

EDMUND TAYLOR WHITTAKER

1873-1956

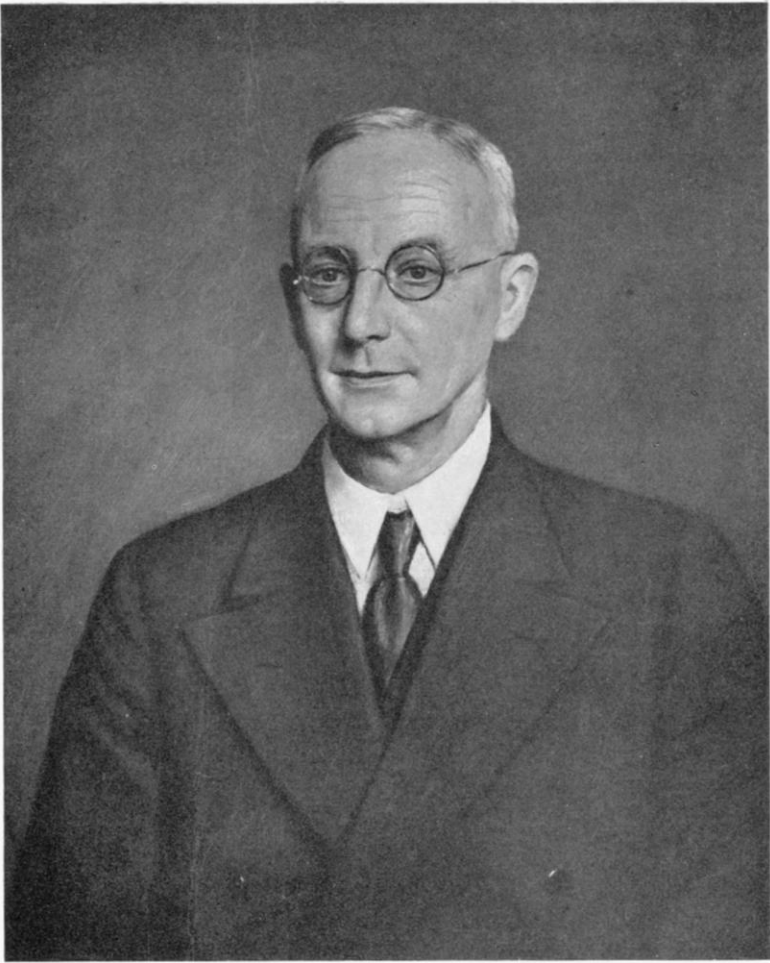
LIFE

EDMUND TAYLOR WHITTAKER was born on 24 October 1873, the eldest child of John Whittaker and his wife Selina, daughter of Edmund Taylor, M.D. The family belonged to the district where the river Ribble forms the boundary between Lancashire and Yorkshire: his father, John Whittaker (1820-1910) was the youngest son of the first marriage of Henry Whittaker (1780-1853) of Grindleton near Clitheroe, who was the sixth of the eight sons of Richard Whittaker of Rodhill Gate near Grindleton. Several of Richard's descendants were men of distinction in the late Victorian period, among them the Right Hon. Sir Thomas Palmer Whittaker (1850-1919), for many years M.P. for the Spen Valley division of Yorkshire, and Sir Meredith T. Whittaker (1841-1931), of Scarborough.

His father on his marriage to his mother—whose father practised as a physician at Middleton near Manchester—settled in Southport. Though not well off, they were able to live on their private means: and there Whittaker was born. In childhood he was extremely delicate, and his time was spent almost entirely with his mother, who devoted herself to him and was in his earlier years his only teacher. As he grew older he grew stronger, and at the age of eleven was sent away from home to the Manchester Grammar School, which then dominated education in Lancashire. He was on the classical side, which meant that three-fifths of his time was devoted to Latin and Greek: in the lower forms, where the study was purely linguistic, he did well: but his lack of interest in poetry and drama caused a falling-off when he was promoted to the upper school, and he was glad to escape by electing to specialize in mathematics. Only after he had left school did he discover the field of Latin and Greek learning that really appealed to him—ancient and mediaeval theology, philosophy and science.

He gained an entrance scholarship to Trinity College, Cambridge, in December 1891, and was bracketed second wrangler in the mathematical tripos of 1895. Six Fellowships of the Royal Society were afterwards attained by the men of this tripos—T. J. Bromwich (analyst), J. H. Grace (geometer), A. Young (algebraist), B. Hopkinson (engineer), F. W. Carter (engineer), and himself. He was elected a Fellow of Trinity in October 1896, was put on the staff, and was awarded the First Smith's Prize in 1897.

On 7 August 1901 he was married to Miss Mary Boyd, daughter of the



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Rev. Thomas Boyd, M.A., and granddaughter of Sir Thomas Jamieson Boyd of Edinburgh.

In 1906 Whittaker was appointed professor of astronomy in the University of Dublin, with the title of Royal Astronomer of Ireland. The Observatory attached to the Chair was situated at Dunsink, five miles from Dublin: its equipment was very poor, as there was no fund for the purchase of instruments, and in fact none had been bought for a generation past: it was tacitly understood that the chief function of the professor was to strengthen the school of mathematical physics in the University. He gave courses of advanced lectures, one of his pupils being Eamon de Valera.

In the spring of 1912 he was elected to the professorship of mathematics in the University of Edinburgh. Here he taught for thirty-four years, until his retirement in 1946. Their family of five was completed in Edinburgh by the birth of their youngest child, the two eldest having been born in Cambridge and the third and fourth at Dunsink.

In connexion with the Edinburgh chair, may be mentioned (1) the institution in 1914 of what was probably the first University mathematical laboratory and (2) the research school, many of whose past members are now distinguished men, including Whittaker's own successor, Dr A. C. Aitken, F.R.S.; and (3) the Edinburgh Mathematical Society, whose publications have a high reputation. Whittaker took a considerable share in administrative work, being Dean of the Arts Faculty from 1924 to 1927, and serving for two periods of office as a representative of the Senators on the University Court.

His house in George Square was a great centre of social and intellectual activity where liberal hospitality was dispensed to students of all ages. In this, as indeed in all his work, he was strengthened and supported by the gracious presence of Lady Whittaker. He was wonderfully happy in his home life and was greatly loved by all his family.

From other universities Whittaker received the honorary degrees of LL.D. (St Andrews, 1926, and California, 1934), Sc.D. (Dublin, 1906) and D.Sc. (National University of Ireland, 1939, and Manchester, 1944).

Some of his books and papers (particularly B.8, B.9, B.10 and B.11)* represent lectures or courses of lectures given at other universities. He held the Rouse Ball and Tarner lectureships at Cambridge (in 1926 and 1947, respectively), the Herbert Spencer lectureship at Oxford (1948), the Donnellan lectureship at Dublin (1946), the Riddell lectureship at Durham (Newcastle) (1942), the Selby lectureship at Cardiff (1933), the Hitchcock professorship at the University of California (1934), the Bruce-Preller lectureship of the Royal Society of Edinburgh (1931), the Larmor lectureship of the Royal Irish Academy (1948), and the Guthrie lectureship of the Physical Society (1943).

