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Resolutions of Homology Manifolds, and the Topological Characterization of Manifolds

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The purpose of this paper is to present a proof of the following theorem:

Resolution Theorem. *Suppose X is an ENR homology manifold of dimension $\geqq 5$. Then there is a cell-like map $M \to X$ with domain a manifold (of the same dimension).*

Such a map is called a resolution, by analogy with the resolution of singularities of algebraic varieties. (Quick definitions: *ENR* = Euclidean neighborhood retract; a neighborhood of some closed embedding $X \subset \mathbb{R}^k$ retracts to X. For a finite dimensional homology manifold this is equivalent to locally 1-connected. *Homology manifold* means $H_*(X, X-p; \mathbb{Z}) \simeq H_*(\mathbb{R}^n, \mathbb{R}^n - 0; \mathbb{Z})$ for all $p \in X$. These satisfy all the homological properties of a manifold. A *cell-like map* has point inverses compact, nonempty, and contractible inside any neighborhood.)

This result has a long history and many special cases have been proved. The most advanced required X to already be mostly a manifold (Cannon et al. [5], Quinn [14] Theorem 3.5.2), or $X \times \mathbb{R}^k$ to be resolvable (Quinn [14], Theorem 3.3.2). For a history and discussion see J. Cannon [3, 4], and R.C. Lacher [12]. This has recently been extended to dimension 4, (Quinn [15]).

When combined with an approximation theorem of R.D. Edwards [1], this theorem implies:

Characterization of Manifolds. *A space is a manifold of dimension $\geqq 5$ if and only if it is an ENR homology manifold, and satisfies the disjoint disc property.*

A space X satisfies the *disjoint disc property* if any two maps $i, j: D^2 \to X$ can be approximated by maps with disjoint images. This property was first formulated by J. Cannon [3, 4], who conjectured the final result. Edwards' theorem asserts that if X satisfies the disjoint disc property then a resolution $M \to X$ can be approximated by a homeomorphism. A proof of this will be included in a forthcoming book by R. Davermann [8].

We remark that the disjoint disc property can fail very badly: Davermann and Walsh [9] have constructed *ENR* homology manifolds X such that any map $D^2 \to X$ which is injective on S^1, contains a nonempty open set in its image!

The methods used here are quite different from, and complementary to, those used by Edwards. In the decomposition theory one fixes a manifold and uses very geometric tools, engulfing and embedding theorems, to study the ways in which it can be decomposed. Since a manifold is required to even start, the basic structure of the theory would seem to preclude the construction of resolutions. By contrast here we fix X and consider manifolds mapping to it, in the aggregate. The tools are ε versions of methods from algebraic topology, homotopy theory, algebraic K-theory, and surgery. Finally, these methods do not seem to give a good enough hold on any one particular manifold to permit a proof of Edwards' theorem.

Section 1 contains a relative version of the theorem, some simple corollaries, and a few remarks on the proof. Sections 2-4 contain the proof of the theorem.

1. Statements and Applications

1.1. **Theorem.** *Suppose $(X, \partial X)$ is an ENR homology manifold pair, ∂X is a manifold, and $\dim X \geqq 5$. Then there is a cell-like map from a manifold $(M, \partial M) \to (X, \partial X)$ which is a homeomorphism on the boundary. Further, if M_1 and M_2 are two such resolutions, then for every $\varepsilon > 0$ there is a homeomorphism such that the diagram*

$$\begin{array}{ccc} M_1 & \simeq & M_2 \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

commutes up to ε, and on the boundary commutes exactly.

In fact this follows from the unbounded case: resolve the interior and then use the boundary collaring results of (Quinn [14]). The uniqueness is given in Quinn [14]. The theorem is extended to dimension 4 in Quinn [15].

It is simple to see, as was first observed by Davermann [7], that if X is an ENR homology manifold then $X \times \mathbb{R}^2$ has the disjoint disc property. Therefore it is a manifold, by the characterization theorem. More surprising is the fact that products $X \times Y$ have this property (C.D. Bass [1]).

1.2. **Corollary.** *Suppose X, Y have dimension $\geqq 2$. Then $X \times Y$ is a manifold if and only if X and Y are ENR homology manifolds.*

For all known examples, in fact $X \times \mathbb{R}$ is a manifold. The outstanding open question in the area (along with the 3 and 4 dimensional versions) is whether $X \times \mathbb{R}$ satisfies the disjoint disc property for all ENR homology manifolds X.

An important goal in decomposition theory for some time was the double suspension problem: characterize those manifolds whose second suspension is homeomorphic to a sphere. We can now identify the *spaces* with this property.

1.3. **Corollary.** *A space X satisfies $\Sigma^2 X \simeq S^{n+2}$ if and only if X is a closed ENR homology manifold, and $H_*(X; \mathbb{Z}) \simeq H_*(S^n; \mathbb{Z})$.*

$\Sigma^2 X$ is a manifold because it satisfies the disjoint disc property. (1.2 applies in the complement of the suspension circle. Near the circle we can deform the discs to cones intersecting the circle in discrete points, and separate these by pushing along the circle.) It is a simply connected homology sphere, so by the generalized Poincaré conjecture is homeomorphic to a sphere.

Remarks on the Proof. ε versions of a number of manifold and algebraic theorems are required. Versions of the h-corbordism and end theorems, and homotopy theory, were developed in Ends of Maps I (Quinn [14]). We will use this material heavily. The new material is given the minimum development required for this proof. The surgery theory for example, is done only in the simply connected $4k$ dimensional case. This is done in Sect. 2, along with an ε version of the theorem of Wall [17] relating chain complexes and CW complexes.

In Sect. 3 we begin a series of reductions. The surgery theorem gives obstructions to finding ε homotopy equivalences $M \to X$. Essentially if we can do this for all $\varepsilon > 0$, then we can use the end theorem of Ends of Maps I to "take the limit $\varepsilon \to 0$" and obtain a resolution. Therefore we must see that these obstructions vanish. Most of them are avoided by a naturality argument. They are essentially shown to form a cohomology class, so when the problem is restricted to a contractible subset only obstructions which occur on a point survive. I am indebted to R.D. Edwards for pointing out that this trick (developed for use elsewhere) could be used here. It considerably simplifies the proof.

The remaining obstruction, a single integer, is harder to avoid. It is locally defined, but globally constant. Therefore being non-zero would imply that no open set in X has a resolution. In Sect. 4 we transfer the problem to a torus (in the tradition of Novikov and Kirby!). There the last obstruction can be recognized as part of the ordinary surgery obstruction, and therefore seen to be zero.

2. Surgery Obstructions

In this section we show that the algebraic obstructions to surgery are ε versions of the ordinary obstructions. Only the "simply connected" $4k$ dimensional case is considered, since that is sufficient for the application. The development of Wall [18, Chap. 1, 5] adapts well to this setting, so we concentrate on the changes necessary for the ε estimates. There are also a few changes in notation (mainly the use of "normal map").

The usual surgery theorem (Wall [18, p. 37], Browder [2, p. 31]) is roughly this:

(1) a degree 1 normal map $f: M^{4k} \to K$ (M a manifold, K Poincaré) has an invariant $\sigma(f)$ defined. This is an equivalence class of nonsingular even symmetric bilinear forms over $\mathbb{Z}$.

(2) If f is normally bordant to $f': M' \to K'$ (allowing both M and K to change) then $\sigma(f') = \sigma(f)$.

(3) If $\sigma(f) = 0$ and K is 1-connected, then f is normally bordant (holding K fixed) to a homotopy equivalence.

Theorem 2.1 is an ε version of this. The most awkward feature is that the ε version of the equivalence relation on forms is not an equivalence relation. ($A \underset{\varepsilon}{\sim} B \underset{\varepsilon}{\sim} C$ only implies $A \underset{2\varepsilon}{\sim} C$). Therefore instead of an equivalence class, the obstruction is a *set* of "associated forms".

New terms used in the statement (eg. ε Poincaré) will be defined below.

