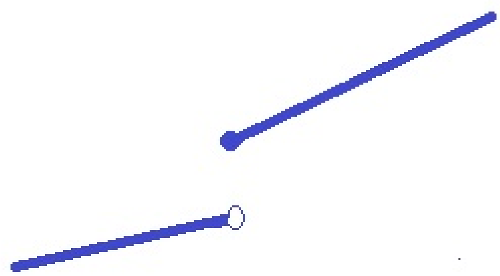
Lesson 2.6: *Differentiability:*

A function is differentiable at a point if it has a derivative there. In other words:
The function f is differentiable at x if

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\text{exists.}$$

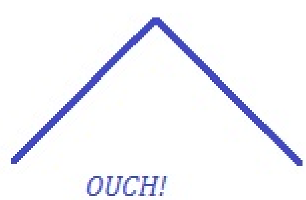
Thus, the graph of f has a non-vertical tangent line at $(x, f(x))$. The value of the limit and the slope of the tangent line are the derivative of f at x_0.
A function can fail to be differentiable at point if:

1. The function is not continuous at the point.

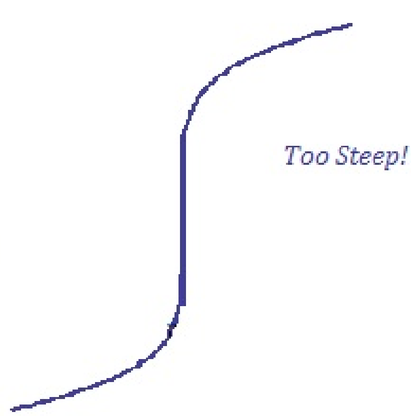


How can you make a tangent line here?

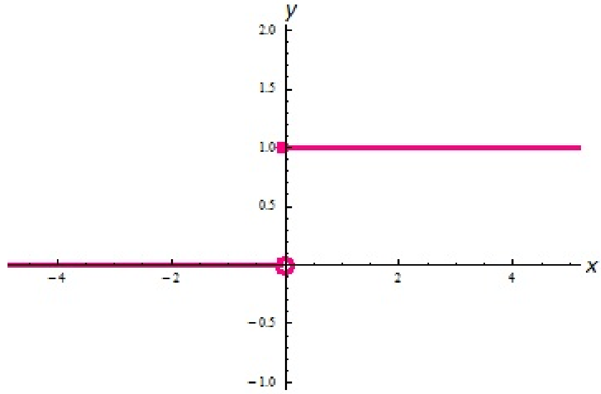
2. The graph has a sharp corner at the point.



3. The graph has a vertical line at the point.



Example 1:



$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

H is not continuous at 0, so it is not differentiable at 0.

The general fact is:

> Theorem 2.1: A differentiable function is continuous:
>
> If $f(x)$ is differentiable at $x = a$, then $f(x)$ is also continuous at $x = a$.

Proof: Since f is differentiable at a,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \text{ exists.}$$

Then

$$\lim_{x \to a}(f(x) - f(a)) = \lim_{x \to a}(x - a) \cdot \frac{f(x) - f(a)}{x - a}$$

This is okay because $x - a \neq 0$ for limit at a.

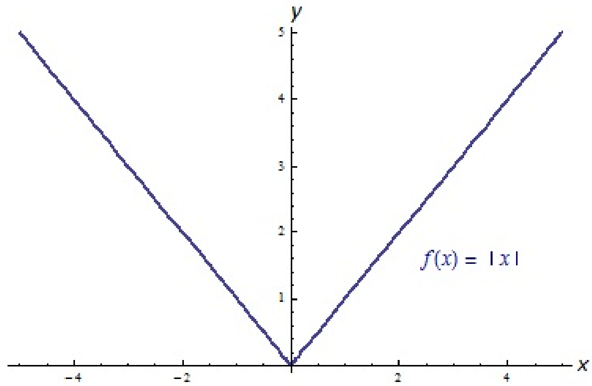
$$= \lim_{x \to a}(x - a) \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = 0$$

Thus, $f'(a) = 0$. Hence, $\lim_{x \to a}(f(x) - f(a)) = 0$, and if we add the constant $f(a)$ to both sides, and use Law 2, we get

$$\lim_{x \to a} f(x) = f(a)$$

which is the definition of continuity of f at $x = a$. $\square$

Example 2:



$$f(x) = \mid x \mid = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

At $x = 0$, there is a corner at $(0, 0)$. Picture is different to the left and right of $(0, 0$. Suggests we try left- and right-hand limits.

$$\begin{aligned} \lim_{h \to 0_+} \frac{f(h) - f(0)}{h} &= \lim_{h \to 0_+} \frac{h - 0}{h} \qquad \text{cancellation of } h \text{ okay, since } h \neq 0 \text{ for limit at } 0 \\ &= \lim_{h \to 0_+} 1 \\ &= 1 \end{aligned}$$

But

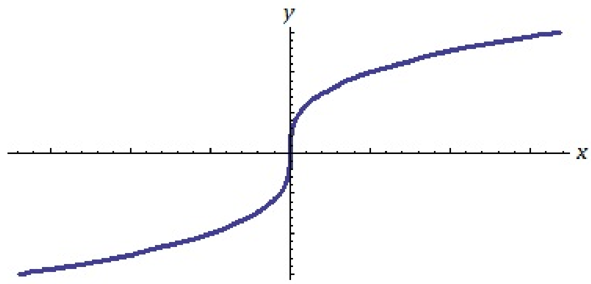
$$\begin{aligned} \lim_{h \to 0_-} \frac{f(h) - f(0)}{h} &= \lim_{h \to 0_-} \frac{-h - 0}{h} \qquad \text{cancellation of } h \text{ okay, since } h \neq 0 \text{ for limit at } 0 \\ &= \lim_{h \to 0_-} -1 \\ &= -1 \end{aligned}$$

Since the left- and right-hand limits do not agree,

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

Does not exits, and so $\mid x \mid$ is not differentiable at $x = 0$.

Example 3:



$$g(x) = x^{1/3}$$

The graph is smooth at $x = 0$, but does appear to have a vertical tangent.

$$\lim_{h \to 0} \frac{(0+h)^{1/3} - 0^{1/3}}{h} = \lim_{h \to 0} \frac{(h)^{1/3}}{h} = \lim_{h \to 0} \frac{1}{h^{2/3}}$$

As $h \to 0$, the denominator becomes small, so the fraction grows without bound. Hence g is not differentiable at $x = 0$.

Example 4: To be discussed in Class

$$g(x) = \begin{cases} x+1 & x \leq 1 \\ 3x-1 & x > 1 \end{cases}$$