Whittaker was elected a Fellow of the Royal Society in 1905, served on the

* The references are to the Bibliography at the end of this notice, the letter B indicating the lists of Books and Monographs, and the letter R the list of Research Papers.

Council in 1911-12 and 1933-35, was awarded the Sylvester Medal in 1931 and the Copley Medal in 1954. With the Royal Society of Edinburgh he had continuous contact, being Gunning Prizeman in 1929 and President 1939-1944. At the end of his tenure of the Presidency, a bronze portrait head, executed by Mr Benno Schotz, R.S.A., was subscribed for by the Fellows and placed in the Society's House. He was President of the Mathematical Association in 1920-21, of the Mathematical and Physical Section of the British Association in 1927, and of the London Mathematical Society in 1928-29, being awarded its De Morgan Medal in 1935.

He was an Honorary Fellow or Foreign Member of the Accademia dei Lincei (1922), the Societa Reale di Napoli (1936), the American Philosophical Society (1944), the Académie Royale de Belgique (1946), the Faculty of Actuaries (1918), the Benares Mathematical Society (1920), the Indian Mathematical Society (1924), and the Mathematical Association (1935). Only twelve days before his death he was elected as a corresponding member of the French Academy of Sciences for the section of Geometry.

He was received into the Roman Catholic Church in 1930, and was Honorary President of the Newman Association (the society of Catholic graduates of Universities) in 1943-45. H.H. Pope Pius XI conferred on him the Cross *Pro Ecclesia et Pontifice* in 1935, and appointed him a member of the Pontifical Academy of Sciences in 1936.

Whittaker died on 24 March 1956. He is survived by his widow and all his five children.

WORK

1. Algebra

As a result of reading the first two volumes of Sir Thomas Muir's great work *The theory of determinants in the historical order of development*, Whittaker was stimulated in the years 1914-1922 to publish half a dozen papers on algebraical topics (R.22, R.28, R.29, R.31, R.33, R.34). Perhaps the most interesting of the theorems obtained was the following formula for the solution of algebraic equations (R.33). The root of the equation

$$0 = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$$

which is the smallest in absolute value is given by the series

$$-\frac{a_0}{a_1} - \frac{a_0^2}{a_1} \begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix} - \frac{a_0^3}{a_1} \begin{vmatrix} a_2 & a_3 \\ a_1 & a_2 \end{vmatrix} - \frac{a_0^4}{a_1} \begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix} - \dots$$

a_0^4	a_2	a_3	a_4						
	a_1	a_2	a_3						
	a_0	a_1	a_2						
a_1	a_2	a_3	a_1	a_2	a_3	a_4			
a_0	a_1	a_2	a_0	a_1	a_2	a_3			
0	a_0	a_1	0	a_0	a_1	a_2			
0	a_0	a_1	0	0	0	a_0	a_1		

2. Interpolation

A small group of papers owe their existence to the fact that Edinburgh is one of the chief centres of Life Assurance. Whittaker formed friendships with many of the actuaries engaged in it, who told him of various mathematical problems that had arisen in their experience, mostly relating to interpolation, curve-fitting, and probability. Becoming interested in the theory of interpolation, he attempted to answer some of its fundamental questions.

If the values of an analytic function $f(x)$ which correspond to a set of equidistant values of its argument (say $a, a+w, a-w, a+2w, a-2w, a+3w, \dots$) are known, then we can write out a 'table of differences' for the function. All the analytic functions which give rise to the same difference-table may be said to be *cotabular*; any two cotabular functions are equal to each other when the argument has any one of the values, $a, a+w, a-w, \dots$ but they are not in general equal to each other when the argument has a value not included in this set.

In the theory of interpolation certain expansions are introduced in order to represent the function $f(x)$, for general values of x , in terms of the quantities occurring in the difference-table: for example, the Newton-Gauss formula

$$f(x) = (fa) + \frac{x-a}{w} \Delta f(a) + \frac{(x-a)(x-a-w)}{2!w^2} \Delta^2 f(a) + \dots$$

There is no reason *a priori* why this expansion should represent $f(x)$ in preference to any other function of the set cotabular with $f(x)$; and thus two questions arise, namely:

- (1) Which one of the functions of the cotabular set is represented by the Newton-Gauss expansion?
- (2) Given any one function $f(x)$ belonging to the cotabular set, is it possible to construct from $f(x)$ a simple analytical expression for that function of the cotabular set which is represented by the Newton-Gauss expansion?

These questions were answered in a paper published in 1915 (R.27). It was shown that there exists a function which belongs to the cotabular set, and which, when subjected to Fourier analysis, has no periodic constituents of period less than twice the tabular interval w : and that this function is uniquely defined by the set. He called it the *cardinal function* of the set. Then the answer to the first question is that the function represented by the Newton–Gauss expansion is the cardinal function: the answer to the second question is that the cardinal function is represented by the series

$$\sum_{n=-\infty}^{\infty} \frac{f(a+nw) \sin \left\{ \frac{\pi}{w}(x-a-nw) \right\}}{\frac{\pi}{w}(x-a-nw)}$$

Some years later Whittaker discussed (R.39, R.40) another problem which was derived from his actuarial friends, namely that of *graduation*: this may be described as follows. A set of numbers $u_1, u_2, u_3, \dots, u_n$ is supposed to have been obtained from observations or statistics of some kind. These numbers would represent the values of a variable u_x corresponding to the values $1, 2, 3, \dots, n$ of its argument x , were it not that they are affected by accidental irregularities due to errors of observation, or to the imperfections of statistics. If we form a table of the differences $\Delta u_1 = u_2 - u_1, \Delta u_2 = u_3 - u_2, \Delta^2 u_1 = \Delta u_2 - \Delta u_1$, etc., it will often be found that these differences are so irregular that the difference-table cannot be used for the purposes to which a difference-table is usually put; before we can use the difference-table, we must perform a process of ‘smoothing’, that is to say, we must find another sequence, $u_1', u_2', u_3', \dots, u_n'$, whose terms differ as little as possible from the terms of the sequence, $u_1, u_2, u_3, \dots, u_n$, but which has regular differences. This smoothing process is called the *graduation* or *adjustment* of the observations.

