THE ALEXANDER HORNED SPHERE AND BING'S HOOKED RUG

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1 Introduction

In this chapter we will describe some examples of wildly embedded 2-spheres in $\mathbb{R}^3$. We have already seen the Fox-Artin sphere, which is the image of an embedding $S^2 \hookrightarrow \mathbb{R}^3$ that is wild at exactly one point. We will now consider two other examples of wild embeddings, where the set of wild points is much larger in both cases. The first example will be the Alexander horned sphere. After constructing the sphere we will see that the set of wild points forms a Cantor set, i.e. there are embeddings $C \hookrightarrow [0,1] \hookrightarrow S^2 \hookrightarrow \mathbb{R}^3$, such that the image of the last embedding is the Alexander horned sphere, C is a Cantor set and the image of C under the composition of the three embeddings is exactly the set of wild points in the Alexander horned sphere.

The second example will be Bing's hooked rug, which is an embedding of a sphere that is wild at every point of the sphere.

This chapter is divided into two different parts. In the first part, we will construct the Alexander horned sphere and prove that the result of the construction is really a sphere. Then we will prove that there are wild points and give an argument why these wild points form the above Cantor set.

The second part is arranged in a very similar manner. After constructing Bing's hooked rug, we will again prove that it is an embedded sphere and that the embedding is wild at every point. Here we will use the fact that for a locally flat codimension-one embedding of a manifold M into a manifold N, the interior $\operatorname{Int} M$ of M is $k-$locally co-connected for all $k \geq 1$.

2 The Alexander horned sphere

Before starting the construction of the Alexander horned sphere, we need the following definition.

Definition 2.1 (Pillbox [DV09])**.** A *pillbox* is a copy C of $D^2 \times [0,1]$ containing linked solid tori T_1 and T_2 as shown in Figure 1 such that $T_1 \cap \partial C = \tau$ and $T_2 \cap \partial C = \beta$, where we call $D^1 \times \{1\}$ the top disc τ and $D^2 \times \{0\}$ the bottom disc β.

A pillbox is shown in Figure 1.

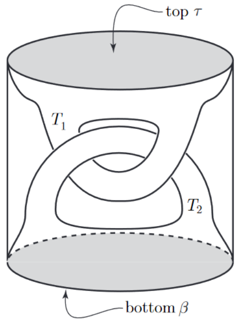

FIGURE 1. A pillbox. This picture is from [DV09, p. 48].

2.1 Construction

We begin with a solid ball B_0 and attach a handle $D^2 \times [0,1]$ along $D^2 \times (\{0\} \cup \{1\})$. The result is a solid torus X_0. Now we remove a pillbox from X_0, i.e. we remove a cylindrical 3-cell (containing two linked tori) and call the resulting manifold B_1. A picture of B_1 is shown in Figure 2 and one can see that it is homeomorphic to a solid ball, so in particular $B_1 \cong B_0$. To B_1 we now add the linked solid tori T_1 and T_2 which we removed earlier and call the result X_1 (Figure 2).

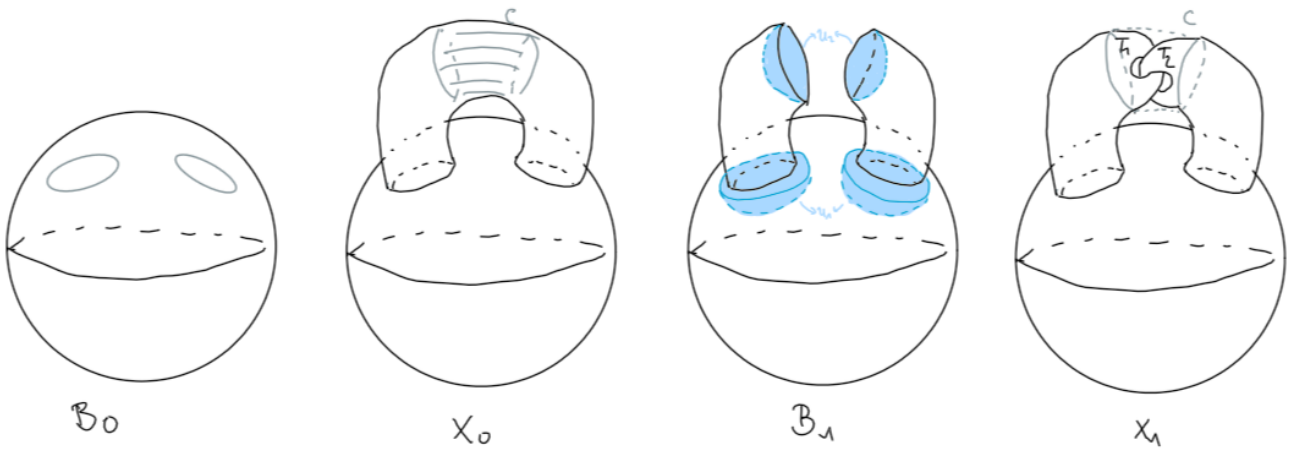

FIGURE 2. The step B_0 to X_1. Note that every space is solid here. Theneighborhoods U_1 and U_2 are indicated in blue.

