Discrete Mathematics
Predicates and Quantifiers

Predicates

Propositional logic is not enough to express the meaning of all statements in mathematics and natural language.

Examples:
Is "$x > 1$" True or False?

Subject Property of the subject = Predicate

Is "x is a great tennis player" True or False?

Predicate Logic

- Variables: x, y, z, etc.
- Predicates: $P(x)$, $Q(x)$, etc.
- Quantifiers: Universal and Existential.
- Connectives from propositional logic carry over to predicate logic.

A **predicate** $P(x)$ is a declarative sentence whose truth value depends on one or more variables.

$P(x)$ is also said to be the value of the **propositional function** P at x.

$P(x)$ becomes a **proposition** when a value of x is assigned from the domain U.

Examples (Propositional Functions):

1. Let $P(x)$ be "$x \geq 1$." Determine the truth value of

   a. $P(2)$

   $2 \geq 1$ True

   b. $P(-2) \to P(1)$

   F T True

2. Let $R(x, y, z)$ be "$x + y = z$." Find these truth values:

   a. $R(2, -1, 5)$

   $2 + (-1) = 5$

   False

   b. $R(x, 3, z)$

   NOT a proposition!

   Cannot determine

Quantifiers

We need *quantifiers* to express the meaning of English words including *all* and *some*:

- "All students in this class are computer science majors"
- "There is a math major student in this class"

The two most important quantifiers are:

- *Universal Quantifier*, "For all," symbol: $\forall$
- *Existential Quantifier*, "There exists," symbol: $\exists$

We write as in $\forall x\, P(x)$ and $\exists x\, P(x)$.

- $\forall x\, P(x)$ asserts $P(x)$ is true for <u>every</u> x in the *domain*.

  If = $\{x_1, x_2, \ldots, x_n\}$, then $\forall x\, P(x) = P(x_1) \wedge P(x_2) \ldots \wedge P(x_n)$. conj.

- $\exists x\, P(x)$ asserts $P(x)$ is true for <u>some</u> x in the *domain*.

  If = $\{x_1, x_2, \ldots, x_n\}$, then $\exists x\, P(x) = P(x_1) \vee P(x_2) \ldots \vee P(x_n)$. disj.

Examples:

1. Let $P(x)$: "$x > -x$" with the domain of all positive real numbers.
   Find the truth value of $\forall x\, P(x)$. statement!

   True

2. Let $P(x)$: "$x > -x$" with the domain of all real numbers.
   Find the truth value of $\forall x\, P(x)$.

   False $x = -1$ $-1 > -(-1)$
   $-1 > 1$ False

- The truth value of $\exists x\, P(x)$ and $\forall x\, P(x)$ depends BOTH on the propositional function $P(x)$ and on the domain U.

Quantifiers		
Statement	***When True?***	***When False?***
$\forall x\, P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x\, P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Example: Suppose the domain of the propositional function $P(x)$: $x^2 \leq x$ consists of $\{1, 2, 3\}$. Write out each of the following propositions using conjunction or disjunction and determine its truth value.

1. $\forall x\, P(x) = P(1) \wedge P(2) \wedge P(3)$
   $T \wedge F \wedge F$
   False
2. $\exists x\, P(x) = P(1) \vee P(2) \vee P(3)$
   $T \vee F \vee F$
   True

An element for which $P(x)$ is false is called a **counterexample of** $\forall x\, P(x)$

Precedence of Quantifiers

The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators.

Example: $\forall x\, P(x) \vee Q(x)$ means $(\forall x\, P(x)) \vee Q(x)$. $\forall x\, (P(x) \vee Q(x))$ means something different.

Negating Quantifiers

De Morgan laws for quantifiers (the rules for negating quantifiers) are:

$$\boxed{\begin{array}{c} \neg \forall x\, P(x) \equiv \exists x \neg P(x) \\ \neg \exists x\, P(x) \equiv \forall x \neg\, P(x) \end{array}}$$

Example: Express each of these statements using quantifiers. Then form a negation of the statement, so that no negation is left of a quantifier. Next, express the negation in simple English.

1. "Some old dogs can learn new tricks."

$T(x) = x$ can learn new tricks

Domain = old dogs

$\exists x\, T(x)$

$\neg \exists x\, T(x) \equiv \forall x \neg T(x)$

No old dog can learn new tricks.

All old dogs cannot learn new tricks.

2. "Every bird can fly."

$F(x) = x$ can fly

Domain = birds

$\forall x F(x)$

$\neg \forall x F(x) \equiv \exists x \neg F(x)$

Some birds cannot fly.

There exists a bird that cannot fly.

3. $\forall x(x^2 > x)$

$\neg \forall x(x^2 > x) \equiv \exists x \neg (x^2 > x) \equiv \exists x (x^2 \leq x)$

Translating from English into Logical Expressions

Examples: Translate the statements into the logical symbols. Let x be in set of all students in this class.

1. Someone in your class can speak Hindi.
2. Everyone in your class is friendly.
3. There is a student in your class who was not born in California.

$H(x) =$ " x *speaks Hindi*", $F(x) =$ " x *is friendly,*" $C(x) =$ " x *was born in California.*"

1. $\exists x H(x)$

2. $\forall x F(x)$

3. $\exists x \neg C(x) \equiv \underline{\neg \forall x\, C(x)}$

Not everyone in your class was born in California

Example: Translate the following sentence into predicate logic and give its negation:

"Every student in this class has taken a course in Java."

Solution:

First, decide on the domain U!

Solution 1: If U is all students in this class, define a propositional function J(x) denoting "x has taken a course in Java" and translate as $\forall x J(x)$

Solution 2: But if U is all people, also define a propositional function S(x) denoting "x is a student in this class" and translate as

$$\forall x\, (S(x) \to J(x))$$

$$\neg \forall x (S(x) \to J(x)) \equiv \exists x \neg (\overset{p}{S(x)} \overset{\to}{\to} \overset{q}{J(x)})$$

$$\equiv \exists x \neg (\neg S(x) \vee J(x))$$

$$\equiv \exists x \neg (\neg S(x)) \wedge \neg J(x))$$

$$\equiv \exists x (S(x) \wedge \neg J(x))$$

$$p \to q \equiv \neg p \vee q$$

Example: Translate the following sentence into predicate logic:

"Some student in this class has taken a course in Java."

Solution:

First, decide on the domain *U*!

Solution 1: If *U* is all students in this class, translate as $\exists x J(x)$

Solution 2: But if *U* is all people, then translate as

$$\exists x (S(x) \wedge J(x))$$

$$\neg \exists x (S(x) \wedge J(x))$$

$$\equiv \forall x \neg (S(x) \wedge J(x))$$

$$\equiv \forall x (\neg S(x) \vee \neg J(x))$$