Various ways of performing it were described in the actuarial text-books, but these had no logical basis. The standpoint which Whittaker now proposed was to recognize that the problem belonged essentially to the mathematical theory of probability, and that a logical solution could be found for it. To effect this, he took as a measure of the *smoothness* of the graduated values, the sum of the squares of their third differences,

$$S = (u_4' - 3u_3' + 3u_2' - u_1')^2 + (u_5' - 3u_4' + 3u_3' - u_2')^2 + \dots,$$

while as a measure of the *fidelity* of the graduated to the ungraduated values he took the sum of the squares of their difference

$$F = (u_1 - u_1')^2 + (u_2 - u_2')^2 + \dots + (u_n - u_n')^2.$$

Then by an application of the theory of inductive probability, he showed that

the most probable set $(u_1', u_2', \dots, u_n')$ of graduated values is that which makes

$$\lambda^2 S + F$$

a minimum, where λ is a constant which expresses the degree to which we are justified in sacrificing fidelity in order to obtain smoothness. On the basis of this rule, the practical details of performing the graduation were worked out. The method has the advantage that the total of the u 's and their first and second moments are the same in the graduated table as in the actual statistics on which it was based.

3. Automorphic functions

Whittaker's first long paper, 'On the connexion of algebraic functions with automorphic functions', (R.2) was the definitive edition of his Trinity fellowship thesis and Smith's Price essay. If u is a *many-valued* function of z , defined by an equation $f(u, z) = 0$, then it may be possible to find a variable t , such that both u and z are *one-valued* functions of t ; t is then called a *uniformizing variable* for the equation $f(u, z) = 0$. The simplest many-valued functions are the *algebraic* functions; for these Poincaré and Klein have proved a general existence theorem, namely, that if u is an algebraic function of z , then a uniformizing variable t exists, and u and z are *automorphic* functions of t ; automorphic functions are the natural generalization of circular and elliptic functions. The existence theorem however did not connect t analytically with u and z .

Now algebraic functions are classified according to their *genus*: and when the genus is zero or unity, t can be found, for the automorphic functions required are rational functions and doubly periodic functions, in the two cases respectively; but no class of automorphic functions with simply connected regions (analogous to period parallelograms) had been found hitherto, which was applicable to the uniformization of algebraic functions whose genus is greater than unity. In the paper under review, a new class of discontinuous groups of transformations of a variable t was obtained as a subgroup of the group generated by elliptic transformations each of period 2 i.e. substitutions of the form

$$\frac{t' - \bar{\alpha}}{t' - \alpha} = - \frac{t - \bar{\alpha}}{t - \alpha},$$

α and $\bar{\alpha}$ being the conjugate double points. A method was given for dividing the plane into regions which are transformed into one another, by the transformations of such a 'Whittaker group'. The regions are curvilinear polygons whose sides may be regarded as straight lines in the sense of non-

Euclidean geometry, with the 'distance' between two points A and B specified by the integral,

$$\int \frac{|du+idv|}{v}$$

taken along the circle orthogonal to the real axis and joining A and B . The scheme is therefore analogous to the familiar partition of the plane into period parallelograms. It was shown that the automorphic functions of a Whittaker group can affect the uniformization of algebraic functions of any genus whatever.

If $f(u_1z)$ is an algebraic form of genus 2 it can be brought by a birational transformation into the normal form

$$u^2=(z-e_1)(z-e_2)(z-e_3)(z-e_4)(z-e_5),$$

and Whittaker proved that the uniformizing variable t for this equation is the quotient of two solutions of a linear differential equation of the form

$$\frac{d^2y}{dz^2} + \frac{3}{16} Iy = 0,$$

where

$$I = \sum_{n=1}^5 \frac{1}{(z-e_n)^2} - \frac{4z^2+c_1z^2+c_2z+c_3}{(z-e_1)(z-e_2)(z-e_3)(z-e_4)(z-e_5)},$$

where c_1, c_2 and c_3 are constants whose values were unknown.

In a later paper (R.49) Whittaker conjectured that if the normal form of the algebraic equation of genus 2 is written as $s^2 = f(z)$, then the correct form of the differential equation is

$$\frac{d^2y}{dz^2} + \frac{3}{16} \left\{ \frac{f'^2}{f^2} - \frac{6f''}{5f} \right\} = 0,$$

in which all the constants are completely determined. If $s^2 = z^5 + 1$, this equation can be integrated in terms of hypergeometric functions, and the conjecture was triumphantly verified.

A little later J. M. Whittaker (*J. Lond. Math. Soc.* **5**, 1930, p. 150) proved that the conjecture is universally true for the form $u^2 = f(z) = (z-e_1) \dots (z-e_n)$ if the invariant I in the differential equation has the form

$$I = \sum \frac{1}{(z-e_2)^2} - \frac{p(z)}{f(z)},$$

where $p(z)$ is a polynomial.

A minor but interesting contribution (R.7) to the theory of automorphic

functions gives an explicit formula for a function which is analogous to Weierstrass's sigma-function in the form

$$\sigma(z, \alpha) = (z - \alpha) \prod_{n=1}^{\infty} (1 - c_n) \exp \frac{1}{2} c_n,$$

where

$$c_n = \frac{(z - \alpha)(z_n - \alpha_n)}{(z - z_n)(\alpha - \alpha_n)},$$

and $z_n = (a_n z + b_n)/(c_n z + d_n)$ is a typical transformation of the group.

4. *Astronomy*

In 1899, Whittaker prepared for the British Association a monumental 'Report on the progress of the solution of the problem of three bodies' (B.1) which covers the period 1868-1898, and which provides a complete summary of the development of the 'new dynamical astronomy'.

While Whittaker was Astronomer Royal of Ireland he wrote a few papers on various astronomical subjects.

A short note in 1906 (R.15) gives a simple and completely general proof that the resolving power of any dispersive apparatus is equal to the number of separate pulses into which an incident pulse is broken up by the apparatus.

Another brief communication in 1908 (R.17) marshals the evidence then available that the stratum of the solar atmosphere, which gives rise to the selective and general absorption characteristic of the spectrum of sun spots, is a region of high pressure and density rather than of high temperature.

There are only two papers on observational astronomy. The first (R.19) analyzes the light curve of the variable star RW Cassiopeiae; while the second (R.20) discusses the variations in the irregular variable SS Cygni with a view to determining possible periodicities.

5. *Potential theory and special functions*

Perhaps the most important section of Whittaker's researches is that which contains the magnificent series of papers on the special functions of mathematical physics regarded as constituents of potential functions.

Towards the end of 1902 Whittaker communicated a note to the Royal Astronomical Society (R.8), later expanded into a substantial paper and published in *Math. Ann.* (R.10), on the general solution of Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

which was obtained in the form

$$V = \int_0^{2\pi} f(x \cos \alpha + y \sin \alpha + iz, \alpha) d\alpha,$$

where f is an arbitrary function of its two arguments ($x \cos \alpha + y \sin \alpha + iz$) and α . This result brought a new unity into potential-theory, since all harmonics must be expressible in this form, and in fact the expression of an arbitrary solution of Laplace's equation as a series of spherical, cylindrical harmonics, etc., is equivalent to the expansion of the function f of the general solution into power-series, Fourier series, Dirichlet series, etc., with respect to each of its arguments.