In the next step we remove a pillbox from each of the solid tori T_1 and T_2 to get a manifold $B_2 \cong B_1 \cong B_0$. Then we replace the pillboxes by the solid tori inside and call the resulting manifold X_2. We continue inductively. So in step n we remove 2^{n-1} pillboxes from X_{n-1} and call the result B_n. Then we attach 2^n linked solid tori and call the resulting manifold X_n.

One can summarize that X_n arises from X_{n-1} by removing something, namely the complement of the linked solid tori in 2^{n-1} disjoint pillboxes.

$$\begin{tikzcd} X_{n-1} \arrow[r, "\text{remove } 2^{n-1} \text{ pillboxes}"] \arrow[rr, bend right, hook, "\text{remove } C\backslash(T_1 \cup T_2)\ 2^{n-1} \text{ times}"'] & B_n \arrow[r, hook, "\text{attach } 2^n \text{ solid tori}"] & X_n \end{tikzcd}$$

In contrast to this, we attach 2^n horns to obtain B_n from B_{n-1}, coming from 2^{n-1} attached solid tori with pillboxes removed. These attached horns are the reason for the name of the Alexander horned sphere.

$$\begin{tikzcd} B_{n-1} \arrow[r, "\text{attach } 2^{n-1} \text{ tori}"] \arrow[rr, bend right, hook, "\text{attach } 2^n \text{ horns}"'] & X_{n-1} \arrow[r, hook, "\text{remove } 2^{n-1} \text{ pillboxes}"] & B_n \end{tikzcd}$$

So in the end we constructed a nested sequence $B_0 \subseteq B_1 \subseteq B_2 \subseteq \dots$ of solid balls as well as a nested sequence $X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots$ of 3-manifolds with boundary. We can now define the *Alexander horned ball* to be

$$B \coloneqq \bigcap_{i=0}^{\infty} X_i.$$

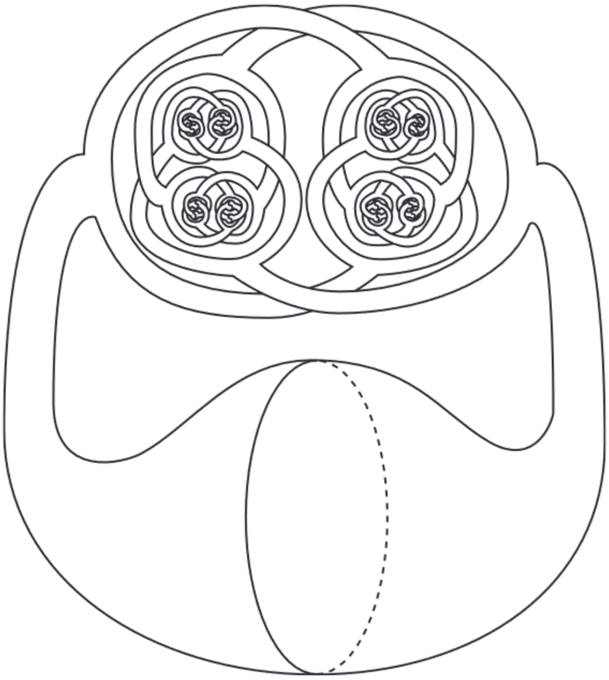

FIGURE 3. The Alexander horned sphere. Picture from [DV09, p.50]

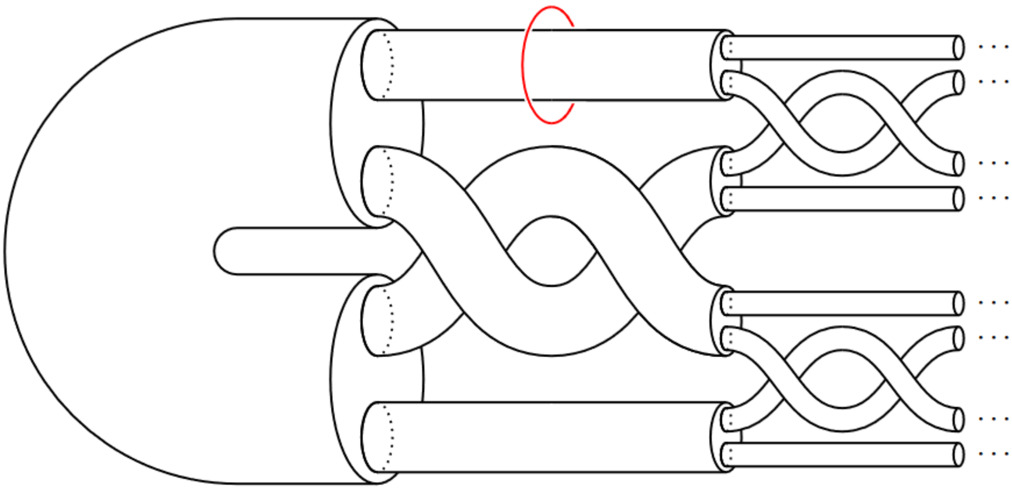

FIGURE 4. Another picture of the Alexander horned sphere. The length of the tubes needs to decrease to zero so that the limit points are reached. Picture from [?, p.72]

We define the *Alexander horned sphere* $A \coloneqq \partial B$. Here we just mean the topological boundary since we do not know yet that B is a manifold. But in the next paragraph we will prove that B is homeomorphic to D^3 and that A is therefore homeomorphic to a sphere.