2.1. **Theorem.** *Suppose X is a locally compact metric ANR, $Y \subseteq X$ is compact, and $k > 1$.*

1) *Given $\varepsilon > 0$ there is $\delta > 0$ such that if $f: (M^{+k}, \partial M) \to (K, \partial K)$ is a proper degree 1 normal map and $p: K \to X$ is proper, satisfying over Y: K is $(\delta, 1)$ connected, ∂f is a δ homotopy equivalence, and $(K, \partial K)$ has a δ Poincaré structure of (total) dimension $\leqq 100$* k then *there is a (nonempty) set of "associated" (see 2.6) even symmetric ε bilinear forms on geometric $\mathbb{Z}$ modules over X, which are ε nonsingular over $Y^{-\varepsilon}$.*

2) *Given $\gamma > 0$ there is $\gamma > \varepsilon > 0$ such that if there is an ε normal bordism $g: (N; \partial_0 N, \partial_1 N) \to (L; \partial_0 L, \partial_1 L)$ with L ε, 1-connected over Y and* $\dim(N) \leqq 100$ k *then forms ε associated to $\partial_0 g, \partial_1 g$ over $Y^{-\varepsilon}$ are γ bordant over $Y^{-\gamma}$.*

3) *Given $\alpha > 0$ there is $\varepsilon > 0$ such that if δ, $f: (M, \partial M) \to (K, \partial K)$ satisfy the conditions of* (1), *and a form associated to f is ε bordant to the trivial form over $Y^{-\varepsilon}$,* then *M is normally bordant* rel M *to $f': (M', \partial M) \to (K, \partial K)$ which is an α homotopy equivalence over $Y^{-\alpha}$.*

We recall and define the terms used, beginning with the algebra.

Geometric modules, and size conditions on homomorphisms, are defined in Ends I, p. 321. We will use *radius*, as in Ends II, rather than the *diameter* notion of Ends I, but the difference is unimportant. A bilinear form $\lambda: G \times G \to \mathbb{Z}$ has *radius* $< \varepsilon$ if for basis elements a, b of G, $\lambda(a, b) = 0$ if $d(a, b) \geqq \varepsilon$. Or equivalently, if the adjoint $\lambda^a: G \to G^*$ has radius $< \varepsilon$. λ is *ε nonsingular* over Y if the adjoint λ^a has an inverse of radius $< \varepsilon$ over Y. A *hyperbolic* form is the form $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ on a module $G \oplus G$. Finally, two forms (A_1, λ_1), (A_2, λ_2) are ε *bordant* over Y if there is a G so that $(A_1, \lambda_1) \oplus (A_2, -\lambda_2) \oplus \left(G \oplus G, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$ is ε isomorphic over Y to a hyperbolic form.

A degree 1 normal map is the standard item (Wall [18, p. 15]) except that degree 1 means with locally finite coefficients if M, K are not compact. The notion of an ε Poincaré space was introduced in Ends of Maps II. We repeat it here with slight modifications.

2.2. **Definition.** *Suppose $(K, \partial K)$ is a locally compact ANR pair, and $p: K \to X$ is proper. Then an ε Poincaré structure of (total) dimension $n + k$ for $(K, \partial K)$ over Y consists of the following:*

1) *A mapping cylinder neighborhood $(U, \partial_0 U)$ of a proper embedding $(K, \partial K) \subset (\mathbb{R}^{n+k-1} \times [0, \infty), \mathbb{R}^{n+k-1} \times \{0\})$. Let $\partial_1 U$ denote the closure of $U - \partial_0 U$.*

2) *A spherical fibration $S^{k-1} \to S(\xi) \to K$.*

3) *A map $(U; \partial_0 U, \partial_1 U) \to (D(\xi); D(\xi | \partial K), S(\xi))$ which is an ε homotopy equivalence over Y.*

Here if ξ is a spherical fibration then $S(\xi)$, $D(\xi)$ denote the total space and associated disc bundle respectively. Notice that the Poincaré duality dimension of $(K, \partial K)$ is n, different from the total dimension of the structure.

2.3. **Example.** *Suppose $(X, \partial X)$ is an ANR homology manifold pair of dimension n. Then for every $\varepsilon > 0$ there is an ε Poincaré structure for $(X, \partial X)$, over X, of total dimension $2n + 1$.*

Proof. Since the dimension of X is n, there is a proper $1-LC$ embedding $(X, \partial X) \subset (\mathbb{R}^{2n} \times [0, \infty), \mathbb{R}^{2n} \times \{0\})$. Let $v: \partial_1 U \to X$ be the map in a mapping cylinder neighborhood of this embedding (one exists by Ends I, 3.1). Alexander duality shows that v is an approximate fibration with fiber $\simeq S^n$ (see Ends I, 3.3). The associated Hurewitz fibration ξ is therefore a S^n fibration, and the natural inclusions $(U;\ \partial_0 U, \partial_1 U) \subset (D\xi; D(\xi | \partial X),\ S(X))$ are ε homotopy equivalences for all $\varepsilon > 0$, over X.

This completes the proof of 2.3.

The next step is to extend Wall's connection [17] between chain and CW complexes to the ε situation. We will use this as a replacement for homology groups, which do not extend.

Suppose (K, L) is a relative CW complex, and $p: K \to X$ is such that the image of each cell has diameter $< \varepsilon$. Choose a basepoint in each cell, and let $C_n^c(K, L)$ be the geometric $\mathbb{Z}$ module generated by the images of basepoints of n-cells. These are the cellular chain groups. The boundary homomorphisms $\partial: C_n^c \to C_{n-1}^c$ can be defined by intersection numbers, so have radius $< 2\varepsilon$. The result we want is that changes in this chain complex (up to chain equivalence) can be realized by changes in the CW complex structure of (K, L).

2.4. **Proposition.** *Suppose $X \supset Y$ as usual (X locally compact metric ANR, Y compact) and n, ε are given. Then there exists $\delta > 0$ so that given the data*

1) *a δ CW pair $(K, L) \to X$ with K, L both $(\delta, 1)$ connected over Y and* $\dim(K - L) \leqq n$,

2) *a geometric $\mathbb{Z}$ chain complex A_* over X of radius $< \delta$ and with $A_j = 0$ for $j \neq 0, 1$ or $j > n$,*

3) *a chain map $f: A_* \to C_*^c(K, L)$ which is a δ chain equivalence over Y,*

then *there is an ε CW pair (K', L), a map $g: (K', L) \to (K, L)$ and a basis preserving ε isomorphism $\theta: A_* \to C_*^c(K', L)$ such that $f = g_* \theta$ over $Y^{-\varepsilon}$.*

Proof. (See the proof of Theorem 2 in Wall [17], Part II.) We show by induction that given δ_k there is δ small enough so there is $(K'_k, L) \to (K, L)$ so that (K, K'_k) is (δ_k, k) connected, and

$$C_*^c(K'_k, L) = \begin{cases} A_*, & * \leqq k \\ 0, & * > k \end{cases}$$

Note that when $k=n$ the theorem is complete, and that we can start with $k=1$ by the 1-connected hypotheses.

Assume that (K'_k, L) satisfy the hypotheses above. Assume (by taking a mapping cylinder) that K'_k is a subcomplex of K. There is a natural δ_k chain equivalence $(A_*, *>k)\to C^c_*(K, K'_k)$. By using the inverse we can represent the image of generators of A_{k+1} by small algebraic sums of $k+1$ cells. Adding copies gives maps $(D^{k+1}, S^k)\to(K, K'_k\cup K_k)$. But (K_k, K'_k) is (δ_k, k) connected so these deform to maps to (K, K'_k). Use the boundaries to attach cells to K'_k; the result is K'_{k+1}. The maps $D^{k+1}\to K$ give a map $K'_{k+1}\to K$ which satisfies the chain complex hypotheses above.

To complete the induction step we must show that (K, K'_{k+1}) is $(\delta_{k+1}, k+1)$ connected. The relative chains are equivalent to $(A_*, *>k+1)$, so it is homologically $(\delta_n, k+1)$ connected (Ends I, p. 302). Therefore by the eventual Hurewicz Theorem (Ends I, p. 302) with all $(A_i, B_i)=(K, K'_{k+1})$ there is δ_k small enough so that (K, K'_{k+1}) is $(\delta_{n+1}, k+1)$ connected.

This completes the proof of 2.4. Notice that a corollary (which has a much shorter proof) is that if (K, L) is (δ, k) connected, then it is ε equivalent to (K', L) with no relative cells in dimensions $\leqq k$.

We begin the proof of 2.1 with an ε analog of Wall [2, Chap. 1]; surgery "below the middle dimension".

