Lecture 27 11.7 Power Series 11.8 Differentiation and Integration of Power Series

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1 Power Series

1.1 Geometric Series and Variations

Geometric Series
Geometric Series: $\sum_{k=0}^{\infty} x^k$

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots \begin{cases} \dfrac{1}{1-x}, & \text{if } |x| < 1, \\ \text{diverges}, & \text{if } |x| \geq 1. \end{cases}$$

Power Series
Define a function f on the interval $(-1, 1)$

$$f(x) = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \quad \text{for } |x| < 1$$

As the Limit
f can be viewed as the limit of a sequence of polynomials:

$$f(x) = \lim_{n\to\infty} p_n(x),$$

where $p_n(x) = 1 + x + x^2 + x^3 + \cdots + x^n$.

Variations on the Geometric Series (I)
Closed forms for many power series can be found by relating the series to the geometric series
Examples 1.

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} (-1)^k x^k = 1 - x + x^2 - x^3 + \cdots \\ &= \sum_{k=0}^{\infty} (-x)^k = \frac{1}{1-(-x)} = \frac{1}{1+x}, \quad \text{for } |x| < 1. \end{aligned}$$

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} 2^k x^{k+2} = x^2 + 2x^3 + 4x^4 + 8x^5 + \cdots \\ &= x^2 \sum_{k=0}^{\infty} (2x)^k = \frac{x^2}{1-2x} \quad \text{for } |2x| < 1. \end{aligned}$$

Variations on the Geometric Series (II)

Closed forms for many power series can be found by relating the series to the geometric series

Examples 2.

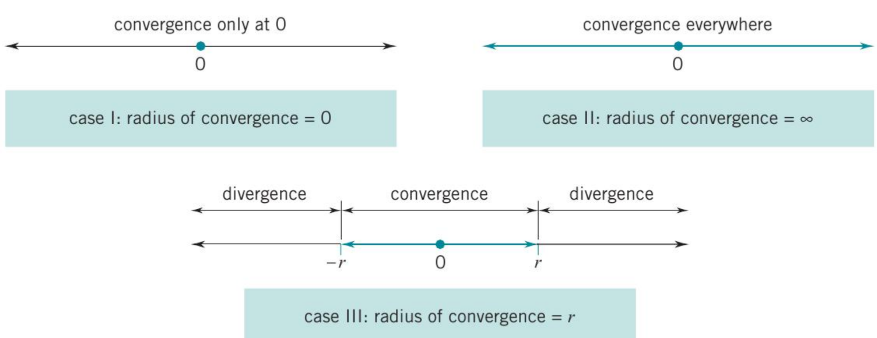
$$f(x) = \sum_{k=0}^{\infty} (-1)^k x^{2k} = 1 - x^2 + x^4 - x^6 + \cdots$$
$$= \sum_{k=0}^{\infty} (-x^2)^k = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}, \quad \text{for } |x| < 1.$$

$$f(x) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{3^k} = x + \frac{1}{3}x^3 + \frac{1}{9}x^5 + \frac{1}{27}x^7 + \cdots$$
$$= x \sum_{k=0}^{\infty} \left(\frac{x^2}{3}\right)^k = \frac{x}{1-(x^2/3)} = \frac{3x}{3-x^2} \quad \text{for } |x^2/3| < 1.$$

1.2 Radius of Convergence

Radius of Convergence

There are exactly three possibilities for a power series: $\sum a_k x^k$.



Radius of Convergence: Ratio Test (I)

The radius of convergence of a power series can usually be found by applying the ratio test. In some cases the root test is easier.

Example 3.

$$f(x) = \sum_{k=1}^{\infty} k^2 x^k = x + 4x^2 + 9x^3 + \cdots$$

$$\text{Ratio Test} : \left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{(k+1)^2 x^{k+1}}{k^2 x^k}\right|$$

$$= \frac{(k+1)^2}{k^2}|x| \to |x| \quad \text{as } k \to \infty$$

Thus the series converges absolutely when $|x| < 1$ and diverges when $|x| > 1$.