Below, there are two pictures of the Alexander horned sphere, see Figure 3 and Figure 4.

2.2 The Alexander horned sphere is a sphere

To show that the Alexander horned sphere is indeed a sphere, we first remark that $B = \overline{\bigcup_{i=0}^{\infty} B_i}$, which holds by construction. Now note that there are homeomorphisms $h_i\colon B_{i-1} \to B_i$ that restrict to the identity on a neighbourhood of the bases of the attached horns because we can contract the horns homeomorphically into neighbourhoods of their base. In particular, for every index n there is a neighborhood U_n of $B_n \setminus B_{n-1} \subseteq B_n$ as indicated in Figure 2 such that $h_k|_{U_n}$ is the identity on $U_n \subset B_k \setminus B_{n-1}$ for any $k \geq n$. Define $f_n\colon B_0 \to B_n$ to be the composition $f_n = h_n \circ \ldots \circ h_1$. The map f_1 equals h_1 and moves nothing outside the horns and U_1. The map f_2 differs from f_1 just on U_2. We can say that the horns get smaller in each step if we consider the Alexander horned sphere as subset of $\mathbb{R}^3$ with its standard metric. So we can choose the neighborhoods U_k so that they get smaller and therefore the map f_n differs from f_{n-1} on a very small neighborhood for larger n. Thus, the maps $\{f_n\}$ converge uniformly to a continuous map $f\colon B_0 \to B$. We want to show that f is bijective.

Define $C \coloneqq f^{-1}(B \setminus \bigcup_{i=0}^{\infty} B_i) \subseteq B_0$. So C is the preimage of the set of points that form the limit of the tubes in Figure 4. A point in $f(C)$ is uniquely determined by the sequence of choices one would make when choosing a path starting at a point in the left half of a torus in Figure 4 and ending at the point in $f(C)$. Whenever we are in an area of the Alexander horned sphere where two horns are attached we have to decide whether to go along the upper horn or along the lower horn. Two different horns will never lead to the same limit point since two horns are never glued to each other. So $f(C)$ forms a Cantor set.

For a point $x \in B_0 \setminus C$ there is an index N such that $x \in U_N$, so $f|_{U_N} = f_N|_{U_N}$, hence f is bijective on $B_0 \setminus C$. We will now show injectivity and surjectivity of $f|_C$. Consider Figure 4. Any two points in C will be separated by horns, that is there exists n such that $f_n(x)$ and $f_n(y)$ lie in different horns, so that they cannot have the same image under f. This implies that $f|_C$ is injective. Each point of $B \setminus \bigcup_{i=0}^{\infty} B_i$ lies in the image of f since there is a (unique) sequence that encodes the horns leading to that point and so this point is part of the Cantor set $f(C)$. So $f|_C$ is also surjective.

Now, f is a continuous and bijective map from a compact space to a Hausdorff space and is therefore a homeomorphism. Since $B_0 \cong D^3$ by definition, the Alexander horned ball is indeed a ball, which implies that the Alexander horned sphere is a sphere.

2.3 Wildness of the Alexander horned sphere

We will prove with the help of Lemma 2.2 below that the fundamental group of the comlpement of the Alexander horned sphere is not trivial. It will follow by the Schoenflies theorem then, that the embedding $A \hookrightarrow \mathbb{R}^3$ cannot be locally flat.

Lemma 2.2. *([DV09, lemma 2.1.9]) Let C be a pillbox and Y be a closed subset of $\mathbb{R}^3$ such that $Y \cap C = \tau \cup \beta$, and let J be a 1-sphere in $\mathbb{R}^3 \setminus (Y \cup C)$ as shown in Figure 5. If $\pi_1(J) \to \pi_1(\mathbb{R}^3 \setminus (Y \cup C))$ is injective, then $\pi_1(\mathbb{R}^3 \setminus (Y \cup C)) \to \pi_1(\mathbb{R}^3 \setminus (Y \cup T_1 \cup T_2))$ is also injective.*

Proof. The proof follows Bing's paper [Bin61]. Assume, that $\pi_1(J) \to \pi_1(\mathbb{R}^3 \setminus (Y \cup C))$ is injective, so J is not null-homotopic in $\mathbb{R}^3 \setminus (Y \cup C)$. We now consider a loop $K \subseteq \mathbb{R}^3 \setminus (Y \cup C)$ that is null-homotopic in $\mathbb{R}^3 \setminus (Y \cup T_1 \cup T_2)$ and show that it is also null-homotopic in $\mathbb{R}^3 \setminus (Y \cup C)$.

Since K is null-homotopic in $\mathbb{R}^3 \setminus (Y \cup T_1 \cup T_2)$, there is a map $f\colon D^2 \to \mathbb{R}^3 \setminus (Y \cup T_1 \cup T_2)$ that maps ∂D^2 homeomorphically onto K. We will now consider the preimage of ∂C under f. If it is empty, then $f(D^2)$ lies entirely inside $\mathbb{R}^3 \setminus (Y \cup C)$, so K would be null-homotopic in $\mathbb{R}^3 \setminus (Y \cup C)$ and we are done. So we consider the case where $f^{-1}(\partial C)$ is non-empty. By transversality, $f^{-1}(\partial C)$ is a finite union of closed submanifolds of dimension 1, so it is a finite union of embedded $S^1 \hookrightarrow D^2$. We will now use the so-called "innermost disc argument" to adjust f so that the preimage of ∂C under this new f is non-empty.