2.5. **Lemma.** *Suppose $X\supset Y$ as usual (locally compact metric ANR, Y compact), $k>0$, and $\varepsilon>0$. Then there is $\delta>0$ so that if $(K, \partial K)\to X$ is a δ CW pair of dimension $\leqq 100$ k, K and ∂K are $(\delta, 1)$ connected over Y and $f: (M, \partial M)\to(K, \partial K)$ is a normal map, then*

1) *if $\dim M=2k$, and ∂f is $(\delta, k-1)$ connected over Y then f is normally bordant rel ∂M to f' which is (ε, k) connected over $Y^{-\varepsilon}$.*

2) *if $\dim M=2k+1$, and ∂f is (δ, k) connected over Y, then f is normally bordant to $f': (M', \partial M')\to(K, \partial K)$ with f' (ε, k) connected over $Y^{-\varepsilon}$, $(K, M'\cup(\partial K))$ $(\varepsilon, k+1)$ connected over $Y^{-\varepsilon}$, and $\partial M'$ differs from ∂M by small trivial surgeries on $k-1$ spheres.*

Proof. This proceeds by induction, showing that if $j<k$ and $\delta_{j+1}>0$ then there is $\delta_j>0$ such that a map which is (δ_j, j) connected over $Y^{-\delta_j}$ can be made $(\delta_{j+1}, j+1)$ connected over $Y^{-\delta_{j+1}}$ (by surgery). We indicate modifications necessary in Wall's treatment.

First the maps used for surgery, $(D^{j+1}, S^j)\to(K, M)$ must be small. For this use 2.4 to represent (K, M) by a CW complex with no cells of dimension $\leqq j$ and use the $j+1$ cells of this complex. To represent $S^j\to M$ by a small immersion use the ordinary immersion theorem in the inverse image of some small open set in X containing the image of D^{j+1}. Now general position gives small embeddings, and small surgeries can be performed. By comparing with the complex obtained above with no cells of dimension $\leqq j$, we can verify the connectivity conclusion.

This ends the discussion of 2.5. We begin the proof of 2.1 with the construction of "associated forms".

Suppose $f: (M, \partial M)\to(K, \partial K)$ is a map as in 2.1.1 (in particular $\dim M=4k$). Then by 2.5 there is a normal bordism rel ∂M to a $(\delta', 2k)$ connected

At this point in the standard case, Poincaré duality is used to show that $H_*(K, M')=0$ if $*\neq 2k+1$. We obtain a similar conclusion on the chain level. First, there is a δ' chain equivalence $C^c_*(K, M')\to A_*$, where $A_*=0$ for $*\leqq 2k$. Next, the usual duality argument applied locally shows that if $Z\subset Y$ then for $j>2k+1$ $H_j(K(Z), M'(Z))\to H_j(K(Z^{2\delta}), M'(Z^{2\delta}))$ is zero. Here $K(Z)$ means the inverse image of Z in K. This implies a similar fact about A_*, which can be used to construct a small chain contraction of A_*, $*>2k+1$. The standard folding process uses the contraction to give a chain equivalence of A_* to a complex of the form $\partial: B_{2k+1}\to B_{2k}$, such that there is a right inverse $j\partial=1_{2k}$.

Next do surgery on M' on small $2k$ spheres corresponding to a bases of B_{2k}. This gives a map $f'': M''\to K$ with chains equivalent to $(\partial, 0, 0): B_{2k+1}\oplus B_{2k}\oplus B_{2k}\to B_{2k}$. This complex is equivalent to $B_{2k+1}\oplus B_{2k}$ concentrated in dimension $2k+1$. Now we can apply 2.4 to find an δ'' homotopy equivalent CW pair $(K', M'')\simeq(K, M'')$ which has cells only in dimension $2k+1$. Denote the chain group by A $(=B_{2k+1}\oplus B_{2k}$ above).

There is an even symmetric $2\delta''$ bilinear form defined on A: if a, b are basis elements, they correspond to cells in K' attached to M'' by maps $S^{2k}\to M^{4k}$, and $\lambda(a, b)$ is the intersection number of these spheres in M. The standard arguments show that it is even and symmetric. It has radius $<2\delta''$ because the cells of K' have diameter $<\delta''$. Finally the standard Poincaré duality argument applied over subset $Z\subset Y$ as above, shows that it is $4\delta''$ nonsingular.

2.6. **Definition.** *An ε form (A, λ) over $Y^{-\varepsilon}$ is* associated *to the map f if it arises by the construction above: there is an ε normal bordism rel ∂M to $f'': M''\to K$, an ε equivalence over Y $(K', M'')\to(K, M'')$ such that (K', M'') has cells only in dimension $2k+1$, $C^c_{2k+1}(K', M'')=A$, and λ is given by intersection numbers in M''.*

According to the discussion above, such associated ε forms exist if the initial δ is small enough. Therefore 2.1.1 is complete.

Now suppose (as in 2.1.2) that f and g are normally bordant, and we are given associated ε forms. As part of the data for the forms we have bordism to highly connected maps f'', g''. Glue these bordisms together to obtain a normal map $F: (W, \partial W)\to(J, \partial J)$ with $\dim W=4k+1$. ∂F is the union of f'', $-g''$, and the bordism of ∂M to ∂N which is a δ homotopy equivalence. It is therefore $(\varepsilon, 2k)$ connected. We can apply surgery below the middle dimension (2.5) to obtain $F': (W', \partial W')\to(J, \partial J)$ which is $(\varepsilon', 2k)$ connected, relatively $(\varepsilon', 2k+1)$ connected, and whose boundary differs from ∂F by trivial surgeries in dimension $2k$.

We can use the previous data to get an ε' equivalence $((\partial J)', \partial W')\to(\partial J, \partial W')$ so that $\partial J'$ is $\partial W'$ union small $2k+1$ cells. These cells come from the form data for f'', g'', and the new surgeries, so the form induced on the chain group is (form of f'')+($-$form of g'')+(hyperbolic). The goal therefore is to show that the form for $\partial F'$ is isomorphic to a hyperbolic form.

At this point in Wall [18, p. 52] one considers the exact sequence

$$0\to H_{2k+2}(J, W'\cup\partial J)\to H_{2k+1}(\partial J, \partial W')\to H_{2k+1}(J, W')\to 0.$$

The end groups are dual, so this can be written as

$$0\to B \xrightarrow{\ j\ } A \xrightarrow{\ h\ } B^*\to 0,$$

and $h=j^*\lambda^a$. Since B is free based (a geometric module) there is an isomorphism $B\simeq B^*$. Composing this with a splitting of h, $v\colon B^*\to A$, and adding to j gives an isomorphism $B\oplus B\to A$. The form λ composed with this isomorphism has matrix $\begin{bmatrix}0 & I\\ I & H\end{bmatrix}$. Since H is even symmetric it can be written as $H_1+H_1^t$ (t = transpose). Changing the splitting v to $(-H_1, 1)$ changes the isomorphism $B+B\to A$ to an isometry from $\begin{bmatrix}0&I\\I&0\end{bmatrix}$ to λ. This is the required isomorphism with a hyperbolic form.

We must reformulate this argument on the chain level, with control.

First we can use local homology and duality to argue, as in the construction of associated forms, that $C^c_*(J,W')$ and $C^c_*(J,W'\cup\partial J)$ have chain contractions except in dimensions $2k+1$, $2k+2$ respectively. Boundary connected sums with small copies of $S^{2k}\times D^{2k+1}$ can be used to stabilize both complexes simultaneously, after which they are equivalent to complexes concentrated in a single dimension. Denote these by $C^c_*(J,W'\cup\partial J)\simeq B_{2k+2}$, $C^c_*(J,W')\simeq D_{2k+1}$.

Next note that because Poincaré duality can be defined using intersection numbers, the usual global duality homomorphism gives an ε'' isomorphism over $Y^{-\varepsilon''}$; $B_{2k+2}\simeq D^*_{2k+1}$. This satisfies the relation denoted by $h=j^*\lambda^a$ above.

