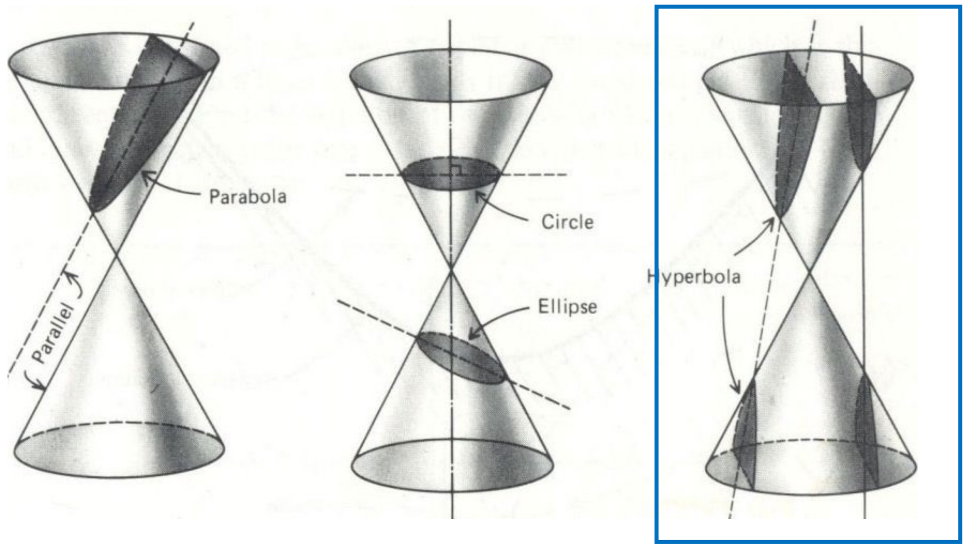
Math.1330 – Section 8.3
Hyperbolas

A **hyperbola** is a conic section formed by intersecting a right circular cone with a plane at an angle such that both halves of the cone are intersected. This intersection produces two separate unbounded curves that are mirror images of each other.



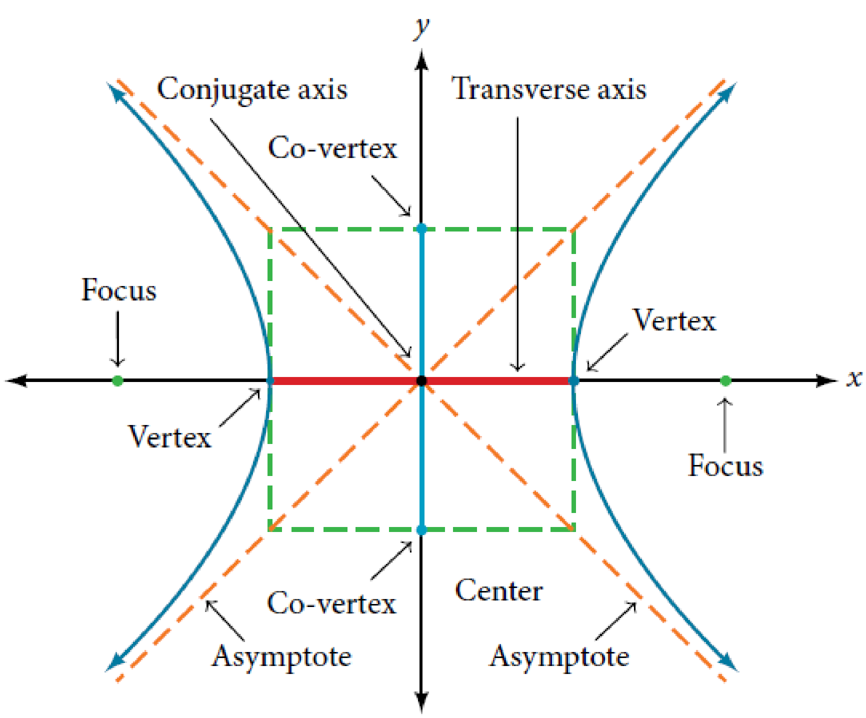
Definition: A *hyperbola* is the set of all points, the difference of whose distances from two fixed points is constant. Each fixed point is called a *focus* (plural = *foci*).

