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# Extending Baker's theorem

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*Abstract.* We summarize the combinatorial properties of congruence generation in congruence distributive varieties which are relevant to Baker's finite basis theorem, explain the extent to which these properties survive in congruence meet-semidistributive varieties, indicate our approach to extending Baker's theorem to the latter varieties, and pose several problems which our approach does not answer.

## 1. Introduction

An algebra is *finitely subdirectly irreducible* if its least congruence is not the intersection of two nonzero congruences. If $\mathcal{K}$ is a class of algebras, then $\mathcal{K}_{\mathrm{FSI}}$ denotes the class of finitely subdirectly irreducible members of $\mathcal{K}$. Similarly, $\mathcal{K}_{\mathrm{SI}}$ denotes the class of subdirectly irreducible members of $\mathcal{K}$. For $m < \omega$, we say that a variety $\mathcal{V}$ is *residually bounded by m* if every member of $\mathcal{V}_{\mathrm{SI}}$ has cardinality less than $m$.

In 1972 K. Baker announced the following theorem.

THEOREM 1.1. ([1]) *Let $\mathcal{V}$ be a congruence distributive variety in a finite language. If $\mathcal{V}$ is residually bounded by some $m < \omega$, then $\mathcal{V}$ is finitely based.*

Building on ideas of C. Herrmann [6] and Baker [2, Theorem 9.1], B. Jónsson obtained the following improvement.

THEOREM 1.2. ([7]) *Let $\mathcal{V}$ be a congruence distributive variety in a finite language. If $\mathcal{V}_{\mathrm{FSI}}$ is strictly elementary, then $\mathcal{V}$ is finitely based.*

If $\mathcal{V}$ is residually bounded by some $m < \omega$, then $\mathcal{V}_{\mathrm{FSI}} = \mathcal{V}_{\mathrm{SI}}$ (see e.g. [12, proof of Lemma 4.2]), and hence $\mathcal{V}_{\mathrm{FSI}}$ is strictly elementary. Hence Jónsson's theorem contains Baker's. Both theorems are celebrated. It is natural to ask for generalizations. In his Ph.D. thesis [11] of 1976, R. Park conjectured a positive answer to the following question.

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PROBLEM 1.3. Suppose $\mathcal{V}$ is a variety in a finite language. If $\mathcal{V}$ is residually bounded by some $m < \omega$, must $\mathcal{V}$ be finitely based?

In the same year, at a meeting in Oberwolfach, Jónsson posed the following question [13, Problem 9, p. 28].

PROBLEM 1.4. Suppose $\mathcal{V}$ is a variety in a finite language. If $\mathcal{V}_{\mathrm{FSI}}$ is strictly elementary, must $\mathcal{V}$ be finitely based?

Both problems remain open to this day. In this paper we describe our recent solution to Problem 1.3 in the congruence meet-semidistributive case and explain why our proof is a natural extension of Baker's proof. We also attempt to formulate the obstacles that have prevented us from avoiding Baker's Ramsey argument and from generalizing Jónsson's theorem.

This paper is expository, consisting of new formulations of old results. Accordingly, we devote most of the paper to giving precise statements of the facts, leaving the verification of many of these facts to the reader. This article is an extended version of a lecture delivered to the Conference on Lattices and Universal Algebra held at Szeged in August 1998. We wish to thank the organizers of the conference for providing the opportunity to publish a paper of this kind.

## 2. Principal congruence generation

Suppose **A** is an algebra. By a *basic translation* of **A** we mean a unary polynomial of the form $F(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n)$ where $F$ is an $n$-ary fundamental operation of **A**, $1 \le i \le n$, and the $a_j$'s are any elements of $A$. A *k-translation* of **A** is a unary polynomial of **A** which can be expressed as the composition of $k$ or fewer basic translations. In particular, the identity map on $A$ is the unique 0-translation of **A**.