Following on the solution of Laplace's equation, two other papers on analysis were written in 1903. The first (R.11) was a study of the functions defined by the differential equation

$$\frac{dy}{dz^2} + \mathcal{Z}y = 0$$

where \mathcal{Z} is a quadratic function of z . These he named *functions of the parabolic cylinder*: a standard solution denoted by $D_n(z)$ was obtained, and various representations and properties were found. In the second paper (R.12) it was shown that many special functions which had been introduced at different times by various authors were particular cases of a more general function which he denoted by $W_{k,m}(z)$, and for which he obtained various integral properties, asymptotic expansions, recurrence-formulae, etc. When the second edition of *Modern analysis* was published in 1915, Whittaker and Watson resolved to devote an entire chapter to $W_{k,m}(z)$, and a name had to be found for it: after some discussion, they settled on the *confluent hypergeometric* function. It has been found useful in applications, and an extensive literature has grown up.

In 1912 Whittaker returned to the topic of special functions with a paper (R.21) on the solutions of the linear differential equations of the second order with periodic coefficients,

$$\frac{d^2y}{dz^2} + (a + k^2 \cos^2 z)y = 0,$$

where a and k are constants. This equation, which occurs frequently in mathematical physics, and in connexion with Hill's lunar theory, is of interest because it is (from the point of view of the classification of differential equations according to their singularities) the simplest linear differential equation which is not reducible to a particular or degenerate case of the hypergeometric differential equation. Its solutions are not, in general, periodic functions of z ; but there are an infinite number of solutions which are periodic functions of z , of period 2π . It was with these (which are the solutions

of importance in physics) that the paper under review was concerned: Whittaker called them *Mathieu functions*, and the differential equation *Mathieu's equation*, in honour of Mathieu, who discovered them in 1868: and these names have been generally accepted. The chief result now published was that the Mathieu functions satisfy a homogeneous integral-equation, namely

$$y(z) + \lambda \int_0^{2\pi} \exp(k \cos z \cos \theta) y(\theta) d\theta = 0.$$

This integral-equation does not possess a solution except when λ has one of a certain set of values: and the solutions corresponding to these values of λ are precisely the periodic solutions of the differential equation, that is to say, they are the Mathieu functions required in physics. It was shown how explicit expressions for them can be obtained by solving the integral-equation.

The integral-equation plays much the same part in the theory of Mathieu functions that Bessel's formula

$$\mathcal{J}_n(z) = \frac{1}{\pi} \int_0^\pi \cos(ns - z \sin s) ds$$

does in the theory of Bessel functions. This is an instance of a general theorem which he proved later, namely that integral-equations play the same part in relation to differential equations with *four* regular singularities, that ordinary definite integrals do in relation to differential equations with *three* regular singularities.

This work on Mathieu functions was essentially an application of the 1902 solution of Laplace's equation: the integral-equation had in fact been given to his advanced pupils in Cambridge in 1904, but at that time he appreciated its significance so little that he did not publish it. His pupil, H. Bateman, mentioned it in 1909, ascribing it to Whittaker (*Trans. Camb. Phil. Soc.* **21**): meanwhile it was independently discovered by the Japanese mathematician K. Aichi in 1908.

The general solution of Mathieu's equation is of the form,

$$y = Ae^{\mu z} \phi(z) + Be^{-\mu z} \psi(z),$$

where A and B are arbitrary constants, μ is a constant depending on the constants a and k of the differential equation, while $\phi(z)$ and $\psi(z)$ are periodic functions of z . For certain values of a and k the constant μ vanishes, and the solution is then one of the purely-periodic functions already referred to: but in general μ is different from zero; a paper of 1914 (R.23) was devoted to its determination.

One of the research students who was working with Whittaker in Edinburgh at this time, E. L. Ince, never lost interest in Mathieu's equation, and

did much original work of the highest quality on it many years afterwards. Whittaker himself returned to it in 1928 (R.48), obtaining the property of the Mathieu functions which corresponds to the property (possessed by all functions of the hypergeometric family) of satisfying *recurrence-formulae*.

The work on the integral-equation satisfied by the Mathieu functions led to papers on integral-equations satisfied by the other functions of mathematical physics. It was shown (R.24, R.25) that the functions of Lamé, that is to say, the doubly-periodic solutions of the differential equation

$$\frac{d^2y}{dz^2} = \{n(n+1)k^2\text{sn}^2z + A\}y,$$

are the solutions of the homogeneous integral-equation

$$y(z) = \lambda \int_0^{4K} P_n(k\text{sn}z\text{sn}s)y(s)ds,$$

where P_n is Legendre's polynomial, and k is the modulus of the elliptic functions, whose period is $4K$.

This theorem was applied in the latter part of the paper in order to express the ellipsoidal harmonics (i.e. the solutions of Laplace's equation which are appropriate to the ellipsoid) in the form which he had found in 1902 for the general solution of Laplace's equation. He now showed that any ellipsoidal harmonic can be expressed in the form

$$\int_0^{2\pi} P_n \left(\frac{x \cos \theta + y \sin \theta + iz}{\sqrt{(z^2 - b^2 \sin^2 \theta)}} \right) f(\theta) d\theta,$$

where P_n denotes Legendre's polynomial: or, fixing the function $f(\theta)$ more precisely, any ellipsoidal harmonic can be expressed in the form

$$\int_0^{4K} P_n \left\{ \frac{1}{k'c} (k'x\text{sn}s + y\text{cn}s + iz\text{dn}s) \right\} E(s) ds,$$

where $E(s)$ denotes a Lamé function.

A further paper of 1914 (R.26) introduced a class of functions which bear the same relation to the Mathieu functions that the associated Legendre functions $P_n^m(z)$ bear to the Bessel function $J_m(z)$. The differential equation of these functions is

$$\frac{d^2y}{dz^2} + \left\{ A - (n+1)l \cos 2z + \frac{1}{8}l^2 \cos 4z \right\} y = 0,$$

which is a case of Hill's equation in the lunar theory: and it was shown that

the periodic solutions of this equation are the solutions of the homogeneous integral-equation

$$y(z) = \lambda \int_0^{2\pi} \cos^n(y-s) \exp \frac{1}{2} l(\sin^2 s + \sin^2 z) y(s) ds.$$

