EQUAZIONI DIFFERENZIALI ESERCIZI

MARTEDÌ 15/1

Esercizio

Calcolare la soluzione del problema di Cauchy

$$\begin{cases} y' = (2\,\text{sen}\,t \cos t)\,y + 3e^{\text{sen}^2 t} \\ y(0) = 0 \end{cases}$$

$a(t) = 2\,\text{sen}\,t \cos t =$
$= \text{sen}(2t)$

$b(t) = 3e^{\text{sen}^2 t}$

$$y' - (2 \operatorname{sen} t \cos t) y = 3 e^{\operatorname{sen}^2 t}$$

$$A(t) = \int_0^t 2 \operatorname{sen} s \cos s \, ds = \operatorname{sen}^2 s \Big|_0^t =$$

$$= \operatorname{sen}^2 t$$

$$y' - (2 \operatorname{sen} t \cos t) y = 3 e^{\operatorname{sen}^2 t} \quad / \cdot e^{-\operatorname{sen}^2 t}$$

$$\underbrace{e^{-\operatorname{sen}^2 t} y'(t) - 2 \operatorname{sen} t \cos t \, e^{-\operatorname{sen}^2 t} y(t)}_{\frac{d}{dt}\left(e^{-\operatorname{sen}^2 t} y(t)\right)} = 3$$

$$\frac{d}{dt}\left(e^{-\operatorname{sen}^2 t} y(t)\right) = 3$$

Integro tra $t_0 = 0$ e t

$$\int_0^t \frac{d}{ds}\left(e^{-\operatorname{sen}^2 s} y(s)\right) ds = \int_0^t 3 \, ds$$

$$e^{-\operatorname{sen}^2 s} y(s) \Big|_0^t = 3t$$

$$e^{-\operatorname{sen}^2 t} y(t) - e^{-\operatorname{sen}^2 0} \underbrace{y(0)}_{=0} = 3t$$

$$y(t) = 3t \, e^{\operatorname{sen}^2 t} \qquad //$$

Esercizio

Sia $y = y(t)$ la soluzione del problema di Cauchy

$$\begin{cases} y' = -\dfrac{2}{t+1}\, y + \dfrac{t}{t+1}, & t>0 \\ y(1) = 0 \end{cases}$$

Calcolare $\displaystyle\lim_{t\to+\infty} \frac{y(t)}{t}$

$a(t) = -\dfrac{2}{t+1}$

$b(t) = \dfrac{t}{t+1}$

$t_0 = 1$

$y_0 = 0$

$$y' + \frac{2}{t+1}\, y = \frac{t}{t+1}$$

$$A(t) = \int_1^t -\frac{2}{s+1}\, ds = -2\ln(s+1)\Big|_1^t =$$

$$= -2\ln(t+1) + 2\ln 2 =$$

$$= \ln\left(\frac{2}{t+1}\right)^2$$

$$e^{-A(t)} = e^{-\ln\left(\frac{2}{t+1}\right)^2} = \left(\frac{t+1}{2}\right)^2$$

$$\frac{1}{e^{\ln\left(\frac{2}{t+1}\right)^2}} = \frac{1}{\left(\frac{2}{t+1}\right)^2}$$

$$y' + \frac{2}{t+1}\, y = \frac{t}{t+1} \quad \Big/ \cdot\ e^{A(t)} = \frac{(t+1)^2}{4}$$

$$\frac{(t+1)^2}{4}\, y'(t) + \frac{(t+1)}{2}\, y = \frac{t\,(t+1)}{4}$$

$$\underbrace{(t+1)^2\, y'(t) + 2(t+1)\, y}_{= \frac{d}{dt}\left((t+1)^2 y(t)\right)} = t\,(t+1)$$

$$\int_1^t \frac{d}{ds}\left((s+1)^2 y(s)\right) ds = \int_1^t s(s+1)\, ds$$

$$(t+1)^2 y(t) - (1+1)^2 \underbrace{y(1)}_{=0} = \left(\frac{1}{3}s^3 + \frac{1}{2}s^2\right|_1^t$$