$A^{(2)}$ denotes the set of all 2-element subsets of $A$. If $\{a, b\}, \{c, d\} \in A^{(2)}$ and $k < \omega$, then we write $\{a, b\} \rightarrow_k \{c, d\}$ to mean that there exists a $k$-translation $f$ such that $\{f(a), f(b)\} = \{c, d\}$. Similarly, if $k, n < \omega$ then we define $\{a, b\} \Rightarrow_{k,n} \{c, d\}$ to mean that there exists a sequence $c = c_0, c_1, \ldots, c_m = d$ with $m \le n$ such that $\{a, b\} \rightarrow_k \{c_i, c_{i+1}\}$ for all $i < m$. Note that $\rightarrow_k$ and $\Rightarrow_{k,1}$ mean the same thing.

In a class of algebras, the 4-ary relations $\{z, w\} \Rightarrow_{k,n} \{x, y\}$ serve as uniform approximations to the relation $(x, y) \in \mathrm{Cg}(z, w)$. Another well-known family of approximations is the one defined by the principal congruence formulas (by which we mean disjunctions of finitely many positive primitive formulas of the kind described in [4, §V.3]). In case the language under consideration is finite, the first family is a cofinal subset of the second family in the following sense:

1. For all $k, n < \omega$ there is a principal congruence formula $\pi_{k,n}(x, y, z, w)$ such that $\pi_{k,n}(x, y, z, w) \wedge (x \neq y)$ defines the relation $\{z, w\} \Rightarrow_{k,n} \{x, y\}$ in all algebras in the language under consideration.
2. For every principal congruence formula $\pi(x, y, z.w)$ in the language under consideration, there exist $k, n < \omega$ such that $\vdash \pi \to \pi_{k,n}$.

Thus in elementary classes, relations built from $(x, y) \in \mathrm{Cg}(z, w)$, if first-order definable at all, will be definable in terms of the relations $\Rightarrow_{k,n}$. This is one justification of our focus on these relations; another is that they play a peculiar role in the proofs of Baker's and Jónsson's theorems.

We now formulate three infinite families of properties; each property asserts uniform witnesses to certain intersections of principal congruences.

DEFINITION 2.1. Let $\mathcal{K}$ be a class of algebras and $m < \omega$.

1. $\mathcal{K}$ **satisfies Property J$_m$** (written $\mathcal{K} \models \mathrm{J}_m$) if there exist $k, n < \omega$ so that the following is true: for all $\mathbf{A} \in \mathcal{K}$ and all $\{a_1, b_1\}, \ldots, \{a_m, b_m\} \in A^{(2)}$, if $\bigcap_{i=1}^{m} \mathrm{Cg}^{\mathbf{A}}(a_i, b_i) \neq 0_A$, then there exists $\{c, d\} \in A^{(2)}$ such that $\{a_i, b_i\} \Rightarrow_{k,n} \{c, d\}$ for all $i = 1, \ldots, m$.
2. $\mathcal{K}$ **satisfies Property C$_m$** (written $\mathcal{K} \models \mathrm{C}_m$) if there exist $k, n < \omega$ so that the following is true: for all $\mathbf{A} \in \mathcal{K}$ and all $S \subseteq A$ with $|S| = m$, if
$$\bigcap_{\{s,t\} \in S^{(2)}} \mathrm{Cg}^{\mathbf{A}}(s, t) \neq 0_A,$$
then there exists $\{c, d\} \in A^{(2)}$ such that $\{s, t\} \Rightarrow_{k,n} \{c, d\}$ for all $\{s, t\} \in S^{(2)}$.
3. $\mathcal{K}$ **satisfies Property B$_m$** (written $\mathcal{K} \models \mathrm{B}_m$) if there exist $k, n < \omega$ so that the following is true: for all $\mathbf{A} \in \mathcal{K}$, if there exists $S \subseteq A$ with $|S| = m$ and
$$\bigcap_{\{s,t\} \in S^{(2)}} \mathrm{Cg}^{\mathbf{A}}(s, t) \neq 0_A,$$
then there exist $\hat{S} \subseteq A$ and $\{c, d\} \in A^{(2)}$ such that $|\hat{S}| = m$ and $\{s, t\} \Rightarrow_{k,n} \{c, d\}$ for all $\{s, t\} \in \hat{S}^{(2)}$.