For use in the last step we define ε exact sequences. A pair of homomorphisms of geometric modules over X, $A\xrightarrow{i} B\xrightarrow{j} C$ is *ε exact over* Y if they have radius $<\varepsilon$, $ji=0$, and if for every $K\subset Y$, $(\ker j)\cap(B|K)\subseteq i(A|K^\varepsilon)$. Since the cells of J are small, the short exact sequence of chain complexes of the triad $(J,W'\cup\partial J,W')$ is actually ε short exact. The chain equivalence over Y of these chain complexes constructed above gives an ε'' short exact sequence over $Y^{-\varepsilon''}$

$$0\to B_{2k+2}\xrightarrow{\ j\ } A_{2k+1}\xrightarrow{\ h\ } D_{2k+1}\to 0.$$

Now the argument outlined above for the $X=$(point) case (considered by Wall) applies with ε estimates. This gives an ε'' isometry of the form on A_{2k+1} with a hyperbolic one. As observed above, this proves 2.1(2).

We now consider part 2.1(3), and suppose an associated form is ε bordant to 0. Part of this data is a normal bordism of f to an $(\varepsilon,2k)$ connected map. Connected sums of M with small copies of $S^{2k}\times S^{2k}$ changes the form by addition of a hyperbolic form. By replacing M by this sum we may assume that the form of f is itself (rather than stably) isomorphic to a hyperbolic form. This is $(A,\lambda)\simeq\left(G\oplus G;\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)$ in the notation above.

There is also a CW complex equivalent to (K,M) with only $(2k+1)$-cells ... the basis for $G+\{0\}$ in A by ma

$(D^{2k+1},S^{2k})\to(K,M)$. All intersections and selfintersection numbers of the boundaries $S^{2k}\to M$ are zero, and $\dim M\geq 5$, so the Whitney trick given in Wall [18, Chap. 3] approximates these by disjoint framed embeddings. Since $K\to X$ is $(\delta,1)$ connected, the 2-discs for the Whitney trick can be chosen of diameter $<\varepsilon$. Therefore the embeddings are within ε of the original maps. As in Wall [18, p. 51] use these embeddings and maps to do surgery on f. Then local homology calculations like those of Wall show that the result is an α homotopy equivalence, provided δ is chosen small enough to begin with.

This completes the proof of Theorem 2.1.

The section closes with a realization theorem for the obstructions. Theorem 2.1 as a special case of a more general situation in which the 1-connectedness of $K\to X$ is relaxed; and modules over group rings $\mathbb{Z}\pi$ are encountered. Proposition 2.7 is more specialized to the 1-connected situation, in the conclusion that it can be done with manifolds without boundary.

2.7. Proposition. *Fix $k>1$, and consider locally compact metric ANRs of dimensions $\leq k$. Then from a form λ on such an ANR X we construct a proper degree 1 normal map $g_\lambda\colon P_\lambda\to Q_\lambda\to X$ of dimension $4k$, 1-connected over X, with $\partial Q_\lambda=\emptyset$ and such that λ is an associated form of g_λ. The construction depends only on the bordism class of λ, is natural with respect to restriction to open sets, and natural up to normal bordism with respect to proper maps $X_1\to X_2$. More precisely:*

1) *If $Y\subset X$ is compact and $\varepsilon>0$, then there is $\delta>0$ such that given a δ symmetric even nonsingular form (G,λ) on Y there is a canonical proper degree 1 normal map $g_\lambda\colon P_\lambda\to Q_\lambda\to Y^{-\varepsilon}$ with Q_λ $(\varepsilon,1)$-connected over $Y^{-2\varepsilon}$, $\partial Q_\lambda=\emptyset$, and (G,λ) is associated to g_λ over $Y^{-2\varepsilon}$.*

2) *Given $\gamma>0$ there is $\varepsilon>0$ so that if (G,λ), (H,τ) are δ forms as above, and are ε bordant over $Y^{-\varepsilon}$, then over $Y^{-\gamma}$ there are γ homeomorphisms $P_\lambda\simeq P_\tau$, $Q_\lambda\simeq Q_\tau$ such that the diagram γ homotopy commutes.*

3) *Given $Y\supset Z$ and $\gamma>0$, there is $\varepsilon>0$ so that if δ is as in (1) for both Y and Z then the realization of the restriction $g_{\lambda|Z}$ is γ homeomorphic (as in (2)) to the restriction $(g_\lambda)|_Z$, over $Z^{-\gamma}$.*

4) *If $f\colon X_1\to X_2$ is proper, and $f^{-1}(Y_2)\subset Y_1$, then given $\gamma>0$ there is $\delta>0$ so that if (G,λ) is a δ form on Y_1 then there is a $(\gamma,1)$-connected normal bordism with $\partial=\emptyset$ between the realization of the image $g_{(f_*\lambda)}$ and the image of the realization $f_*(g_\lambda)$, over $Y_2^{-\gamma}$.*

Proof. Let $k=\dim X$, and let $X\subset W$ be a mapping cylinder neighborhood of 1-LC embedding of X in R^{4k-1}. Denote the projection by $p\colon W\to X$, then this is $(\delta,1)$ connected for all δ. Since $X\subset W$ we can consider (G,λ) as δ form over W, nonsingular over $p^{-1}(Y)$. We apply the realization procedure of Wall [17, p. 53]: represent the basis of G by small embedded $2k$ discs in W. Then construct regular homotopes of the boundary spheres so that the tracks of these homotopies in $W\times I$ have intersection numbers given by λ. Finally add $2k$ handles to the resulting embeddings of S^{2k-1} in the top of $W\times I$. This gives a degree 1 normal map $M\to W\times I$.

By construction the normal map is a homeomorphism of the parts of ∂M lying over $(\partial M)\times I\cup M\times\{0\}$. We claim that it can be approximated to be a homeomorphism on $\partial M \times \{1\}$ also. More precisely, if $\alpha > 0$ there is $\delta > 0$ so that the result of the construction can be α approximated by a homeomorphism over $(\partial M \times \{1\}) \cap p^{-1}(Y^{-\alpha})$. To see this, first note that the homology calculations of Wall used locally show that since λ is nonsingular over $p^{-1}(Y)$, the boundary of M over $W \times \{1\}$ is homologically δ equivalent to $W \times \{1\}$ (δ measured in W). The eventual Hurewitz theorem (Ends I, 5.2) shows that if δ is small enough, it will be a γ homotopy equivalence over $p^{-1}(Y^{-\alpha})$, for given $\gamma > 0$. Finally Chapman and Ferry [6] have shown that if W, α are given, then there is $\gamma > 0$ so that a γ homotopy equivalence to W (measured in W) is α homotopic to a homeomorphism.

Since $f\colon M \to W \times I$ is a homeomorphism on the boundary (over $Y^{-\alpha}$), we can extend it by the identity map to obtain $g\colon M \cup_\partial W \times I \to (W \times I) \cup_\partial (W \times I)$. (The target space is the double of $W \times I$.) This is a degree 1 normal map without boundary (over $Y^{-\alpha}$ at least). Therefore g will satisfy the conclusion of the theorem if we show that (G, λ) is an ε associated form.

In fact g satisfies the conditions required to define the form: $M \cup_\partial W \times I$ is equivalent to the double $(W \times I) \cup_\partial (W \times I)$ wedge small $2k$ spheres corresponding to the handles added. Attaching $2k+1$ discs to these gives the equivalent CW complex with only $2k+1$ cells. The cellular chain group is exactly G. The boundaries in $M \cup_\partial W \times I$ are composed of the handles added to $W \times I$, union the track of the regular homotopies in $W \times I$, union the discs in $W \times \{0\}$. Therefore by construction of the homotopies the intersections are given by λ. This shows that (G, λ) is associated to g, and completes 2.7(1).

For statement (2), suppose forms λ, τ are bordant. This means that after stabilization by hyperbolic forms they are isomorphic. According to Ends I, 9.4, the isomorphism after further stabilization is equivalent to a deformation. Stabilization corresponds to adding cancelling pairs of handles, so does not change the underlying manifold. Similarly deformations can be realized by handle moves which do not change the manifold. We are therefore comparing two handlebodies formed by adding $(2k)$-handles to $W \times I$, on spheres which extend to discs in $W \times I$. There is a bijective correspondence between the two sets of discs, so that corresponding discs have the same intersection numbers.