$$(t+1)^2\, y(t) = \frac{1}{3}t^3 + \frac{1}{2}t^2 - \frac{5}{6}$$

$$y(t) = \frac{1/3\, t^3 + \frac{1}{2}t^2 - 5/6}{(t+1)^2}$$

$$\lim_{t \to +\infty} \frac{y(t)}{t} = \frac{1}{3}$$

(Si poteva osservare che, se $\exists \lim\limits_{t \to +\infty} \frac{y(t)}{t}$ e $\lim\limits_{t \to +\infty} y'(t) \Rightarrow \lim\limits_{t \to +\infty} \frac{y(t)}{t} = \frac{1}{3}$

$$y'(t) = -\frac{2}{t+1}\, y(t) + \frac{t}{t+1}$$

$$\lim_{t\to+\infty} \frac{y(t)}{t} \overset{(H)}{=} \lim_{t\to+\infty} y'(t) = \ell$$

$$\frac{y(t)}{t} \cdot \frac{t}{t+1} \to \ell$$

$$y'(t) = -2\,\frac{y(t)}{t+1} + \frac{t}{t+1}$$

$t \to +\infty$

$$\ell = -2\ell + 1$$

$$\Rightarrow 3\ell = 1, \quad \ell = \frac{1}{3}$$

MERCOLEDì 16/1

Esercizio

Calcolare, se esiste, la soluzione del problema

$$\begin{cases} y' + \frac{2}{t}\, y = \operatorname{sen} t \\ \lim_{t\to 0^+} y(t) = 0 \end{cases}$$

se $t > 0$

$a(t) = -\frac{2}{t}$

$b(t) = \operatorname{sen} t$

$I =]0, +\infty[$

Mi procuro tutte le soluzioni di

$y' + \frac{2}{t} y = \operatorname{sen} t$ (integrale generale) e

fra queste cerco quella (o quelle) tale che $\lim_{t \to 0^+} y(t) = 0$

$A'(t) = -\frac{2}{t}$ $\qquad A(t) = -2 \ln t = -\ln t^2$

$e^{-A(t)} = t^2$

$\hookrightarrow e^{\ln t^2} = t^2$

$y' + \frac{2}{t} y = \operatorname{sen} t \qquad / \cdot t^2$

$t^2 y' + 2t y = t^2 \operatorname{sen} t$

$\frac{d}{dt}\left(t^2 y(t)\right) = t^2 \operatorname{sen} t$

$\Rightarrow t^2 y(t) = \int t^2 \operatorname{sen} t \, dt$ lo calcolo

$\int t^2 \operatorname{sen} t \, dt = -t^2 \cos t + 2 \int t \cos t \, dt =$

$= -t^2 \cos t + 2\left[t \operatorname{sen} t - \int \operatorname{sen} t \, dt\right] =$

$$= -t^2\cos t + 2t\,\mathrm{sen}\,t + 2\cos t + c$$

$$\Rightarrow t^2 y(t) = -t^2\cos t + 2t\,\mathrm{sen}\,t + 2\cos t + c$$

$$y(t) = -\cos t + \frac{2\,\mathrm{sen}\,t}{t} + \frac{2\cos t}{t^2} + \frac{c}{t^2}$$

$$t > 0$$

Al variare di $c \in \mathbb{R}$ ottengo tutte le soluzioni dell'equazione data

$\Rightarrow$ devo trovare c affinché $\lim_{t \to 0^+} y(t) = 0$

$$0 = \lim_{t\to 0^+} y(t) = \lim_{t\to 0^+}\Big[\underbrace{-\cos t}_{\downarrow\, -1} + \underbrace{\frac{2\,\mathrm{sen}\,t}{t}}_{\downarrow\, 2} + \frac{c+2\cos t}{t^2}\Big]$$