B is for Baker; J is for Jónsson; C is between B and J. We also let $\mathcal{K}_{[m]}$ denote the class of all $\mathbf{A} \in \mathcal{K}$ for which there exists $S \subseteq A$ with $|S| = m$ and such that $\bigcap_{\{s,t\} \in S^{(2)}} \mathrm{Cg}^{\mathbf{A}}(s, t) \neq 0_A$, and write $\underset{\sim}{A} \models \Delta_m(k, n)$ if there exist $S \subseteq A$ and $\{c, d\} \in A^{(2)}$ such that $|S| = m$ and $\{s, t\} \Rightarrow_{k,n} \{c, d\}$ for all $\{s, t\} \in S^{(2)}$. Thus $\mathcal{K} \models \mathrm{B}_m$ iff $\mathcal{K}_{[m]} \models \Delta_m(k, n)$ for some $k, n < \omega$.

The following assertions hold for any class $\mathcal{K}$ of algebras.

1. $\mathcal{K} \models \mathrm{C}_m$ implies $\mathcal{K} \models \mathrm{B}_m$. If $p = \binom{m}{2}$, then $\mathcal{K} \models \mathrm{J}_p$ implies $\mathcal{K} \models \mathrm{C}_m$.
2. If $\mathcal{K} = \mathcal{K}_{\mathrm{FSI}}$ and $\mathcal{K} \models \mathrm{J}_2$, then $\mathcal{K} \models \mathrm{J}_m$ for all $m < \omega$.

3. If $\mathcal{V}$ is a variety and is residually bounded by $m$, then $\mathcal{V}_{[m]} = \varnothing$ and hence $\mathcal{V} \models \mathrm{C}_m$ vacuously.
4. If $\mathcal{K}$ is elementary, $\mathcal{K} = \mathcal{K}_{\mathrm{FSI}}$, and the language of $\mathcal{K}$ is finite, then $\mathcal{K} \models \mathrm{J}_m$ for all $m < \omega$ (by a compactness argument).

$\mathrm{B}_m$ and $\mathrm{J}_2$ are connected to Baker's and Jónsson's finite basis theorems by the next lemma, whose proof is a good exercise in the compactness theorem.

LEMMA 2.2. *Let $\mathcal{V}$ be a variety in a finite language and $m < \omega$.*

1. *Suppose $\mathcal{V}$ is residually bounded by $m$. If there exists a strictly elementary class $\mathcal{K}$ such that $\mathcal{V} \subseteq \mathcal{K}$ and $\mathcal{K}_{\mathrm{SI}} \models \mathrm{B}_m$, then $\mathcal{V}$ is finitely based.*
2. *Suppose $\mathcal{V}_{\mathrm{FSI}}$ is elementary (and hence strictly elementary). If there exists a strictly elementary class $\mathcal{K}$ such that $\mathcal{V} \subseteq \mathcal{K}$ and $\mathcal{K} \models \mathrm{J}_2$, then $\mathcal{V}$ is finitely based.*

*Sketch of proof.* (1) Choose $k, n < \omega$ so that $(\mathcal{K}_{\mathrm{SI}})_{[m]} \models \Delta_m(k, n)$. All members of $\mathcal{V}$ satisfy the assertion "I am in $\mathcal{K}$ & $\neg\Delta_m(k, n)$". This assertion is expressible by a first-order sentence. Any subvariety $\mathcal{W}$ of $\mathcal{K}$ which satisfies this assertion also satisfies $\mathcal{W}_{[m]} = \varnothing$, is residually bounded by $m$, and has only finitely many subvarieties.

