar{\Delta} + t\bar{A})$ for $t \in [0, \varepsilon)$ then it is a semi-ample model i.e., $K_{\bar{Y}} + \bar{\Gamma} + t\bar{B}$ is semiample for $t \in [0, \varepsilon)$ $\Rightarrow$ $K_{\bar{Y}} + \bar{\Gamma}$ is semi-ample and so

$\Rightarrow$ $K_Y + \Gamma$ is semiample

(Y, Γ) is a good minimal model

<u>Left to show</u> (Lemma 2.9.1)

(X, Δ) dlt $\mathbb{Q}$-factorial projective

If (X, Δ) has a weak lc-model then (X, Δ) has a semi-ample model

$\Leftrightarrow$ (X, Δ) has a good minimal model.

See previous lemma (page 9) for the proof of the part of Lemma 2.9.1 which is used here.