The *focal axis* is the line passing through the foci.

Notice that the definition of a hyperbola is very similar to that of an ellipse.
The distinction is that the hyperbola is defined in terms of the *difference* of two distances, whereas the ellipse is defined in terms of the *sum* of two distances.

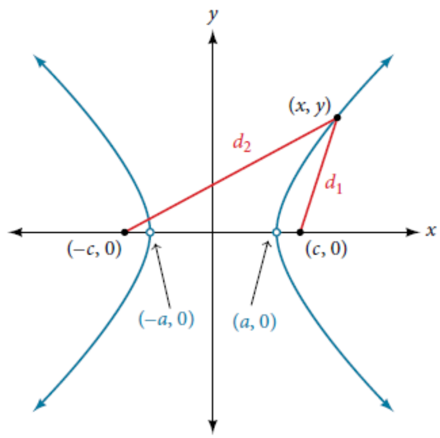
As with the ellipse, every hyperbola has two axes of symmetry.

- The **transverse axis** is a line segment that passes through the center of the hyperbola and has vertices as its endpoints.
- The **foci** lie on the line that contains the transverse axis.
- The **conjugate axis** is perpendicular to the transverse axis and has the co-vertices as its endpoints.
- The **center of a hyperbola** is the midpoint of both the transverse and conjugate axes, where they intersect.
- Every hyperbola also has two **asymptotes** that pass through its center. As a hyperbola recedes from the center, its branches approach these asymptotes.
- The **central rectangle** of the hyperbola is centered at the origin with sides that pass through each vertex and co-vertex; it is a useful tool for graphing the hyperbola and its asymptotes. To sketch the asymptotes of the hyperbola, simply sketch and extend the diagonals of the central rectangle



Deriving the Equation of a Hyperbola Centered at the Origin

Let $(-c, 0)$ and $(c, 0)$ be the foci of a hyperbola centered at the origin. The hyperbola is the set of all points (x, y) such that the difference of the distances from (x, y) to the foci is constant, as shown below



If $(a, 0)$ is a vertex of the hyperbola, the distance from $(-c, 0)$ to $(a, 0)$ is $a-(-c) = a+c$. The distance from $(c, 0)$ to $(a, 0)$ is $c-a$.
The difference of distances from the foci to the vertex is $(a+c)-(c-a) = 2a$.
If (x, y) is a point on the hyperbola, then we can define the following variables:

$d_2 =$ the distance from $(-c, 0)$ to (x, y)

$d_1 =$ the distance from $(c, 0)$ to (x, y)

By the definition of a hyperbola, $d_2 - d_1$ is constant for any point (x, y) on the hyperbola. We know that the difference of these distances is $2a$ for vertex $(a, 0)$. It follows that $d_2 - d_1 = 2a$ for any point on the hyperbola. We will begin the derivation by applying the distance formula. The rest of the derivation is algebraic.

$$d_2 - d_1 = \sqrt{\left(x-(-c)\right)^2 + (y-0)^2} - \sqrt{\left(x-(c)\right)^2 + (y-0)^2} = 2a$$

$$\sqrt{\left(x+(c)\right)^2 + y^2} - \sqrt{\left(x-(c)\right)^2 + y^2} = 2a$$

After a lot of algebraic calculations and using $b^2 = c^2 - a^2$, one can derive

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Basic Horizontal Hyperbola centered at (0,0):