(2) Since $\mathcal{K} \models \mathrm{J}_2$, the relation "$\mathrm{Cg}(x, y) \cap \mathrm{Cg}(z, w) \neq 0$" is definable in $\mathcal{K}$ by a first-order formula; hence $\mathcal{K}_{\mathrm{FSI}}$ is strictly elementary. $\mathcal{V}$ is the unique subvariety $\mathcal{W}$ of $\mathcal{K}$ such that $\mathcal{V} \subseteq \mathcal{W}$ and every member of $\mathcal{W}$ satisfies "I am in $\mathcal{K}$ & if I am in $\mathcal{K}_{\mathrm{FSI}}$ then I am in $\mathcal{V}_{\mathrm{FSI}}$." The latter assertion is expressible by a first-order sentence.

## 3. Congruence distributive varieties

In hindsight, Baker's and Jónsson's theorems follow from Lemma 2.2, the assertions which precede Lemma 2.2, and the following two propositions.

PROPOSITION 3.1. (Baker) *For every congruence distributive variety $\mathcal{V}$ in a finite language, $\mathcal{V}_{\mathrm{FSI}} \models \mathrm{B}_m$ for all $m < \omega$.*

PROPOSITION 3.2. (Jónsson) *If $\mathcal{V}$ is a congruence distributive variety in a finite language and $\mathcal{V}_{\mathrm{SI}} \models \mathrm{J}_2$, then there exists a strictly elementary class $\mathcal{K}$ satisfying $\mathcal{V} \subseteq \mathcal{K}$ and $\mathcal{K} \models \mathrm{J}_2$.*

In fact Baker [2, Lemma 7.1] claimed only that $\mathcal{V}_{\mathrm{SI}} \models \mathrm{B}_m$, but his proof establishes $\mathcal{V}_{\mathrm{FSI}} \models \mathrm{B}_m$. (The first sentence of [2, §7.3] is the only point in Baker's proof of [2, Lemma 7.1] at which the assumption of subdirect irreducibility is used.) Proposition 3.2. can be inferred from Jónsson's proofs of [7, Lemma 3.4, Theorem 4.1] and our discussion in the third paragraph following Lemma 3.4.

The next argument is *not* how Baker established his finite basis theorem (he used an effective version of Lemma 2.2(1) assuming $\mathcal{K}$ is a congruence distributive variety), but still faithfully reflects the organization of his argument. The proof of Jónsson's theorem which follows *is* essentially equivalent to Jónsson's argument.

*Proof of Theorem* 1.1. Given $\mathcal{V}$ which is residually bounded by $m$, choose Jónsson terms for $\mathcal{V}$ and let $\mathcal{K}$ be the variety defined by the Jónsson identities for the chosen terms; thus $\mathcal{K}$ is congruence distributive and finitely based. $\mathcal{K}_{\mathrm{FSI}} \models \mathrm{B}_m$ by Proposition 3.1, so $\mathcal{V}$ is finitely based by Lemma 2.2(1).

*Proof of Theorem* 1.2. Given $\mathcal{V}$ with $\mathcal{V}_{\mathrm{FSI}}$ elementary, note that $\mathcal{V}_{\mathrm{FSI}} \models \mathrm{J}_2$ by a previously mentioned compactness argument, and hence $\mathcal{V}_{\mathrm{SI}} \models \mathrm{J}_2$. Now use Proposition 3.2 and Lemma 2.2(2).

Now let us discuss how Propositions 3.1 and 3.2 are proved. Baker's proof of Proposition 3.1 rests on the following fact, which is a simple consequence of the Jónsson identities for congruence distributivity.

LEMMA 3.3. (Baker's Single-sequence lemma) *If $\mathcal{V}$ is a congruence distributive variety in a finite language, then $\mathcal{V}$ is term-equivalent to a variety $\mathcal{W}$, also in a finite language, such that every $\mathbf{A} \in \mathcal{W}$ has the following property:*

*For all $a_0, a_1, \dots, a_n \in A$ with $a_0 \neq a_n$ there exist $i < n$ and $\{c, d\} \in A^{(2)}$ such that $\{a_0, a_n\} \rightarrow_1 \{c, d\}$ and $\{a_i, a_{i+1}\} \rightarrow_1 \{c, d\}$.*

Let us call an algebra $\mathbf{A}$ which satisfies the property displayed in the above lemma a *Baker* algebra. Baker's argument in effect proves property $\mathrm{B}_m$ for the class of all subdirectly irreducible Baker algebras (in any language). The argument requires a difficult Ramsey argument, which we shall not describe here. However, one consequence of being a Baker algebra is worth mentioning (cf. [9, Corollary 3.4]).