The culmination of the series of papers on the solution of differential equations by definite integrals and by integral-equations was reached some years later in the following theorem (R.50) which seems to comprehend most of the known results in this field of analysis, and relates to partial as well as ordinary differential equations. Consider any contact-transformation from a set of variables $(q_1, q_2, \dots, q_n, p_1, p_2, \dots, p_n)$ to a set of variables $(Q_1, Q_2, \dots, Q_n, P_1, P_2, \dots, P_n)$. Suppose that the latter when expressed in terms of the former are denoted by $Q_r(q_1, q_2, \dots, q_n, p_1, p_2, \dots, p_n)$, $P_r(q_1, q_2, \dots, q_n, p_1, p_2, \dots, p_n)$, $(r=1, 2, \dots, n)$. Then the set of partial differential equations

$$\left. \begin{aligned} Q_r \left(q_1, q_2, \dots, q_n, \frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \dots, \frac{\partial}{\partial q_n} \right) \chi &= t_r \chi, \\ P_r \left(q_1, q_2, \dots, q_n, \frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \dots, \frac{\partial}{\partial q_n} \right) \chi &= -\frac{\partial \chi}{\partial t_r}, \end{aligned} \right\} (r = 1, 2, \dots, n),$$

is a compatible set of differential equations, and possesses a solution $\chi(q_1, q_2, \dots, q_n, t_1, t_2, \dots, t_n)$. Now let it be required to solve a set of n compatible linear partial differential equations in n independent variables, say

$$F_r \left(q_1, q_2, \dots, q_n, \frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \dots, \frac{\partial}{\partial q_n} \right) \psi = 0 \quad (r = 1, 2, \dots, n), \quad (A)$$

where $\psi(q_1, q_2, \dots, q_n)$ is the function to be determined. Suppose that when the q 's and p 's are replaced by their values in terms of the Q 's and P 's, the function $F_r(q_1, q_2, \dots, q_n, p_1, p_2, \dots, p_n)$ becomes $G_r(Q_1, Q_2, \dots, Q_n, P_1, P_2, \dots, P_n)$. Then the solution of the partial differential equations (A) is furnished by definite integrals of the type

$$\begin{aligned} &\psi(q_1, q_2, \dots, q_n) \\ &= \iiint \dots \int \chi(q_1, q_2, \dots, q_n, t_1, t_2, \dots, t_n) q(t_1, t_2, \dots, t_n) dt_1 dt_2 \dots dt_n \end{aligned}$$

where $q(t_1, t_2, \dots, t_n)$ is the solution of the set of n compatible linear partial differential equations

$$G_r \left(t_1, t_2, \dots, t_n, \frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \dots, \frac{\partial}{\partial t_n} \right) q = 0 \quad (r = 1, 2, \dots, n).$$

Among the applications of this theorem worked out in the paper were several

new properties of the hypergeometric function, the solution of the ordinary linear differential equation of the third order with four regular singularities, and an extension to partial differential equations of Laplace's method for solving ordinary differential equations.

6. Dynamics

Two early papers (R.1 and R.3) are devoted to a new method of obtaining the equation of planetary theory by the use of Lagrange brackets, and to a general process for reducing the order of the differential equations of a dynamical system by means of the integral of energy. Paper (R.18) discusses the vibrations of a compound system formed of two equal sub-systems which are coupled together by a connexion of 'gyroscopic' type. It was shown that the actual frequencies of such a compound system (*not* the squares of the frequencies, which are the quantities usually occurring in the solution of dynamical problems) can be expressed by a simple formula of Deslandres' type,

$$n = p^2 f(q^2, r^2) + Bq^2 + \phi(r^2)$$

in terms of numbers, p , q , r , which assume in succession all integer values, so that the spectrum of frequencies is of the 'banded' type.

But Whittaker's principal contributions to analytical dynamics are his well-known treatise (discussed below), his theory of periodic orbits and his elucidation of planetary theory by means of the 'adelphic integral'.

In (R.4) a new method of detecting periodic orbits was introduced in terms of the sign taken by the function

$$I = \frac{h - f(x, y)}{\rho} - \frac{1}{2} \cos \gamma \frac{\partial f}{\partial x} - \frac{1}{2} \sin \gamma \frac{\partial f}{\partial y}$$

on certain closed curves C . Here $f(x, y)$ and h are the potential energy at (x, y) and the total energy, ρ is the radius of curvature of C , and $(\cos \gamma, \sin \gamma)$ the direction cosines of the normal to C . It was shown that if one closed curve C_+ is contained within another closed curve C_- , and if I is positive at all points of C_+ and negative at all points of C_- , then in the ring shaped region between the two curves there exists a periodic orbit of the dynamical problem. Furthermore it was proved that the value of the integral

$$\frac{1}{2\pi} \iint \left\{ \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \log \{h - f(x, y)\} \right\} dx dy$$

taken over the interior of any periodic orbit, is equal to the number of centres of force enclosed by the orbit, diminished by two. A subsequent paper (R.6) extended this result to periodic orbits described under *moving* centres of force and, in particular, to the restricted problem of three bodies.

Whittaker's classical researches on the integration of a system of Hamiltonian equations open in 1902 with a paper (R.5) which describes a process of successive transformations which yields a solution in terms of trigonometric series, the first terms of which correspond to a position of stable equilibrium and to small oscillations about its position. This work was completed in 1916 by the discovery of the 'adelphic' integral of a Hamiltonian system (R.30). In this paper Whittaker examined the difficulties indicated by Poincaré's celebrated theorem that the series of celestial mechanics, if they converge at all, cannot converge uniformly for all values of the time on the one hand, and on the other hand for all values of the constants comprised between certain limits.

The ideas of the paper may be illustrated most readily by considering a particular problem, namely that of the motion of a particle on the surface of a smooth ellipsoid under no external forces. The particle describes a geodesic on the surface, so the periodic solutions are simply those geodesics which are closed curves. Now a geodesic on an ellipsoid satisfies the equation

$$pd = \text{constant}$$

where p denotes the perpendicular from the centre of the ellipsoid on the tangent-plane at the point, and d is the diameter parallel to the tangent to the geodesic at the point. The same equation holds for the lines of curvature on the ellipsoid: so that every geodesic may be associated with a line of curvature, namely, that line of curvature for which pd has the same value as it has for the geodesic: and we may speak of the geodesic as 'belonging to' the line of curvature. There is only one line of curvature having a prescribed value for pd , but there is an infinite number of geodesics having this value for pd , so that an infinite number of geodesics belong to each line of curvature. The line of curvature consists of two closed curves on the ellipsoid: the region between these two portions of the line of curvature is a belt extending round the ellipsoid: and all the geodesics which belong to the line of curvature are comprised within this belt, and touch the two portions of the line of curvature alternately.

In order that the geodesic may be closed, it is necessary (as in all poristic problems) that a certain parameter (depending in this case on the value of the constant pd of the line of curvature) should be a rational number: the geodesic is unclosed if this parameter is an irrational number. If it is closed, then there are ∞^1 other geodesics which belong to the same line of curvature and which are also closed; but if it is not closed, then no other geodesic belonging to this particular line of curvature can be a closed geodesic.