Deve essere $\lim_{t\to 0^+} \frac{c+2\cos t}{t^2} = -1$

$$\lim_{t\to 0^+}\frac{c+2\cos t}{t^2} = \lim_{t\to 0^+}\frac{c+2\left(1-\frac{t^2}{2}+o(t^2)\right)}{t^2} \overset{\text{PdS}}{=}$$

$$= \lim_{t\to 0^+}\frac{c+2-t^2}{t^2} = \begin{cases} \pm\infty & \text{se } c \neq -2 \\ -1 & \text{se } \boxed{c=-2} \end{cases}$$

La soluzione cercata è

$$y(t) = -\cos t + 2\frac{\operatorname{sen} t}{t} + 2\frac{\cos t - 1}{t^2}$$ //

Esercizio

Sia $u = u(t)$ la soluzione di

$$\begin{cases} u' + \frac{2}{t+1}u = \operatorname{arctan} t \\ u(0) = 0 \end{cases}$$

Calcolare, se esiste, $\lim_{t \to +\infty} u(t)$

Calcoliamo prima l'integrale generale $u' + \frac{2}{t+1}u = \operatorname{arctan} t$

e, poi, fra tutte le soluzioni cerco quella che soddisfa $u(0) = 0$

$$u' + \frac{2}{t+1}u = \operatorname{arctan} t \quad / \cdot (t+1)^2$$

$a(t) = -\frac{2}{t+1}$

$b(t) = \operatorname{arctan} t$

$$\underbrace{(t+1)^2 u' + 2(t+1)u}_{\frac{d}{dt}\left((t+1)^2 u(t)\right)} = (t+1)^2 \operatorname{arctan} t$$

$$(t+1)^2 u(t) = \int (t+1)^2 \arctan t \, dt$$

lo calcolo

$$\int (t+1)^2 \arctan t \, dt = \frac{1}{3}(t+1)^3 \arctan t +$$
$$- \frac{1}{3}\int \frac{(t+1)^3}{1+t^2}\, dt$$

$$(t+1)^3 = (t+3)(t^2+1) + 2t - 2$$

$$\frac{(t+1)^3}{1+t^2} = t+3 + \frac{2t-2}{1+t^2}$$

$$\int \frac{(t+1)^3}{1+t^2}\, dt = \int \left(t+3 + \frac{2t}{1+t^2} - \frac{2}{1+t^2} \right) dt =$$
$$= \frac{1}{2}(t+3)^2 + \ln(1+t^2) - 2\arctan t + c$$

$$\int (t+1)^2 \arctan t = \frac{1}{3}(t+1)^3 \arctan t +$$
$$- \frac{1}{6}(t+3)^2 - \frac{1}{3}\ln(1+t^2) + \frac{2}{3}\arctan t + c$$

$$(t+1)^2 u(t) = \frac{1}{3}(t+1)^3 \arctan t - \frac{1}{6}(t+3)^2 +$$
$$- \frac{1}{3}\ln(1+t^2) + \frac{2}{3}\arctan t + c$$

$$u(t) = \frac{1}{(t+1)^2}\left[\frac{1}{3}(t+1)^3 \arctan t - \frac{1}{6}(t+3)^2 + \right.$$
$$\left. - \frac{1}{3}\ln(1+t^2) + \frac{2}{3}\arctan t + c\right]$$

Trovo c che verifica $u(0) = 0$

$\Rightarrow$ al variare di $c \in \mathbb{R}$ ottengo tutte le soluzioni

$$\Rightarrow u(0) = -\frac{9}{6} + c = 0 \quad \Rightarrow c = \frac{3}{2}$$

$$\Rightarrow u(t) = \frac{1}{(t+1)^2}\left[\frac{1}{3}(t+1)^3 \arctan t - \frac{1}{6}(t+3)^2 - \frac{1}{3}\ln(1+t^2) + \right.$$
$$\left. + \frac{2}{3}\arctan t + \frac{3}{2}\right]$$