LEMMA 3.4. (Baker) *Suppose $\mathbf{A}$ is a Baker algebra, $\{a_1, b_1\}, \dots, \{a_m, b_m\}, \{u, v\} \in A^{(2)}$ and $\ell, n < \omega$ are such that $\{a_i, b_i\} \Rightarrow_{\ell,n} \{u, v\}$ for all $i = 1, \dots, m$. Then there exists $\{c, d\} \in A^{(2)}$ such that $\{a_i, b_i\} \rightarrow_{\ell+m} \{c, d\}$ for all $i = 1, \dots, m$.*

In particular, if $\mathrm{Cg}^{\mathbf{A}}(a_1, b_1) \cap \mathrm{Cg}^{\mathbf{A}}(a_2, b_2) \neq 0_A$ in a Baker algebra $\mathbf{A}$, then this can be witnessed by relations $\Rightarrow_{k,n}$ with $k$ possibly very large but $n = 1$.

Now we turn to Jónsson's proof of Proposition 3.2. The proof passes through two stages: first lift $\mathrm{J}_2$ from $\mathcal{V}_{\mathrm{SI}}$ to $\mathcal{V}$, and then lift $\mathrm{J}_2$ from $\mathcal{V}$ to a (provably) strictly elementary class $\mathcal{K} \supseteq \mathcal{V}$. The first stage is established by a clever argument [7, Lemma 3.4] which uses the previous lemma and $\mathrm{J}_2$ as in the next paragraph, together with an observation that essentially requires the following consequence of congruence distributivity: if $\mathrm{Cg}(x, y) \cap \mathrm{Cg}(x', y') = 0$, then $(x, y) \in \mathrm{Cg}(x, x') \vee \mathrm{Cg}(y, y')$.

The second stage, on the other hand, is a consequence of Lemma 3.4 alone, as we now explain. Without loss of generality, $\mathcal{V} \subseteq \mathcal{W}$ where $\mathcal{W}$ is a finitely based variety of Baker algebras. Since $\mathcal{V} \models \mathrm{J}_2$ and using Lemma 3.4, we see that $\mathcal{V} \models \mathrm{J}_2$ is witnessed by some $k < \omega$ and $n = 1$. For any $\ell < \omega$ and $\{a_1, b_1\}, \{a_2, b_2\} \in A^{(2)}$, let us say that $\{a_1, b_1\}, \{a_2, b_2\}$ are **$\ell$-bounded** if there exists $\{c, d\} \in A^{(2)}$ such that $\{a_i, b_i\} \to_\ell \{c, d\}$ for $i = 1, 2$. We say that $\{a_1, b_1\}, \{a_2, b_2\}$ are **bounded** if they are $\ell$-bounded for some $\ell < \omega$. Now on the one hand, in every $\mathbf{A} \in \mathcal{W}$ the relation "$\mathrm{Cg}^{\mathbf{A}}(x, y) \cap \mathrm{Cg}^{\mathbf{A}}(x', y') \neq 0_A$" is equivalent to "$\{x, y\}, \{x', y'\}$ are bounded." On the other hand, in any algebra the assertion "Any bounded pair of two-element sets is $k$-bounded" is equivalent to "any $(k+1)$-bounded pair is $k$-bounded." The latter assertion is expressible by a first-order sentence. Thus if $\mathcal{K}$ is defined to be the class of all $\mathbf{A} \in \mathcal{W}$ in which any $(k+1)$-bounded pair is $k$-bounded, then $\mathcal{K}$ has the required properties.

Incidentally, a similar argument shows that if $\mathcal{V}$ is congruence distributive and $\mathcal{V} \models \mathrm{J}_m$, then $\mathrm{J}_m$ can be lifted to a strictly elementary class $\mathcal{K} \supseteq \mathcal{V}$. Analogous arguments for lifting $\mathrm{C}_m$ (or $\mathrm{B}_m$) do not seem possible by this method; it is instructive to see at what step the argument fails.