Equation: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

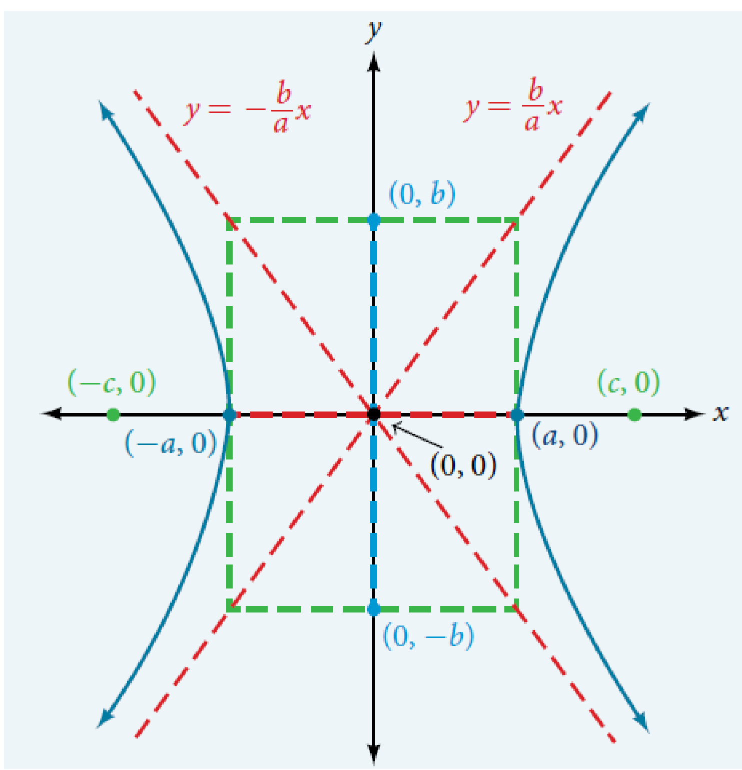
Asymptotes: $y = \pm \frac{b}{a} x$

Foci: $(\pm c, 0)$, where $b^2 = c^2 - a^2$

Vertices: $(\pm a, 0)$
Length of transverse axis is $2a$.

Co-Vertices: $(0, \pm b)$
Length of conjugate axis is $2b$.

Eccentricity: $\frac{c}{a}$ (> 1)



Basic Vertical Hyperbola centered at (0,0):

Equation: $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

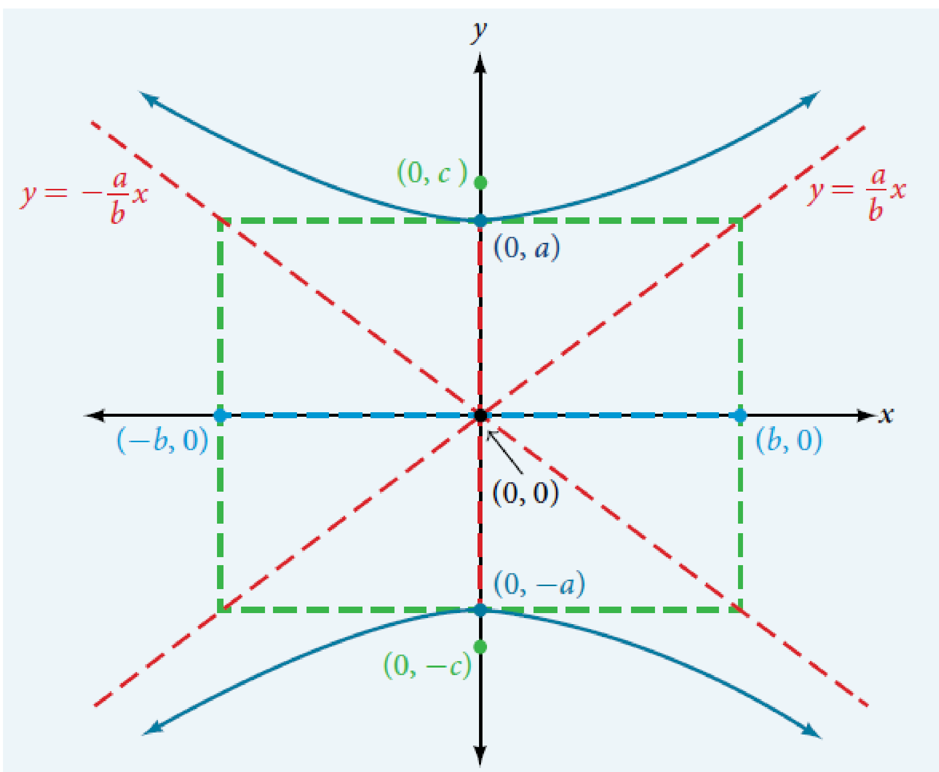
Asymptotes: $y = \pm\frac{a}{b}x$

Foci: $(0, \pm c)$, where $b^2 = c^2 - a^2$

Vertices: $(0, \pm a)$
Length of transverse axis is $2a$.