Poiché $-\frac{1}{6}(t+3)^2 - \frac{1}{3}\ln(1+t^2) + \frac{2}{3}\arctan t +$

$$+ \frac{3}{2} = o\left((t+1)^3 \arctan t\right)$$

per $t \to +\infty$

$$\lim_{t\to+\infty} u(t) \overset{PdS}{=} \lim_{t\to+\infty} \frac{1}{3} \frac{(t+1)^3 \arctan t}{(t+1)^2} = +\infty \quad //$$

Esercizio

Calcolare la soluzione del problema di Cauchy

$$\begin{cases} y' = \dfrac{1}{y} \\ y(0) = -2 \end{cases}$$

$f(y) = \frac{1}{y}$

$g(t) = 1$

$t_0 = 0$

$y_0 = -2$

1) $f(y_0) = 0$?? No

(Se fosse $f(y_0) = 0$, avrei finito perché $y(t) = y_0 \; \forall t$ sarebbe soluzione)

2) $f(-2) \neq 0$

Separo le variabili

$$y \, y' = 1$$

$\to$ primitiva di $\frac{1}{f(y)} = y$

$$F(y) = \int y \, dy = \frac{1}{2} y^2$$

$\to$ primitiva di $g(t) = 1 \Rightarrow G(t) = t$

$\Rightarrow$ le soluzioni soddisfano

$$F(y(t)) = G(t) + c \Rightarrow \frac{1}{2} (y(t))^2 = t + c$$

Cerco c in modo che $y(0) = -2$

$$\frac{1}{2}(y(t))^2 = t + c$$

$$\frac{1}{2}(y(0))^2 = c \qquad y(0) = -2$$

$$\Rightarrow c = 2$$

La soluzione cercata soddisfa

$$\frac{1}{2}(y(t))^2 = t + 2$$

$$\Rightarrow (y(t))^2 = 2(t+2) \begin{cases} \to y(t) = \sqrt{2(t+2)} \\ \to y(t) = -\sqrt{2(t+2)} \end{cases}$$

$$\downarrow$$

$$t \geq -2$$

Essendo $y(0) = -2 < 0$, la soluzione corretta è $y(t) = -\sqrt{2(t+2)}$ //

Esercizio

Calcolare la soluzione del problema di Cauchy

$$\begin{cases} y' = 2t\sqrt{1-y^2} \\ y(\sqrt{\pi}) = \frac{1}{2} \end{cases}$$

$f(y) = \sqrt{1-y^2}$

$g(t) = 2t$

$t_0 = \sqrt{\pi}$

$y_0 = \frac{1}{2}$

$f(y_0) = f\left(\frac{1}{2}\right) \neq 0 \Rightarrow$ procediamo per separazione di variabili

$\rightarrow$ primitiva di $\frac{1}{f(y)} = \frac{1}{\sqrt{1-y^2}}$

$\Rightarrow F(y) = \arcsin y$ $\quad y \in]-1,1[$

$\rightarrow$ primitiva di $p(t) = 2t$

$\Rightarrow G(t) = t^2$

$$\arcsin y(t) = t^2 + c$$

Cerco c che verifichi $y(\sqrt{\pi}) = \frac{1}{2}$

$$\arcsin y(\sqrt{\pi}) = (\sqrt{\pi})^2 + c$$

$$\arcsin \frac{1}{2} = \pi + c$$

$$\frac{\pi}{6} = \pi + c \quad \Rightarrow c = -\frac{5}{6}\pi$$

La soluzione cercata soddisfa

$$\arcsin y(t) = t^2 - \frac{5}{6}\pi \quad \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$$

perché $y(t) \in]-1,1[$

$\frac{5}{6}\pi - \frac{\pi}{2} < t^2 < \frac{5}{6}\pi + \pi/2$ $\rightarrow$ $\sqrt{\frac{5}{6}\pi - \frac{\pi}{2}} < t < \sqrt{\frac{5}{6}\pi + \frac{\pi}{2}}$ $\quad t_0 = \sqrt{\pi}$