Now consider the connexion between the ∞^1 members of the family of geodesics which belong to the same line of curvature. It is well known that if we have a dynamical system with two degrees of freedom, and if we denote the generalized co-ordinates by (q_1, q_2) and the momenta by (p_1, p_2) , and if

$$\phi(q_1, q_2, p_1, p_2) = \text{constant}$$

is an integral of the system, then the infinitesimal contact-transformation which is defined by the equations

$$\delta q_1 = \epsilon \frac{\partial \phi}{\partial p_1}, \quad \delta q_2 = \epsilon \frac{\partial \phi}{\partial p_2}, \quad \delta p_1 = -\epsilon \frac{\partial \phi}{\partial p_1}, \quad \delta p_2 = -\epsilon \frac{\partial \phi}{\partial p_2},$$

(where ϵ is a small constant) transforms any trajectory into an adjacent curve which is also a trajectory. If we apply this theorem to the motion on the ellipsoid, we find that the infinitesimal transformation which corresponds to the integral $pd = \text{constant}$ transforms any geodesic into another geodesic which belongs to the same line of curvature. Thus the ∞^2 geodesics on an ellipsoid may be classified into ∞^1 families, each family consisting of geodesics: the members of any one family are either all closed or all unclosed, and a certain continuous group of transformations, which is associated with the integral $pd = \text{constant}$, transforms any geodesic into all the geodesics which belong to the same family. Features essentially the same as these are to be found quite generally in dynamical systems with two degrees of freedom which possess an integral of energy. There exists an integral of the system (called by Whittaker the *adelphic integral*) which plays the same part as the integral $pd = \text{constant}$ does in the case of the ellipsoid: namely the infinitesimal transformation associated with it changes every trajectory of the system into an adjacent trajectory, in such a way that every periodic solution is changed into an adjacent periodic solution having the same period and the same constant of energy. This led to a method for finding the adelphic integral, and became the basis from which to complete the integration of the system. It appeared that although the adelphic integral can always be found (as an infinite series), it cannot in general be represented by a single analytical expression which is valid for all values of the parameters involved in the problem: its form is different in different cases, depending on whether a certain parameter has a rational or irrational value. This fact is the underlying reason for the difficulty formulated in Poincaré's theorem: for as the parameter changes continuously, it is perpetually passing through rational and irrational values, and at each change from rational to irrational, or *vice versa*, the form of the adelphic integral is abruptly changed; so that no one series-solution of the dynamical problem can be valid over the whole range. But when the rationality or irrationality of the parameter is given, the adelphic integral can be constructed and the solution can then be completed: so that Poincaré's difficulty is overcome.

7. Relativity and electromagnetic theory

Whittaker's first paper on theoretical physics (excluding dynamics) was published in 1904 (R.13). This paper shows that any electromagnetic field can be specified in terms of the derivatives of *two* real scalar wave functions, F and G , by means of the formulae,

$$\begin{aligned}\mathbf{E} &= \text{curl curl } \mathbf{F} + \text{curl } c^{-1} \dot{\mathbf{G}} \\ \mathbf{H} &= \text{curl } c^{-1} \dot{\mathbf{F}} - \text{curl curl } \mathbf{G} \\ \mathbf{F} &= (0, 0, F) \quad , \quad \mathbf{G} = (0, 0, G)\end{aligned}$$

(The usual specification in terms of the Maxwellian scalar-potential ϕ and vector-potential A is equivalent to *four* real functions connected by the relation $\dot{\phi} + c \text{ div } \mathbf{A} = 0$.)

These two scalar wave-functions were evaluated in terms of the charges and co-ordinates of the electrons generating the field.

From 1921 onwards he published a succession of papers on relativity. The first of them (R.36) was a relativistic generalization of Faraday's theory of lines and tubes of force. Surfaces in four-dimensional space-time were introduced which reduce to Faraday's tubes of electric force when the field is purely electrostatic and which reduce to Faraday's tubes of magnetic force when the field is purely magnetostatic, and which between these extreme types provide a continuous transition. The way in which the generalization is made may be illustrated by the following statement. With Faraday's electrostatic tubes, as we proceed along a tube, the magnitude of the electric vector at any point is inversely proportional to the area of the cross section of the tube, and the three components of the electric vector are to each other in the same ratio as the areas of the projections of this cross-section on the three co-ordinate planes. With the surfaces introduced in the paper under review (to which Whittaker gave the name *calamoids*), the quantity

$$\sqrt{\{(\text{electric vector})^2 - (\text{magnetic vector})^2\}}$$

(which, as is well known, is covariant with respect to all Lorentz transformations) is inversely proportional to the area of the cross-section of the calamoid, and the six components of the electric and magnetic forces are connected in a simple way with the areas of the six projections of this cross-section on the six co-ordinate planes of yz, zx, xy, xt, yt, zt . The calamoids are covariant with respect to Lorentz transformations, so they are the same, whatever be the observer whose measure of electric and magnetic force are used in constructing them.

This was followed by four other papers (R.44, R.45, R.46, R.47) concerned with electromagnetic phenomena in the curved space of general relativity. The simplifying assumption was made, that the gravitational field is statical, so that the curvature of space at any point does not vary with the time. In the distorted space of this fixed gravitational field, Whittaker supposed an electromagnetic field to exist: strictly speaking, the electromagnetic field has itself a gravitational effect, but this effect is in general small and he treated the ideal case in which it is ignored, so the metric is simply that of the gravitational field originally postulated. The problem is, therefore, to study

the existence and propagation of electromagnetic fields in a medium whose properties (i.e. the distortion of space) vary from point to point: and Whittaker showed that the problem is mathematically identical with that of the Maxwellian electromagnetic field in a medium whose dielectric constant and magnetic permeability have a particular kind of anisotropy. Many special problems were studied, such as the capture and imprisonment of radiation by the intense gravitational field surrounding a point-mass, the form of the equipotential surfaces due to an electric charge which is at rest in a gravitational field, and the expression of the electro-magnetic four-vector potential due to electrons moving in any manner. It was found that an electron at rest in a varying gravitational field must emit radiation: and that any electromagnetic disturbance which is filiform must necessarily have the form of a null geodesic of space-time.

In 1931 he discussed (R.51) the definition of distance in the curved space of General Relativity, and the displacement of the spectral lines of distant sources. There are certain ambiguities involved in the use of the terms 'time', 'spatial distance', and 'velocity'. when applied by an observer to an object which is remote from him in curved space-time. The 'interval' which is defined by $ds^2 = \Sigma g_{pq} dx^p dx^q$ involves space and time blended together: and although any particular observer at any instant perceives in his immediate neighbourhood an 'instantaneous three-dimensional space', yet this space cannot be defined beyond his immediate neighbourhood. The concept of 'spatial distance between two material particles' is, however, not really dependent on the concept of 'simultaneity'. When the astronomer asserts that 'The distance of the Andromeda nebula is a million light-years', he is stating a relation between the world-point occupied by ourselves at the present instant and the world-point occupied by the Andromeda nebula at the instant when the light left it which arrives here now: that is, between two world-points which lie on the same null geodesic. In order to define 'spatial distance' conformably to this idea, in a general Riemannian metric, Whittaker translated into the language of differential geometry the principle by which astronomers actually calculate the 'distance' of very remote objects such as the spiral nebulae, and so arrived at the following definition. He took a 'star' A and an observer B , which are on the same null geodesic (so that a ray of light can pass from A to B) and considered a thin pencil of null geodesics which issue from A and pass near B . This pencil intersects the observer B 's 'instantaneous three-dimensional space', giving a two-dimensional cross-section: the 'spatial distance AB ' is then defined to be proportional to the square root of this cross-section. Distance, as thus specified, is independent of the choice of the co-ordinate system. On the basis of this definition, an analytical theory was worked out and applied to the displacement of the spectral lines of distant sources and to the history of the observation of a star in the de Sitter world.