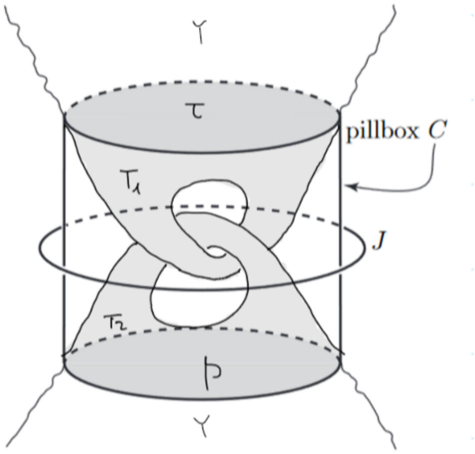

FIGURE 5. Parts of this picture are from [DV09].

Among all the embedded S^1s in the preimage of ∂C under f, there will be at least one, call it $\mathcal{L}_1$, that bounds an innermost disc. That is, it bounds a disc $\operatorname{Int} D_1$ such that $f(\operatorname{Int} D_1) \cap \partial C$ is empty.

Claim. *$f(D_1)$ can be shrunk to a point on $\partial C \setminus (\tau \cup \beta)$.*

Assume, that the claim holds. Then we can shrink $f(D_1)$ to a point and push this point slightly away from ∂C into $\mathbb{R} \setminus (Y \cup C)$. Like this, we got rid of one innermost disc. Since we just have finitely many closed curves in the preimage of ∂C, we can repeat this process until there are no curves in $f^{-1}(\partial C)$ anymore, and we are done.

Proof of Claim. We have to check two cases, namely the case that $\operatorname{Int} D_1 \subseteq \operatorname{Int} C$ and the case that $D_1 \cap \operatorname{Int} C = \emptyset$.

Case 1: $D_1 \subseteq \operatorname{Int} C$.
Define $M_1 \coloneqq C \setminus (T_1 \cup T_2)$. This is a manifold with boundary. Note that T_1 and T_2 are closed, so that the manifold boundary of M_1 is exactly $\partial C \setminus (\tau \cup \beta)$ which is homeomorphic to $S^1 \times (0, 1)$.

We compute $\pi_1(M_1)$. It is equivalent to the fundamental group of the complement of the finite graph G shown in Figure 6.

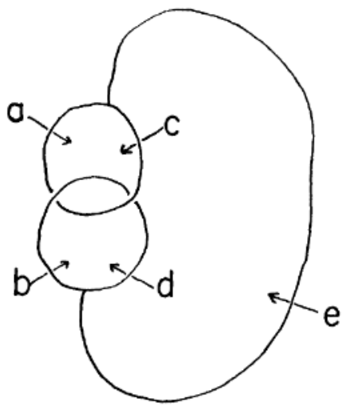

FIGURE 6. A finite graph with $\pi_1(\mathbb{R}^3 \setminus G) \cong \pi_1(M_1)$. Picture from [Bin61].

We can easily compute the Wirtinger presentation of $\pi_1(\mathbb{R}^3 \setminus G)$ and see that

$$\pi_1(\mathbb{R}^3 \setminus G) = \langle a, b, c, d, e | a = ec, cb = dc, ab = bc, b = ed\rangle \cong \langle a, b\rangle \quad \text{where } e = ab^{-1}a^{-1}b \neq 1.$$

It is the free group on two generators and a loop corresponding to e is not trivial in $\pi_1(M_1)$. This latter loop L is given by one that circles $\partial M_1 \cong S^1 \times (0, 1)$ once. So it corresponds to the generator of $\pi_1(\partial M_1) \cong \mathbb{Z}$. In particular, any loop circling ∂M_1 k times for $0 \neq k \in \mathbb{Z}$ is not null-homotopic since it corresponds to the element $k \cdot e \in \pi_1(M_1)$ which is not trivial. We conclude that any loop α on the boundary of M_1 that is null-homotopic in M_1 is already null-homotopic on the boundary of M_1.

Since $\mathcal{L}_1$ bounds a disc inside C, it is null-homotopic inside M_1 and therefore also on $\partial M_1 = \partial C \setminus \tau \cup \beta$ which is what we had to show.

Case 2: $f(\operatorname{Int} D_1) \cap \operatorname{Int} C = \emptyset$.
Define $M_2 \coloneqq \mathbb{R}^3 \setminus (Y \cup \operatorname{Int} C)$. It is a 3-dimensional manifold with boundary. Since we remove Y entirely and keep the boundary of C, its manifold boundary is given by $\partial M_2 = \partial C \setminus \tau \cup \beta \cong S^1 \times (0, 1)$.

We apply the loop theorem to M_2: if there exists a closed curve γ on ∂M_2 such that $\gamma \simeq *$ in M_2 but $\gamma \not\simeq *$ on ∂M_2, then there exists a simple closed curve with the same property.

