RESEARCH STATEMENT

ZACHARY SMITH

1. INTRODUCTION

My research interests cover a broad spectrum fields, including complex dynamics, Teichmüller theory, Fuchsian groups, and classical complex analysis. The unifying theme of my work is Thurston's theory of postcritically-finite rational maps and questions that arise there from.

Before describing how these subjects tie together, some historical context is needed. Complex dynamics is the study of dynamical systems obtained by iterating holomorphic maps. Most would say complex dynamics as a field began with the seminal work of Julia and Fatou in the late 1910s. While some work continued in the years after, interest in the subject waned in the midcentury. This changed in the 1980s. The discovery of the Mandelbrot set, the advent computer graphics, and multiple major breakthroughs sparked a renaissance in complex dynamics, and the field has had a vibrant and active research community ever since.

One of the breakthroughs of 1980s was the development of combinatorial models of polynomials and rational maps to better understand their dynamics. One such model is Thurston's model for postcritically-finite rational maps. A *Thurston map* is an orientation-preserving branched cover $f\colon S^2 \to S^2$ that is not a homeomorphism and such that each of its critical points has finite forward orbit; it is considered together with the data of a finite set of marked points A that contains said forward orbits. In 1982 Thurston presented his celebrated characterization theorem, which gives necessary and sufficient conditions for when such a map is "realized" by a rational map in a suitable sense. The proof, as explicated by Douady and Hubbard [DH93], uses an analytic map on Teichmüller space $\sigma_f\colon \mathcal{T}_A \to \mathcal{T}_A$ which is induced by f by pulling back complex structures. The question of whether f is realized by a rational map then reduces to whether σ_f has a fixed point in $\mathcal{T}_A$.

Interestingly, the main requirement in Thurston's theorem is the (non)existence of Jordan curves in $S^2 \setminus A$ with certain invariance properties. More generally, every Thurston map induces a pullback relation on isotopy classes of Jordan curves in $S^2 \setminus A$. The restriction $f : S^2 \setminus f^{-1}(A) \to S^2 \setminus A$ is a covering map, so if $\gamma \subset S^2 \setminus A$ is a Jordan curve, then a component $\widetilde{\gamma}$ of $f^{-1}(\gamma)$ will also be a Jordan curve in $S^2 \setminus A$. We say that $\widetilde{\gamma}$ is a *pullback* of γ by f. Lifting isotopies shows that the set of isotopy classes of $f^{-1}(\gamma)$ rel. A depends only on the isotopy class of γ rel. A.

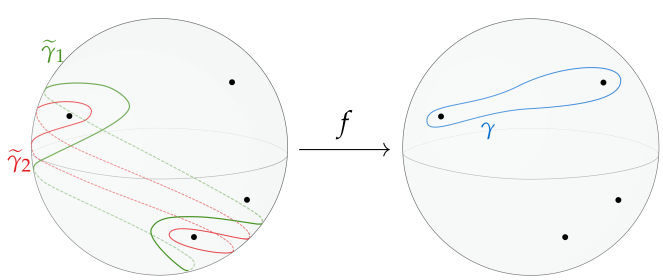

FIGURE 1. Generic picture of some curve pullbacks in a sphere with four marked points.

With this in mind, understanding the dynamics of Thurston maps f, finding suitable invariants by which to classify such maps, and understanding the dynamics of the associated pullback map σ_f are all closely related to the curve pullback relation described above. See, for example, [BEKP09],[Sel12], [Koc13], [KPS16], and [Pil22].

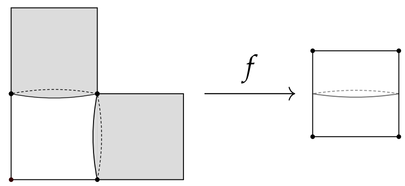

FIGURE 2. A combinatorial depiction of a Thurston map with four marked points. White tiles are mapped to the front face of the "pillow" on the right (which is a copy of S^2), and gray tiles are mapped to the back face.