We construct an h-cobordism between the handlebodies. Consider one of the sets of discs as lying in $(W \times I, W \times \{0\})$, the other in $(W \times I, W \times \{1\})$. Changing ends reverses orientation, so the collections now have opposite intersections. Taking connected sums of corresponding discs on each side gives framed embeddings $S^{2k-1} \times I \subset W \times I$ with zero intersections and selfintersections. Use the Whitney trick to remove these intersections, and obtain embeddings. Use these embeddings to attach (handles)$\times I$; $D^{2k} \times D^{2k} \times I$, $S^{2k-1} \times D^{2k} \times I$ to these embeddings in the top of $(W \times I) \times I$. The result is an (ε) h-cobordism, with the original handlebodies on the ends. The thin h-cobordism theorem therefore implies that they are homeomorphic.

The remaining choice was of a homeomorphism from the upper boundary to the lower. But since the homeomorphism group is locally contractible (Edwards and Kirby [11]), if ε is small enough these will be isotopic.

The statement (3) should be clear. For (4) one observes that the map f gives a codimension 0 embedding of the regular neighborhood of X_1 in the neigh-

borhood of X_2. The bordism between W_1 and W_2 is given by $W_1 \times [0,1] \cup_{W_1 \times \{1\}} W_2 \times [1,2]$, and the handlebody and homeomorphism on W_1 are extended to W_2 by the identity.

This completes the proof of 2.7.

3. Reduction to the Single Obstruction

In this section the resolution problem is reduced to a single integer obstruction.

3.1. First Reduction. *To show that X of dimension ≥ 5 has a resolution, it is sufficient to show that for some k each point in $X \times \mathbb{R}^k$ has a neighborhood which has a resolution.*

Proof. Since $X \times \mathbb{R}^k$ ($k \geq 2$) satisfies the 2-disc condition (Daverman [7]) local resolvability implies that it is a manifold (Edwards [10]). But according to Ends I, 3.23, this implies that X itself is resolvable if $\dim X \geq 5$.

3.2. Definition of the Obstruction. Suppose X is an ANR homology manifold. By crossing with some R^j we may assume that the dimension of X is $4k$, some k. Choose a point x_0 in X. The obstruction will be to the construction of a resolution of a neighborhood of x_0.

Suppose $X \subseteq \mathbb{R}^{n-4k}$ is a proper $1-LC$ embedding. Then there is a mapping cylinder neighborhood W, and the projection $v\colon W \to X$ is an approximate $n-1$ sphere fibration. Let X_1 be a neighborhood of x_0 which is contractible in X. Then the restriction of this approximate fibration to X_1 has a fiber homotopy trivialization. Considered as a smooth structure on the Spivak normal fibration, this defines in the traditional way a proper degree 1 normal map from a smooth manifold to X_1. Explicitly, let $\partial W_1 = v^{-1}(X_1)$, and W_1 the mapping cylinder of this. Then the fiber homotopy trivialization defines a map $(W_1, \partial W_1) \to (D^n, S^{n-1})$. Approximate this rel boundary to be transverse to $0 \in D^n$. The inverse image of 0 is then a smooth manifold $M \subset W$, and the composition $M \subset W \to X_1$ is a proper degree 1 normal map.

Now we apply the surgery Theorem 2.1. Choose $X_2 \subset X_1$ a compact neighborhood of x_0. Note $\partial M = \emptyset$, X_1 is $(\delta, 1)$ connected, and is δ Poincare over X_1, for every $\delta > 0$ (Example 2.3). Therefore for every $\alpha > 0$ there is $\varepsilon > 0$ and ε associated bilinear forms which are obstructions to obtaining an α homotopy equivalence over X_2^{α}. Let (G, λ) be one of these forms. Then by the realization Theorem 2.7 there is a proper degree 1 normal map $P \to Q \to X_2$ with $\dim P = 16k$, $\partial P = \partial Q = \emptyset$, and such that $P \to Q$ has (G, λ) as an associated form over $X_2^{-\varepsilon}$.

The fiber of $(W_2, \partial W_2) \to X_1$ is homotopy equivalent to $(D^n, S^{n-1}) \to (W_2, \partial W_2)$. The composition $Q \to X_2 \subset W_2$ is disjoint from ∂W_2, and therefore the image of S^{n-1}. Since these are all smooth manifolds we can closely approximate the maps to be transverse regular. Making P transverse gives a pullback diagram

$$\begin{array}{ccccc} P' & \longrightarrow & Q' & \longrightarrow & (D^n, S^{n-1}) \\ \downarrow & & \downarrow & & \downarrow \\ P & \longrightarrow & Q & \longrightarrow & (W_2, W_2). \end{array}$$

$P' \to Q'$ is a degree 1 normal map of closed smooth manifolds of dimension $12k$. Since this is a multiple of 4, the surgery obstruction is an integer ($=1/8$ (index P' − index Q')). This integer is the obstruction we associate to $x_0 \in X$.

This completes the Definition 3.2. Note that the point of going through forms and the realization theorem is to obtain a map with manifold range, so that transversality can be applied.

3.3. **Second Reduction.** *Suppose the obstruction defined in 3.2 is zero. Then $x_0 \times \{0\}$ has a neighborhood in $X \times \mathbb{R}^3$ which has a manifold resolution.*

Proof. The first step is to show that there is a neighborhood $X_4 \subset X_2$ such that for every $\varepsilon > 0$ the proper degree 1 normal map $M \to X_1$ constructed in 3.2 is normally bordant to $f_\varepsilon: M_\varepsilon \to X_1$ which is an ε homotopy equivalence over $X_4^{-\varepsilon}$.

Choose X_3 to be a neighborhood of x_0 in X_2 which is contractible in X_2. Let X_4 be a neighborhood whose closure is in X_3. As above let W_i be the part of W lying over X_i. The inclusion $(W_3, \partial W_3) \subset (W_2, \partial W_2)$ is homotopy equivalent to $X_3, \subset X_2$ crossed with (D^n, S^{n-1}), so the contraction of X_3 gives a factorization up to homotopy

$$\begin{array}{ccc} & (D^n, S^{n-1}) & \\ \nearrow & & \searrow \\ (W_3, \partial W_3) & \longrightarrow & (W_2, \partial W_2). \end{array}$$

We will use this homotopy to construct a normal bordism of $P \to Q$.

Choose a smooth triangulation of W_2 so that the simplices have images in X of diameter $< \varepsilon$, and so that there is a compact PL submanifold $W_3 \supset K \supset W_4$. Triangulate D^n so that $(D^n, S^{n-1}) \to (W_4, \partial W_4)$ is simplicial.

Next make the map $Q \to W_2$ transverse to this triangulation (transverse to each simplex), and make $P \to Q$ transverse to the resulting partition of Q. This breaks $P \to Q$ up into many small degree 1 normal maps. The inverse image of a simplex of W_2 in Q is a manifold, with boundary the inverse image of the boundary of the simplex.

The next step fits nicely into the geometric description of surgery developed in the author's thesis (see Quinn [13] or Wall [18, Chap. 17A]).

There is a simplicial complex NM (Δ-set actually; see Rourke and Sanderson [16]) whose simplices are degree 1 normal maps of manifolds, with boundary split up into pieces like the boundary of a simplex. The face operation ∂_j in the Δ-set corresponds to taking the part of the boundary of the normal map corresponding to $\partial_j \Delta$. The Δ-set NM was first described by C. Rourke in unpublished notes on Sullivan's work on surgery.

The association of a simplex in W_2 to the piece of the normal map $P \to Q \to W_2$ lying over the simplex, defines a map $W_2 \to NM$. Since ∂Q is empty, and the image of Q is disjoint from ∂W_2, ∂W_2 maps to the empty normal map $\emptyset \in NM$.

Next restrict this map to the neighborhood W_3. The homotopy factorization constructed above gives a homotopy rel ∂W_3 to a map which factors

$$(W_3, \partial W_3) \to (D^n, S^{n-1}) \to |NM, \emptyset|.$$

Using the simplicial approximation theorem and the Kan condition we can get simplicial maps

$$(W_3, \partial W_3) \to (\Delta^n, \partial \Delta^n) \to (NM, \emptyset).$$

The simplicial map $(\Delta^n, \partial \Delta^n) \to (NM, \emptyset)$ corresponds to a degree one normal map of closed manifolds, specifically the transverse pullback $P' \to Q'$ constructed in 3.2.