Co-Vertices: $(\pm b, 0)$
Length of conjugate axis is $2b$.

Eccentricity: $\frac{c}{a}$ (> 1)



Graphing hyperbolas:

To graph a hyperbola with center at the origin:

- Rearrange into the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ or $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$.

- Decide if it's a "horizontal" or "vertical" hyperbola.
    - if x^2 comes first, it's horizontal (vertices are on x-axis).
    - If y^2 comes first, it's vertical (vertices are on y-axis).
- Use the square root of the number under x^2 to determine how far to measure in x-direction.

- Use the square root of the number under y^2 to determine how far to measure in y-direction.

- Draw a box with these measurements.

- Draw diagonals through the box. These are the asymptotes. Use the dimensions of the box to determine the slope and write the equations of the asymptotes.

- Put the vertices at the edge of the box on the correct axis.

- Then draw a hyperbola, making sure it approaches the asymptotes smoothly.

- $c^2 = a^2 + b^2$ where a^2 and b^2 are the denominators.

- The foci are located c units from the center, on the same axis as the vertices.

When graphing hyperbolas, you will need to find the orientation, center, values for a, b and c, lengths of transverse and converse axes, vertices, foci, equations of the asymptotes, and eccentricity.

Example 1: Find all relevant information and graph $\frac{x^2}{36} - \frac{y^2}{4} = 1$.

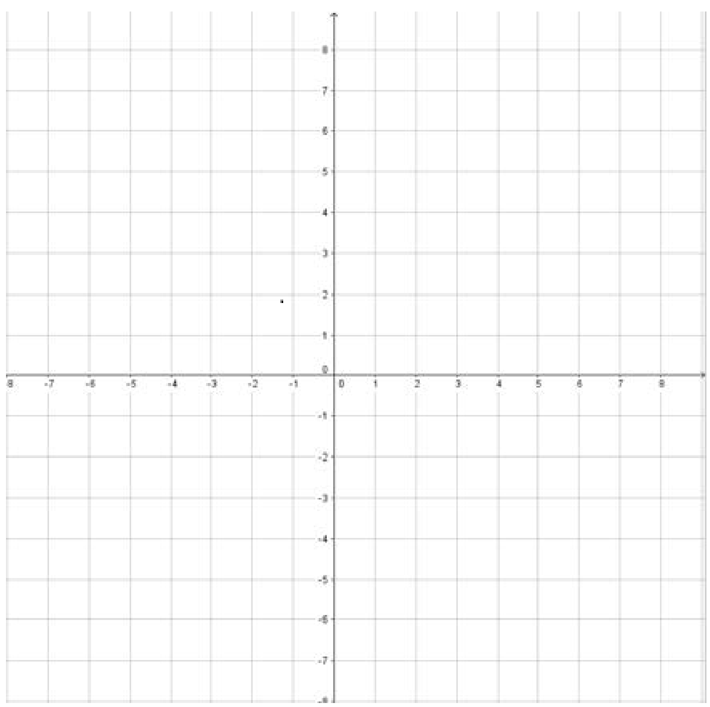
Vertices:
Transverse Axis:
Length of transverse axis:

Co-Vertices:
Conjugate axis:
Length of conjugate axis:

Foci:

Slant Asymptotes:

Eccentricity:



Example 2: Find all relevant information and graph $\frac{y^2}{16} - \frac{x^2}{9} = 1.$.

