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CARLEMAN ESTIMATES FOR THE LAMÉ SYSTEM WITH STRESS BOUNDARY CONDITION AND THE APPLICATION TO AN INVERSE PROBLEM

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Abstract. In this paper, for functions without compact supports, we established Carleman estimates for the two-dimensional non-stationary Lamé system with the stress boundary condition. Using this estimate, we proved the uniqueness and the stability for the inverse problem of determination of coefficients for the Lamé system by a single measurement.

§1. Introduction and the main results.

In this paper, we establish Carleman estimates for the two-dimensional non-stationary Lamé system with stress boundary condition:

$$\begin{aligned}&P(x,D)\mathbf{u} \equiv (P_1(x,D)\mathbf{u}, P_2(x,D)\mathbf{u})^T\\ =&\rho(\widetilde{x})\frac{\partial^2\mathbf{u}}{\partial x_0^2} - \mu(\widetilde{x})\Delta\mathbf{u} - (\mu(\widetilde{x})+\lambda(\widetilde{x}))\nabla_{\widetilde{x}}\mathrm{div}\,\mathbf{u}\\ -&(\mathrm{div}\,\mathbf{u})\nabla_{\widetilde{x}}\lambda(\widetilde{x}) - (\nabla_{\widetilde{x}}\mathbf{u}+(\nabla_{\widetilde{x}}\mathbf{u})^T)\nabla_{\widetilde{x}}\mu(\widetilde{x}) = \mathbf{f} \quad \text{in } Q=(0,T)\times\Omega,\end{aligned} \tag{1.1}$$

$$\mathbb{B}(x,D)\mathbf{u} = \left(\sum_{j=1}^{2} n_j\sigma_{j1}, \sum_{j=1}^{2} n_j\sigma_{j2}\right)^T = \mathbf{g} \quad \text{on } (0,T)\times\partial\Omega,$$

$$\mathbf{u}(T,\widetilde{x}) = \frac{\partial\mathbf{u}}{\partial x_0}(T,\widetilde{x}) = \mathbf{u}(0,\widetilde{x}) = \frac{\partial\mathbf{u}}{\partial x_0}(0,\widetilde{x}) = 0, \tag{1.2}$$

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where $\mathbf{u} = (u_1, u_2)^T$, $\mathbf{f} = (f_1, f_2)^T$ are the vector functions, $\mathbf{u}^T$ denotes the transpose of the vector $\mathbf{u}$, Ω is a bounded domain in $\mathbb{R}^2$ with $\partial\Omega \in C^3, x = (x_0, \widetilde{x}), \widetilde{x} = (x_1, x_2)$, $(n_1, n_2)^T$ is the unit outward normal vector to $\partial\Omega$ and

$$\sigma_{jk} = \lambda(\widetilde{x})\delta_{jk}\mathrm{div}\, \mathbf{u} + \mu(\widetilde{x}) \left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j} \right).$$

The boundary condition in (1.2) describes the surface stress. In (1.1), the coefficients $\rho, \mu, \lambda \in C^2(\overline{\Omega})$ are assumed to satisfy

$$\rho(\widetilde{x}) > 0, \quad \mu(\widetilde{x}) > 0, \quad \mu(\widetilde{x}) + \lambda(\widetilde{x}) > 0, \quad \widetilde{x} \in \overline{\Omega}. \tag{1.3}$$

Physically λ and μ are the Lamé coefficients of the isotropic medium occupying domain Ω, and ρ is the density.

A Carleman estimate is an inequality for solutions to a partial differential equation with weighted L^2-norm and is a strong tool for proving the uniqueness in the Cauchy problem or the unique continuation for a partial differential equation with non-analytic coefficients. Moreover Carleman estimates have been applied essentially to estimation of energy of solutions (e.g., [KK]) and to inverse problems of determining coefficients by boundary observations (e.g., [Buk], [K] as initiaing works).

As a pioneering work, we refer to Carleman [Ca] which proved a Carleman estimate to apply it for proving the uniqueness in the Cauchy problem for a two-dimensional elliptic equation. Since [Ca], the theory of Carleman estimates has been studied extensively. We refer to Hörmander [Hö] in the case where the symbol of a partial differential equation is isotropic and functions under consideration have compact supports (that is, they and their derivatives of suitable orders vanish on the boundary of a domain). Later Carleman estimates for functions with compact supports have been obtained for partial differential operators with anisotropic symbols by Isakov ([Is2], [Is3]). Moreover for functions without compact supports, see [Ta].

Our main task of establishing a Carleman estimate for (1.1) - (1.2) is difficult twofold: Firstly, in (1.1) the highest order derivatives are coupled and secondly (1.2) contains a boundary condition of the non-Dirichlet type.

First difficulty: As for Carleman estimates for strongly coupled systems, there are not many works. In fact, all the above-mentioned papers discuss single partial differential equations. As long as the unique continuation is concerned, to our best knowledge, the most general result for such systems of partial differential equations is Calderón's uniqueness theorem (see e.g., [E], [Zui]). However, the non-stationary Lamé system does not satisfy all conditions of that theorem. More precisely, the eigenvalues of the matrix associated with the principal symbol of the Lamé system change the multiplicities and at some points of cotangent bundle, they are not smooth, which break the assumptions in Caledrón's uniqueness theorem. On the other hand, for solving the unique continuation, the Lamé system can be decoupled (modulo low order terms) for example by introducing a new function $\mathrm{div}\,\mathbf{u}$ and applying to the new system the technique developed for the scalar partial differential equations (see e.g., [EINT]). This method produces a Carleman estimate for the Lamé system, but the displacement function $\mathbf{u}$ is required to have a compact support, so that method does not work for (1.1) and (1.2) if $\mathbf{u}$ does not have a compact support. In [IY4] and [IY7], we have established Carleman estimates for the Dirichlet case where the stress boundary condition in (1.2) is replaced by $\mathbf{u} = \mathbf{g}$

on $(0,T)\times\partial\Omega$. It is known that there are two types of the interior waves for the Lamé system: the longitudinal wave with the velocity $\sqrt{\frac{\lambda+2\mu}{\rho}}$ and the transverse wave with the velocity $\sqrt{\frac{\mu}{\rho}}$. Thus a weight function in the Carleman estimate is assumed to be pseudoconvex with respect to the two symbols (see Condition 1.1).

Second difficulty: The essential difference between the stress boundary condition and the Dirichlet boundary condition which was studied by the authors in [IY7], is that the stress boundary condition requires us to deal with the new phenomena - the Rayleigh boundary waves. In order to treat the boundary waves, we have to assume additionally that a weight function is strictly pseudoconvex with respect to the pseudodifferential operator whose principal symbol is given by the Lopatinskii determinant (see Condition 1.2). Furthermore, from the practical point of view (e.g., in view of the seismology), the stress boundary condition is very important and well describes the reality such as the surface wave, so that the associated inverse problems and energy estimation are highly requested to be studied.

Under Conditions 1.1 and 1.2, we state our main results - Carleman estimates (Theorem 1.1 and Corollaries 1.1 and 1.2). Among applications of the Carleman estimates obtained in this paper, we mention the sharp unique continuation/conditional stability results for the Cauchy problem for (1.1), the exact controllability of the Lamé system with locally distributed or boundary control, and applications to the inverse problems. However, in this paper, we will discuss only one application to an inverse problem. That is, in Section 5, using the Carleman estimate, we establish the uniqueness and conditional stability results for the inverse problem of determining the three coefficients ρ, λ, μ.

In this paper, we exclusively consider the two-dimensional case, and the higher dimensional case is more difficult. Really, as is shown in [Y], in the case where the spatial dimension is greater than 2, the Lopatinskii determinant equals zero at some point, so that we cannot satisfy Condition 1.2 which is crucial for the Carleman estimate.

Among related papers, we refer to Bellassoued [B1] - [B3], Dehman and Robbiano [DR], and Imanuvilov and Yamamoto [IY6], where Carleman estimates for the stationary Lamé system were obtained. Also see Weck [W] for the unique continuation for the stationary Lamé system.

Throughout this paper, we use:

Notations. $\vec{e}_1 = (1,0)$, $\vec{e}_2 = (0,1)$, $\vec{n} = (n_1, n_2)$, $x = (x_0, x_1, x_2) = (x_0, \widetilde{x})$, $\widetilde{x} = (x_1, x_2)$, $y = (y_0, y_1, y_2)$, $y' = (y_0, y_1)$, $\xi = (\xi_0, \xi_1, \xi_2)$, $\xi' = (\xi_0, \xi_1)$, $\partial_{y_j}\phi = \phi_{y_j} = \frac{\partial \phi}{\partial y_j}$, $\partial_{x_j}\phi = \phi_{x_j} = \frac{\partial \phi}{\partial x_j}$, $\phi_{x_j x_k} = \partial^2_{x_j x_k}\phi = \partial_{x_j}\partial_{x_k}\phi$, $\nabla = (\partial_{x_0}, \partial_{x_1}, \partial_{x_2})$ or $\nabla = (\partial_{y_0}, \partial_{y_1}, \partial_{y_2})$ if there is no fear of confusion. (Otherwise we will add the subscript x or y.) $\nabla_{\widetilde{x}} = (\partial_{x_1}, \partial_{x_2})$, $\operatorname{div}\mathbf{u} = \partial_{x_1}u_1 + \partial_{x_2}u_2$ for $\mathbf{u} = (u_1, u_2)^T$, $\mathbf{D}_{y_j} = \frac{1}{i}\frac{\partial}{\partial y_j} + is\partial_{y_j}\phi$, $\mathbf{D}' = (\mathbf{D}_{y_0}, \mathbf{D}_{y_1})$, $\mathbf{D} = (\mathbf{D}_{y_0}, \mathbf{D}_{y_1}, \mathbf{D}_{y_2})$, $\nabla_{y'} = (\partial_{y_0}, \partial_{y_1})$, $D' = (D_{y_0}, D_{y_1})$, $D_{y_j} = \frac{1}{i}\frac{\partial}{\partial y_j}$, $\alpha = (\alpha_0, \alpha_1, \alpha_2)$, $\alpha_j \in \mathbb{N}\cup\{0\}$, $\partial^\alpha_x = \partial^{\alpha_0}_{x_0}\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_3}$, $\zeta = (s, \xi_1, \xi_2)$, $\zeta' = (s, \xi_0, \xi_1)$. For a domain Q in the $x-$ space, $H^{1,s}(Q)$ is the Sobolev space of scalar-valued functions equipped with the norm

$$\|u\|_{H^{m,s}(Q)} = \left(\sum_{|\alpha|\le m} s^{2m-2|\alpha|}\|\partial^\alpha_x u\|^2_{L^2(Q)}\right)^{\frac{1}{2}},$$

$\mathbf{H}^{1,s}(Q) = H^{1,s}(Q)\times\cdots\times H^{1,s}(Q)$ is the corresponding space of vector-valued functions $\mathbf{u}$. For a domain Ω in the $\widetilde{x}$-space, we will similarly define the Sobolev spaces $H^{1,s}(\Omega)$ and $\mathbf{H}^{1,s}(\Omega)$. Let $[A, B] = AB - BA$, and let $\epsilon(\delta)$ be a nonnegative function such that $\epsilon(\delta) \to +0$ as $\delta \to +0$. By $\mathcal{O}(\delta_1)$, we denote the conic neighbourhood of the point ζ^*: $\mathcal{O}(\delta_1) = \left\{\zeta; \left|\frac{\zeta}{|\zeta|} - \zeta^*\right| \le \delta_1\right\}$, $B_\delta(y^*) = \{y; |y - y^*| < \delta\}$ is the ball centred at y^* with radius δ. $\mathcal{L}(X_1, X_2)$ is the space of linear operators from a normed space X_1 to a normed space X_2, E_k is the $k \times k$ unit matrix.