Now we assemble these maps. A simplicial map $K \to NM$ assigns to each simplex of K a normal map, with boundary divided into pieces corresponding to the faces of the simplex. If two simplices have a common face, then the corresponding parts of the normal maps are equal. Therefore we can fit them together. Take the disjoint union of the normal maps corresponding to nondegenerate simplices of K, and identify pieces of the boundaries corresponding to common faces. If K is a PL manifold then the result is a normal map of PL manifolds. (If K is a PL homology manifold the assembly is a topological normal map; see Ends I, 3.3.1 Part 2.)

The original map $W_2 \to NM$ was defined by splitting $P \to Q$ into pieces over simplices of W_2. Therefore the map assembles to give back $P \to Q$. The homotopy of $W_3 \to NM$ assembles to give a normal bordism of the restriction of $P \to Q$ to W_3. The result of the homotopy factors simplicially through Δ^n, so can be described as follows: a simplical map $(W_3, \partial W_3) \to (\Delta^n, \partial \Delta^n)$ is automatically transverse to the barycenter of Δ^n. Let $N \subset W_3$ be the inverse image manifold. Then the factored map assembles to the product $N \times (P' \to Q')$. We have therefore constructed a normal bordism from $P \to Q$ (over W_3) to $N \times (P' \to Q')$.

The next step is to apply the hypothesis of 3.3 that the surgery obstruction of $P' \to Q'$ is trivial. This means that $P' \to Q'$ is normally cobordant to a homotopy equivalence $P'' \to Q''$. Crossing with N and adding to the previous normal bordism gives a normal bordism of $P \to Q$ (over W_3) to a homotopy equivalence. Finally since it maps to X_3 by projection $N \times (P'' \to Q'') \to N \subset W_3 \to X_3$, it is an ε homotopy equivalence.

Actually because of simplicial technicalities the map is not quite the topological projection. Since the simplices were arranged to have diameter $< \varepsilon$ in X, it differs from the projection by only ε, so is still an ε homotopy equivalence.

Now we apply the uniqueness part of the surgery Theorem, 2.1(2). Since $P \to Q$ over X_3 is normally bordant to an ε homotopy equivalence, the form (G, λ) is δ bordant to the trivial form over $X_3{}^{\gamma}$. The bordism constructed may not satisfy the $(\varepsilon, 1)$ connected condition of 2.1(2), but this can easily be arranged; before the maps to NM are assembled, do surgery on the image of each simplex to make it a normal map of 1-connected manifolds. On the space level this is a deformation into the subcomplex NM' of 1-connected normal maps. The dimensions of the manifolds involved are large enough (>4) for this surgery to be done.

Finally since the form (G, λ) is bordant to the trivial form, Theorem 2.1(3) implies (if δ is small enough) that the original degree 1 map $M_3 \to X_3$ is normally bordant to an α homotopy equivalence over $X_3^{-\alpha}$ for $\alpha > 0$.

This completes the first step in the proof of 3.3: normal bordisms of f to $f_\varepsilon: M_\varepsilon \to X_1$ which are ε homotopy equivalences over X_4. The next step is to construct h-cobordisms between these, when stabilized: we will show that there is $X_6 \subset X_4$ a neighborhood of x_0 such that if $\varepsilon > 0$ so that if $\alpha, \rho < \delta$ and f_α, f_β are the maps constricted above, then there is a proper normal bordism $F_\varepsilon: N_\varepsilon \to X_1 \times \mathbb{R}^3$ so that $\partial_0 F = f_\alpha \times 1_{R^3}$, $\partial_1 F = f_\beta \times 1_{R^3}$, and over $X_6 \times B^3$, F is an (ε, h) cobordism from $M_\alpha \times \mathbb{R}^3$ to $M_\beta \times \mathbb{R}^3$.

As above let $X_5 \subset X_4$ be a neighborhood which contracts in X_4, and X_6 a neighborhood whose closure is contained in X_5. A normal bordism from f_α to f_β can be considered as a normal map to $X \times I$. Crossing with $\mathbb{R}^3$ gives a normal map of dimension $4(k+1)$, $F: N \to X \times I \times \mathbb{R}^3$, whose boundary is a $\max(\alpha, \beta)$ equivalence over $X_4 \times \mathbb{R}^3$.

Applying the surgery theorem to this gives an associated form (G, λ) over $X_4 \times 2B^3$. As above we apply the realization theorem, and use the space NM to analyze the restriction to $X_5 \times 2B^3$. It is equivalent to a product $N \times (P' \to Q')$ as above. Now however, since $X \times \mathbb{R}^3$ has odd dimension, $P' \to Q'$ has odd dimension. Since odd dimensional simply connected normal maps are bordant to homotopy equivalences ($L_{2i+1}(\mathbb{Z}[1]) = 0$), the argument proceeds without obstruction to give an ε equivalence. This completes the second step.

The final step is an application of the end theorem of Ends I. Choose $\delta_i > 0$ so that $f_{\delta_i} \times 1_{\mathbb{R}^3}$ is $(1/2^i, h)$ cobordant to $f_{\delta_{i+1}} \times 1_{\mathbb{R}^3}$ over $X_6 \times B^3$, by step 2. Taking the union of these h-cobordisms gives a manifold $S \to X_1 \times \mathbb{R}^3$ with an end, which is tame and 1-connected over $X_6 \times B^3$. By Ends I, Theorem 1.4 there is a completion of S over $X_6 \times B^3$. The levels M_{δ_i} are approximate completions of this end, and are δ_i homotopy equivalent to $X_6 \times B^3$. It follows that the new boundary of the completion is a δ homotopy equivalence for all $\delta > 0$, hence a resolution of $X_6 \times B^3$.

This complete the proof of 3.3, and reduces the main theorem to the single obstruction.

4. The Last Obstruction

We show that the obstruction defined in 3.2 vanishes. The obstruction is defined by transversality on a manifold degree 1 normal map with the same form as $f: M \to X$. The analogous transversality construction on f itself would be: take a manifold point $p \in X$, make f transverse, and take the surgery obstruction of $f^{-1}(p) \to p$. This is a degree 1 map of a discrete set of points, so has obstruction zero. Therefore the obstruction will vanish if we can show that it depends only on the form, not the particular degree 1 normal map. This is done by transferring the problem to a torus, where it can be recognized as part of the ordinary surgery obstruction $L(\mathbb{Z}[\mathbb{Z}^n])$ (forget ε). The hard step will be extending an ε Poincaré space across the puncture in a punctured torus.

4.1. Reduction to Poincaré Problems. As in 3.2 we let X_1 be a neighborhood of x_0 whose mapping cylinder neighborhood in $\mathbb{R}^{4k+n}$ is fiber homotopically trivial. Fix $j: X_1^* \to (\mathbb{R}^{4k})^*$, a degree 1 map of 1-point compactifications, such that $g(\infty) = \infty$ and $g(x_0) = 0$. Obstruction theory shows that there is one. Let B be an open ball about 0 in $\mathbb{R}^{4k}$, and identify it with a ball in the torus T^{4k}.

First Reduction. *Suppose that for every $\delta > 0$ there is $h: Y \to T^{4k}$, a $(\delta, 1)$ connected degree 1 normal map, with Y δ Poincaré over T^{4k}, and $(Z: X_B, Y_B) \to B$ which over $B^{-\delta}$ is a $(\delta, 1)$ connected δ Poincaré normal bordism. Then the obstruction of 3.2 is zero. Here X_B, Y_B denote $j^{-1}(B)$, $h^{-1}(B)$ respectively.*

Proof. Note that normal map, normal bordism here just mean that the (relative) mapping cylinder neighborhoods in Euclidean space are δ equivalent to the product with (D^j, δ^{j-1}) for appropriate j. Proceeding as in 3.2 we make projection to D^j transverse to 0, and do surgery below the middle dimension. Over X_B this gives the ε form used in 3.2. The map $Y \to T^n$ defines an ε form over T^n. As in the uniqueness result for forms, 2.1(2), the δ bordism from X_B to Y_B over $B^{-\varepsilon}$ gives a bordism of the forms, over $B^{-\varepsilon}$.