Radius of Convergence: Ratio Test (II)

The radius of convergence of a power series can usually be found by applying the ratio test. In some cases the root test is easier.
Example 4.

$$f(x) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} x^k = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \cdots = e^{-x}$$

$$\text{Ratio Test} : \left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{x^{k+1}/(k+1)!}{x^k/k!}\right|$$

$$= \frac{k!}{(k+1)!}\left|\frac{x^{k+1}}{x^k}\right| = \frac{1}{k+1}|x| \to 0 < 1 \quad \text{for all } x$$

Thus the series converges absolutely for all x.

Radius of Convergence: Ratio Test (III)

The radius of convergence of a power series can usually be found by applying the ratio test. In some cases the root test is easier.
Example 5.

$$f(x) = \sum_{k=1}^{\infty} \left(\frac{k+1}{k}\right)^{k^2} x^k = 2x + (3/2)^4 x^2 + (4/3)^9 x^3 + \cdots$$

$$\text{Ratio Test} :(|a_k|)^{\frac{1}{k}} = \left(\left(\frac{k+1}{k}\right)^{k^2} |x|^k\right)^{\frac{1}{k}} = \left(\frac{k+1}{k}\right)^k |x|$$

$$= \left(1 + \frac{1}{k}\right)^k |x| \to e|x| < 1 \quad \text{if } |x| < 1/e$$

Thus the series converges absolutely when $|x| < 1/e$ and diverges when $|x| > 1/e$.

Interval of Convergence

For a series with radius of convergence r, the interval of convergence can be $[-r, r]$, $(-r, r]$, $[-r, r)$, or $(-r, r)$.

Example 6. In general, *the behavior of a power series at $-r$ and at r is not predictable.* For example, the series

$$\sum x^k, \quad \sum \frac{(-1)^k}{k} x^k, \quad \sum \frac{1}{k} x^k, \quad \sum \frac{1}{k^2} x^k$$

all have radius of convergence 1, but the first series converges only on $(-1, 1)$, the second converges on $(-1, 1]$, but the third converges on $[-1, 1)$, the fourth on $[-1, 1]$.

Interval of Convergence

Example 7.

$$f(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k$$

$$\text{Ratio Test} : \left| \frac{a_{k+1}}{a_k} \right| = \left| \frac{x^{k+1}/(k+1)}{x^k/k} \right| = \frac{k}{k+1} |x| \to |x|$$

Thus the series converges absolutely when $|x| < 1$ and diverges when $|x| > 1$. So the radius of convergence is 1

$$x = -1 : \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (-1)^k = \sum_{k=1}^{\infty} \frac{-1}{k} \text{ diverges}$$

$$x = 1 : \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (1)^k = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \text{ converges conditionally}$$

The interval of convergence is $(-1, 1]$.

2 Differentiation and Integration

2.1 Differentiation and Integration

Differentiation and Integration Theorem

Let $f(x) = \sum a_k x^k$ be a power series with a nonzero radius of convergence r. Then

$$f'(x) = \sum a_k \, k \, x^{k-1} \quad \text{for } |x| < r$$

$$\int f(x)\, dx = \sum \frac{a_k}{k+1} x^{k+1} + C \quad \text{for } |x| < r$$

Geometric series: $\dfrac{1}{1-x} = \sum_{k=0}^{\infty} x^k \quad \text{for } |x| < 1$

Differentiation: $\dfrac{1}{(1-x)^2} = \sum_{k=0}^{\infty} k\, x^{k-1} \sum_{k=0}^{\infty} (k+1)\, x^k \ \text{ for } |x| < 1$

Integration: $-\ln(1-x) = \sum_{k=0}^{\infty} \dfrac{1}{k+1}\, x^{k+1} = \sum_{k=1}^{\infty} \dfrac{1}{k}\, x^k \ \text{ for } |x| < 1$

2.2 Examples

Power Series Expansion of $\ln(1+x)$

$$\text{Note: } \frac{d}{dx}\ln(1+x) = \frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k \ \text{ for } |x| < 1$$

$$\text{Integration: } \ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}\, x^{k+1} (+C = 0)$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k}{k}\, x^k = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

The interval of convergence is $(-1, 1]$. At $x = 1$,

$$\ln 2 = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Power Series Expansion of $\tan^{-1} x$

$$\text{Note: } \frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k} \ \text{ for } |x| < 1$$

$$\text{Integration: } \tan^{-1} x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\, x^{2k+1} (+C = 0)$$

$$= x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$$

The interval of convergence is $(-1, 1]$. At $x = 1$,

$$\tan^{-1} 1 = \sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

Outline

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