The first main purpose is to establish Carleman estimates for system (1.1) - (1.2) for $\mathbf{u}$ having no compact supports. Let $\omega \subset \Omega$ be an arbitrarily fixed open set which is not necessarily connected. Denote by $\vec{n}$ and $\vec{t}$, the outward unit normal vector and the unit counterclockwise oriented tangential vector on $\partial\Omega$, and we set $\frac{\partial u}{\partial \vec{n}} = \nabla_{\widetilde{x}} u \cdot \vec{n}$ and $\frac{\partial u}{\partial \vec{t}} = \nabla_{\widetilde{x}} u \cdot \vec{t}$. By Q_ω we denote the cylindrical domain $Q_\omega = (0,T) \times \omega$. We set

$$p_1(x,\xi) = \rho(\widetilde{x})\xi_0^2 - \mu(\widetilde{x})(\xi_1^2 + \xi_2^2), \quad p_2(x,\xi) = \rho(\widetilde{x})\xi_0^2 - (\lambda(\widetilde{x}) + 2\mu(\widetilde{x}))(\xi_1^2 + \xi_2^2).$$

For arbitrary smooth functions $\phi(x,\xi)$ and $\psi(x,\xi)$, we define the Poisson bracket by $\{\phi, \psi\} = \sum_{j=0}^{2} \left(\frac{\partial \phi}{\partial \xi_j} \frac{\partial \psi}{\partial x_j} - \frac{\partial \phi}{\partial x_j} \frac{\partial \psi}{\partial \xi_j} \right)$. We assume that the coefficients μ, λ, ρ and Ω, ω satisfy the following conditions:

Condition 1.1. *There exists a function $\psi \in C^2(\overline{Q})$ such that (i), (ii) and (1.6) hold:*

(i)

$$\{p_k, \{p_k, \psi\}\}(x,\xi) > 0, \quad \forall k \in \{1,2\} \tag{1.4}$$

if $\xi \in \mathbb{R}^3 \setminus \{0\}$ and $x \in \overline{Q \setminus Q_\omega}$ satisfy $p_k(x,\xi) =< \nabla_\xi p_k, \nabla_x \psi >= 0$.

(ii)

$$\frac{1}{2is}\{p_k(x, \xi - is\nabla_x \psi(x)), p_k(x, \xi + is\nabla_x\psi(x))\} > 0 \tag{1.5}$$

if $\xi \in \mathbb{R}^3 \setminus \{0\}$, $s > 0$ and $x \in \overline{Q \setminus Q_\omega}$ satisfy $p_k(x, \xi + is\nabla_x\psi(x))$ $=< \nabla_\xi p_k(x, \xi + is\nabla_x\psi(x)), \nabla_x \psi >= 0$.

On the lateral boundary we assume

$$p_1(x,\nabla_x\psi)<0,\quad \forall x\in\overline{(0,T)\times(\partial\Omega\setminus\partial\omega)},\quad \frac{\partial\psi}{\partial\vec n}<0\quad \textit{on } (0,T)\times\overline{(\partial\Omega\setminus\partial\omega)},$$
$$\frac{\partial\psi}{\partial\vec t}\neq 0\quad \textit{on } \overline{[0,T]\times(\partial\Omega\setminus\partial\omega)}. \tag{1.6}$$

Using ψ in Condition 1.1, we introduce the function $\phi(x)$ by

$$\phi(x)=e^{\tau\psi(x)},\quad \tau>1, \tag{1.7}$$

where the parameter τ will be fixed below. In order to deal with surface waves, we additionally need Condition 1.2 on the function ψ. We formulate that assumptions below as (1.23), and for the statement, we need boundary differential operators by means of a new local coordinate.

For an arbitrarily fixed point $(x_1^0,x_2^0)\in\partial\Omega$, we set $\widehat{x}_1=x_1-x_1^0$ and $\widehat{x}_2=x_2-x_2^0$. We consider (1.1) and (1.2) in the new coordinates $(\widehat{x_1},\widehat{x_2})$. Since (1.1) and (1.2) are invariant with respect to the translation by the constant vector (x_1^0,x_2^0), we use the same notations x_1,x_2 instead of $\widehat{x_1}$, $\widehat{x_2}$. Therefore we may assume that $(0,0)\in\partial\Omega$ and that locally near $(0,0)$, the boundary $\partial\Omega$ is given by an equation $x_2-\ell(x_1)=0$, where $\ell=\ell(x_1)$ is a C^3-function. Moreover, since the function $\widetilde{\mathbf{u}}=\mathcal{O}\mathbf{u}(x_0,\mathcal{O}^{-1}\widetilde{x})$ satisfies system (1.1) and (1.2) with $\widetilde{\mathbf{f}}=\mathcal{O}\mathbf{f}(x_0,\mathcal{O}^{-1}\widetilde{x})$ for any orthogonal matrix $\mathcal{O}$, we may assume that

$$\ell'(0)\equiv\frac{d\ell}{dx_1}(0)=0.$$

We make the change of variables $y=(y_0,y_1,y_2)=Y(x)\equiv(x_0,x_1,x_2-\ell(x_1))$.

Then we reduce equations (1.1) to

$$\begin{aligned} P_1(y,D)\mathbf{u} \equiv \rho\frac{\partial^2 u_1}{\partial y_0^2} - \mu\left\{\frac{\partial^2 u_1}{\partial y_1^2} - 2\ell'(y_1)\frac{\partial^2 u_1}{\partial y_1\partial y_2} + (1+|\ell'(y_1)|^2)\frac{\partial^2 u_1}{\partial y_2^2}\right\} \\ +\mu\ell''(y_1)\frac{\partial u_1}{\partial y_2} - (\lambda+\mu)\frac{\partial}{\partial y_1}\left(\operatorname{div}\mathbf{u} - \frac{\partial u_1}{\partial y_2}\ell'\right) + (\lambda+\mu)\frac{\partial}{\partial y_2}\left(\operatorname{div}\mathbf{u} - \frac{\partial u_1}{\partial y_2}\ell'\right)\ell' \\ +\widetilde{K}_1(y,D)\mathbf{u} = f_1 \qquad \text{in } \mathcal{G}, \end{aligned} \tag{1.8}$$

$$\begin{aligned} P_2(y,D)\mathbf{u} \equiv \rho\frac{\partial^2 u_2}{\partial y_0^2} - \mu\left\{\frac{\partial^2 u_2}{\partial y_1^2} - 2\ell'(y_1)\frac{\partial^2 u_2}{\partial y_1\partial y_2} + (1+|\ell'(y_1)|^2)\frac{\partial^2 u_2}{\partial y_2^2}\right\} \\ +\mu\ell''(y_1)\frac{\partial u_2}{\partial y_2} - (\lambda+\mu)\frac{\partial}{\partial y_2}\left(\operatorname{div}\mathbf{u} - \frac{\partial u_1}{\partial y_2}\ell'\right) + \widetilde{K}_2(y,D)\mathbf{u} = f_2 \qquad \text{in } \mathcal{G}, \end{aligned} \tag{1.9}$$

where we set

$$\mathcal{G} = \{y;\, y_2 \ge 0, y \in Y((0,T)\times B_\varepsilon(0,0))\}$$

with some $\varepsilon > 0$, and we keep the same notations P_1, P_2, $\mathbf{u}, \mathbf{f}$ after the change of variables, and $\widetilde{K}_j(y,D)$ are first order differential operators with C^1-coefficients. We set $P(y,D) = (P_1(y,D), P_2(y,D))$.

The stress boundary condition in (1.2) has the form

$$\begin{aligned} n_1\left\{\lambda\left(\frac{\partial u_1}{\partial y_1} + \frac{\partial u_1}{\partial y_2}(-\ell') + \frac{\partial u_2}{\partial y_2}\right) + 2\mu\left(\frac{\partial u_1}{\partial y_1} + \frac{\partial u_1}{\partial y_2}(-\ell')\right)\right\} \\ +n_2\mu\left(\frac{\partial u_2}{\partial y_1} + \frac{\partial u_2}{\partial y_2}(-\ell') + \frac{\partial u_1}{\partial y_2}\right) = g_1, \end{aligned} \tag{1.10}$$

$$n_1\mu\left\{\frac{\partial u_1}{\partial y_2} + \frac{\partial u_2}{\partial y_1} + \frac{\partial u_2}{\partial y_2}(-\ell')\right\} + n_2\left\{\lambda\left(\frac{\partial u_1}{\partial y_1} + \frac{\partial u_1}{\partial y_2}(-\ell') + \frac{\partial u_2}{\partial y_2}\right) + 2\mu\frac{\partial u_2}{\partial y_2}\right\} = g_2. \tag{1.11}$$

Here we use the same notations n_1, n_2 after the change of the variables.

We can solve system (1.10) and (1.11) with respect to $\left(\frac{\partial u_1}{\partial y_2}, \frac{\partial u_2}{\partial y_2}\right)$ in the form:

$$\begin{pmatrix} \frac{\partial u_1}{\partial y_2} \\ \frac{\partial u_2}{\partial y_2} \end{pmatrix} = A(y_1)\begin{pmatrix} \frac{\partial u_1}{\partial y_1} \\ \frac{\partial u_2}{\partial y_1} \end{pmatrix} + \widetilde{A}(y_1)\mathbf{g}, \quad A(0) = \begin{pmatrix} 0 & -1 \\ -\frac{\lambda}{\lambda+2\mu}(0,0) & 0 \end{pmatrix}, \qquad y \in \partial\mathcal{G}, \tag{1.12}$$

and $\widetilde{A}(y_1)$ is a C^2 matrix-valued function. By A_1 and $\widetilde{A_1}$, we denote the first rows of the matrices A and $\widetilde{A}$ respectively, and the second by A_2 and $\widetilde{A_2}$: $A_j = (a_{j1}, a_{j2})$ and $\widetilde{A_j} = (\widetilde{a_{j1}}, \widetilde{a_{j2}})$, $j = 1, 2$.

After the change of variables, the functions $z_1 \equiv \mathrm{rot}\,\mathbf{u} = \partial_{x_1} u_2 - \partial_{x_2} u_1$ and $z_2 \equiv \mathrm{div}\,\mathbf{u}$ have the form

$$z_1(y) = \frac{\partial u_2}{\partial y_1} - \frac{\partial u_2}{\partial y_2}\ell'(y_1) - \frac{\partial u_1}{\partial y_2}, \quad z_2(y) = \frac{\partial u_1}{\partial y_1} + \frac{\partial u_2}{\partial y_2} - \frac{\partial u_1}{\partial y_2}\ell'(y_1).$$

Using (1.12), we can transform them as follows:

$$\mathrm{rot}\,\mathbf{u} = z_1(y) = \frac{\partial u_2}{\partial y_1} - \ell'(y_1)A_2(y_1)\frac{\partial \mathbf{u}}{\partial y_1} - A_1(y_1)\frac{\partial \mathbf{u}}{\partial y_1} - \ell'(y_1)\widetilde{A}_2(y_1)\mathbf{g} - \widetilde{A}_1(y_1)\mathbf{g}$$
$$\equiv b_{11}(y_1, D')u_1 + b_{12}(y_1, D')u_2 + \widetilde{C}_1(y_1)\mathbf{g}, \quad y \in \partial\mathcal{G}, \tag{1.13}$$

where

$$b_{11}(y_1, \xi) = i(-\ell'(y_1)a_{21}(y_1) - a_{11}(y_1))\xi_1, \quad b_{12}(y_1, \xi) = i(1 - a_{22}(y_1)\ell'(y_1) - a_{12}(y_1))\xi_1. \tag{1.14}$$

For the function $z_2(y)$, we have

$$\mathrm{div}\,\mathbf{u}(y) = z_2(y) = \frac{\partial u_1}{\partial y_1} + A_2(y_1)\frac{\partial \mathbf{u}}{\partial y_1} - \ell'(y_1)A_1(y_1)\frac{\partial \mathbf{u}}{\partial y_1}$$
$$+\widetilde{A}_2(y_1)\mathbf{g} - \widetilde{A}_1(y_1)\mathbf{g}\ell'(y_1) \equiv b_{21}(y_1, D')u_1 + b_{22}(y_1, D')u_2 + \widetilde{C}_2(y_1)\mathbf{g}, \quad \forall y \in \partial\mathcal{G}, \tag{1.15}$$

where

$$b_{21}(y_1, \xi) = i(\xi_1 + a_{21}(y_1)\xi_1 - \ell'(y_1)a_{11}(y_1)\xi_1), \quad b_{22}(y_1, \xi) = i(a_{22}(y_1)\xi_1 - a_{12}(y_1)\xi_1\ell'(y_1)), \tag{1.16}$$

and $\widetilde{C}_j$ are C^2 matrix-valued functions.

Denote

$$p_\beta(y,s,\xi_0,\xi_1,\xi_2) = -\rho(\xi_0 + is\partial_{y_0}\phi)^2$$

$$+\beta[(\xi_1 + is\partial_{y_1}\phi)^2 - 2\ell'(\xi_1 + is\partial_{y_1}\phi)(\xi_2 + is\partial_{y_2}\phi) + (\xi_2 + is\partial_{y_2}\phi)^2|G|], \tag{1.17}$$

where $|G| = 1 + (\ell'(y_1))^2$, $\beta \in \{\mu, \lambda + 2\mu\}$ and s is a positive parameter.