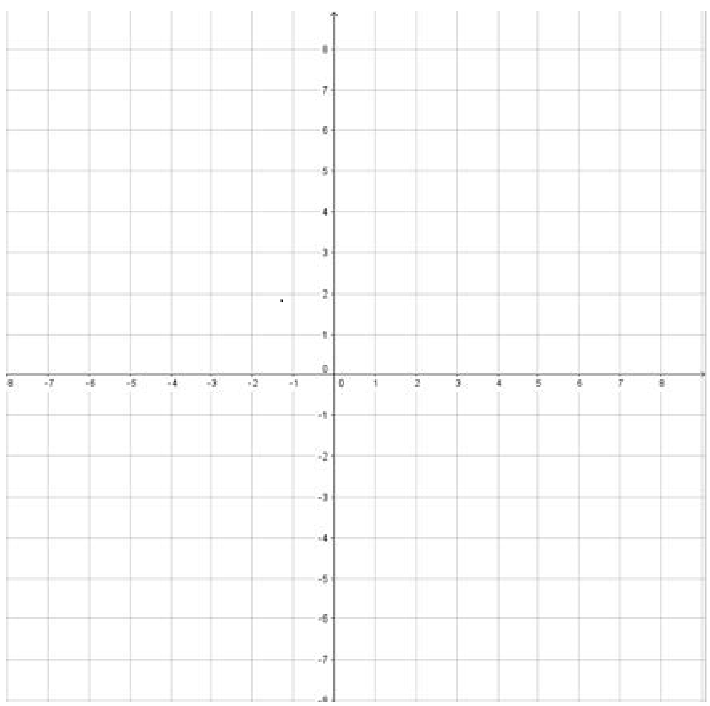
Vertices:
Transverse Axis:
Length of transverse axis:

Co-Vertices:
Conjugate axis:
Length of conjugate axis:

Foci:

Slant Asymptotes:

Eccentricity:



The equation of a hyperbola with center (h, k) not at the origin:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \quad or \quad \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1.$$

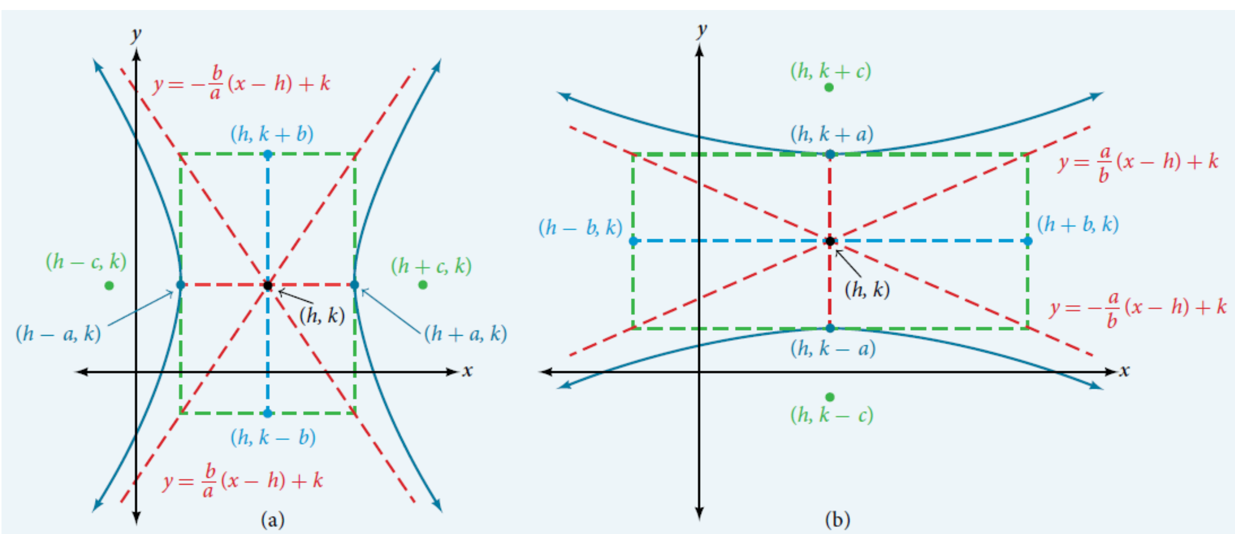
To graph a hyperbola with center not at the origin:

- Rearrange (complete the square if necessary) to look like

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \quad or \quad \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1.$$

- Start at the center (h,k) and then graph it as before.

- To write down the equations of the asymptotes, start with the equations of the asymptotes for the similar hyperbola with center at the origin. Then replace x with $x-h$ and replace y with $y-k$.



