NONDIMENSIONALIZATION OF THE EQUATIONS OF FLUID FLOW

In this lesson, we will:

- Assign **Characteristic Scales** or **Scaling Parameters** and use them to **Nondimensionalize** the equations of fluid flow
- Explain the difference between **Nondimensionalization** and **Normalization**
- Do an example problem

How to Solve the Navier-Stokes Equation

We have three ways to solve the differential equations of fluid flow (continuity and N-S):

1. **Analytically** [solve exactly, but only for very simple problems]
2. **Numerically** [use CFD on a computer to solve for thousands of cells]
3. **Approximately** [ignore some terms in the N-S equation, then solve]

Nondimensionalization of the Equations

GOAL: To re-write the eqs of fluid flow in nondimensional form

Why? — So we can compare terms in the N-S eq to see which (if any) are negligibly small compared to other terms

If so, we can solve an approximate (simpler) eq. set

CONSIDER INCOMPRESSIBLE FLOW OF A NEWTONIAN FLUID

Continuity

$$\boxed{\vec{\nabla}\cdot\vec{V}=0}$$

cont.

$$\boxed{\rho\frac{D\vec{V}}{Dt}=\rho\left[\frac{\partial\vec{V}}{\partial t}+\left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}\right]=-\vec{\nabla}P+\rho\vec{g}+\mu\nabla^2\vec{V}}$$

N-S

$$\vec{\nabla}=\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k}$$

$$\vec{V}=u\vec{i}+v\vec{j}+w\vec{k}$$

$$\vec{x}=x\vec{i}+y\vec{j}+z\vec{k}$$

use * superscript for all nondimen. variables

INTRODUCE SCALING PARAMETERS & NONDIMENSIONAL VARIABLES

Let L = some characteristic length scale in the flow

then $\boxed{\vec{x}^*=\frac{\vec{x}}{L}}\Rightarrow x^*=\frac{x}{L},\ y^*=\frac{y}{L},\ z^*=\frac{z}{L}$

Let V = some characteristic velocity scale

$$\vec{V} = \vec{V}^* V \leftarrow \boxed{\vec{V}^* = \frac{\vec{V}}{V}} \Rightarrow u^* = \frac{u}{V}, \quad v^* = \frac{v}{V}, \quad w^* = \frac{w}{V}$$

$$\{\vec{\nabla}\} = \left\{\frac{1}{L}\right\} \quad \text{since} \quad \vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

So, define $\boxed{\vec{\nabla}^* = L\vec{\nabla}} \rightarrow \left(\vec{\nabla} = \frac{\vec{\nabla}^*}{L}\right)$

a nondimensional form of $\vec{\nabla}$

Thus, continuity eq $\vec{\nabla} \cdot \vec{V} = 0$ becomes

$$\frac{\vec{\nabla}^*}{L} \cdot \vec{V}^* V \rightarrow \frac{V}{L} \vec{\nabla}^* \cdot \vec{V}^* = 0$$

mult by $\frac{L}{V} \rightarrow \boxed{\vec{\nabla}^* \cdot \vec{V}^* = 0}$

NONDIMENSIONALIZED CONTINUITY EQUATION

Navier-Stokes Equation

$$\rho \frac{D\vec{V}}{Dt} = \rho\left[\frac{\partial \vec{V}}{\partial t} + \left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}\right] = -\vec{\nabla}P + \rho\vec{g} + \mu\nabla^2\vec{V}$$

This equation is *dimensional*, and each variable or property (ρ, $\vec{V}$, t, μ, etc.) is also *dimensional*. What are the primary dimensions (in terms of {m}, {L}, {t}, {T}, etc.) of each term in this equation?

$$\{\rho g\} = \left\{\frac{m}{L^3}\frac{L}{t^2}\right\} = \left\{\frac{m}{L^2t^2}\right\} \qquad \{\vec{\nabla}P\} = \left\{\frac{1}{L}\frac{mL}{t^2}\frac{1}{L^2}\right\} = \left\{\frac{m}{L^2t^2}\right\}$$

Pressure differences are important in a fluid flow, not absolute magnitude of pressure

To nondimensionalize the above equation, we choose scaling parameters as follows:

Scaling Parameter	Description	Primary Dimensions
L	Characteristic length	$\{L\}$
V	Characteristic speed	$\{Lt^{-1}\}$
f	Characteristic frequency	$\{t^{-1}\}$
$P_0 - P_\infty$	Reference pressure difference	$\{mL^{-1}t^{-2}\}$
g	Gravitational acceleration	$\{Lt^{-2}\}$

All images from Çengel and Cimbala, Ed. 4.