We remark in passing that, by changing a few words in Baker's proof of Proposition 3.1, one gets a proof of the following stronger conclusion.

PROPOSITION 3.5. *If $\mathcal{V}$ is a congruence distributive variety and $m < \omega$, then there exist $k, n < \omega$ and a partition of $\mathcal{V}$ into two subclasses $\mathcal{K}$ and $\mathcal{K}^c$ so that $\mathcal{K} \models \mathrm{J}_2$ while $\mathcal{K}^c \models \Delta_m(k, n)$.*

(Cf. [12, Lemma 3.4] where the Ramsey argument for CSD($\wedge$) varieties is explicitly formulated in this way.)

Suppose $\mathcal{V}$ is residually bounded by $m$ and let $\mathcal{W}$ be a finitely based congruence distributive variety containing $\mathcal{V}$. Let $\mathcal{W}$ be partitioned into $\mathcal{K}$ and $\mathcal{K}^c$ as in the previous proposition. As $\mathcal{V}_{[m]} = \varnothing$, we must have $\mathcal{V} \cap \mathcal{K}^c = \varnothing$ and hence $\mathcal{V} \subseteq \mathcal{K} \models \mathrm{J}_2$. This observation with Lemma 2.2(2) and the second stage of Jónsson's proof gives yet another proof of Baker's theorem.

In summary, let $\mathcal{V}$ be an arbitrary variety in a finite language and assume that $\mathcal{V}$ is residually bounded by $m$. $\mathcal{V}$ automatically satisfies $\mathrm{C}_m$ and hence $\mathrm{B}_m$, and $\mathcal{V}_{\mathrm{SI}}$ automatically satisfies $\mathrm{J}_2$. To prove that $\mathcal{V}$ is finitely based it suffices to do either of the following two things:

1. Lift $\mathrm{B}_m$ (or $\mathrm{C}_m$) from $\mathcal{V}$ to $\mathcal{K}_{\mathrm{FSI}}$ for some strictly elementary class $\mathcal{K} \supseteq \mathcal{V}$; or
2. Lift $\mathrm{J}_2$
   (a) from $\mathcal{V}_{\mathrm{SI}}$ to $\mathcal{V}$, and then
   (b) from $\mathcal{V}$ to some strictly elementary class $\mathcal{K} \supseteq \mathcal{V}$.

In case $\mathcal{V}$ is congruence distributive, there is an elementary argument establishing item 2(b). Baker's Ramsey argument establishes items 1 and 2(a), but the argument is overkill. Jónsson's argument establishes item 2(a) even without the assumption that $\mathcal{V}$ is residually bounded by $m$, and is simpler than the Ramsey argument, but requires a special consequence of congruence distributivity.

## 4. Congruence meet-semidistributive varieties

A variety is *congruence meet-semidistributive*, or CSD($\wedge$) for short, if the congruence lattice of each member satisfies the meet-semidistributive law $x \wedge y = x \wedge z \rightarrow x \wedge y = x \wedge (y \vee z)$. Congruence meet-semidistributive varieties include congruence distributive varieties as well as any variety whose clone contains a semilattice operation. G. Czédli proved in 1982 that congruence meet-semidistributivity is a weak Mal'cev property of varieties [5]. That is, he discovered an infinite sequence of Mal'cev conditions and proved that a variety is CSD($\wedge$) if and only if it satisfies all of the Mal'cev conditions in the sequence. Recently, K. Kearnes and Á. Szendrei [8] and P. Lipparini [10] proved that the first Mal'cev condition in the sequence already characterizes CSD($\wedge$). While studying this Mal'cev condition, we discovered that CSD($\wedge$) varieties in a finite language almost satisfy Baker's single-sequence lemma (Lemma 3.3), in the following sense.