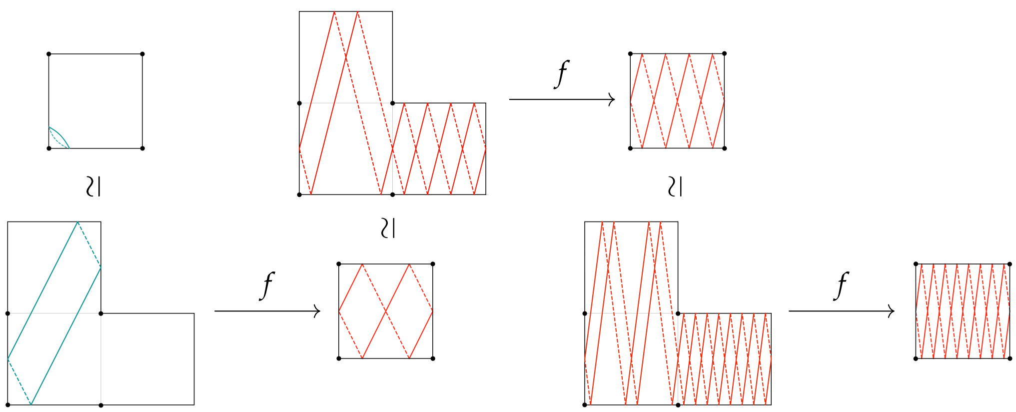

FIGURE 3. Some iterates of the pullback relation for the previous map, where the isotopy class relative to the marked points is redrawn in the square pillow after each pullback. Using the bijection between essential curve classes and the extended rationals described later, this represents $\mu_f\colon 8/1 \mapsto 4/1 \mapsto 2/1 \mapsto o$, where o denotes the "trivial" curve classes.

One of the major open problems in the study of the pullback relation on curves is the following (see [Lod13] and [Pil22]):

Conjecture 1.1 (Finite Global Curve Attractor Conjecture). *Let $f : S^2 \to S^2$ be a Thurston map with hyperbolic orbifold that is realized by a rational map. Then there is a finite set $\mathcal{A}(f)$ of Jordan curves in $S^2 \setminus A$ with the following property: for every Jordan curve γ in $S^2 \setminus A$ there is a positive integer $N(\gamma)$ such that, for $n \geqslant N(\gamma)$, all pullbacks $\widetilde{\gamma}$ of γ under f^n are contained in $\mathcal{A}(f)$ up to isotopy* rel. A.

Stated less formally, if one iterates the pullback relation on curves, then one eventually lands in a finite set of isotopy classes. This problem appears to be quite difficult, as even if one restricts to the case of four marked points (so $|A| = 4$) there are relatively few classes of rational maps for which it has been verified. Searching for partial solutions of this conjecture is the primary focus of my research and I discuss it, and my results, further in Section 2.

It turns out that the Thurston pullback map and the pullback operation on curves are related to each other in a significant way: the map $\sigma_f\colon \mathcal{T}_A \to \mathcal{T}_A$ admits an extension to the *Weil-Petersson boundary* of Teichmüller space, and the pullback operation on curves is then encoded in the boundary behavior of this extension. This correspondence is demonstrated by Selinger in [Sel12]. In the case of $|A| = 4$, there is the pleasingly simple description of all these objects. The Teichmüller space is just the upper half-plane $\mathbb{H}$ and the Weil-Petersson boundary as just the extended rationals $\widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$. On the other hand, the isotopy classes $[\gamma]$ of (essential) Jordan curves of S^2 are in bijection with $\widehat{\mathbb{Q}}$ (one may think of these as representing curves with rational slopes), and the pullback operation on curves is just a function $\mu_f\colon \widehat{\mathbb{Q}} \to \widehat{\mathbb{Q}} \cup \{o\}$, where o represents isotopy classes of "trivial curves". These two pullback maps are essentially the same on the extended rationals (up to conjugation by negative inversion; see, e.g., [CFPP12, Section 6]).

