PROBLEM SET 5

Hand in all.

(1) Find the Galois group of $f = x^4 - 2$ over $\mathbb{Q}$, $\mathbb{Z}_3$, and $\mathbb{Z}_7$.

(2) [Jacobson, p. 243 #1] Show that in the subgroup-subfield correspondence in the Fundamental Theorem of Galois theory, intersections of subgroups correspond to compositums of subfields, and intersections of subfields corresponds to the group generated by their "Galois groups".

(3) Given a finite group G, show that there exists a Galois extension L/K such that $Aut(L/K) \cong G$.

(4) [Jacobson, p. 243 #9] Show that $E = \mathbb{Q}(\sqrt{2}, \sqrt{3}, u)$ where $u^2 = (9 - 5\sqrt{3})(2 - \sqrt{2})$, is normal. Determine $\mathrm{Gal}(E/\mathbb{Q})$.

(5) Factorize $x^{p^p} - x$ over $\mathbb{Z}_p$.

(6) Let $p < q$ be primes, $p \nmid q - 1$. Show that there is an extension $L/\mathbb{Z}_q$ which is a splitting field extension for each of the polynomials $x^p - a$ ($a \in \mathbb{Z}_q^*$).

(7) Show that the simple transcendental extension $K(t)/K$ has infinitely many intermediate fields.

(8) [Jacobson, p. 151 #19] Prove that if $\varphi(n)$ is the Euler phi-function, then $\varphi(n) = \sum_{d|n} \mu(\frac{n}{d})d$.