We also have fluid properties ρ and μ in the Navier-Stokes equation, so these must also be known to complete the analysis.

We define ***nondimensional variables***, using the scaling parameters in the above table:

$$\{t^*\} = \left\{\frac{1}{t}\cdot t\right\} = \{1\}$$

$$t^* = ft \qquad \vec{x}^* = \frac{\vec{x}}{L} \qquad \vec{V}^* = \frac{\vec{V}}{V}$$

$$P^* = \frac{P - P_\infty}{P_0 - P_\infty} \qquad \vec{g}^* = \frac{\vec{g}}{g} \qquad \vec{\nabla}^* = L\vec{\nabla}$$

To plug these nondimensional variables into the Navier-Stokes equation, we need to first rewrite them as ***dimensional*** variables,

$$t = \frac{1}{f}t^* \qquad \vec{x} = L\vec{x}^* \qquad \vec{V} = V\vec{V}^*$$

$$P = P_\infty + \left(P_0 - P_\infty\right)P^* \qquad \vec{g} = g\vec{g}^* \qquad \vec{\nabla} = \frac{1}{L}\vec{\nabla}^*$$

Now we substitute the above into the Navier-Stokes equation, rearranging the first two terms,

$$\rho \frac{\partial \vec{V}}{\partial t} + \rho\left(\vec{V}\cdot\vec{\nabla}\right)\vec{V} = -\vec{\nabla}P + \rho\vec{g} + \mu\nabla^2\vec{V}$$

$$\rho \frac{\partial\left(V\vec{V}^*\right)}{\partial\left(t^*/f\right)} + \rho\left(V\vec{V}^*\cdot\frac{1}{L}\vec{\nabla}^*\right)V\vec{V}^* =$$

$$\rho V f \frac{\partial \vec{V}^*}{\partial t^*} + \frac{\rho V^2}{L}\left(\vec{V}^*\cdot\vec{\nabla}^*\right)\vec{V}^* =$$

$$\rho V f \frac{\partial \vec{V}^*}{\partial t^*} + \frac{\rho V^2}{L}\left(\vec{V}^*\cdot\vec{\nabla}^*\right)\vec{V}^* = -\frac{P_0 - P_\infty}{L}\vec{\nabla}^* P^* + \rho g \vec{g}^* + \frac{\mu V}{L^2}\nabla^{*2}\vec{V}^*$$

Every additive term in the above equation has primary dimensions $\{m^1L^{-2}t^{-2}\}$. To nondimensionalize the equation, we multiply every term by constant $L/(\rho V^2)$, which has primary dimensions $\{m^{-1}L^2t^2\}$, so that the dimensions cancel. After some rearrangement,

$$\boxed{\left[\frac{fL}{V}\right]}\frac{\partial \vec{V}^*}{\partial t^*} + \left(\vec{V}^*\cdot\vec{\nabla}^*\right)\vec{V}^* = -\boxed{\left[\frac{P_0 - P_\infty}{\rho V^2}\right]}\vec{\nabla}^* P^* + \boxed{\left[\frac{gL}{V^2}\right]}\vec{g}^* + \boxed{\left[\frac{\mu}{\rho V L}\right]}\nabla^{*2}\vec{V}^*$$

Strouhal number, where $\mathrm{St} = \dfrac{fL}{V}$

Euler number, where $\mathrm{Eu} = \dfrac{P_0 - P_\infty}{\rho V^2}$

Inverse of Froude number squared, where $\mathrm{Fr} = \dfrac{V}{\sqrt{gL}}$

Inverse of Reynolds number, where $\mathrm{Re} = \dfrac{\rho V L}{\mu}$

* EVERY TERM IS NOW NONDIMENSIONAL *

The above equation can therefore be written as

Navier-Stokes Equation in Nondimensional Form:

$$\boxed{[\mathrm{St}]\frac{\partial \vec{V}^*}{\partial t^*} + (\vec{V}^*\cdot\vec{\nabla}^*)\vec{V}^* = -[Eu]\vec{\nabla}^* P^* + \left[\frac{1}{\mathrm{Fr}^2}\right]\vec{g}^* + \left[\frac{1}{\mathrm{Re}}\right]\nabla^{*2}\vec{V}^*}$$

UNSTEADY TERM

INERTIAL TERM

PRESSURE TERM

GRAVITATIONAL TERM

VISCOUS TERM

Nondimensionalization vs. Normalization

The above equation is *nondimensional*, but not necessarily *normalized*. What is the difference?