In 1935 (R.53) he extended to general relativity the well-known theorem of Gauss on the Newtonian potential, namely that the total flux of gravita-

tional force through a simple closed surface is equal to $(-4\pi) \times$ the total gravitating mass contained within the surface. In the extended theorem, the Newtonian concept of 'gravitating mass' is replaced by that of the energy-tensor, which does not in general consist solely of the 'material' energy-tensor and need not involve any 'matter' at all. The concept of the 'potential energy' of a particle in a statical field in general relativity was then elucidated. In Newtonian dynamics the equation of conservation of energy is

$$(\text{kinetic energy}) + (\text{potential energy}) = C$$

where the potential energy is (for a single particle) the product of the mass—a fixed quantity—into a function which depends only on the position of the particle, and where C is a constant which depends on the initial circumstances. In general relativity with statical fields on the other hand, the equation of conservation of energy for a single particle of proper-mass m is of the form

$$(\text{kinetic energy}) = (\text{lost potential energy})$$

where now the lost potential energy is the product of $m(1-w^2/c^2)^{-\frac{1}{2}}$ into a function which depends only on the position of the particle, w denoting the velocity which the particle would have after escaping from the gravitational field and arriving at the Galilean space-time at infinity: the constant w corresponds to the Newtonian constant C , but enters into the equation in a wholly different manner. He called $m(1-w^2/c^2)^{-\frac{1}{2}}$ the *potential mass*, and showed that this definition of potential mass enables us to express the generalized Gauss theorem, in the case when the energy-tensor is due to actual masses, by a simple statement almost identical with the original Gauss theorem of Newtonian theory. Finally it was shown that the electrostatic form of Gauss theorem in classical physics, namely that the total strength of the tubes of force issuing from a closed surface is equal to the total electric charge within the surface, can also be extended to general relativity, but that this extension is different in character from the gravitational theorem.

8. *Quantum theory*

Whittaker wrote a few papers on quantum theory, three of which show how certain characteristic quantum phenomena can be exhibited by classical systems. In (R.37) a magnetic structure was described, towards which an electron was supposed to be moving, and the encounter between them was investigated. It was found that if the initial velocity of the electron is less than a certain quantity, the electron, after advancing to a finite distance from the structure, comes to a stop and is then repelled, returning to its original position without any permanent exchange of energy: the electron and the structure have had an 'elastic impact'. If however the electron has a sufficiently great initial velocity, it is able to pass completely through and out of the structure, so as ultimately to be free from its influence: and in the process

it gives up a certain definite amount of energy to the structure, and retains the rest. The other two papers (R.41) and (R.42) provide models of light quanta in the form of exact solutions of Maxwell's equations, suitably generalized to include magnetic currents or magnetic dipoles moving with the speed of light. At this time Whittaker was obviously fascinated by the concept of magnetic currents and even wrote a paper (R.43) to show that Hilbert's world function could be generalized to include such hypothetical entities.

Perhaps two other papers on quantum theory are likely to be of more permanent interest—(R.54) which investigates some relations between the tensor calculus and spinor calculus with especial reference to electromagnetic theory, and (R.55) which discusses the theory which corresponds in quantum mechanics to the theory of Hamilton's principal function in classical dynamics. It will be remembered that if (taking for simplicity a problem with one degree of freedom) the co-ordinate and momentum are (q, p) at the instant t and are (Q, P) at a previous instant T , then Hamilton's principal function W is a function of q, Q and $t - T$, which satisfies the equations

$$\frac{\partial W}{\partial q} = p, \quad \frac{\partial W}{\partial Q} = -P, \quad \frac{\partial W}{\partial t} = -H, \tag{A}$$

where H is the Hamiltonian function. In the paper, a quantum-mechanical problem was considered, specified by a Hamiltonian $H(q, p)$. The variables q and p are now no longer ordinary algebraic quantities, but non-commuting operators (which may be indicated by arrows pointing from right to left).

The quantity \bar{Q} , which represents the co-ordinate at the instant T , is also an operator, which does not commute with \bar{q} or \bar{p} . A function of \bar{q} and \bar{Q} is said to be well-ordered, when it is arranged (as it can be, by use of the commutation rule) as a sum of terms each of the form $f(\bar{q})g(\bar{Q})$. Then it was shown that there exists a well-ordered function $\bar{U}(\bar{q}, \bar{Q}, t - T)$ which satisfies formally exactly the same equations as (A), namely:

$$\frac{\partial \bar{U}}{\partial \bar{q}} = \bar{p}, \quad \frac{\partial \bar{U}}{\partial \bar{Q}} = -\bar{P}, \quad \frac{\partial \bar{U}}{\partial t} = -\bar{H}. \tag{B}$$

\bar{U} may be called the quantum-mechanical Principal Function. Although equations (B) are the same as (A), the expression of the function \bar{U} in terms of $\bar{q}, \bar{Q}, t - T$, is quite different from the expression of the classical function W in terms of $q, Q, t - T$; the reason being, that equations (B) are true only on the understanding that all the quantities in them are well-ordered in \bar{q} and \bar{Q} :

but on substituting the well-ordered expression for \overleftarrow{p} in the Hamiltonian $H(\overleftarrow{q}, \overleftarrow{p})$ we shall need to invert the order of the factors in many terms, by use of the commutation rules, in order to reduce H to be a well-ordered function of \overleftarrow{q} and \overleftarrow{Q} : and this introduces new terms which do not occur in equations (A). This explains why, although the quantum-mechanical equations are formally identical with those of classical mechanics, the solutions in the two cases are altogether different.

He also introduced a third function $R(q, Q, t-T)$ which was obtained by taking the function \overleftarrow{U} and replacing the operators \overleftarrow{q} and \overleftarrow{Q} by ordinary algebraic quantities q and Q . R was called the *Third Principal Fraction*. Writing

$$S(q, Q, t-T) = \exp iR(q, Q, t-T)/\hbar,$$

it was then shown that $S(q, Q, t-T)$ satisfies Schrödinger's differential equation for the wave-function belonging to the Hamiltonian $H(q, p)$. Many of the early derivations of Schrödinger's equation started from the principal function of classical dynamics, and after transforming a partial differential equation, which it satisfies, by writing $\psi = \exp iW/\hbar$, obtained Schrödinger's equation for ψ by omitting certain terms which were shown to be small because \hbar is small. It now appeared, however, that no terms need be omitted, and that Hamilton's partial differential equation for the principal function is rigorously equivalent to Schrödinger's wave-equation, provided the principal function is understood to be the third principal function R and not Hamilton's classical principal function W .