The loop $f(\mathcal{L}_1)$ is a simple closed curve on ∂M" that bounds a disc in M_2. If $f(\mathcal{L}_1)$ could not be shrunk to a point on ∂M_2, then by the loop theorem there is a simple closed curve with the same property. A simple closed curve is an embedded S^1. We will now consider simple closed curves on ∂M_2. Any simple closed curve γ on ∂M_2 such that $\gamma \not\simeq *$ on ∂M_2 is homotopic to L which was the loop corresponding to the generator of $\pi_1(M_1) = \pi_1(\partial C \setminus (\tau \cup \beta)) = \pi_1(M_2)$. If $L \simeq *$ in M_2, then $\pi_1(\partial M_2)$ would be trivial in $\pi_1(M_2)$. This means, $\pi_1(J)$ would be trivial in $\pi_1(\mathbb{R}^3 \setminus (Y \cup C))$ which is not the case by assumption. We conclude that any loop on ∂M_2 that is not null-homotopic in M_2 is already null-homotopic on ∂M_2 and can finish the proof with the same argument as in Case 1. □

The proof of the claim finishes the proof of Lemma 2.2. □

Now if we find an essential loop in $\mathbb{R}^3 \setminus X_0$, Lemma 2.2 tells us that the loop will also be essential in $\mathbb{R}^3 \setminus X_n$ for every $n \geq 0$. The complements of the X_n form a nested sequence of open sets $\mathbb{R}^3 \setminus X_0 \subseteq \mathbb{R}^3 \setminus X_1 \subseteq \dots$. Consider any null-homotopic loop in $\mathbb{R}^3 \setminus A$. The image of the homotopy that contracts the loop to a point is compact in $\mathbb{R}^3 \setminus A$ and lies therefore already in $\mathbb{R}^3 \setminus X_n$ for some n. Thus, the loop is already null-homotopic in the complement of some X_n. We conclude that every essential loop in $\mathbb{R}^3 \setminus X_n$ for some n is also essential in $\mathbb{R}^3 \setminus A$. So we can derive from lemma Lemma 2.2 that $\pi_1(\mathbb{R}^3 \setminus A)$ is not trivial since J is one example of an essential loop in X_0 if we choose Y to be $X_0 \setminus C$. By the Schoenflies theorem, the embedding is therefore not locally flat, hence wild.

Remark 2.3. We have already seen an embedding of a Cantor set into the sphere and know that the Alexander horned sphere is locally flat outside the image of the embedded Cantor set. But we did not see why this Cantor set is exactly the set of wild points. One way to think about this is to consider neighborhoods U_x of any point x in the embedded Cantor set on the Alexander horned sphere. Such a point lies in the intersection of an infinite sequence of solid tori $(T_k)_{1\leq k\in\mathbb{Z}}$ lying inside pillboxes $P_k \supseteq T_k$ and U_x contains a part of this sequence, say $(T_k)_{k\geq l}$ for some $l \geq 1$. We find for any n a neighborhood V_x of x that looks exactly the same as U_x such that V_x just contains $(T_k)_{k\geq n+l}$. But if $\pi_1(\mathbb{R}^3 \setminus U_x)$ is not trivial then $\pi_1(\mathbb{R}^3 \setminus V_x)$ cannot be trivial as well. Since we can choose U_x to be A itself, we can find for every neighborhood V of x a neighborhood $V_x \subseteq V$ such that $\pi_1(\mathbb{R}^3 \setminus V_x)$ is not trivial. Therefore there is not neighborhood W of x such that $W \setminus A$ is homeomorphic to $\mathbb{R}^3 \setminus R^2$ which implies that x is not embedded in a locally flat way. More details on this can be found in [Hat02].

3 Bing's hooked rug

As already mentioned above, the set of wild points of the Alexander horned sphere A is a Cantor set in the sense that there is an embedding of a Cantor set $C \hookrightarrow A$ such that the image of C is exactly the set of wild points of A. We now construct an example of an embedded 2-sphere in $\mathbb{R}^3$ such that the embedding is wild at every point of the sphere. This sphere was originally constructed by R.H. Bing in [Bin61] in order to give an counterexample to the conjecture that an embedding $S^2 \hookrightarrow \mathbb{R}^3$ is locally flat if each arc in the image of the sphere is locally flat. However, we will not prove that each arc in Bing's hooked rug is tame since we would have to develop some tools that would go beyond the scope of this chapter. But we will see that Bing's hooked rug is somehow a 'very' wild sphere, in the sense that it is wild at every point.

For the construction of Bing's hooked rug we need the definition of an *eyebolt*.

Definition 3.1 (Eyebolt [DV09])**.** An *eyebolt* is the union of a tube with a solid torus. A *plug* for the eyebolt is a copy of $D^2 \times (0,1)$ embedded into the solid torus part of the eyebolt.

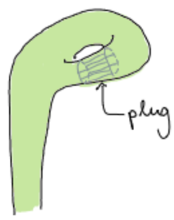

FIGURE 7. A plug and an eyebolt.

Now we can start the construction.

3.1 Construction

We start with the standard solid ball F_0 in $\mathbb{R}^3$.

Step 1. In the first step of the construction, we cover F_0 with discs $E_1, \ldots, E_n$ where $n > 0$ is an integer of our choice. The discs should satisfy the following two properties:

(1) $\operatorname{Int} E_i \cap \operatorname{Int} E_j = \emptyset$ for any $i \neq j$,
(2) the discs are arranged in a circular pattern, i.e. $E_i \cap E_{i+1}$ is an arc in the boundary of each for $i \leq n-1$. The same holds for $E_n \cap E_1$.