LEMMA 4.1. ([12, Lemma 3.1]) *If $\mathcal{V}$ is a congruence meet-semidistributive variety in a finite language, then $\mathcal{V}$ is term-equivalent to a variety $\mathcal{W}$, also in a finite language, such that every $\mathbf{A} \in \mathcal{W}$ has the following property:*

> *For all $a_0, a_1, \ldots, a_n \in A$ with $a_0 \neq a_n$ there exist $i < n$ and $\{c, d\} \in A^{(2)}$ such that $\{a_0, a_n\} \Rightarrow_{1,2} \{c, d\}$ and $\{a_i, a_{i+1}\} \Rightarrow_{1,2} \{c, d\}$.*

The converse is also true: if $\mathcal{V}$ is term-equivalent to a variety $\mathcal{W}$ satisfying the displayed property, then $\mathcal{V}$ is CSD($\wedge$) [12, Theorem 2.1(7 $\Rightarrow$ 1)].

From Lemma 4.1 we deduced the following version of Baker's Lemma 3.4.

LEMMA 4.2. ([12, Corollary 3.3]) *Suppose the algebra $\mathbf{A}$ satisfies the condition displayed in Lemma* 4.1. *Suppose also that $\{a_1, b_1\}, \ldots, \{a_m, b_m\}, \{u, v\} \in A^{(2)}$ and $\ell, n < \omega$ are such that $\{a_i, b_i\} \Rightarrow_{\ell,n} \{u, v\}$ for all $i = 1, \ldots, m$. Then there exist $\{u_1, v_1\}, \ldots, \{u_m, v_m\}, \{c, d\} \in A^{(2)}$ such that $\{a_i, b_i\} \rightarrow_{\ell} \{u_i, v_i\} \Rightarrow_{m,2^m} \{c, d\}$ for all $i = 1, \ldots, m$.*

Intrigued, we wondered if Baker's or Jónsson's arguments could be simulated with these weakened versions of Lemmas 3.3 and 3.4. The reader should see that, because CSD($\wedge$) is a Mal'cev property and using Lemma 4.2, the second stage of the proof of Proposition 3.2 can be mimicked to give the following.

PROPOSITION 4.3. *If $\mathcal{V}$ is a* CSD($\wedge$) *variety in a finite language and* $\mathcal{V} \models \mathrm{J}_2$, *then there exists a stricly elementary class $\mathcal{K}$ satisfying $\mathcal{V} \subseteq \mathcal{K}$ and $\mathcal{K} \models \mathrm{J}_2$.*

Baker's Ramsey argument can also be simulated, though we do not claim that this is obvious.

THEOREM 4.4. ([12, Lemma 3.4, Corollary 3.5]) *Suppose $\mathcal{V}$ is a* CSD($\wedge$) *variety in a finite language, and $m < \omega$. Then the following are true.*

1. $\mathcal{V}_{\mathrm{FSI}} \models \mathrm{B}_m$.
2. *There exist $k, n < \omega$ and a partition of $\mathcal{V}$ into two subclasses $\mathcal{K}$ and $\mathcal{K}^c$ so that $\mathcal{K} \models \mathrm{J}_2$ and $\mathcal{K}^c \models \Delta_m(k, n)$.*
3. *Hence if $\mathcal{V}$ is residually bounded by $m$, then $\mathcal{V} \models \mathrm{J}_2$.*

Hence we get two proofs (one using Proposition 3.1, the other using Proposition 3.2) of the following extension of Baker's finite basis theorem.

COROLLARY 4.5. ([12, Theorem 4.3]). *If $\mathcal{V}$ is* CSD($\wedge$) *and residually bounded by some $m < \omega$, then $\mathcal{V}$ is finitely based.*

However, we are unable to see how to avoid the consequence of congruence distributivity that Jónsson used in the first stage of his proof of Proposition 3.2. Hence we do not see how to avoid the Ramsey argument in proving Corollary 4.5, nor do we see how to extend the corollary to CSD($\wedge$) varieties $\mathcal{V}$ for which it is assumed only that $\mathcal{V}_{\mathrm{FSI}}$ is elementary. Thus we pose the following problems.