9. *Scientific books and monographs*

Although Whittaker's original researches in mathematics and theoretical physics have had a profound effect by reason of their great range, depth and fertility, they are rivalled, if not surpassed, in interest, importance and influence by his scientific books and monographs. Three of these works call for especial mention—the treatises on analysis (B.2), on analytical dynamics (B.3), and on the calculus of observations (B.7).

A course of modern analysis, published in 1902, and in many subsequent editions with the collaboration of Professor G. N. Watson, F.R.S., was the first, and for many years almost the only book in English to provide students with an introductory account of functions of a complex variable, of the methods of analysis, and of the special functions of mathematical physics. This great work embodied many of Whittaker's own discoveries on potential functions, and it is still an invaluable work of reference. Tribute should also be

paid to Lady Whittaker who made herself sufficiently acquainted with the mathematical concepts and symbols to write out a fair copy for press.

The great *Treatise on the analytical dynamics of particle and rigid bodies*, subsequently translated into German, remains the standard work on the mathematical theory and methods. The comprehensive and magisterial character of the first part can be gauged from the titles of two chapters on 'The soluble problems of particle dynamics', and 'The soluble problems of rigid dynamics'. The second part provides an introduction to the problem of three bodies and a systematic account of the use of contact transformations. Much of this work is an exposition of Whittaker's own fundamental researches.

The calculus of observations embodied the practical experience and the theoretical work which he carried on at Edinburgh in the Mathematical Institute and as a result of his association with the actuaries in that great centre of insurance. Like the other two works just described it was a pioneer effort which opened up a new field of mathematical exploration and provided a lucid introduction to methods of practical computation.

A fourth great work stands in a class apart—the monumental treatise on *A history of the theories of Aether and electricity* (B.5) first published in one volume in 1910, and subsequently in a greatly enlarged edition of which only two volumes have been published (in 1951 and 1953). Whittaker had begun to collect materials for the third volume which would have covered the period 1926-1956, but failing health did not allow him to complete this work. The *History* provides a complete, systematic and critical account of the development of the physical theories of electromagnetism, atomic structure and of the quantum theory from their remote beginnings up to the year 1926. Of the first edition Whittaker wrote that it 'involved an immense amount of reading and historical research, which was made possible by the freedom and comparative leisure of Dunsink'. The second edition was the fruit of his years of retirement at Edinburgh and will remain as an outstanding achievement by reason of the clarity of the exposition, the comprehension of its range and the penetration of its criticism, which give it all the force and authority of an original investigation.

Two other works which were written at Dunsink are *The theory of optical instruments* (B.4) and an article for the *Encyklopädie du mathematische Wissenschaften* on perturbation theory and orbits (B.6).

10. *Historical and philosophical writings*

Whittaker's famous treatise on *A history of the theories of aether and electricity* required an enormous amount of historical research into the origins of mathematical physics, and this activity, combined with the author's intense interest in ancient, mediaeval and modern philosophy, made him an unrivalled authority on what may be called the philosophical history of physical science. He was also gifted with a peculiarly lucid and persuasive manner of exposition which enabled him to present the most abstruse and difficult conceptions in a

simple and elegant language, in which accuracy and fidelity were united with clarity and charm.

The numerous philosophical and historical papers, listed in the bibliography at the end of this notice, all bear the marks of the author's learning, literary powers and critical judgment. The twenty magnificent obituaries which he wrote of distinguished physicists and mathematicians set a new standard in this peculiarly difficult form of biography, and they provide invaluable material for the historian of physical science. Painstakingly exact and complete, eminently fair and just, urbanely critical where criticism is enlightening, and written with charming lucidity, they are also an unconscious revelation of Whittaker's scholarship, clarity, judgement and culture.

In connexion with Whittaker's writings on historical and philosophical subjects it is necessary to mention that after the death of Sir Arthur Eddington in November 1944, his sister Miss Eddington entrusted Whittaker with the editing and publication of Eddington's *Fundamental theory*, which appeared in 1946. The study of this difficult and fascinating work had a considerable influence on Whittaker and he published a number of introductory studies of Eddington's theory which did much to make it known, at least in outline, to mathematical physicists.

But Whittaker's major contributions to the philosophical history of science are to be found in his Tarner lectures at Cambridge, the Herbert Spencer lectures at Oxford, the Riddell lectures at Durham and the Donnellan lectures at Dublin.

The Tarner lectures (B.11) trace the development of theories in natural philosophy from Euclid to Eddington and provide in a modest compass a history of the great concepts and principles, especially those which have provoked lengthy and unresolved controversies. From Plato and Aristotle to Einstein and Eddington, these lectures summarize the great speculations of natural philosophers on space, time and movement, and expound in an easy and familiar manner relativistic and quantum theories together with the new 'fundamental theory' of Eddington.

The Herbert Spencer lecture (B.10) is a plea for the rehabilitation of the Cartesian ideal of philosophic certitude by means of the new discoveries in physics and the new logical apparatus developed by mathematicians.

In his Riddell Memorial Lectures (B.8) Whittaker marshalled the physical and astronomical evidence which is relevant to the enquiry whether the world had a beginning and whether it will come to an end, and emphasized with great power and religious feeling the significance of modern cosmogony for the philosopher and theologian. Without claiming syllogistic rigour for his argument he was content to exhibit the congruence of modern physical theories with the doctrines of Christian natural theology.

The Donnellan lectures (B.9) are frankly entitled 'Space and spirit, theories of the universe and arguments for the existence of God'. Whittaker had always been humorously critical of the influence of philosophy on science, and in these lectures he passed sentence on the influence of philosophy in

natural theology. The classical 'Quinque Viae' in which St Thomas Aquinas summarized the arguments for the existence of God were subjected to severe examination, on the ground that with these famous proofs 'there are entwined more or less closely, certain doctrines regarding motion, causality, cosmology and teleology, which were derived from Aristotelian sources', and which have been discredited by modern science. Whittaker undertook to restate the argument starting from the evidence for the finite age of the universe provided by contemporary cosmogony. His restatement is extraordinarily interesting to the scientist even if not entirely convincing to the theologian.*

INFLUENCE

The influence of Whittaker on mathematics and mathematicians has been remarkable both for the number of topics which he enriched by his researches and the multitude of students who have been stimulated by his teaching. It is not only his extraordinary output of original researches, authoritative monographs, historical and philosophical studies which give him such an outstanding place in the history of mathematics, but the personal influence which he exerted in the Mathematical Institute at Edinburgh, and in his famous lectures to wider audiences.

He had a peculiar facility for coining names for analytical concepts and entities, many of which have obtained such currency in the language of mathematicians that their origin and authorship is forgotten. Consider as examples, 'isometric circle', 'adelphic integral', 'cotabular functions', 'cardinal function', 'congruent hypergeometric function', 'Mathieu function', 'calamoids'. The vocabulary ranges easily over a vast field of modern mathematical physics and forms an unexpected tribute to the widespread influence of Whittaker.

G. TEMPLE

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