The roots of $p_\beta = 0$ with respect to the variable ξ_2 are

$$\Gamma^\pm_\beta(y,s,\xi_0,\xi_1) = -is\partial_{y_2}\phi + \alpha^\pm_\beta(y,s,\xi_0,\xi_1), \tag{1.18}$$

$$\alpha^\pm_\beta(y,s,\xi_0,\xi_1) = \frac{(\xi_1 + is\partial_{y_1}\phi)\ell'(y_1)}{|G|} \pm \sqrt{r_\beta(y,s,\xi_0,\xi_1)}, \tag{1.19}$$

$$r_\beta(y,s,\xi_0,\xi_1) = \frac{(\rho(\xi_0 + is\partial_{y_0}\phi)^2 - \beta(\xi_1 + is\partial_{y_1}\phi)^2)|G| + \beta(\xi_1 + is\partial_{y_1}\phi)^2(\ell')^2}{\beta|G|^2}. \tag{1.20}$$

Henceforth, fixing $\zeta^* \in \mathbb{R}^3$ such that $|\zeta^*| = 1$ arbitrarily, and set $\mathbf{y}^* = (y_0, 0, 0)$ and $\gamma = (\mathbf{y}^*, \zeta^*)$. Suppose that $|r_\beta(\gamma)| \ge 2\widehat{\delta} > 0$. In [IY7], it was shown that there exists $\delta_0(\widehat{\delta}) > 0$ such that for all $\delta, \delta_1 \in (0, \delta_0)$, there exists a constant $C_1 > 0$ such that for one of the roots of the polynomial (1.17), which we denote by Γ^-_β, we have

$$-\operatorname{Im}\Gamma^-_\beta(y,s,\xi_0,\xi_1) \ge sC_1, \quad \forall y \in B_\delta(y_0,0,0),\ (s,\xi_0,\xi_1) \in \mathcal{O}(\delta_1). \tag{1.21}$$

Set

$$\mathcal{B}(y', s, D') = \begin{pmatrix} \mathcal{B}_{11}(y',s,D') & \mathcal{B}_{12}(y',s,D') \\ \mathcal{B}_{21}(y',s,D') & \mathcal{B}_{22}(y',s,D') \end{pmatrix}, \quad y \in \partial\mathcal{G}, \tag{1.22}$$

where

$$\mathcal{B}_{11}(y',s,D') = -\rho\mathbf{D}^2_{y_0} - \mu i\alpha^+_\mu(y',0,s,D')b_{11}(y_1,\mathbf{D}')$$

$$-(\lambda+2\mu)\{i\mathbf{D}_{y_1} - (\lambda+2\mu)(i\mathbf{D}_{y_1} - \ell'(y_1)i\alpha^+_{\lambda+2\mu}(y',0,s,D'))\}b_{21}(y_1,\mathbf{D}'),$$

$$\mathcal{B}_{12}(y',s,D') = -(\lambda+2\mu)\{i\mathbf{D}_{y_1} - (\lambda+2\mu)(i\mathbf{D}_{y_1} - \ell'(y_1)i\alpha^+_{\lambda+2\mu}(y',0,s,D'))\}b_{22}(y_1,\mathbf{D}')$$

$$-\mu i\alpha^+_\mu(y',0,s,D')b_{12}(y_1,\mathbf{D}'),$$

$$\mathcal{B}_{21}(y',s,D') = -(\lambda+2\mu)i\alpha^+_{\lambda+2\mu}(y',0,s,D')b_{21}(y_1,\mathbf{D}')$$

$$+\mu(i\mathbf{D}_{y_1} - \ell'(y_1)i\alpha^+_\mu(y',0,s,D'))b_{11}(y_1,\mathbf{D}'),$$

$$\mathcal{B}_{22}(y',s,D') = -\rho\mathbf{D}^2_{y_0} - (\lambda+2\mu)i\alpha^+_{\lambda+2\mu}(y',0,s,D')b_{22}(y_1,\mathbf{D}')$$

$$+\mu(i\mathbf{D}_{y_1} - \ell'(y_1)i\alpha^+_\mu(y',0,s,D'))b_{12}(y_1,\mathbf{D}').$$

Now we formulate a condition which allow us to observe the surface waves. For this purpose, we use the operator $\mathcal{B}$ which was introduced in the local coordinates. For an arbitrary point $x^0 \equiv (x^0_0, x^0_1, x^1_2) \in [0,T]\times\overline{(\partial\Omega\setminus\omega)}$, we rotate and translate Ω such that after the rotation and the translation, the normal vector to the boundary at x^0 is $(0,0,-1)$. Then by $\mathcal{Y}(x)$, we denote the transform involved with the rotation and the translation. Now we are ready to state the condition:

Condition 1.2. *Let $x \in [0,T]\times\overline{(\partial\Omega\setminus\omega)}$ be an arbitrary point and $y = \mathcal{Y}(x)$. We assume that*

$$Im\frac{1}{s}\sum_{j=0}^{1}\frac{\partial det\mathcal{B}(y',s,\xi_0,\xi_1)}{\partial y_j}\overline{\frac{\partial det\mathcal{B}(y',s,\xi_0,\xi_1)}{\partial \xi_j}} > 0 \qquad (1.23)$$

for any $(y,s,\xi_0,\xi_1) \in \{(y,s,\xi_0,\xi_1) \in \partial\mathcal{G}\times S^2;\ det\ \mathcal{B}(y',s,\xi_0,\xi_1) = 0,\ s>0,\ y_0 \in (0,T),\ Im\,\Gamma^+_\beta(y',0,s,\xi_0,\xi_1)/s \ge 0,\ \forall\beta\in\{\mu,\lambda+2\mu\},\ \xi_0 \ne 0\}$.

Now, under Conditions 1.1 and 1.2, we are ready to state our Carleman estimates:

Theorem 1.1. *We assume (1.3), Conditions 1.1 and 1.2. Let $\mathbf{f}\in\mathbf{H}^1(Q)$, $\mathbf{g}\in\mathbf{H}^{\frac{3}{2}}(\partial Q)$ and let the function ϕ be given by (1.7). Then there exists $\widehat{\tau}>0$ such* *that for any $\tau > \widehat{\tau}$, we can choose $s_0(\tau) > 0$ such that for any solution $\mathbf{u} \in \mathbf{H}^1(Q) \cap L^2(0,T;\mathbf{H}^2(\Omega))$ to problem (1.1) - (1.2), the following estimate holds true:*

$$\int_Q \sum_{|\alpha|=0}^{2} s^{4-2|\alpha|}|\partial_x^\alpha \mathbf{u}|^2 e^{2s\phi}dx + s\left\|\left(\mathbf{u}e^{s\phi}, \frac{\partial \mathbf{u}}{\partial \vec{n}}e^{s\phi}\right)\right\|^2_{\mathbf{H}^{\frac{3}{2},s}(\partial Q)\times \mathbf{H}^{\frac{1}{2},s}(\partial Q)}$$
$$\le C\left(\|\mathbf{f}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q)} + s\|\mathbf{g}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(\partial Q)} + \int_{Q_\omega}\sum_{|\alpha|=0}^{2} s^{4-2|\alpha|}|\partial_x^\alpha \mathbf{u}|^2 e^{2s\phi}dx\right), \quad \forall s \ge s_0(\tau), \tag{1.24}$$

where the constant $C = C(\tau) > 0$ is independent of s.

Assume in addition that

$$\partial_{x_0}\phi(0,\cdot) > 0 \quad \text{and} \quad \partial_{x_0}\phi(T,\cdot) < 0 \qquad \text{on } \overline{\Omega}. \tag{1.25}$$

Then we can show Carleman estimates whose right hand sides are estimated in $\mathbf{L}^2(Q)$ and $\mathbf{H}^{-1}(Q)$

Corollary 1.1. *We assume (1.3), (1.25), Conditions 1.1 and 1.2. Let $\mathbf{f} \in \mathbf{L}^2(Q)$, $\mathbf{g} = 0$ and let the function ϕ be given by (1.7). Then there exists $\widehat{\tau} > 0$ such that for any $\tau > \widehat{\tau}$, we can choose $s_0(\tau) > 0$ such that for any solution $\mathbf{u} \in \mathbf{H}^1(Q)$ to problem (1.1) - (1.2), the following estimate holds true:*

$$\|\mathbf{u}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q)} \le C(\|\mathbf{f}e^{s\phi}\|^2_{\mathbf{L}^2(Q)} + \|\mathbf{u}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q_\omega)}), \quad \forall s \ge s_0(\tau).$$

Here $C = C(\tau) > 0$ is independent of s.

Corollary 1.2. *We assume (1.3), (1.25), Conditions 1.1 and 1.2. Let $\mathbf{f} = \mathbf{f}_{-1} + \sum_{j=0}^2 \partial_{x_j}\mathbf{f}_j$ where $\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2 \in \mathbf{H}^1_0(Q)$, $\mathbf{f}_{-1} \in \mathbf{H}^{-1}(Q)$, $supp\,\mathbf{f}_{-1} \subset Q$, $\mathbf{g} = 0$, and let the function ϕ be given by (1.7). Then there exists $\widehat{\tau} > 0$ such that for any $\tau > \widehat{\tau}$, we can choose $s_0(\tau) > 0$ such that for any solution $\mathbf{u} \in \mathbf{H}^1(Q)$ to problem (1.1) -*

(1.2), the following estimate holds true:

$$\int_Q |\mathbf{u}|^2 e^{2s\phi} dx \le C \left(\|\mathbf{f}_{-1} e^{s\phi}\|^2_{\mathbf{H}^{-1}(Q)} + \sum_{j=0}^{2} \|\mathbf{f}_j e^{s\phi}\|^2_{\mathbf{L}^2(Q)} + \int_{Q_\omega} |\mathbf{u}|^2 e^{2s\phi} dx \right), \quad \forall s \ge s_0(\tau).$$

Here $C = C(\tau) > 0$ is independent of s.

The proofs of Corollaries 1.1 and 1.2 are similar to the proof of Theorems 2.2 and 2.3 in [IY7], and we omit them. As for parabolic operators, an H^{-1}-Carleman estimate is proved in [IY5] on the basis of [Im1].

Condition 1.1 is a usual condition for the pseudoconvexity, while the realization of Condition 1.2 requires us extra calculations. Next we try to find a simple sufficient condition which implies Condition 1.2.

For any fixed $\widetilde{x} \in \partial\Omega$, we define a cubic polynomial by

$$H(t) = t^3 - t^2 \left(8\frac{\mu}{\rho}\right)(\widetilde{x}) + t\left(\frac{24\mu^2}{\rho^2} - \frac{16\mu^3}{\rho^2(\lambda+2\mu)}\right)(\widetilde{x}) - \left(\frac{16\mu^3(\lambda+\mu)}{\rho^3(\lambda+2\mu)}\right)(\widetilde{x}). \tag{1.26}$$

Then we can directly verify that $H'(t) > 0$ if $t \le 0$, $H(0) < 0$, $H\left(\frac{\mu}{\rho}\right) = \frac{\mu^3}{\rho^3} > 0$ and $H''\left(\frac{8\mu}{3\rho}\right) = 0$. Therefore we can prove that $H(t) = 0$ possesses a unique simple root t in the interval $\left(0, \left(\frac{\mu}{\rho}\right)(\widetilde{x})\right)$ for any $\widetilde{x} \in \partial\Omega$, and by $\mathcal{C} = \mathcal{C}(\widetilde{x})$ we denote the root. Moreover the rest real roots are greater than $\left(\frac{\mu}{\rho}\right)(\widetilde{x})$ if there exists other real roots.

Remark. By means of the Cardano formula, we can give $\mathcal{C} = \mathcal{C}(\widetilde{x})$ explicitly. We set

$$\widetilde{a_1} = -8\frac{\mu}{\rho}, \quad \widetilde{a_2} = 24\frac{\mu^2}{\rho^2} - \frac{16\mu^3}{\rho^2(\lambda+2\mu)},$$
$$\widetilde{a_3} = -\frac{16\mu^3(\lambda+\mu)}{\rho^3(\lambda+2\mu)}.$$

That is, $H(t) = t^3 + \widetilde{a}_1 t^2 + \widetilde{a}_2 t + \widetilde{a}_3$. Moreover we put

$$\widetilde{b}_1 = \frac{\widetilde{a}_1{}^3}{27} - \frac{\widetilde{a}_1\widetilde{a}_2}{6} + \frac{\widetilde{a}_3}{2}, \quad \widetilde{b}_2 = \frac{1}{9}(3\widetilde{a}_2 - \widetilde{a}_1{}^2),$$
$$\widetilde{b}_3 = \widetilde{b}_1{}^2 + \widetilde{b}_2{}^3, \quad \widetilde{b}_4 = \mathrm{sign}\,(\widetilde{b}_1)|\widetilde{b}_2|^{\frac{1}{2}}.$$

Then we have:

$$\mathcal{C} = -\frac{\widetilde{a}_1}{3} - 2\widetilde{b}_4 \cosh\left(\frac{\theta}{3}\right)$$

if $\widetilde{b}_3 > 0$ and $\widetilde{b}_2 < 0$ where θ solves the equation $\cosh\theta = \frac{\widetilde{b}_1}{\widetilde{b}_4{}^3}$.

$$\mathcal{C} = -\frac{\widetilde{a}_1}{3} - 2\widetilde{b}_4 \sinh\left(\frac{\theta}{3}\right)$$

if $\widetilde{b}_2 > 0$ where θ solves the equation $\sinh\theta = \frac{\widetilde{b}_1}{\widetilde{b}_4{}^3}$.