We attach an eyebolt g_i on each of the discs E_i and "hook" g_i to the base of g_{i+1} (and g_n to the base of g_1), as indicated in Figure 8. We shrink the resulting 3-manifold slightly such that it lies inside F_0 and call it H_1. Now, we remove a plug from each of the eyebolts to get a manifold F_1 that is homeomorphic to a solid ball, so $F_1 \cong F_0$.

Step 2. Now, we cover F_1 with closed discs $E'_1, \ldots, E'_n$ that have the same boundaries as the image of $E_1, \ldots, E_n$ under a homeomorphism $F_0 \cong F_1$. Afterwards, we cover each E'_i with discs $E^i_1, \ldots, E^i_k$ for a number $k \geq 2$ that we can freely choose, that satisfy properties 1 and 2. Now, we attach an eyebolt g^i_j to each E^i_j and hook g^i_j to the base of g^i_{j+1} for $j \leq n-1$, and g^i_n to the base of g^i_1. After shrinking the result slightly, we again obtain a 3-manifold with boundary which we call $H_2 \subseteq H_1$. We remove a plug from each eyebolt on H_2 and call the resulting manifold $F_2 \cong F_1 \cong F_0$. Parts of H_2 are shown in Figure 9.

We continue inductively. Here is an instruction for step k, that shows how we obtain H_k and F_k from F_{k-1}.

FIGURE 8. The first stage H_1 of the construction of Bing's hooked rug. This picture is from [DV09].

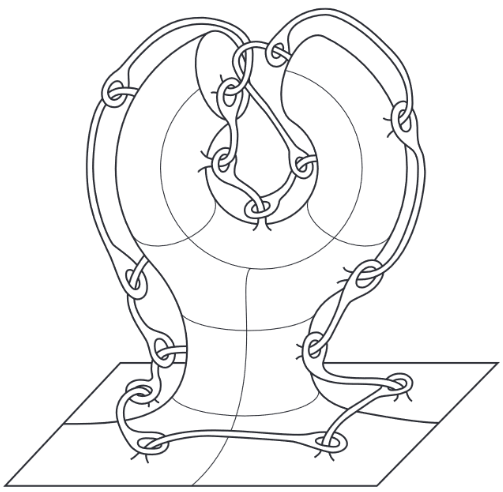

FIGURE 9. Parts of the manifold H_2. The picture is from [DV09].

Step k. F_{k-1} is covered with discs $E_1, \dots, E_n$ where on each disc there is an eyebolt with a plug removed and which is not covered by the E_i. Here we take the same notation for the cover as in earlier steps since another naming would be complicated and the name of the discs will not be very important later. Cover F_{k-1} with discs $E'_1, \dots, E'_n$, that have the same boundaries as $E_1, \dots, E_n$ and cover each of these discs with n discs that satisfy properties 1 and 2. Attach an eyebolt to each disc and hook it to the base of the eyebolt on the next disc. Shrink the result and call it H_k. Now remove a plug from each eyebolt and call the resulting space F_k. It is homeomorphic to F_i for any $i \leq k$.

We constructed a nested sequence of 3-manifolds with boundary $H_1 \supseteq H_2 \supseteq \dots$ and we define

$$H \coloneqq \bigcap_{i=1}^{\infty} H_i.$$

Bing's hooked rug is now defined as ∂H.

As for the Alexander horned sphere, our aim is now to prove that Bing's hooked rug is indeed homeomorphic to a sphere as well as that it is wildly embedded into $\mathbb{R}^3$. This will be proven in the following two sections.

3.2 Bing's hooked rug is an embedded sphere

As already mentioned, there are homeomorphisms $h_i\colon F_{i-1} \to F_i$ that can be controlled in their size by the number and size of the covering discs. This is the reason why we can choose the discs and the sequence $(f_n)_{n\in\mathbb{N}}$ such that it converges uniformly, where $f_n \coloneqq h_n \circ \ldots \circ h_1$. Thus the limit map $f\colon F_0 \to H$ will be continuous. Again, we just have to show bijectivity of f to conclude that f is a homeomorphism.

For the proof of bijectivity of f we can choose the shrinking applied in each step to be the radial shrinking which is a homeomorphism. Therefore we will drop this step from now on. In step k of the construction we subdivide the covering of F_{k-1} such that the union of all boundaries of the old discs form a subset of the union of the boundaries of all discs in the subdivision. From F_{k-1} to F_k we do not change anything on the boundary of the discs E_i so we can choose h_k such that it fixes the boundaries of the discs on F_{k-1}.

Two disjoint discs of the covering of F_n will be disjoint under f since we do not change their boundary and the discs have disjoint interiors by construction. For two different points $x, y \in F_0$ there will be an index k such that $f_k(x)$ and $f_k(y)$ lie on different discs of the covering of F_k. So $f(x)$ and $f(y)$ will be distinct points and we can conclude that f is injective.

We have proved that f is a continuous and bijective map from a compact space to a Hausdorff space. By the compact Hausdorff lemma $f\colon F_0 \to H$ is a homeomorphism which shows that H is homeomorphic to a solid ball. Thus $B = \partial H$ is homeomorphic to a sphere.