Thus, in the case where $|A| = 4$, the existence of a finite global curve attractor is equivalent to the existence of a finite attractor on rational cusps for $\sigma_f\colon \mathbb{H} \to \mathbb{H}$. I mention this setting not only because I employ it in Section 2, but also because it is the starting point for the other major component of my research. The map $\sigma_f\colon \mathbb{H} \to \mathbb{H}$ carries additional structure in the form of a family of a functional identities:

$$\sigma_f \circ g = \varphi_f(g) \circ \sigma_f$$

where g ranges over a finite index subgroup H of $G = \mathrm{PMod}(S^2, A)$, thinking of the latter as a discrete subgroup of $\mathrm{Aut}(\mathbb{H})$, and $\varphi_f : H \to G$ is a homomorphism (see [KPS16]). Analytic maps on the upper half-plane $\mathbb{H}$ satisfying such a collection of functional identities (with g ranging over a Fuchsian group) are called *polymorphic maps* in the classical literature. The notion of polymorphic maps goes all the way back to Fricke and Klein, who originally studied them in relation to elliptic modular forms (see, e.g., [Fri12]). More recently they have been the subject of investigations by Hejhal [Hej75, Hej76], Mejía and Pommerenke [MP12a, MP12b, MP08], and many others. The framework of polymorphic maps has not, to my knowledge, been invoked in the context of the Thurston pullback before. Using it, I have obtained a new proof of Thurston's characterization theorem in the $|A| = 4$ case which uses only classical complex analysis and the fact that σ_f is polymorphic, which is remarkably different in flavor from Thurston's original proof as presented in [DH93]. I have also obtained further partial progress on the curve attractor problem using these methods. I elaborate on this result and others in Section 3.

2. The Finite Global Curve Attractor Problem

We will begin this section with a more careful description of Thurston maps. Let $f\colon S^2 \to S^2$ be an orientation-preserving branched covering map with degree $\deg(f) \geqslant 2$. A point $c \in S^2$ where the local mapping degree $\deg(f, c)$ is at least 2 is a *critical point*, and we denote the set of critical points by C_f. A *postcritical point* of f is a point $p \in S^2$ of the form $p = f^n(c)$ where c is a critical point of f and f^n denotes the nth iterate of f with n a nonnegative integer. We denote the set of all postcritical points P_f, so that

$$P_f = \bigcup_{n \geqslant 1} \{f^n(c) : c \in C_f\}.$$

If $|P_f|$ is finite then f is said to be *postcritically-finite*.

Definition 2.1. Let $f\colon S^2 \to S^2$ be an orientation-preserving postcritically-finite branched covering map with $\deg(f) \geqslant 2$. Let $A \subset S^2$ be a finite set of marked points with the properties $P_f \subset A$ and $f(A) \subset A$. We call the map of pairs $f\colon (S^2, A) \to (S^2, A)$ a *Thurston map*.

We remark that our definition is a slight extension of the most basic one, where one takes $A = P_f$.

We have already described the pullback relation on curves and the finite global curve attractor conjecture in the introduction. We give a brief overview of the cases for which it has been verified. Koch, Pilgrim, and Selinger have shown it holds when the virtual endomorphism is contracting [KPS16]. Belk, Lanier, Margalit, and Winarski have proven it for all postcritically-finite polynomials [BLMW22]. Hlushchanka has proven it for rational maps where every critical point is a fixed point [Hlu19]. In the case of four postcritical points it is also known for certain NET maps [Lod13, FKK$^+$17], for all quadratic non-Lattès maps [KL19], and for maps obtained from a certain blowup of 2×2-Lattès maps [BHI21]. The general conjecture—even in the simplest nontrivial case of four marked points—still remains open.

I now describe my results. I have proven

Theorem 2.2 (Smith '23)**.** *Let $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational Thurston map with a set A of four marked points. If the postcritical set $P_f \subset A$ has at most three points, then (f, A) has a finite global curve attractor.*

Here are some of the ideas behind the proof.

As we described in the introduction, it is sufficient to show σ_f has a *finite cusp attractor* (FCA). An FCA is a finite subset $\mathcal{A} \subset \widehat{\mathbb{Q}}$ such that, for each $r \in \widehat{\mathbb{Q}}$, either $\sigma_f^N(r) \in \mathbb{H}$ for some N or $\sigma_f^n(r) \in \mathcal{A}$ for all n sufficiently. We thus wish to understand the boundary dynamics of the map σ_f.