Example 3: Find all relevant information and graph $\frac{(x-1)^2}{16} - \frac{(y-3)^2}{9} = 1$

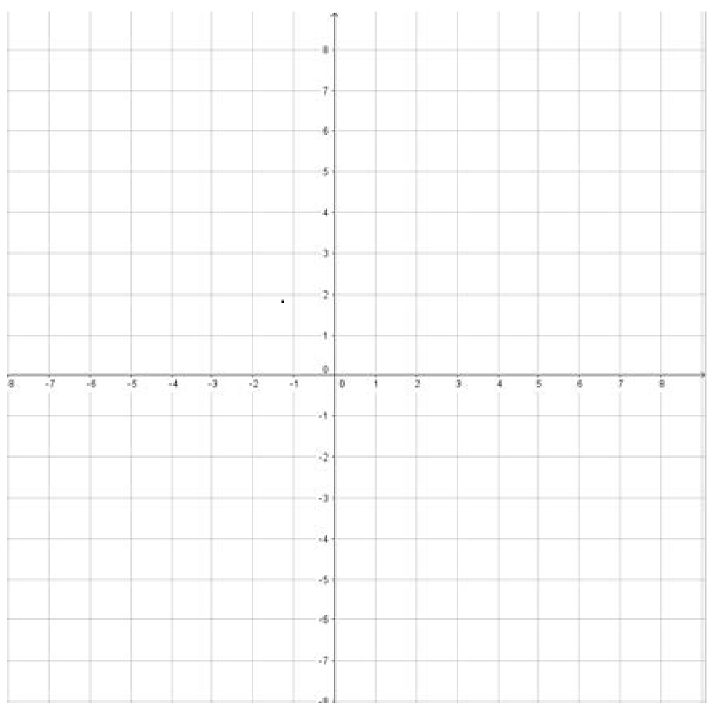
Vertices:
Transverse Axis:
Length of transverse axis:

Co-Vertices:
Conjugate axis:
Length of conjugate axis:

Foci:

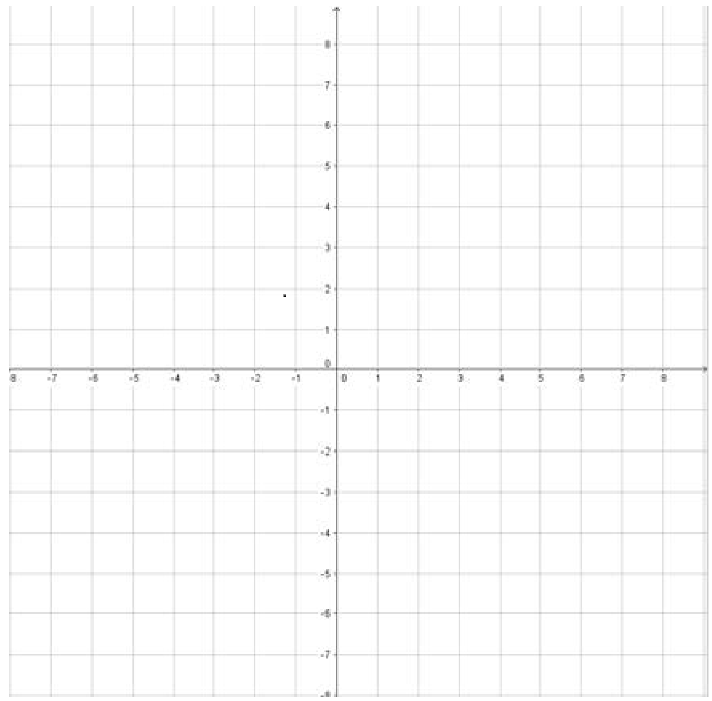
Slant Asymptotes:

Eccentricity:



Example 4: Write the equation in standard form, find all relevant information and graph

$9x^2 - 16y^2 - 18x + 96y = 279$.



Vertices:
Transverse Axis:
Length of transverse axis:

Co-Vertices:
Conjugate axis:
Length of conjugate axis:

Foci:

Slant Asymptotes:

Eccentricity:

Example 5: Write an equation of the hyperbola with center at (-2, 3), one vertex is at (-2, -2) and eccentricity is 2.

Example 6: Write an equation of the hyperbola if the vertices are (4, 0) and (4, 8) and the asymptotes have slopes ± 1.