3.3 Wildness of Bing's hooked rug

We will prove that the embedding $B \hookrightarrow \mathbb{R}^3$ is wild at every point by showing that it is not 1-LCC at every point of the embedding.

Recall that a co-dimension one embedding $A \hookrightarrow X$ that is locally flat at a point $a \in A$ is k-LCC for $k \geq 1$ at that point. The definition of k-LCC can be found in **??**.

We will prove a similar lemma to Lemma 2.2 and can conclude from it that the embedding is not 1-LCC at every point by finding an index i and an essential loop in $\mathbb{R}^3 \setminus H_i$ circling the base of an eyebolt in H_i and proving that it is essential in $\mathbb{R}^3 \setminus H_k$ for any $k \geq i$. By construction, the eyebolts will be spread densely over B in the end, so we can find such a loop in every neighborhood of a point.

Lemma 3.2. *Let C be a 3-cell in $\mathbb{R}^3$ and let B_1, B_2 and B_3 be three disjoint discs on ∂C. Let T be a solid torus in C such that $T \cap \partial C = B_1$ and let S be a 3-cell in C such that $S \cap \partial C = B_1 \cup B_2$. Assume T and S are linked as indicated in Figure 10. Let Y be a closed subset of $\mathbb{R}^3$ such that $Y \cap C = B_1 \cup B_2 \cup B_3$.*

If $\pi_1(\partial C \setminus (B_1 \cup B_2 \cup B_3)) \to \pi_1(\mathbb{R}^3 \setminus (Y \cup \operatorname{Int} C))$ is injective, then $\pi_1(\mathbb{R}^2 \setminus (Y \cup C)) \to \pi_1(\mathbb{R}^3 \setminus (Y \cup S \cup T))$ is injective.

Proof. The proof of this lemma is similar to the proof of Lemma 2.2. Assume, $\pi_1(\partial C \setminus (B_1 \cup B_2 \cup B_3)) \to \pi_1(\mathbb{R}^3 \setminus (Y \cup \operatorname{Int} C))$ is injective. Now choose a loop $J \subseteq \mathbb{R}^3 \setminus (Y \cup S \cup T)$ that is null-homotopic, i.e. that bounds a disc. We have a map $f\colon D^2 \to \mathbb{R}^3 \setminus (Y \cup S \cup T)$ such that $f(\partial D^2) \cong J$. We consider again $f^{-1}(\partial C)$ and use transversality to see that it is a finite union of simple closed curves. We choose an innermost disc bounded by $\mathcal{L}_1$ and will prove now that $\mathcal{L}_1$ is nullhomotopic on $\partial C \setminus (B_1 \cup B_2 \cup B_3)$. Again, we have to consider the two cases where the disc bounded by $\mathcal{L}_1$ lies inside or outside C.

Case 1: Define M_1 to be $C \setminus (T \cup S)$. It is a 3-dimensional manifold with boundary $\partial M_1 = \partial C \setminus (B_1 \cup B_2 \cup B_3)$. We compute $\pi_1(M_1)$ which is a fre group of two generators. Note that

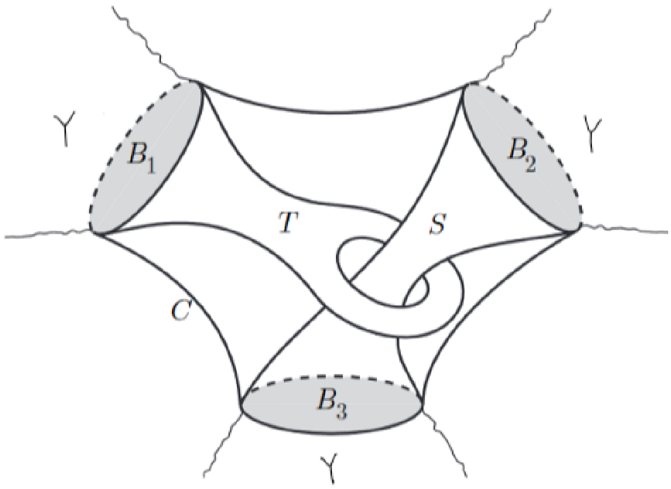

FIGURE 10. The situation of Lemma 3.2. Picture from [DV09, p.88]

the boundary of any of the three discs B_i is not null-homotopic in ∂M_1 and that the three boundaries of the discs even generate the fundamental group of ∂M_1. Now assume there is a closed curve γ on ∂M_1 such that γ is null-homotopic in M_1 but not on ∂M_1. By the loop theorem, there is a simple closed curve with the same property. This could, up to homotopy, only be ∂B_i for some $i = 1, 2, 3$. But ∂B_i is not null-homotopic in M_1. So there is no such closed curve $\gamma \subseteq \partial M_1$. Thus, any loop on ∂M_1 that is null-homotopic in M_1 is null-homotopic on ∂M_1. In particular, $\mathcal{L}_1$ is null-homotopic on ∂M_1. Thus in this case, $\mathcal{L}_1$ is null-homotopic on ∂M_1 as we wanted to show.