If $\widetilde{b}_2 < 0$ and $\widetilde{b}_3 \le 0$, then we define $\mathcal{C}$ by the one of the three zeros of the polynomial H which belongs to the interval $[0, \mu/\rho]$: $t_1 = -\frac{\widetilde{a}_1}{3} - 2\widetilde{b}_4 \cos\left(\frac{\theta}{3}\right)$, $t_2 = -\frac{\widetilde{a}_1}{3} + 2\widetilde{b}_4 \cos\left(\frac{\pi}{3} - \frac{\theta}{3}\right)$, $t_3 = -\frac{\widetilde{a}_1}{3} + 2\widetilde{b}_4 \cos\left(\frac{\pi}{3} + \frac{\theta}{3}\right)$, where θ solves the equation $\cos\theta = \frac{\widetilde{b}_1}{\widetilde{b}_4{}^3}$.

In terms of $\mathcal{C}(\widetilde{x})$, we can state one sufficient condition:

Proposition 1.1. *Let $\psi \in C^2(\overline{\Omega})$, and*

$$\partial_{x_0}\psi(x) \pm \sqrt{\mathcal{C}(\widetilde{x})}\frac{\partial\psi}{\partial\vec{t}}(x) \ne 0 \tag{1.27}$$

for any $x \in (0,T) \times \overline{(\partial\Omega \setminus \partial\omega)}$. Then there exists $\tau_0 > 0$ such that Condition 1.2 holds for $\phi = e^{\tau\psi}$ if $\tau > \tau_0$.

In this section, first we give

Proof of Proposition 1.1.

For this, it suffices to prove : *Let* $\psi \in C^2(\overline{Q})$ *satisfy* $\partial_{y_1}\psi \neq 0$ *on* $\overline{Q}$, *and for* $(x_1^0, x_2^0) \in \overline{\partial\Omega \setminus \partial\omega}$, *let the local coordinate* $\widetilde{y} = (y_1, y_2)$ *be introduced by the local representation* $x_2 = \ell(x_1)$ *of* $\partial\Omega$. *We assume*

$$\partial_{y_0}\psi(\mathbf{y}^*) \pm \sqrt{\mathcal{C}(0,0)}\partial_{y_1}\psi(\mathbf{y}^*) \neq 0.$$

for any $(x_1^0, x_2^0) \in \overline{\partial\Omega \setminus \partial\omega}$ *and* $y_0 \in (0,T)$. *Then there exists* $\tau_0 > 0$ *such that Condition 1.2 holds for the function* $\phi = e^{\tau\psi}$ *if* $\tau > \tau_0$.

We recall that $\mathbf{y}^* = (y_0, 0, 0)$. The principal symbol of the operator $\mathcal{B}$ at the point $\mathbf{y}^*$ is

$$\mathcal{B}(\mathbf{y}^*, \zeta') = \begin{pmatrix} -\rho\widetilde{\zeta}_0^2 + 2\mu\widetilde{\zeta}_1^2 & -2\mu\alpha_\mu^+(\mathbf{y}^*, \zeta')\widetilde{\zeta}_1 \\ 2\mu\alpha_{\lambda+2\mu}^+(\mathbf{y}^*, \zeta')\widetilde{\zeta}_1 & -\rho\widetilde{\zeta}_0^2 + 2\mu\widetilde{\zeta}_1^2 \end{pmatrix},$$

where $\zeta' = (s, \xi_0, \xi_1)$ and $\widetilde{\zeta}_j = \xi_j + is\phi_{y_j}(\mathbf{y}^*)$. Obviously

$$\det \mathcal{B}(\mathbf{y}^*, \zeta') = \rho^2 \left(-\widetilde{\zeta}_0^2 + 2\frac{\mu}{\rho}\widetilde{\zeta}_1^2\right)^2 + 4\mu^2\alpha_{\lambda+2\mu}^+(\mathbf{y}^*, \zeta')\alpha_\mu^+(\mathbf{y}^*, \zeta')\widetilde{\zeta}_1^2. \tag{1.28}$$

We study the structure of the set

$$\Psi = \left\{\zeta' \in \mathbb{R}^3 \setminus \{0\};\, \det \mathcal{B}(\mathbf{y}^*, \zeta') = 0, \mathrm{Im}\frac{\Gamma_\mu^+(\mathbf{y}^*, \zeta')}{s} \geq 0,\ \mathrm{Im}\frac{\Gamma_{\lambda+2\mu}^+(\mathbf{y}^*, \zeta')}{s} \geq 0\right\}. \tag{1.29}$$

Then

Lemma 1.1. *Let (1.3) hold true and let* $\partial_{y_1}\psi(y^*) \neq 0$. *Then*

$$\Psi \subset \Psi_1 \cup \Psi_2, \qquad \mathit{dist}\,(\Psi_1, \Psi_2) > 0,$$

where $\Psi_1 = \{\zeta' = (s, \xi_0, \xi_1) \in S^2; \xi_0 + is\phi_{y_0}(\mathbf{y}^*) = 0\}$ *and* $\Psi_2 = \{\zeta' \in S^2; \xi_0 + is\phi_{y_0}(\mathbf{y}^*) = \pm\sqrt{\mathcal{C}}(\xi_1 + is\phi_{y_1}(\mathbf{y}^*)), Im\, \frac{\Gamma_\mu^+(\mathbf{y}^*,\zeta')}{s} \geq 0,\quad Im\, \frac{\Gamma_{\lambda+2\mu}^+(\mathbf{y}^*,\zeta')}{s} \geq 0\}$.

Proof. We can directly see from the definition that $\mathrm{dist}\,(\Psi_1, \Psi_2) > 0$. Taking into account that $(\alpha^+_\beta(\mathbf{y}^*, \zeta'))^2 = \frac{\rho}{\beta}\widetilde{\zeta}_0^2 - \widetilde{\zeta}_1^2$, we obtain

$$\Psi \subset \Bigg\{\zeta' \in S^2;\ \mathcal{F}(\widetilde{\zeta}_0^2, \widetilde{\zeta}_1^2) \equiv \left(\widetilde{\zeta}_0^2 - 2\frac{\mu}{\rho}\widetilde{\zeta}_1^2\right)^4$$

$$-16\frac{\mu^3}{\rho^2(\lambda+2\mu)}\widetilde{\zeta}_1^4\left(\widetilde{\zeta}_0^2 - \frac{\mu}{\rho}\widetilde{\zeta}_1^2\right)\left(\widetilde{\zeta}_0^2 - \frac{\lambda+2\mu}{\rho}\widetilde{\zeta}_1^2\right) = 0\Bigg\}.$$

We fix ρ and by $t_2 = t_2(\lambda(0,0), \mu(0,0))$ and $t_3 = t_3(\lambda(0,0), \mu(0,0))$ we denote the roots of $H(t)$ with $\widetilde{x} = (0,0)$ which are distinct from $\mathcal{C}(0,0)$. Then we have

$$t_2, t_3 > \frac{\mu(0,0)}{\rho(0,0)}$$

if they are real.

Therefore, noting that $\mathcal{F}(\widetilde{\zeta}_0^2, \widetilde{\zeta}_1^2) = \widetilde{\zeta}_0^2\widetilde{\zeta}_1^6 H(t)$ with $\widetilde{\zeta}_0^2 = t\widetilde{\zeta}_1^2$, we have only to prove that $\widetilde{\zeta}_0^2 = t_j\widetilde{\zeta}_1^2$ for $j = 2, 3$ are impossible, that is,

$$\det \mathcal{B}(\mathbf{y}^*, \zeta') \ne 0$$
$$\text{if} \quad (\xi_0 + is\phi_{y_0}(\mathbf{y}^*))^2 = t_j(\xi_1 + is\phi_{y_1}(\mathbf{y}^*))^2, \qquad \forall j \in \{2, 3\}. \tag{1.30}$$

Moreover we have only the two cases: $t_2, t_3 \in \mathbb{R}$ or $t_2, t_3 \notin \mathbb{R}$.

First we consider the case of $t_2, t_3 \in \mathbb{R}$. Really $\left(\widetilde{\zeta}_0^2 - 2\frac{\mu}{\rho}\widetilde{\zeta}_1^2\right)^2 = \widetilde{\zeta}_1^4\left(t_j - 2\frac{\mu}{\rho}\right)^2$ and

$$\alpha^+_\mu(\mathbf{y}^*, \zeta') = \sqrt{\widetilde{\zeta}_1^2(t_j\rho/\mu - 1)} = \mathrm{sign}(\phi_{y_1}(\mathbf{y}^*))\widetilde{\zeta}_1\sqrt{t_j\rho/\mu - 1},$$

where we used the fact that $t_j\rho/\mu - 1 > 0$. If $t_j\rho/(\lambda+2\mu) - 1 > 0$, then

$$\alpha^+_{\lambda+2\mu}(\mathbf{y}^*, \zeta') = \mathrm{sign}(\phi_{y_1}(\mathbf{y}^*))\widetilde{\zeta}_1\sqrt{t_j\rho/(\lambda+2\mu) - 1}$$

and we have $\det \mathcal{B}(\mathbf{y}^*, \zeta') = \widetilde{\zeta}_1^4\left\{\rho^2\left(t_j - 2\frac{\mu}{\rho}\right)^2 + 4\mu^2\sqrt{t_j\rho/\mu - 1}\sqrt{t_j\rho/(\lambda+2\mu) - 1}\right\} \ne$

0. If $t_j\rho/(\lambda+2\mu)-1<0$, then $\alpha^+_{\lambda+2\mu}(\mathbf{y}^*,\zeta')=\mathrm{sign}(\xi_1)i\widetilde{\zeta}_1\sqrt{-t_j\rho/(\lambda+2\mu)+1}$ and

$$\begin{aligned}&\det\mathcal{B}(\mathbf{y}^*,\zeta')\\=&\widetilde{\zeta}_1^4\left\{\rho^2\left(t_j-2\frac{\mu}{\rho}\right)^2+4\mu^2 i\mathrm{sign}(\xi_1)\mathrm{sign}(\phi_{y_1}(\mathbf{y}^*))\sqrt{t_j\rho/\mu-1}\sqrt{-t_j\rho/(\lambda+2\mu)+1}\right\}\neq 0.\end{aligned}$$

Next we will consider the case of $t_j\notin\mathbb{R}$, $j=2,3$. We set $\rho_0=\rho(0,0)$. Henceforth in place of $H(t)$ defined by (1.26), we consider

$$H(t)=t^3-\frac{8b}{\rho_0}t^2+t\left(\frac{24b^2}{\rho_0^2}-\frac{16b^3}{\rho_0^2(a+2b)}\right)-\frac{16b^3(a+b)}{\rho_0^3(a+2b)}\tag{1.31}$$

for all $(a,b)\in\mathbb{R}^2$ such that $a,b>0$. We note that if we set $a=\lambda(0,0)$ and $b=\mu(0,0)$, then this coincides with $H(t)$ by (1.26). Moreover, without fear of confusion, by $t_2=t_2(a,b)$ and $t_3=t_3(a,b)$, we may denote the roots which are not in the interval $\left[0,\frac{b}{\rho_0}\right]$ of $H(t)$ defined by (1.31). We set $\mathbf{z}=(z_0,z_1)\in\mathbb{C}^2$ and for factorizing of $\mathcal{F}(\widetilde{\zeta_0}^2,\widetilde{\zeta_1}^2)$ with $\lambda=a$, $\mu=b$ and $\rho=\rho_0$, we introduce two functions

$$\mathcal{H}^{\pm}(\mathbf{z})=\left(z_0^2-2\frac{b}{\rho_0}z_1^2\right)^2\pm 4\frac{b^2}{\rho_0^2}z_1^2\alpha^+_{a+2b}(\mathbf{z})\alpha^+_b(\mathbf{z}),$$

and $\alpha^+_\beta(\mathbf{z})=\sqrt{\frac{\rho_0}{\beta}z_0^2-z_1^2}$ with $\beta\in\{a+2b,b\}$. Henceforth we set

$$Dom\,\alpha^+_\beta=\left\{\mathbf{z}=(z_0,z_1);\ \frac{\rho_0}{\beta}z_0^2-z_1^2\notin\mathbb{R}_+\right\}.$$

For $\mathbf{z}\in Dom\,\alpha^+_\beta$, we take the complex root $\sqrt{\frac{\rho_0}{\beta}z_0^2-z_1^2}$ in such a way that $\mathrm{Im}\,\alpha^+_\beta(\mathbf{z})>0$. We set $Dom\,\mathcal{H}^+=Dom\,\mathcal{H}^-=Dom\,\alpha^+_b\cap Dom\,\alpha^+_{a+2b}$. It is sufficient to prove

$$\mathcal{H}^-(\mathbf{z}^{(j)}(\lambda(0,0),\mu(0,0),z_1))=0,\qquad\forall j\in\{2,3\}\tag{1.32}$$

if

$$\mathbf{z}^{(j)}(a,b,z_1)\equiv(\pm\sqrt{t_j(a,b)}z_1,z_1)\in Dom\,\mathcal{H}^+\quad\text{for } j\in\{2,3\},\ z_1\in\mathbb{C}.$$

In fact, if (1.32) will be proved, then we have (1.30). Because assume contrarily that $\det \mathcal{B}(y^*,\zeta') = 0$. Then, in terms of (1.28), we obtain $\widetilde{\zeta_0}^2 = \frac{2\mu(0,0)}{\rho_0}\widetilde{\zeta_1}^2$, which contradicts that $\widetilde{\zeta_0}^2 = t_j\widetilde{\zeta_1}^2$, $j=2,3$ where $t_j \notin \mathbb{R}$.