In our case, the pullback map on Teichmüller space σ_f covers a *map* on moduli space, and this map will essentially be the same as f itself. This allows one to explicitly relate the dynamics of σ_f to the dynamics of f. The other major idea is what I call the "leashing argument". Roughly speaking, we affix a horoball at each rational cusp, and then attach this horoball to the fixed point of Teichmüller space by a "leash". With respect to a clever choice of metric (specifically, a pullback of a suitably chosen orbifold metric), the length of this leash coarsely contracts, and hence "tightens" by iterating σ_f. There will only be finitely many horoballs that we may land on after this procedure, and the base cusps of these horoballs will be exactly the desired attractor on cusps.

This result admits several possible extensions to the four postcritical point case. The easiest one to state uses a criterion due to Kelsey and Lodge for detecting when a rational Thurston map with four postcritical points is obtained by post-composing a map satisfying the previous theorem by a Möbius transformation (see [KL19, Proposition 2.5]). In particular:

Theorem 2.3 (Smith '23)**.** *Let $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational Thurston map with four postcritical points. If one of the postcritical points of f is statically trivial, then (f, P_f) has a finite global curve attractor.*

It is also possible to construct examples which have a finite global curve attractor by post-composing maps that satisfy Theorem 2.2 by flexible Lattès maps, but detecting this situation seems difficult.

Question 2.4. Is there a criterion which detects when a Thurston map is obtained by composing a map (g, A) satisfying the previous theorems with a flexible Lattès map?

Question 2.5. Can any of these techniques be bootstrapped into the setting where there are four postcritical points and $A = P_f$?

3. Polymorphic Maps

In this section I will discuss some of my work in the theory of polymorphic maps as it relates to Thurston theory. Here is the formal definition that I will use:

Definition 3.1. Let G be a finite coarea Fuchsian group, and let $\varphi\colon G \to \mathrm{Aut}(\mathbb{H})$ be a homomorphism. We will say a nonconstant holomorphic function $\sigma\colon \mathbb{H} \to \mathbb{H}$ is *φ-polymorphic* if σ satsifies the intertwining relation

$$\sigma \circ g = \varphi(g) \circ \sigma$$

for all $g \in G$.

It should be noted that this definition is somewhat restricted compared to the definition used by other authors. A more general definition weakens f to being a meromorphic function $\mathbb{H} \to \widehat{\mathbb{C}}$, and only requires the image of the homomorphism φ to be a Möbius transformation rather than specifically an automorphism of $\mathbb{H}$.

The study of polymorphic maps has a venerable history going all the way back to Fricke and Klein (see, e.g., [Fri12]). The term "polymorphic" was coined by Fricke (see [BWFH21, pp. 432]) so as to contrast these functions with automorphic forms.

As stated in the introduction, I have been able to use the framework of polymorphic maps to produce a new proof of Thurston's theorem in the case of four postcritical points. Here is a precise formulation of the theorem in this case:

Theorem 3.2 (Thurston's Characterization Theorem)**.** *Let $f : S^2 \to S^2$ be a Thurston map with hyperbolic orbifold and $|P_f| = 4$. Then f is realized by a rational map if and only if every invariant essential Jordan curve γ has $\lambda_f(\gamma) < 1$, i.e., f has no Thurston obstruction.*

I will say a few words about the strategy of the argument in order to convince the reader my proof really is different in flavor from the standard one. The main idea is to apply the classical Denjoy-Wolff theorem, which says that if a holomorphic map $\sigma\colon \mathbb{H} \to \mathbb{H}$ is not the identity, then it has unique fixed point $\tau_0 \in \overline{\mathbb{H}}$ with the property $|\sigma'(\tau_0)| \leqslant 1$, where this derivative is interpreted in the sense of angular derivatives when $\tau_0 \in \partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$. The point τ_0 is called the Denjoy-Wolff (DW) point. Meanwhile, a Thurston map f is realized by a rational map when $\sigma_f\colon \mathbb{H} \to \mathbb{H}$ has a fixed point in $\mathbb{H}$. In light of the aforementioned theorem, it suffices to show that the DW point of σ_f occurs on the boundary. This can be established by making a careful study of the angular derivatives of polymorphic maps. The only particulars about the Thurston pullback that we need beyond the fact that σ_f is polymorphic is some knowledge about the lifts of Dehn twists by f, and the data we need is exactly encoded by the Thurston multiplier.