Now we use the naturality properties of the realization construction 2.7. The construction gives $g_0: P_0 \to Q_0 \to X_B$ realizing the form for X_B, and $g_1: P_1 \to Q_1 \to T^n$ realizing the form for Y. Applying the naturality 2.7(4) gives a normal bordism from g_0 to the realization of the image form over B. The bordism of forms over B and naturality with respect to restriction 2.7(3) identify this image as the restriction of g_1. Therefore over the mapping cylinder $h: X_B \to B$ we have an ε normal bordism $G: P \to Q \to B_h$ with $\partial_0 G = g_0$, $\partial_1 G = g_1 | B^{-\varepsilon}$.

The next step is to make G transverse to something. As in the construction of the single obstruction (3.2) we let $(W; \partial_0 W, \partial_1 W) \supset (B_h, X_B, B)$ be a mapping cylinder neighborhood of a proper relative embedding into $\mathbb{R}^{4k+n} \times [0, 1]$. As part of the data of δ Poincaré duality and normal maps, we have a δ homotopy equivalence $(W; \partial_0 W, \partial_1 W, \partial_2 W) \to (B_h \times D^n; X_B \times D^n, B \times D^n, B_h \times S^{n-1})$, over $B^{-\varepsilon}$. Consider the inverse applied to the disc crossed with the arc from x_0 to 0. This is a map $(D^n \times I; D^n \times \{0\}, D^n \times \{1\}, S^{n-1} \times I) \to (W; \partial_0 W, \partial_1 W, \partial_2 W)$. Make the normal bordism $G: P \to Q \to B_h \subset W$ transverse to this map. This gives a normal bordism of closed manifold normal maps. On the end over X is the normal map used to define the single invariant in 3.2. The invariant is therefore equal to the surgery obstruction at the other end, over $B \subset T^{4k}$.

Over T^{4k}, the mapping cylinder neighborhood is $T^{4k} \times D^n$. Transversality to the disc is therefore the same as the transverse inverse image of a point in $g_1: P_1 \to Q_1 \to T^{4k}$.

The surgery obstruction group $L_{4*}(\mathbb{Z}[\mathbb{Z}^{4k}])$ has a summand $L_{4*}(\mathbb{Z}[1]) \simeq \mathbb{Z}$, which for closed manifold surgery problems is detected by the surgery obstruction of the inverse image of a point in T^{4k} (Wall [18]). Therefore the integer that we want is part of the total surgery obstruction $\sigma(g_1) \in L_{4*}(\mathbb{Z}[\mathbb{Z}^{4k}])$. But the surgery obstruction depends only on the form on the middle dimension the surgery below the middle dimension. Therefore it is the same as the after surgery below the middle dimension. Therefore it is the same as the degree 1 normal map $M \to Y$ used to find the form for g_1. Finally, as pointed out in the introduction of the section, the appropriate part of this obstruction vanishes; $Y \to T^{4k}$ is itself degree 1, so the transverse image of a point is 0-dimensional.

Second Reduction. We must show that the δ Poincaré space $X_B \to B \subset T^{4k}$ extends over all of T^{4k}. The objective here is to reduce this to a problem over an interval.

Let p be a point in $T^{4k} - B$, and let $\alpha: T^{4k} - p \to \mathbb{R}^{4k}$ be a smooth immersion which is the identity on B. Then the pullback

$$\begin{array}{ccc} X' & \longrightarrow & T^{4k} - p \\ \downarrow & & \downarrow \alpha \\ j^{-1}(\mathbb{R}^{4k}) & \xrightarrow{j} & \mathbb{R}^{4k} \end{array}$$

gives a proper degree 1 normal map, and over B, $X = X'$. Let $\delta > 0$. We must find a $(\delta, 1)$ connected normal bordism to a $(\delta, 1)$ connected Poincaré space over $T^{4k} - p$, and then extend it over p.

For the first step, for any $\gamma > 0$ there is a γ homotopy equivalence $X'' \to X'$ (measured in X') such that X'' has a manifold neighborhood of a γ 1-skeleton. This is the ε version of Wall [19], Corollary 2.3.2, and is proved in the same way using the ε chain complex material of 2.4.

Now take a γ 2-skeleton of $T^{4k} - p$, deform the image of the 1-skeleton of X'' into it, and form the mapping cylinder. This gives a relative 2-complex which is universal for the $(\delta, 1)$-connected lifting problem. Attach 0, 1 and 2 handles to the top of $X'' \times [0, 1]$ corresponding to the cells of this 2-complex. This gives a $(\delta, 1)$ connected normal bordism of X'' to $Y_0 \to T^{4k} - p$, which is also $(\delta, 1)$ connected.

Define $T^{4k} - p \to T^{4k} \cup_p [0, 1)$ by: choose an open collar $S^{4k-1} \times [-1, 1)$. It is the projection to $[0, 1)$ on $S^{4k-1} \times [0, 1)$, is radial expansion on $S^{4k-1} \times [-1, 0) \to S^{4k-1} \times [-1, 1)$, and is the identity outside the collar. Composing gives $Y_0 \to T^{4k} \cup_p [0, 1)$ which is proper, degree 1 (on T^{4k}), normal, δ Poincaré, and $(\delta, 1)$ connected over $T^{4k} \cup [0, 1-\delta)$. Finally we can project this to $[0, 1)$ by mapping T^{4k} to 0.

4.2. **Lemma.** *Given $k > 0$ and $\frac{1}{4} > \varepsilon > 0$ there is $\delta > 0$ such that if $Y \to [0, 1)$ is proper, over $[0, 1-\delta)$ is a δ Poincaré CW complex of dimension $4k$ with fiber homotopically trivial normal fibration, and is $\delta, 1$ connected over $(\delta, 1-\delta)$, then there is $Y_1 \to [0, 1]$ which is a compact δ Poincaré space with trivial normal fibration which is $\varepsilon, 1$ connected over $(\varepsilon, 1]$ and the restrictions $Y_1 | [0, \frac{1}{2}]$ and $Y | [0, \frac{1}{2}]$ are equal (homeomorphic commuting with the map to I).*

Completion of the Reduction. Apply the lemma to the situation preceeding it. There is a map $Y_1 \to T^{4k} \cup_p [0, 1]$ since $Y_0 = Y_1$ near 0. Project to T^{4k} by collapsing $[0, 1]$ to p, then the composition $Y_1 \to T^{4k}$ satisfies all the requirements of the reduction.

Proof of 4.2. We "double" Y to get something which is not Poincaré, but has a trivial normal bundle. Smooth transversality gives a manifold M mapping to this object. We use the surgery theory of Section 2 to make M ε equivalent to Y near $\frac{3}{4}$, and then use the chain complex material 2.4 to patch together M and Y.

The homotopy and Poincaré data give the following: Let U be a regular neighborhood of Y in $\mathbb{R}^{4k+n}$, which is a mapping cylinder of $p: \partial U \to Y$. There is $V \subset \partial U$ containing $p^{-1}(Y | [0, 1-2\delta])$, and a map $(U, V) \to (Y \times D^n, Y \times S^{n-1})$ which is projection on the Y factor and δ homotopy equivalence over $[0, 1-3\delta]$.

Choose t near $1 - 4\delta$ so that $Y \to [0, 1]$ is transverse to t (we may assume p is PL). Let Y' be the inverse of $[0, t]$, Z the inverse of t. Then the double $p^{-1}(Y') \cup_{p^{-1}(Z)} p^{-1}(Y') \to Y' \cup_Z Y'$ gives a regular neighborhood in $\mathbb{R}^{n+4k}$. Since $p^{-1}(Y') \subset V$, the map to D^n gives a map rel boundary of this neighborhood to D^n, S^{n-1}. Make this transverse to 0 to give a manifold $M \to Y' \cup_Z Y'$, which is closed since $Y' \cup_Z Y'$ is compact.

Now map $Y' \cup_Z Y' \to Y'/Z$ by collapsing the second piece to a point. Since Z is a point inverse, we get $Y'/Z \to [0, 1]$. Over $[0, 1-5\delta]$, $Y'/Z = Y$. Since M was constructed using the Poincaré data of Y, $M \to Y'/Z$ is a degree 1 normal map over $[0, 1-5\delta]$. Since Y is δ, 1 connected over $[\delta, 1-5\delta]$, and of dimension $4k$, we can apply the surgery Theorem 2.1. The obstruction is a form which we can analyse as in Sect. 3: realize it as a map of manifolds, make transverse to get a map $(\varepsilon, 1-\varepsilon) \to NM$. Since the interval is contractible the only obstruction is the inverse over a single point. But the dimension of this is $4k-1$, so has surgery obstruction 0. The form is therefore bordant to 0, and we can do surgery to obtain an ε equivalence over $(\varepsilon, 1-\varepsilon)$, if δ was small enough.