Proof of (1.32). Let $\delta_2 > 0$ be an arbitrary but fixed number. We introduce the sets

$$\Pi_1 = \{(a,b);\ a,b > \delta_2, \quad \text{there exists } z_1 \in \mathbb{C} \text{ with } |z_1| = 1 \text{ such that}$$
$$\mathcal{H}^+(\mathbf{z}^{(j)}(a,b,z_1)) = 0\, for\ j=2\ or\ j=3\}$$

and

$$\Pi_2 = \{(a,b);\ a,b > \delta_2,$$
$$\mathcal{H}^+(\mathbf{z}^{(j)}(a,b,z_1)) \ne 0 \quad \text{for any } z_1 \in \mathbb{C} \text{ with } |z_1| = 1 \text{ and any } j \in \{2,3\}$$
$$\text{such that } \mathbf{z}^{(j)}(a,b,z_1) \in Dom\mathcal{H}^+\}.$$

It suffices to prove that $\Pi_1 = \emptyset$. Let $(\widetilde{a},\widetilde{b}) \in \Pi_2$. Such a point exists because there exist a_0, b_0 such that $t_j(a_0,b_0) \in \mathbb{R}$ for $j=2,3$ and then we have already shown that $\det \mathcal{B}(\mathbf{y}^*,\zeta') \ne 0$, so that $(a_0,b_0) \in \Pi_2$ in terms of (1.28) and the definition of $\mathcal{H}^{\pm}$.

Assume contrarily that the set $\Pi_1 \ne \emptyset$. Then $\text{dist}\,((\widetilde{a},\widetilde{b}),\Pi_1) > 0$. There exist sequences $\{(a_n,b_n)\}_{n=1}^\infty \subset \Pi_1$ and $\{z_{1,n}\}_{n=1}^\infty \in \mathbb{C}$ such that $|z_{1,n}| = 1$, $\lim_{n\to\infty}(a_n,b_n) = (\widehat{a},\widehat{b})$, $\lim_{n\to\infty} z_{1,n} = \widehat{z}_1$,

$$\text{dist}\,((\widetilde{a},\widetilde{b}),(\widehat{a},\widehat{b})) = \text{dist}\,((\widetilde{a},\widetilde{b}),\Pi_1)$$

and $\mathcal{H}^+(\mathbf{z}^{j_1}{}_n) = 0$ where we set

$$\mathbf{z}^{j_1}{}_n = (\pm\sqrt{t_{j_1}(a_n,b_n)}z_{1,n}, z_{1,n}),$$

for some $j_1 \in \{2,3\}$ and some $z_{1,n} \in \mathbb{C}$ with $|z_{1,n}| = 1$.

Let us show that there exists $\widetilde{z}_1 \in \mathbb{C}$ such that $|z_1| = 1$ and $\widetilde{\mathbf{z}} \equiv (\pm\sqrt{t_{j_1}(\hat{a}, \widehat{b})}\widetilde{z}_1, \widetilde{z}_1) \in Dom\mathcal{H}^+$. Really if $\widehat{\mathbf{z}} \equiv (\pm\sqrt{t_{j_1}(\widehat{a}, \widehat{b})}\widehat{z}_1, \widehat{z}_1) \in Dom\mathcal{H}^+$, then we can take $\widetilde{\mathbf{z}} = \widehat{\mathbf{z}}$. On the other hand if $\widehat{\mathbf{z}} \notin Dom\mathcal{H}^\pm$, then

$$\widehat{\mathbf{z}} \notin Dom\,\alpha^+_{\widehat{b}} \quad \text{or} \quad \widehat{\mathbf{z}} \notin Dom\,\alpha^+_{\widehat{a}+2\widehat{b}}.$$

Let us assume for example that

$$\widehat{\mathbf{z}} \notin Dom\,\alpha^+_{\widehat{b}}.$$

Then $\sqrt{t_{j_1}(\widehat{a}, \widehat{b})\rho_0/\widehat{b} - 1} = re^{i\widehat{\theta}}$ and $\sqrt{t_{j_1}(\widehat{a}, \widehat{b})\rho_0/(\widehat{a} + 2\widehat{b}) - 1} = r_0 e^{i\widehat{\theta}_0}$. Since the imaginary part of t_{j_1} is not zero, we have $\widehat{\theta}_0 \ne \widehat{\theta} \pmod{2\pi}$. Then either

$$\mathcal{H}^+(\widetilde{\mathbf{z}}) = 0 \quad \text{for } \beta \in \{\widehat{b}, \widehat{a} + 2\widehat{b}\} \text{ with } \widetilde{\mathbf{z}} = (\pm\sqrt{t_{j_1}(\widehat{a}, \widehat{b})}\widetilde{z}_1, \widetilde{z}_1),\ \widetilde{z}_1 = e^{\frac{-i}{2}(\widehat{\theta}+\widehat{\theta}_0)},$$

or

$$\mathcal{H}^+(\widetilde{\mathbf{z}}) = 0 \quad \text{for } \beta \in \{\widehat{b}, \widehat{a} + 2\widehat{b}\} \text{ with } \widetilde{\mathbf{z}} = (\pm\sqrt{t_{j_1}(\widehat{a}, \widehat{b})}\widetilde{z}_1, \widetilde{z}_1),\ \widetilde{z}_1 = e^{\frac{-i}{2}(\widehat{\theta}+\widehat{\theta}_0)+i\pi}.$$

In the first case, we set $\widetilde{z}_1 = e^{\frac{-i}{2}(\widehat{\theta}+\widehat{\theta}_0)}$, and in the second case, $\widetilde{z}_1 = e^{\frac{-i}{2}(\widehat{\theta}+\widehat{\theta}_0)+i\pi}$.

On the other hand, we have

$$[(\widetilde{a}, \widetilde{b}), (\widehat{a}, \widehat{b})) \equiv \{\varepsilon(\widetilde{a}, \widetilde{b}) + (1-\varepsilon)(\widehat{a}, \widehat{b});\ 0 < \varepsilon \le 1\} \subset \Pi_2. \tag{1.33}$$

In fact, noting that $\{(a, b);\ a, b > \delta_2\}$ is convex and is contained in $\Pi_1 \cup \Pi_2$, we see that $[(\widetilde{a}, \widetilde{b}), (\widehat{a}, \widehat{b})) \subset \Pi_1 \cup \Pi_2$. Assume contrarily that there exists $(a^*, b^*) \in \Pi_1$ such that (a^*, b^*) is in the open segment $((\widetilde{a}, \widetilde{b}), (\widehat{a}, \widehat{b}))$. Then $\mathrm{dist}\,((\widetilde{a}, \widetilde{b}), (a^*, b^*)) < \mathrm{dist}\,((\widetilde{a}, \widetilde{b}), (\widehat{a}, \widehat{b})) = \mathrm{dist}\,((\widetilde{a}, \widetilde{b}), \Pi_1)$, which is a contradiction. Thus we have proved that $[(\widetilde{a}, \widetilde{b}), (\widehat{a}, \widehat{b})) \subset \Pi_2$.

We set $(a_\varepsilon, b_\varepsilon) = \varepsilon(\widetilde{a}, \widetilde{b}) + (1-\varepsilon)(\widehat{a}, \widehat{b})$. Then, for sufficiently small $\varepsilon > 0$, we have

$$\widetilde{\mathbf{z}}_\varepsilon = (\pm\sqrt{t_{\widehat{j}}(a_\varepsilon, b_\varepsilon)}\widetilde{z}_1, \widetilde{z}_1) \in Dom\mathcal{H}^+.$$

Then, by (1.33), we have $\lim_{\varepsilon\to+0}\mathcal{H}^-(\widetilde{\mathbf{z}}_\varepsilon) = \mathcal{H}^-(\widetilde{\mathbf{z}}) = 0$. Moreover by the choice of $\widetilde{\mathbf{z}}$, we have $\mathcal{H}^+(\widetilde{\mathbf{z}})$. Hence $\mathcal{H}^\pm(\widetilde{\mathbf{z}}) = 0$. This implies that $t_{j_1}(\widehat{a}, \widehat{b}) = \frac{2\widehat{b}}{\rho_0}$ and this is impossible because the left hand side is not real and the right hand side is real. Thus we have a contradiction. ■

Now we proceed to completion of

Proof of Proposition 1.1. Let $\mathcal{C} \in [0, \mu/\rho]$ be the zero of the polynomial H. By Lemma 1.2 for any $\zeta' \in S^2$, the set of all the possible solutions to the equation

$$\det\mathcal{B}(\mathbf{y}^*, \zeta') = 0, \quad \mathrm{Im}\,\frac{\Gamma^+_\beta(\mathbf{y}^*, \zeta')}{s} \ge 0,\ \forall\beta\in\{\mu, \lambda+2\mu\},\ \xi_0 \ne 0$$

is given by the formula

$$\xi_0 + is\phi_{y_0}(\mathbf{y}^*) \pm\sqrt{\mathcal{C}}(0)(\xi_1 + is\phi_{y_1}(\mathbf{y}^*)) = 0.$$

Let $(\xi_1^*, s^*) \in S^1$ be an arbitrary but fixed point. Let $z_0^* = \xi_0^* + is^*\phi_{y_0}(\mathbf{y}^*)$ with some $\xi_0^* \ne 0$ and $z_1^* = \xi_1^* + is^*\phi_{y_1}(\mathbf{y}^*)$ satisfy $z_0^* = \pm\sqrt{\mathcal{C}}z_1^*$ and $\frac{1}{s}\mathrm{Im}\,\Gamma^+_\beta(\mathbf{y}^*, s^*, z_0^*, z_1^*) \ge 0$ for $\beta \in \{\mu, \lambda+2\mu\}$. For fixed s and y_0, we consider $\det\mathcal{B}$ as a function of y_1 and two complex variables z_0, z_1: $J(y', z_0, z_1) = \det\mathcal{B}(y', s, z_0, z_1)$. Applying the implicit function theorem, we see that there exists a function $q(y', z_1)$ which is defined in a neighbourhood of $(\mathbf{y}^*, z_1^*)$ and analytic in z_1 such that $(y', q(y', z_1), z_1)$ is a solution to the equation $J(y', z_0, z_1) = 0$. Note that

$$q(\mathbf{y}^*, z_1) = \pm\sqrt{\mathcal{C}}z_1. \tag{1.34}$$

Set $r(y',z) = z_0 - q(y',z_1)$. Since

$$\det \mathcal{B}(y',s,\xi_0,\xi_1) = r(y',\xi_0,\xi_1) \times \frac{\det \mathcal{B}(y',s,\xi_0,\xi_1)}{r(y',\xi_0,\xi_1)} \equiv r(y',\xi_0,\xi_1)\widehat{r}(y',s,\xi_0,\xi_1),$$

and $\widehat{r}$ is smooth and not equal to zero, Condition 1.2 is equivalent to

$$\mathrm{Im}\frac{1}{s}\sum_{k=0}^{1}\overline{\frac{\partial r(y',\xi_0,\xi_1)}{\partial \xi_k}}\frac{\partial r(y',\xi_0,\xi_1)}{\partial y_k} > 0.$$