$\rightarrow -\sqrt{\frac{5}{6}\pi + \frac{\pi}{2}} < t < -\sqrt{\frac{5}{6}\pi - \frac{\pi}{2}}$

$$y(t) = \operatorname{sen}\left(t^2 - \frac{5}{6}\pi\right)$$

$$t \in \left] \sqrt{\frac{5}{6}\pi - \frac{\pi}{2}} \, , \, \sqrt{\frac{5}{6}\pi + \frac{\pi}{2}} \right[$$

Esercizio

Calcolare la soluzione del problema di Cauchy

$$\begin{cases} y' = \dfrac{t+1}{4y^3(t^2+6t+10)} \\ y(0) = 1 \end{cases}$$

$f(y) = \dfrac{1}{4y^3}$

$g(t) = \dfrac{t+1}{t^2+6t+10}$

$t_0 = 0$

$y_0 = 1$

$f(y_0) = f(1) = \frac{1}{4} \neq 0 \Rightarrow$ separiamo le variabili

$$4y^3 y' = \frac{t+1}{t^2+6t+10}$$

$\to$ primitiva di $\dfrac{1}{f(y)} = 4y^3$ è

$F(y) = y^4$

$\to$ primitiva di $g(t) = \dfrac{t+1}{t^2+6t+10}$

dobbiamo calcolarla

$$= \int \frac{t+1}{t^2+6t+10}\, dt$$

$\hookrightarrow \Delta < 0 \qquad t^2+6t+10 = (t+3)^2+1$

$$= \frac{1}{2}\int 2 \cdot \frac{t+1+2-2}{1+(t+3)^2}\, dt =$$

$$= \frac{1}{2}\int 2 \cdot \frac{t+3 \quad -2}{1+(t+3)^2}\, dt =$$

$$= \frac{1}{2}\left[\int \frac{2(t+3)}{1+(t+3)^2}\, dt - 4\int \frac{1}{1+(t+3)^2}\, dt\right] =$$

$$= \underbrace{\frac{1}{2}\left[\ln\left(1+(t+3)^2\right) - 4 \arctan(t+3)\right]}_{G(t)} + c$$

$$\Rightarrow \underbrace{\left(y(t)\right)^4}_{F(y(t))} = \frac{1}{2}\left[\ln\left(1+(t+3)^2\right) - 4\arctan(t+3)\right] + c$$

Calcolo c sfruttando $y(0)=1$

$$\left(y(0)\right)^4 = \frac{1}{2}\left[\ln\left(1+(0+3)^2\right) - 4\arctan(0+3)\right] + c$$

$$1 = \frac{1}{2}\left[\ln 10 - 4\arctan 3\right] + c$$

$$C = 1 - \frac{1}{2}\ln 10 + 2\arctan 3 =$$

$$= 1 - \ln\sqrt{10} + 2\arctan 3$$

$$\Rightarrow (y(t))^4 = \frac{1}{2}\left[\ln\left(1+(t+3)^2\right) - 4\arctan(t+3)\right] + 1 - \ln\sqrt{10} + 2\arctan 3$$

$$y(t) = \left\{\frac{1}{2}\left[\ln\left(1+(t+3)^2\right) - 4\arctan(t+3)\right] + 1 - \ln\sqrt{10} + 2\arctan 3\right\}^{1/4}$$

$y(0) = 1 > 0$

oppure

$$y(t) = -\left\{\frac{1}{2}\left[\ln\left(1+(t+3)^2\right) - 4\arctan(t+3)\right] + 1 - \ln\sqrt{10} + 2\arctan 3\right\}^{1/4}$$

La soluzione cercata è

$$y(t) = \left\{\frac{1}{2}\left[\ln\left(1+(t+3)^2\right) - 4\arctan(t+3)\right] + 1 - \ln\sqrt{10} + 2\arctan 3\right\}^{1/4} \quad //$$