PROBLEM 4.6. Suppose $\mathcal{V}$ is a CSD($\wedge$) variety in a finite language. If $\mathcal{V}_{\mathrm{FSI}}$ is elementary, does it follow that $\mathcal{V} \models \mathrm{J}_2$?

PROBLEM 4.7. Suppose $\mathcal{V}$ is a CSD($\wedge$) variety in a finite language. If $\mathcal{V}_{\mathrm{SI}} \models \mathrm{J}_2$, does it follow that $\mathcal{V} \models \mathrm{J}_2$?

A positive answer to either of these two questions would provide an elementary proof of both Corollary 4.5 and its extension to varieties $\mathcal{V}$ for which $\mathcal{V}_{\mathrm{FSI}}$ is elementary. If these problems turn out to be intractable, or to have negative answers, can the following be achieved by other means?

PROBLEM 4.8. Suppose $\mathcal{V}$ is a CSD($\wedge$) variety in a finite language and $\mathcal{V}$ is residually bounded by $m$. Find an elementary argument that $\mathcal{V} \models \mathrm{J}_2$.

PROBLEM 4.9. Suppose $\mathcal{V}$ is a CSD($\wedge$) variety in a finite language. If $\mathcal{V}_{\mathrm{FSI}}$ is elementary, does it follow that $\mathcal{V}$ is finitely based?

## 5. Related problems

Baker and J. Wang [3] have recently given a new proof of Baker's finite basis theorem. The proof is constructive and elementary. The key lemma in the proof is the following.

LEMMA 5.1. *Suppose $\mathcal{V}$ is a congruence distributive variety and is residually less than some $m < \omega$. Then there exists $k < \omega$ such that the following holds:*

*For all $\mathbf{A} \in \mathcal{V}$ and all $\{a, b\}, \{c, d\} \in A^{(2)}$, if $(c, d) \in \mathrm{Cg}^{\mathbf{A}}(a, b)$ then $\{a, b\} \Rightarrow_{k,n} \{c, d\}$ for some $n < \omega$.*

A variety which satisfies the condition displayed in the previous lemma for some $k < \omega$ is said to have *bounded Mal'cev depth*. Any congruence distributive variety with bounded Mal'cev depth satisfies $\mathrm{J}_2$, by Lemma 3.4. The same is true for CSD($\wedge$) varieties by Lemma 4.2. The following questions immediately arise.

PROBLEM 5.2. Suppose $\mathcal{V}$ is a CSD($\wedge$) variety in a finite language. If $\mathcal{V}$ is residually bounded by some $m < \omega$, must $\mathcal{V}$ have bounded Mal'cev depth?

PROBLEM 5.3. Suppose $\mathcal{V}$ is a CSD($\wedge$) (or even congruence distributive) variety in a finite language. If $\mathcal{V}_{\mathrm{FSI}}$ is elementary, does it follow that $\mathcal{V}$ has bounded Mal'cev depth?

In 1987, R. McKenzie answered Park's Problem 1.3 affirmatively in the congruence modular case. Our Corollary 4.5 answers Problem 1.3 in the CSD($\wedge$) case. It is natural to ask for a common generalization. According to tame congruence theory, the obvious candidate for generalization is the class of varieties which omit type 1.

PROBLEM 5.4. Suppose $\mathcal{V}$ is a variety in a finite language which is residually bounded by some $m < \omega$, and which omits type 1 in the sense of tame congruence theory. Must $\mathcal{V}$ be finitely based?

It seems that no progress has been made on Jónsson's Problem 1.4 beyond the congruence distributive case.

PROBLEM 5.5. Suppose $\mathcal{V}$ is a congruence modular variety in a finite language. If $\mathcal{V}_{\mathrm{FSI}}$ is elementary, must $\mathcal{V}$ be finitely based?

We predict that it will be easier to solve Problem 4.9 than Problem 5.5. But let us be optimistic and suppose that both problems will soon be answered in the positive. We would want a common generalization.

PROBLEM 5.6. What is the smallest Mal'cev class of varieties containing both the congruence modular and the congruence meet-semidistributive varieties?

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