- ***Nondimensionalization*** concerns only the ***dimensions*** of the equation – we can use *any* value of scaling parameters L, V, etc., and we always end up with the above equation.
- ***Normalization*** is more restrictive than nondimensionalization. To *normalize* the equation, we must choose scaling parameters L, V, etc. that are appropriate for the flow being analyzed, such that ***all nondimensional variables*** (t^*, $\vec{V}^*$, P^*, etc.) ***in the above equation are of order of magnitude unity***. In other words, their minimum and maximum values are reasonably close to 1.0,

"$\sim$" = order of magnitude

$$t^* \sim 1, \quad \vec{x}^* \sim 1, \quad \vec{V}^* \sim 1, \quad P^* \sim 1, \quad \vec{g}^* \sim 1, \quad \vec{\nabla}^* \sim 1$$

If we have properly normalized the Navier-Stokes equation, we can compare the ***relative importance*** of various terms in the equation by comparing the ***relative magnitudes*** of the nondimensional parameters St, Eu, Fr, and Re.

Example: Comparison of Magnitudes of Terms in the Navier-Stokes Equation

Given: Water at 20ºC is flowing over an object. $\rightarrow \rho = 998.0 \ \frac{kg}{m^3}$, $\mu = 1.002 \times 10^{-3} \ \frac{kg}{m \cdot s}$

The diameter of the object is about 0.6 m $= L$

The freestream speed is about 4 m/s $= V$

The object vibrates at 4 Hz $= f$

The pressure difference between the stagnation pressure and the static pressure of the freestream is about 8000 Pa $= P_0 - P_\infty$

To do: Compare the magnitudes of each of the five terms in the Navier-Stokes equation.

Solution:

$$\underbrace{[\mathrm{St}]}_{\sim 0.6}\frac{\partial \vec{V}^*}{\partial t^*} + \underbrace{\left(\vec{V}^* \cdot \vec{\nabla}^*\right)\vec{V}^*}_{\sim 1} = -\underbrace{[\mathrm{Eu}]}_{\sim 0.5}\vec{\nabla}^* P^* + \underbrace{\left[\frac{1}{\mathrm{Fr}^2}\right]}_{\sim 0.4}\vec{g}^* + \underbrace{\left[\frac{1}{\mathrm{Re}}\right]}_{\sim 4\times 10^{-7}}\vec{\nabla}^{*2}\vec{V}^*$$

$$St = \frac{fL}{V} = \frac{(4\,\frac{1}{s})(0.6\,m)}{4\,m/s} = 0.6$$

$$Eu = \frac{P_0 - P_\infty}{\rho V^2} = \frac{8000\,\frac{N}{m^2}}{\left(998\,\frac{kg}{m^3}\right)\left(4\,\frac{m}{s}\right)^2}\left(\frac{kg \cdot m}{s^2 \cdot N}\right) = 0.501$$

$$\frac{1}{Fr^2} = \frac{gL}{V^2} = \frac{\left(9.807\,\frac{m}{s^2}\right)(0.6\,m)}{(4\,m/s)^2} = 0.368$$

$$\frac{1}{Re} = \frac{\mu}{\rho V L} = \frac{1.002\times 10^{-3}\,kg/m^3}{\left(998.0\,\frac{kg}{m^3}\right)\left(4\,\frac{m}{s}\right)(0.6\,m)}$$

$$\frac{1}{Re} = 4.18\times 10^{-7}$$

If we ignore viscous terms in the N-S eq., we get

$$\boxed{\rho \frac{\partial \vec{V}}{\partial t} + \rho (\vec{V} \cdot \vec{\nabla}) \vec{V} = -\vec{\nabla} P + \rho \vec{g}} + \cancel{\text{viscous terms}}$$

EULER EQ. (N-S without viscous terms)

AN APPROXIMATE EQUATION

CAUTION: THE SOLUTION WILL BE AN APPROXIMATE SOLUTION WHICH MAY NOT BE ACCURATE