Computing the left hand side of this inequality, we obtain

$$\sum_{k=0}^{1}\mathrm{Im}\frac{1}{s}\overline{\frac{\partial r(y_0,0,\zeta)}{\partial \xi_k}}\frac{\partial r(y_0,0,\zeta)}{\partial y_k}$$
$$=\mathrm{Im}\,\frac{1}{s}\{is\phi_{y_0y_0}(\mathbf{y}^*) - q_{y_0}(y_0,0,\zeta_1) \pm\sqrt{\mathcal{C}}is\phi_{y_0y_1}(\mathbf{y}^*)$$
$$\mp\sqrt{\mathcal{C}}(is\phi_{y_0y_1}(\mathbf{y}^*) - q_{y_1}(y_0,0,\zeta_1) \mp\sqrt{\mathcal{C}}is\phi_{y_1y_1}(\mathbf{y}^*))\}$$
$$=\phi_{y_0y_0}(\mathbf{y}^*) \pm\sqrt{\mathcal{C}}\phi_{y_0y_1}(\mathbf{y}^*) + \mathcal{C}\phi_{y_1y_1}(\mathbf{y}^*) - \mathrm{Im}\,\frac{1}{s}(q_{y_0}(y_0,0,\zeta_1) \pm\sqrt{\mathcal{C}}q_{y_1}(y_0,0,\zeta_1)).$$

By the implicit function theorem, $q_{y_k}(y_0,0,\zeta_1) = -\frac{J_{y_k}(0,z_0^*,z_1^*)}{J_{z_0}(0,z_0^*,z_1^*)} = m_k(\xi_1^*+is^*\phi_{y_1}(\mathbf{y}^*))$, $k=0,1$, where a number m_k depends on the sign in formula (1.34). Using this formula we obtain

$$\mathrm{Im}\,\frac{1}{s}\overline{\frac{\partial r(y_0,0,\zeta)}{\partial \xi_k}}\frac{\partial r(y_0,0,\zeta)}{\partial y_k} = \tau\phi(\tau(\psi_{y_0}(\mathbf{y}^*) \pm\sqrt{\mathcal{C}}\psi_{y_1}(\mathbf{y}^*))^2 + \psi_{y_0y_0}(\mathbf{y}^*)$$
$$\pm\sqrt{\mathcal{C}}\psi_{y_0y_1}(\mathbf{y}^*) + \mathcal{C}\psi_{y_1y_1} - (m_0\psi_{y_0}(\mathbf{y}^*) \pm\sqrt{\mathcal{C}}m_1\psi_{y_1}(\mathbf{y}^*))).$$

Obviously under condition (1.27), for all sufficiently large $\tau > 0$, inequality (1.23) holds true at the point $\mathbf{y}^*$. The proof of Proposition 1.1 is complete. ■

Now we start the proof of Theorem 1.1. Our proof is based on decoupling of the Lamé system into the scalar hyperbolic equations for $\mathrm{rot}\,\mathbf{u}$ and $\mathrm{div}\mathbf{u}$. Then, applying to these equations the standard procedure (e.g., [Hö]) for obtaining a

Carleman estimate, and finally we carefully analyze the boundary integrals, which appear in the previous step by means of the microlocalization technique.

First we show that it suffices to consider only the case where the support of displacement $\mathbf{u}$ is located in a small neightbourhood of the point $\mathbf{y}^*$.

Lemma 1.2. *Under conditions of Theorem 1.1 it suffices to prove (1.24) under the assumption that*

$$supp\,\mathbf{u} \subset B_\delta(\mathbf{y}^*), \tag{1.35}$$

where $\delta > 0$ is an arbitrary small number and $\mathbf{y}^*$ is an arbitrary point in Q.

Proof of Lemma 1.2. Let us consider the finite covering of the domain $\overline{Q}$ by balls $B_\delta(\mathbf{y}_j^*)$. Let $e \in C_0^\infty(\overline{B_{2\delta}(0)})$ be a non-negative function such that $e|_{B_{\frac{5}{4}\delta}(0)} = 1$ and $e(x) < 1$ for all $x \in B_{2\delta}(0) \setminus \overline{B_{\frac{5}{4}\delta}(0)}$and let $\widetilde{e} \in C_0^\infty(\overline{B_{2\delta}(0)})$ be a non-negative function such that $\widetilde{e}|_{B_{\frac{9}{8}\delta}(0)} = 1$ and $\widetilde{e}(x) < 1$ for all $x \in B_{2\delta}(0) \setminus \overline{B_{\frac{9}{8}\delta}(0)}$. We set $e_j(x) = e(x - \mathbf{y}_j^*)$ and $\widetilde{e}_j(x) = e(x - \mathbf{y}_j^*)$. For the function $e_j\mathbf{u}$ we have the following boundary condition

$$\mathbb{B}(x, D)e_j\mathbf{u} = -[e_j, \mathbb{B}]\mathbf{u} + e_j\mathbf{g}. \tag{1.36}$$

Let $\psi_j(x) = \psi(x) + \epsilon(\widetilde{e}_j(x) - 1)$, $\phi_j(x) = e^{\tau\psi_j(x)}$ and $\epsilon \in (0, 1)$. The function ψ_j satisfies Conditions 1.1 and Condition 1.2 for all sufficiently small ϵ. Applying

Carleman estimate (1.24) to the equation $P(x,D)e_j\mathbf{u} = e_j\mathbf{f} - [e_j, P]\mathbf{u}$, we have

$$\int_Q \sum_{|\alpha|=0}^{2} s^{4-2|\alpha|}|\partial_x^\alpha \mathbf{u}|^2 e^{2s\phi}dx + s\left\|\left(\mathbf{u}e^{s\phi}, \frac{\partial \mathbf{u}}{\partial \vec{n}}e^{s\phi}\right)\right\|^2_{\mathbf{H}^{\frac{3}{2},s}(\partial Q)\times\mathbf{H}^{\frac{1}{2},s}(\partial Q)}$$

$$\leq C\sum_j \int_{B_\delta(\mathbf{y}_j)} \sum_{|\alpha|=0}^{2} s^{4-2|\alpha|}|\partial_x^\alpha \mathbf{u}e_j|^2 e^{2s\phi}dx + s\left\|\left(\mathbf{u}e^{s\phi}e_j, \frac{\partial \mathbf{u}}{\partial \vec{n}}e_je^{s\phi}\right)\right\|^2_{\mathbf{H}^{\frac{3}{2},s}(\partial Q)\times\mathbf{H}^{\frac{1}{2},s}(\partial Q))}$$

$$\leq C\sum_j \int_{B_{2\delta}(\mathbf{y}_j)} \sum_{|\alpha|=0}^{2} s^{4-2|\alpha|}|\partial_x^\alpha \mathbf{u}e_j|^2 e^{2s\phi_j}dx + s\left\|\left(\mathbf{u}e^{s\phi_j}e_j, \frac{\partial \mathbf{u}}{\partial \vec{n}}e_je^{s\phi_j}\right)\right\|^2_{\mathbf{H}^{\frac{3}{2},s}(\partial Q)\times\mathbf{H}^{\frac{1}{2},s}(\partial Q)}$$

$$\leq C\Bigg(\|\mathbf{f}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q)} + s\|\mathbf{g}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(\partial Q)} + s\sum_j \|[e_j, \mathbb{B}(x,D)]\mathbf{u}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(\partial Q)}$$

$$+\sum_j \|[e_j,P]\mathbf{u}e^{s\phi_j}\|^2_{\mathbf{H}^{1,s}(Q)} + \int_{Q_\omega}\sum_{|\alpha|=0}^{2} s^{4-2|\alpha|}|\partial_x^\alpha \mathbf{u}|^2 e^{2s\phi}dx\Bigg), \quad \forall s \geq s_0. \tag{1.37}$$

Note that $\|[e_j,P]\mathbf{u}e^{s\phi_j}\|^2_{\mathbf{H}^{1,s}(Q)} = \|[e_j,P]\mathbf{u}e^{s\phi_j}\|^2_{\mathbf{H}^{1,s}(B_{2\delta}(\mathbf{y}_j^*)\setminus B_{\frac{5}{4}\delta}(\mathbf{y}_j^*))}$. Moreover thanks to our choice of the functions ψ_j, we have $\phi_j(x) < \phi(x)$ for all $x \in \overline{B_{2\delta}(\mathbf{y}_j^*) \setminus B_{\frac{5}{4}\delta}(\mathbf{y}_j^*)}$. Therefore increasing s_0 if necessary, we have

$$\sum_j \|[e_j,P]\mathbf{u}e^{s\phi_j}\|^2_{\mathbf{H}^{1,s}(Q)} \leq \frac{1}{s}\int_Q \sum_{|\alpha|=0}^{2} s^{4-2|\alpha|}|\partial_x^\alpha \mathbf{u}|^2 e^{2s\phi}dx, \qquad \forall s \geq s_0.$$

Hence the second term at the right hand side of (1.37) can be absorbed into the left hand side, so that we obtain (1.24). Thus the proof of Lemma 1.2 is complete.

■

Lemma 1.3. *Let the hypotheses of Theorem 1.1 be fulfilled then*

$$\int_Q \sum_{|\alpha|=0}^{2} s^{4-2|\alpha|}|\partial_x^\alpha \mathbf{u}|^2 e^{2s\phi}dx \leq C_1\Bigg\{\|\mathbf{f}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q)} + s\left\|\frac{\partial\, div\, \mathbf{u}}{\partial \vec{n}}e^{s\phi}\right\|^2_{\mathbf{L}^2((0,T)\times\partial\Omega)}$$

$$+s\left\|\frac{\partial\, rot\, \mathbf{u}}{\partial \vec{n}}e^{s\phi}\right\|^2_{\mathbf{L}^2((0,T)\times\partial\Omega)}$$

$$+s\|(div\,\mathbf{u})e^{s\phi}\|^2_{\mathbf{H}^{1,s}((0,T)\times\partial\Omega)} + s\|(rot\,\mathbf{u})e^{s\phi}\|^2_{\mathbf{H}^{1,s}((0,T)\times\partial\Omega)}$$

$$+\int_{Q_\omega}\sum_{|\alpha|=0}^{2} s^{4-2|\alpha|}|\partial_x^\alpha \mathbf{u}|^2 e^{2s\phi}dx\Bigg\} \quad \forall s \geq s_0(\tau), \tag{1.38}$$

where the constant $C_1 > 0$ is independent of s.

In order to prove this lemma, we need the following proposition.

Proposition 1.2. *There exists $\widehat{\tau} > 1$ such that for any $\tau > \widehat{\tau}$ exist $s_0(\tau)$ that for any function $\mathbf{u} \in \mathbf{H}^2(Q)$*

$$\int_Q \left(\frac{1}{s} \sum_{i,j=1}^{2} |\partial^2_{x_i x_j} \mathbf{u}|^2 + s|\nabla_{x'}\mathbf{u}|^2 + s^3|\mathbf{u}|^2 \right) e^{2s\phi} dx$$
$$\le C_2 \Bigg\{ \|(rot\, \mathbf{u})\, e^{s\phi}\|^2_{H^1(Q)} + \|(div\, \mathbf{u}) e^{s\phi}\|^2_{H^1(Q)}$$
$$+ \left\| \left(\mathbf{u}, \frac{\partial \mathbf{u}}{\partial \vec{n}} \right) e^{s\phi} \right\|^2_{\mathbf{H}^{\frac{3}{2},s}(\partial Q) \times \mathbf{H}^{\frac{1}{2},s}(\partial Q)} + \int_{Q_\omega} (s|\nabla_{x'}\mathbf{u}|^2 + s^3|\mathbf{u}|^2) e^{2s\phi} dx \Bigg\}, \qquad \forall s \ge s_0(\tau). \tag{1.39}$$

The proof of this proposition is similar to the proof of Proposition 1.3 which is presented later, and we will omit the proof of Proposition 1.2.