There are other facts about the pullback operation that can be proven using these calculations. Here is another example: let $c(f) = \max(1/\lambda(r) - 1)$ with $r \in \widehat{\mathbb{Q}}$ ranging over all boundary fixed points of σ_f. Assuming the map is unobstructed, then $c(f) > 0$ and one can use an inequality due to Cowen and Pommerenke [CP82, Theorem 4.1(i)] to show

$$|\mathrm{Fix}(\sigma_f) \cap \widehat{\mathbb{Q}}| \leqslant c(f) \frac{1 - |\sigma_f'(\tau_0)|^2}{|1 - \sigma_f'(\tau_0)|^2}$$

where τ_0 is the DW point of σ_f in $\mathbb{H}$. Put differently, this shows that there are finitely many isotopy classes of curves fixed by the pullback operation. This reproves a result of Parry (see [Par18, Theorem 10.1]).

I have also been able to obtain further partial progress on the curve attractor problem by combining the above analysis with the "leashing" argument used to prove Theorem 2.2. For a Thurston map (f, A) with $|A| = 4$, call f *completely unobstructed* if *all* of its Thurston multipliers have $\lambda_f(r) < 1$.

Theorem 3.3 (Smith '23). *If (f, A) is a completely unobstructed Thurston map where A is a set of four marked points, then (f, A) has a finite global curve attractor.*

It seems that much more can be said using these tools. Work is ongoing.

4. Future Directions

As we saw in the previous section, many results in Thurston theory regarding the pullback relation on curves in the $|A| = 4$ can be proven from the perspective of polymorphic functions and appeals to classical complex analysis. One wonders, then, if there is a formulation of the FCA problem for *all* polymorphic functions—not just those which arise as a Thurston pullback map.

Question 4.1. Is there a formulation of the FCA problem for generic polymorphic maps $\sigma\colon \mathbb{H} \to \mathbb{H}$ which is provable using only classical complex analysis techniques?

Here is a first stab at what the analogous problem might look like:

Conjecture 4.2. *Let G_1, G_2 be finite coarea Fuchsian groups with a common set of cusps, let $\varphi\colon G_1 \to G_2$ be a homomorphism, and let $\sigma\colon \mathbb{H} \to \mathbb{H}$ be an analytic φ-polymorphic map with interior DW point. Then σ has a finite cusp attractor.*

Regardless of whether some generalized variant of the FCA problem holds, it seems that the theory of boundary dynamics of polymorphic maps is currently underdeveloped and there should be rich structure in this setting.

Question 4.3. What else can be said about the boundary dynamics of a polymorphic map $\sigma\colon \mathbb{H} \to \mathbb{H}$?

I conclude by describing one last side project. Another important setting in which polymorphic maps appear (using the more general notion of functions from $\mathbb{H}$ to $\widehat{\mathbb{C}}$) is as a ratio $f = \alpha_1/\alpha_2$ of the periods α_1 and α_2 of certain elliptic integrals. Such functions determine a second order Fuchsian ODE which can be calculated by finding its Schwarzian derivative (see, e.g., [Hej75]). On the other hand, differential equations of this type provide a description of f as a conformal mapping of circular arc polygons (see, e.g., [Neh75, Section V.7]). Recently, Bonk [Bon22] was able to use a similar technique to give a precise geometric description of the function $\tau \mapsto \eta_1/\eta_2$, where η_1 and η_2 are the pseudo-periods of the Weierstrass ζ-function and $\tau = \omega_1/\omega_2$ is the ratio of the generators of the underlying lattice. These techniques seem to apply to period ratios of many other naturally appearing elliptic integrals. I am performing computer calculations to study this further.

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