Case 2: Define $M_2 \coloneqq \mathbb{R}^3 \setminus (Y \cup \operatorname{Int} C)$. It is a 3-manifold with boundary $\partial M_2 = \partial C \setminus (B_1 \cup B_2 \cup B_3)$. By using the loop theorem, we see that every loop on ∂M_2 that is null-homotopic in M_2 is also null-homotopic on ∂M_2 since $\pi_1(\partial M_2)$ is generated by the boundaries of the three discs that are not null-homotopic in M_2. Thus $\mathcal{L}_1$ is null-homotopic on ∂M_2.

As in Lemma 2.2 we can now conclude the statement of this lemma. □

To apply the lemma, we will consider the step $H_{i-1} \to H_i$ in the construction of Bing's hooked rug again in more detail. In Figure 11 the step is divided into several substeps that we explain next.

We start in H_{i-1} on the top left. After removing the complement of two linked solid tori inside a pillbox, we constructed H_i'. Now, to get to H_i'', we need two substeps. First, we attach a handle to each of the discs in the subdivision $(\widetilde{H}_i)$. Then, we move one base of each handle including the first two handles on H_i' onto the disc with the next higher index $(\widetilde{H}_i')$. So everything is moved in a circular pattern around the surface. To obtain H_i'', we move the base of a handle, that is now on the disc with higher index, onto the other handle that has a base on the same disc and thicken up the moved base of the handle. Like this, we get such tubes on each disc, that have a bulb at the end where the next tube goes through. Since everything is solid, these intersections do not disturb the manifold property of H_i''. From H_i'' to H_i we cut a hole into the bulbs where the tubes can go through an see that we have constructed hooked eyebolts.

Now we can see what happens to an essential loop in the complement of H_i after passing to H_{i+1}. The loop K shown in Figure 11 is essential in $\mathbb{R}^3 \setminus H_i''$ since it circles an attached handle. Lemma 3.2 implies that K will also be essential in $\mathbb{R}^3 \setminus H_i$. Now we can see that by Lemma 2.2, K is essential in the complement of H_{i+1}'. From H_{i+1}' to H_{i+1}'' we just add some handles that can not trivialize essential loops. So K is essential in $\mathbb{R}^3 \setminus H_{i+1}''$. We continue again inductively.

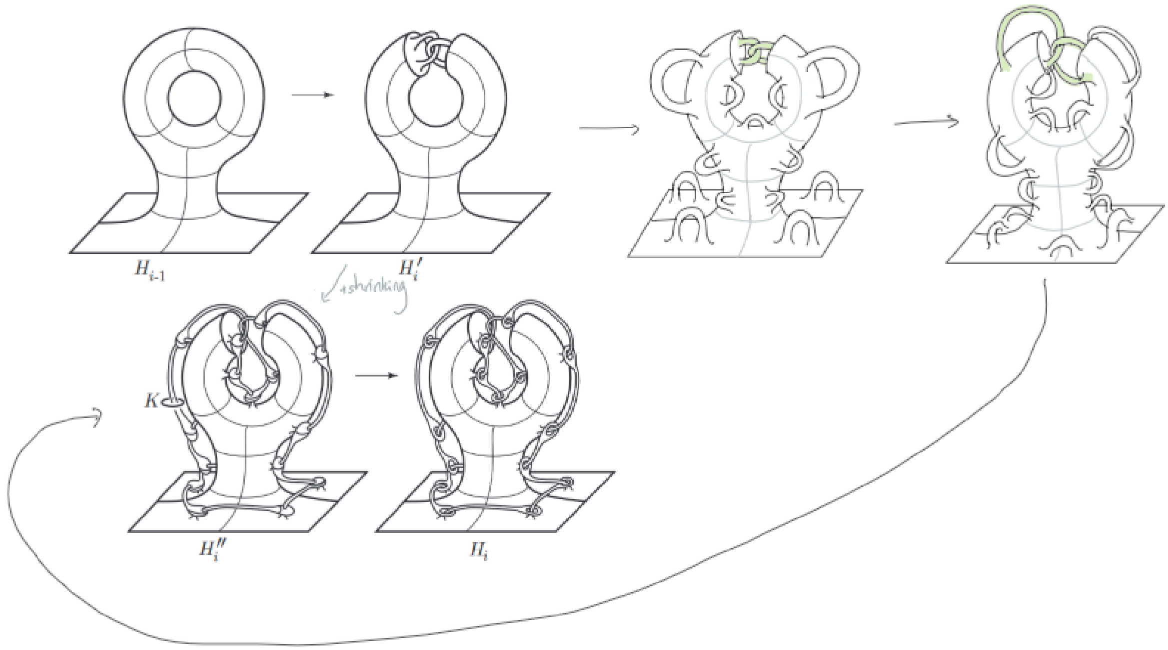

FIGURE 11. The construction step $H_{i-1} \to H_i$ in more detail. Parts of the picture are from [DV09]. Note that what is shown here is always just a part of H_i or H_{i-1}.

As mentioned above, for any neighbourhood $U \subseteq B$ of a point $x \in B$ we will always find N, such that one of the discs E_i covering F_N in stage N lies entirely in U. Therefore, for any neighbourhood U we will find an essential loop in $U \setminus B$. This means that B is not 1-LCC at any point as we wanted to show.

References

[Bin61] R. H. Bing. A wild surface each of whose arcs is tame. *Duke Math. J.*, 28:1–15, 1961.
[DV09] Robert J. Daverman and Gerard A. Venema. *Embeddings in manifolds*, volume 106 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2009.
[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

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