Proof of Lemma 1.3. Without loss of generality, we may assume that $\rho \equiv 1$. Otherwise we introduce new coefficients $\mu_1 = \mu/\rho, \lambda_1 = \lambda/\rho$. It is known that the functions $\mathrm{rot}\,\mathbf{u}$ and $\mathrm{div}\,\mathbf{u}$ satisfy the equations

$$\partial^2_{x_0^2} \mathrm{rot}\ \mathbf{u} - \mu \Delta \mathrm{rot} \mathbf{u} = m_1 \quad \text{in}\, Q, \quad \partial^2_{x_0^2} \mathrm{div}\,\mathbf{u} - (\lambda + 2\mu)\Delta \mathrm{div}\,\mathbf{u} = m_2 \quad \text{in}\, Q, \tag{1.40}$$

$$m_1 = K_1 \mathrm{rot}\,\mathbf{u} + K_2 \mathrm{div}\,\mathbf{u} + \mathcal{K}_1 \mathbf{u} + \mathrm{rot}\,\mathbf{f}, \quad m_2 = K_3 \mathrm{rot}\,\mathbf{u} + K_4 \mathrm{div}\,\mathbf{u} + \mathcal{K}_2 \mathbf{u} + \mathrm{div}\,\mathbf{f},$$

where K_j are first order differential operators with L^∞- coefficients.

Thanks to the pseudoconvexity Condition 1.1 on the weight function ψ there exists $\widehat{\tau}$ such that for all $\tau > \widehat{\tau}$ we have (see e.g., [Ta])

$$s\|(\nabla \mathrm{rot}\ \mathbf{u}) e^{s\phi}\|^2_{\mathbf{L}^2(Q)} + s\|(\nabla \mathrm{div}\ \mathbf{u}) e^{s\phi}\|^2_{\mathbf{L}^2(Q)} + s^3\|(\mathrm{rot}\ \mathbf{u}) e^{s\phi}\|^2_{L^2(Q)} + s^3\|(\mathrm{div}\ \mathbf{u}) e^{s\phi}\|^2_{L^2(Q)}$$
$$\le C_3 \Bigg\{ \|\mathbf{f} e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q)} + s \left\| \frac{\partial\ \mathrm{div}\ \mathbf{u}}{\partial \vec{n}} e^{s\phi} \right\|^2_{L^2((0,T)\times\partial\Omega)} + s \left\| \frac{\partial\ \mathrm{rot}\ \mathbf{u}}{\partial \vec{n}} e^{s\phi} \right\|^2_{L^2((0,T)\times\partial\Omega)}$$
$$+ s\|(\mathrm{div} \mathbf{u}) e^{s\phi}\|^2_{H^{1,s}((0,T)\times\partial\Omega)} + s\|(\mathrm{rot} \mathbf{u}) e^{s\phi}\|^2_{H^{1,s}((0,T)\times\partial\Omega)}$$
$$+ \int_{Q_\omega} \sum_{|\alpha|=0}^{2} s^{4-2|\alpha|} |\partial_x^\alpha \mathbf{u}|^2 e^{2s\phi} dx \Bigg\}, \qquad \forall s \ge s_0(\tau), \tag{1.41}$$

where the constant $C_3 > 0$ is independent of s.

Using (1.39) and (1.41), we estimate the norm $\sum_{|\alpha|=0,\alpha=(0,\alpha_1,\alpha_2)}^{2} \|(\partial_x^\alpha \mathbf{u})e^{s\phi}\|^2_{\mathbf{L}^2(Q)}$ via the right hand side of inequality (1.41). Next using this estimate and equation (1.1), we obtain the estimate for the norm $\|(\partial^2_{x_0^2}\mathbf{u})e^{s\phi}\|^2_{\mathbf{L}^2(Q)}$ via the right hand side of (1.39). Finally we obtain the estimate for $\|(\partial^2_{x_0x_j}\mathbf{u})e^{s\phi}\|^2_{\mathbf{L}^2(Q)}$ and $s^2\|(\partial_{x_0}\mathbf{u})e^{s\phi}\|^2_{\mathbf{L}^2(Q)}$ using the interpolation arguments. Therefore combining these estimates with (1.41) we obtain (1.38). The proof of Lemma 1.3 is complete. ■

If $\mathbf{y}^* \in Q$, then we take $\delta > 0$ sufficiently small and we can assume $B_\delta(\mathbf{y}^*)\cap\partial Q = \emptyset$. In that case all the boundary integrals at the right hand side of (1.38) are zero and we have (1.24). Therefore, thanks to Lemma 1.2, we have to concentrate on the case $\mathbf{y}^* \in \partial Q$. From now on without loss of generality, we assume that $\mathbf{y}^* = (y_0^*, 0, 0)$. We need to estimate the boundary integrals at the right hand side of (1.38). In order to do that, it is convenient to use a weight function φ such that $\varphi|_{\partial\Omega} = \phi|_{\partial\Omega}$ and $\varphi(x) < \phi(x)$ for all x in a neighbourhood of ∂Q. We construct such a function φ locally near the boundary $\partial\Omega$:

$$\varphi(x) = e^{\tau\widetilde{\psi}(x)}, \quad \widetilde{\psi}(x) = \psi(x) - \frac{1}{\sqrt{N}}\ell_1(x) + N\ell_1^2(x), \tag{1.42}$$

where $N > 0$ is a large positive parameter and $\ell_1 \in C^3(\overline{\Omega})$ satisfies

$$\ell_1(x') > 0 \quad \forall x' \in \Omega, \quad \ell_1|_{\partial\Omega} = 0, \ \nabla\ell_1|_{\partial\Omega} \neq 0.$$

Denote $\Omega_N = \{x' \in \Omega | 0 < dist(x', \partial\Omega) < \frac{1}{N^2}\}$. Obviously for any fixed $\widehat{\epsilon} > 0$, there exists $N_0 > 0$ such that

$$\varphi(x) < \phi(x), \qquad \forall x \in [0,T] \times \Omega_N, \ N \in (N_0, \infty). \tag{1.43}$$

We have the following analogue of Proposition 1.2 for the weight function φ.

Proposition 1.3. *There exist $\widehat{\tau} > 1$ and $N_1 > 1$ such that for any $\tau > \widehat{\tau}$, there exists $s_0(\tau, N) > 0$ such that for any function $\mathbf{u} \in \mathbf{H}^2(Q)$ satisfying (1.8) and (1.9), we have*

$$N\|\mathbf{u}e^{s\varphi}\|^2_{\mathbf{H}^{2,s}(Q)} \le C_4\Bigg(N\|\mathbf{f}e^{s\varphi}\|^2_{\mathbf{L}^2(Q)} + s\|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(Q)}$$
$$+N\left\|\left(\mathbf{u}, \frac{\partial \mathbf{u}}{\partial \vec{n}}\right)e^{s\varphi}\right\|^2_{\mathbf{H}^{\frac{3}{2},s}(\partial Q)\times \mathbf{H}^{\frac{1}{2},s}(\partial Q)}\Bigg), \quad s \ge s_0(\tau, N), N \ge N_1$$

and C_4 is independent of N, s.

We give the proof of this proposition in Appendix I.

Our goal is to prove an analogue of (1.24) for the weight function φ instead of ϕ. We make the additional assumption

$$\operatorname{supp}\mathbf{u} \subset B_\delta(\mathbf{y}^*) \cap \mathcal{G} = [0, 1/N^2] \times \mathbb{R}^2. \tag{1.44}$$

Thanks to Lemma 1.2 we can work with the variable y instead of x. By (1.8) and (1.9) on the boundary $\partial\mathcal{G}$, we have

$$\frac{\partial^2 u_1}{\partial y_0^2} - \mu\left(\frac{\partial^2 u_1}{\partial y_1^2} - 2\ell'(y_1)\frac{\partial^2 u_1}{\partial y_1 \partial y_2}\right) + \mu\ell''(y_1)\frac{\partial u_1}{\partial y_2} - (\lambda+\mu)\frac{\partial}{\partial y_1}\left(\operatorname{div}\mathbf{u} - \frac{\partial u_1}{\partial y_2}\ell'(y_1)\right)$$
$$+(\lambda+\mu)\frac{\partial^2 u_1}{\partial y_1\partial y_2}\ell'(y_1)$$
$$=f_1 + \mu(1+|\ell'|^2)\frac{\partial^2 u_1}{\partial y_2^2} - \frac{\partial^2 u_2}{\partial y_2^2}(\lambda+\mu)\ell' + (\lambda+\mu)\frac{\partial^2 u_1}{\partial y_2^2}|\ell'|^2 - \widetilde{K}_1(y, D)\mathbf{u} \tag{1.45}$$

and

$$\frac{\partial^2 u_2}{\partial y_0^2} - \mu\left(\frac{\partial^2 u_2}{\partial y_1^2} - 2\ell'(y_1)\frac{\partial^2 u_2}{\partial y_1\partial y_2}\right) + \mu\ell''(y_1)\frac{\partial u_2}{\partial y_2} - (\lambda+\mu)\frac{\partial^2 u_1}{\partial y_1 \partial y_2}$$
$$=f_2 + \mu(1+|\ell'|^2)\frac{\partial^2 u_2}{\partial y_2^2} + (\lambda+\mu)\left(\frac{\partial^2 u_2}{\partial y_2^2} - \frac{\partial^2 u_1}{\partial y_2^2}\ell'\right) - \widetilde{K}_2(y, D)\mathbf{u}. \tag{1.46}$$

By (1.12) we know that

$$\frac{\partial u_1}{\partial y_2} = \left(A_1(y_1), \frac{\partial \mathbf{u}}{\partial y_1}\right) + (\widetilde{A}_1(y_1), \mathbf{g}), \quad \frac{\partial u_2}{\partial y_2} = \left(A_2(y_1), \frac{\partial \mathbf{u}}{\partial y_1}\right) + (\widetilde{A}_2(y_1), \mathbf{g}). \tag{1.47}$$

Hence

$$\frac{\partial^2 u_1}{\partial y_2 \partial y_1} = \left(A_1(y_1), \frac{\partial^2 \mathbf{u}}{\partial y_1^2}\right) + \left(A_1'(y_1), \frac{\partial \mathbf{u}}{\partial y_1}\right) + \frac{\partial}{\partial y_1}(\widetilde{A}_1(y_1), \mathbf{g}),$$
$$\frac{\partial^2 u_2}{\partial y_2 \partial y_1} = \left(A_2(y_1), \frac{\partial^2 \mathbf{u}}{\partial y_1^2}\right) + \left(A_2'(y_1), \frac{\partial \mathbf{u}}{\partial y_1}\right) + \frac{\partial}{\partial y_1}(\widetilde{A}_2(y_1), \mathbf{g}). \tag{1.48}$$

Using these equations we may transform (1.45) and (1.46) to

$$\begin{aligned} B_1(y', D')\mathbf{u} = {} & \frac{\partial^2 u_1}{\partial y_0^2} - \mu\left\{\frac{\partial^2 u_1}{\partial y_1^2} - 2\ell'(y_1)\left(\left(A_1(y_1), \frac{\partial^2 \mathbf{u}}{\partial y_1^2}\right) + \left(A_1'(y_1), \frac{\partial \mathbf{u}}{\partial y_1}\right)\right)\right\} \\ & -\mu\ell''(y_1)\left(A_1(y_1), \frac{\partial \mathbf{u}}{\partial y_1}\right) - (\lambda+\mu)\frac{\partial^2 u_1}{\partial y_1^2} - (\lambda+\mu)\left\{\left(A_2(y_1), \frac{\partial^2 \mathbf{u}}{\partial y_1^2}\right) + \left(A_2'(y_1), \frac{\partial \mathbf{u}}{\partial y_1}\right)\right\} \\ & +(\lambda+\mu)\left(A_1(y_1), \frac{\partial \mathbf{u}}{\partial y_1}\right)\ell''(y_1) + (\lambda+\mu)\left\{\left(A_1(y_1), \frac{\partial^2 \mathbf{u}}{\partial y_1^2}\right) + \left(A_1'(y_1), \frac{\partial \mathbf{u}}{\partial y_1}\right)\right\}\ell''(y_1) \\ & +(\lambda+\mu)\ell'(y_1)\left\{\left(A_1(y_1), \frac{\partial^2 \mathbf{u}}{\partial y_1^2}\right) + \left(A_1'(y_1), \frac{\partial \mathbf{u}}{\partial y_1}\right)\right\} \\ \equiv {} & f_1 + \mu(1+|\ell'|^2)\frac{\partial^2 u_1}{\partial y_2^2} - \frac{\partial^2 u_2}{\partial y_2^2}(\lambda+\mu)\ell' + (\lambda+\mu)\frac{\partial^2 u_1}{\partial y_2^2}|\ell'|^2 + \widetilde{\mathcal{K}_1}(y, D)\mathbf{u} + K_1\mathbf{g}. \end{aligned} \tag{1.49}$$

and

$$\begin{aligned} B_2(y', D')\mathbf{u} = {} & \frac{\partial^2 u_2}{\partial y_0^2} - \mu\left\{\frac{\partial^2 u_2}{\partial y_1^2} - 2\ell'(y_1)\left(\left(A_2(y_1), \frac{\partial^2 \mathbf{u}}{\partial y_1^2}\right) + \left(A_2'(y_1), \frac{\partial \mathbf{u}}{\partial y_1}\right)\right)\right\} \\ & -\mu\ell''(y_1)\left(A_2(y_1), \frac{\partial \mathbf{u}}{\partial y_1}\right) - (\lambda+\mu)\left\{\left(A_1(y_1), \frac{\partial^2 \mathbf{u}}{\partial y_1^2}\right) + \left(A_1'(y_1), \frac{\partial \mathbf{u}}{\partial y_1}\right)\right\} \\ \equiv {} & f_2 + \mu(1+|\ell'|^2)\frac{\partial^2 u_2}{\partial y_2^2} + (\lambda+\mu)\left(\frac{\partial^2 u_2}{\partial y_2^2} - \frac{\partial^2 u_1}{\partial y_2^2}\ell'\right) + \widetilde{\mathcal{K}_2}(y, D)\mathbf{u} + K_2\mathbf{g}. \end{aligned} \tag{1.50}$$