We now have $M \to Y'/Z$ an ε homotopy equivalence over $(\frac{1}{2}+\varepsilon, 1-\varepsilon)$. Define $X \to [0, 1]$ to be the union of M over $[\frac{3}{4}, 1]$ and Y over $[0, \frac{3}{4}]$. This object fails to be Poincaré at $\frac{3}{4}$. However it maps to Y over $[\frac{1}{2}+\varepsilon, 1-\varepsilon]$ by an ε equivalence except at the ends and near $\frac{3}{4}$. We will add cells to X near $\frac{3}{4}$ to make it an ε' equivalence. The result will be Poincaré, and almost the object needed for the lemma.

Using the δ, 1 connectedness of Y add 0, 1 and 2 cells to X near $\frac{3}{4}$ to make it ε, 1 connected over $(\frac{1}{2}+\varepsilon, 1-\varepsilon)$. The pair (Y, X) then satisfies the hypotheses of the chain complex Lemma 2.4. Therefore to add cells near $\frac{3}{4}$ to X to obtain a complex equivalent to Y, it is sufficient to show that $C_*(Y, X)$ is equivalent to a geometric chain complex concentrated near $\frac{3}{4}$. Since $X \to Y$ is an ε equivalence on regions on each side of $\frac{3}{4}$, $(\frac{1}{2}+\varepsilon, \frac{3}{4}-\varepsilon)$ and $(\frac{3}{4}+\varepsilon, 1-\varepsilon)$ the complex $C_*(Y, X)$ has an ε chain contraction over these regions. We use these to show C_* can be made zero on slightly smaller regions.

The standard folding argument concentrates C_* in two dimensions: if C_j is the lowest nonzero module over (a, b), let $C'_i = C_i$ for $i \neq j, j+2$, $C'_j = C_j | (a, a+\varepsilon) \cup (b-\varepsilon, b)$, $C'_{j+2} = C_{j+2} \oplus (C_j | [a+\varepsilon, b-\varepsilon])$, with boundary homeomorphism $\partial + s$ in dimension $j+2$. Here s is the chain contraction. Notice that the next step takes place on a smaller interval, and has larger radius. The loss is determined by the number of steps necessary, namely dim $Y=4k$. Since this is fixed in advance we can allow for it.

When C_* is concentrated in two dimensions the boundary homeomorphism is an isomorphism. The isomorphism Theorem 8.4 of Ends I shows that this can be stably deforned to be the identity on a smaller interval. Finally a subcomplex on which the boundary is the identity can be deleted from the complex. This eliminates C_* except near $\frac{3}{4}$.

As explained above this shows how to modify X to be equivalent to Y over $(\varepsilon', 1-\varepsilon')$, if δ was small enough. X is globally Poincaré, and satisfies all the conclusions of Lemma 4.3 except ε, 1 connectedness near 1. It is a manifold there, so low dimensional surgery there yields this condition.

This completes the proof of the lemma, and therefore the theorem.

References

1. Bass, C.D.: Some products of topological space which are manifolds. Proc. AMS **81**, 641-646 (1981)
2. Browder, W.: Surgery on simply-connected manifolds. Berlin-Heidelberg-New York: Springer 1972
3. Cannon, J.W.: The recognition problem: what is a topological manifold? Bull. AMS **84**, 832-866 (1978)
4. Cannon, J.W.: The characterization of topological manifolds of dimension ≧5. Proc. Int. Congess of Math. 1978, Acad. Sci. Fennica, Helsinki 1980, pp. 449-454
5. Cannon, J.W., Bryant, J., Lacher, R.C.: The structure of generalized manifolds having nonmanifold set of trivial dimension. In: Geometric Topology. Cantrell, J.C., (ed.). New York: Academic Press 1979, pp. 261-300
6. Chapman, T.A., Ferry, S.: Approximating homotopy equivalences by homeomorphisms. Amer. J. Math. **101**, 583-607 (1979)
7. Davermann, R.J.: Detecting the disjoint disks property. Pacific J. Math. **93**, 277-298 (1981)
8. Davermann, R.J.: Decompositions of manifolds. New York: Academic Press, in preparation
9. Davermann, R.J., Walsh, J.J.: A ghastly generalized manifold. Ill. J. Math.
10. Edwards, R.D.: The topology of manifolds and cell-like maps, Proc. Int. Congress of Math. 1978, Acad. Sci. Fennica, Helsinki 1980, pp. 111-127
11. Edwards, R.D., Kirby, R.C.: Deformations of spaces of embeddings. Ann. Math. **93**, 63-88 (1971)
12. Lacher, R.C.: Cell-like mappings and their generalizations. Bull. AMS **83**, 495-552 (1977)
13. Quinn, F.: A geometric formulation of surgery. In: Topology of Manifolds, Cantrell, J.C., Edwards, C.H., (eds.). Markham 1970, pp. 500-511
14. Quinn, F.: Ends of Maps, I, Ann. Math. **110**, 275-331 (1979)
15. Quinn, F.: Ends of Maps III; dimensions 4 and 5, J. Diff. Geom. 17, 503-521 (1982).
16. Rourke, C.P., Sanderson, B.J.: Δ-sets I, Quart. J. Math Oxford, **2**, (22) 321-328 (1971)
17. Wall, C.T.C.: Finiteness conditions for CW complexes. I. Ann. Math. **81**, 56-69 (1965); II, Proc. Royal Soc. A **295**, 129-139 (1966)
18. Wall, C.T.C.: Surgery on compact manifolds. New York: Academic press 1970
19. Wall, C.T.C.: Poincaré complexes. I. Ann. Math. **86**, 213-245 (1967)

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Invent. math. 72, 285-298 (1983)

Inventiones mathematicae

Differentiability of Minima of Non-Differentiable Functionals

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In this paper we shall consider the problem of the regularity of the derivatives of functions minimizing a variational integral

$$F(u;\Omega)=\int_\Omega f(x,u,Du)\,dx \tag{1.1}$$

where Ω is an open set in $\mathbb{R}^n$, $u:\Omega\to\mathbb{R}^N$, $Du=\{D_\alpha u^i\}$ $\alpha=1,\dots,n$; $i=1,\dots,N$, and $f:\Omega\times\mathbb{R}^N\times\mathbb{R}^{nN}\to\mathbb{R}$ is a Carathéodory function (i.e. measurable in x and continuous in u,p) satisfying

$$\lambda|p|^2-a\leq f(x,u,p)\leq\Lambda|p|^2+a,\qquad \lambda>0. \tag{1.2}$$

A local minimum for the functional F is a function $u\in W^{1,2}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ such that for every $\phi\in W^{1,2}(\Omega,\mathbb{R}^N)$ with supp $\phi\subset\subset\Omega$ we have

$$F(u;\operatorname{supp}\phi)\leq F(u+\phi;\operatorname{supp}\phi).$$

In a recent article [6], we have proved basic regularity results for the local minima of the functional (1.1). In the scalar case ($N=1$) we have shown that every local minimum of F with condition (1.2) is Hölder-continuous in Ω.

In the general case $N\geqq 1$ such a result cannot hold; we proved however that $Du\in L^{2+\sigma}_{\mathrm{loc}}$ for some $\sigma>0$. More generally these results hold for Q-minima (see [7]).

In this paper we investigate the regularity of the first derivatives of the minima of F, under additional hypotheses on the function $f(x,u,p)$. Roughly speaking, we assume that f is twice differentiable and strictly convex in p, and Hölder-continuous in (x,u). We remark that we do not assume the existence of the derivative f_u, and therefore our functionals are in general non differentiable.

As usual, our results will take different form in the scalar and in the vector case. When $N=1$, we prove that every local minimum has Hölder-continuous derivatives in Ω. When $N\geqq 1$, we obtain that for every local minimum u there exists an open set $\Omega_0\subset\Omega$, with mean $s(\Omega-\Omega_0)=0$ such that $u\in C^{1,\alpha}(\Omega_0,\mathbb{R}^N)$