Set

$$B(y', \xi) = \begin{pmatrix} B_{11}(y, \xi) & B_{12}(y, \xi) \\ B_{21}(y, \xi) & B_{22}(y, \xi) \end{pmatrix},$$
$$Q(y') = \begin{pmatrix} \mu + (\lambda+2\mu)|\ell'(y_1)|^2 & -(\lambda+\mu)\ell'(y_1) \\ -(\lambda+\mu)\ell'(y_1) & (\lambda+2\mu) + \mu|\ell'(y_1)|^2 \end{pmatrix}.$$

In terms of the new notations, we may rewrite (1.49) and (1.50) as

$$\widetilde{B}(y', D') \equiv Q^{-1}B(y', D')\mathbf{u} = \frac{\partial^2 \mathbf{u}}{\partial y_2^2} + Q^{-1}\mathbf{f} + Q^{-1}K\mathbf{g}, \quad y \in \partial\mathcal{G}. \tag{1.51}$$

Note by (1.12) that the principal symbol of the operator $\widetilde{B}$ is given by

$$\widetilde{B}_2(\mathbf{y}^*, \xi) = \begin{pmatrix} \left\{-\xi_0^2 + \left(\lambda + 2\mu - \frac{\lambda(\lambda+\mu)}{\lambda+2\mu}\right)\xi_1^2\right\}/\mu & 0 \\ 0 & (-\xi_0^2 - \lambda\xi_1^2)/(\lambda+2\mu) \end{pmatrix}, \tag{1.52}$$

and we note that

$$Q(\mathbf{y}^*) = \begin{pmatrix} \mu & 0 \\ 0 & \lambda + 2\mu \end{pmatrix}.$$

In the y-coordinate, equations (1.40) for $\mathrm{rot}\,\mathbf{u}$ and $\mathrm{div}\,\mathbf{u}$ have the form

$$\begin{aligned} P_\mu(y, D)z_1 =& D_0^2 z_1 - \mu(D_2^2 z_1 - 2\ell'(y_1)D_1D_2z_1 + (1 + |\ell'(y_1)|^2)D_2^2 z_1) \\ & -\mu i\ell''(y_1)D_2z_1 = m_1, \end{aligned} \tag{1.53}$$

$$\begin{aligned} P_{\lambda+2\mu}(y, D)z_2 =& D_0^2 z_2 - (\lambda+2\mu)(D_2^2 z_2 - 2\ell'(y_1)D_1D_2z_2 + (1 + |\ell'(y_1)|^2)D_2^2 z_2) \\ & -(\lambda+2\mu)i\ell''(y_1)D_2z_2 = m_2. \end{aligned} \tag{1.54}$$

After the change of the coordinates, we use the same letters m_1, m_2 as in (1.40).

We consider a finite covering of the unit sphere $S^2 \equiv \{(s, \xi_0, \xi_1); s^2 + \xi_0^2 + \xi_1^2 = 1\}$. That is, $S^2 \subset \cup_{\nu=1}^{K(\delta_1)}\{(s, \xi_0, \xi_1) \in S^2; |\zeta - \zeta_\nu^*| < \delta_1\}$ where $\zeta_\nu^* \in S^2$, and by $\{\chi_\nu(\zeta)\}_{1\le\nu\le K(\delta_1)}$ we denote the corresponding partition of unity: $\sum_{\nu=1}^{K(\delta_1)} \chi_\nu(\zeta) = 1$ for any $\zeta \in S^2$ and $\mathrm{supp}\,\chi_\nu \subset \{\zeta \in S^2; |\zeta - \zeta_\nu^*| < \delta_1\}$. Henceforth we extend χ_ν to the set $\{\zeta; |\zeta| > 1\}$ as the homogeneous function of the order zero in $C^\infty(\mathbb{R}^3)$ such that

$$\mathrm{supp}\,\chi_\nu \subset \mathcal{O}(\delta_1) \equiv \left\{\zeta; \left|\frac{\zeta}{|\zeta|} - \zeta^*\right| < \delta_1\right\}.$$

Moreover we set

$$P_{\mu,s}(y, s, D) = P_\mu(y, \mathbf{D}), \qquad P_{\lambda+2\mu,s}(y, s, D) = P_{\lambda+2\mu}(y, \mathbf{D}).$$

Under some condition, we can factorize the operator $P_{\beta,s}$ as a product of two first order pseudodifferential operators.

Proposition 1.4. *Let $\beta \in \{\mu, \lambda + 2\mu\}$ and $|r_\beta(y,\zeta)| \ge \widehat{\delta} > 0$ for all $(y,\zeta) \in B_\delta(\mathbf{y}^*) \times \mathcal{O}(2\delta_1)$. Then we can factorize the operator $P_{\beta,s}$ into the product of two first order pseudodifferential operators:*

$$\begin{aligned} P_{\beta,s}\chi_\nu(s,D')V =& \beta|G|(D_{y_2} - \Gamma^-_\beta(y,s,D'))(D_{y_2} - \Gamma^+_\beta(y,s,D'))\chi_\nu(s,D')V \\ &+ T_\beta V, \end{aligned} \tag{1.55}$$

where $\operatorname{supp} V \subset B_\delta(\mathbf{y}^*) \cap \mathcal{G}$ and

$$T_\beta \in \mathcal{L}(L^2(0,1;H^{1,s}(\mathbb{R}^3)) \to L^2(0,1;L^2(\mathbb{R}^3))).$$

Let us consider the equation

$$(D_{y_2} - \Gamma^-_\beta(y,s,D'))\chi_\nu(s,D')V = q, \quad V|_{y_2=1} = 0, \quad \text{supp } V \subset B_\delta(\mathbf{y}^*) \cap \mathcal{G}.$$

For solutions of this problem we have an a priori estimate:

Proposition 1.5. *Let $\beta \in \{\mu, \lambda + 2\mu\}$ and $|r_\beta(y,\zeta)| \ge \widehat{\delta} > 0$ for all $(y,\zeta) \in B_\delta(\mathbf{y}^*) \times \mathcal{O}(2\delta_1)$. Then there exists a constant $C_5 > 0$ independent of N such that*

$$\|\sqrt{s}\chi_\nu(s,D')V|_{y_2=0}\|_{L^2(\mathbb{R}^2)} \le C_5\|q\|_{L^2(\mathcal{G})}. \tag{1.56}$$

The proofs of Proposition 1.4 and 1.5 can be found for example in [IY7]. Next we consider the equation

$$(D_{y_2} - \Gamma^+_\beta(y,s,D'))\chi_\nu(s,D')w = g, \quad w|_{y_2=1} = 0, \quad \text{supp } V \subset B_\delta(\mathbf{y}^*) \cap \mathcal{G}.$$

Proposition 1.6. *Let $\beta \in \{\mu, \lambda + 2\mu\}$ and $|r_\beta(y,\zeta)| \ge \widehat{\delta} > 0$ for all $(y,\zeta) \in B_\delta(\mathbf{y}^*) \times \mathcal{O}(\delta_1)$, $s^* \ne 0, \xi_0^* \ne 0$ and $\operatorname{supp}\chi_\nu \subset \mathcal{O}(\delta_1)$. Then for sufficiently small δ, δ_1 there exists a constant $C_6 > 0$ independent of N such that*

$$\|\chi_\nu(s,D')w\|_{H^{1,s}(\mathcal{G})} \le C_6\left(\frac{1}{\sqrt{s}}\|g\|_{H^{1,s}(\mathcal{G})} + s^{\frac{3}{4}}\|\chi_\nu w(\cdot,0)\|_{L^2(\partial\mathcal{G})}\right). \tag{1.57}$$

For the proof of this proposition, we can use the exactly same arguments as in [IP], and we give it in Appendix II for completeness.

Let $\beta > 0$ and $\widetilde{w} = \widetilde{w}(y)$ satisfy a scalar second order hyperbolic equation

$$P_{\beta,s}\widetilde{w} = q \quad \text{in } \mathcal{G}, \quad \frac{\partial \widetilde{w}}{\partial y_2}|_{y_2=1/N^2} = \widetilde{w}|_{y_2=1/N^2} = 0, \quad \text{supp}\, \widetilde{w} \subset B_\delta(\mathbf{y}^*) \times \mathbb{R}^1.$$

Let $P^*_{\beta,s}$ be the formally adjoint operator to $P_{\beta,s}$, where $\beta \in [\mu, \lambda+2\mu]$. Set $L_{+,\beta} = \frac{P_{\beta,s}+P^*_{\beta,s}}{2}$ and $L_{-,\beta} = \frac{P_{\beta,s}-P^*_{\beta,s}}{2}$. One can easily check that the principal part of the operator $L_{-,\beta}$ is given by formula

$$L_{-,\beta}\widetilde{w} = -2s\varphi_{y_0}\frac{\partial \widetilde{w}}{\partial y_0}$$
$$+\beta\left\{2s\varphi_{y_1}\frac{\partial \widetilde{w}}{\partial y_1} - 2s\ell'(y_1)\left(\varphi_{y_2}\frac{\partial \widetilde{w}}{\partial y_1} + \varphi_{y_1}\frac{\partial \widetilde{w}}{\partial y_2}\right) + 2s(1+(\ell'(y_1))^2)\varphi_{y_2}\frac{\partial \widetilde{w}}{\partial y_2}\right\}.$$

Obviously $L_{+,\beta}w + L_{-,\beta}w = q$. For almost all $s \in \mathbb{R}^1$ the following equality holds true:

$$\Sigma_\beta + \|L_{-,\beta}\widetilde{w}\|^2_{L^2(\mathcal{G})} + \|L_{+,\beta}\widetilde{w}\|^2_{L^2(\mathcal{G})} + \text{Re}\int_{\mathcal{G}}([L_{+,\beta}, L_{-,\beta}]\widetilde{w}, \widetilde{w})dy = \|q\|^2_{L^2(\mathcal{G})}, \quad (1.58)$$

where

$$\Sigma_\beta = \int_{\partial\mathcal{G}} \widetilde{p}_\beta(y, \nabla\varphi, (0,-1,0))(s\widetilde{p}_\beta(y, \nabla\widetilde{w}, \overline{\nabla\widetilde{w}}) - s^3 p_\beta(y, \nabla\varphi)|\widetilde{w}|^2)dy_0dy_1$$
$$+\text{Re}\int_{\partial\mathcal{G}} \widetilde{p}_\beta(y, \nabla\widetilde{w}, -\vec{e}_2)\overline{L_{-,\beta}\widetilde{w}}dy_0dy_1, \quad (1.59)$$

and

$$\widetilde{p}_\beta(y, \xi, \widetilde{\xi}) = \xi_0\widetilde{\xi}_0 - \beta(\xi_1\widetilde{\xi}_1 - \ell'(y_1)(\xi_1\widetilde{\xi}_2 + \xi_2\widetilde{\xi}_1) + (1+|\ell'(y_1)|^2)\xi_2\widetilde{\xi}_2).$$

We note that $\phi_{y_k}|_{\partial\mathcal{G}} = \varphi_{y_k}|_{\partial\mathcal{G}}$ for $k \in \{0,1\}$. Therefore on $\partial\mathcal{G}$ the function $\nabla_{y'}\varphi$ is independent of N and $|\nabla\phi(y') - \nabla\varphi(y')| \le C_7/\sqrt{N}$ where the constant C_7 is independent of N. In particular, for all sufficiently large N, we have (1.6). It is convenient for us to rewrite (1.58) in the form

$$\Sigma_\beta = \Sigma_\beta^{(1)} + \Sigma_\beta^{(2)},$$

$$\Sigma_\beta^{(1)} = \mathrm{Re}\int_{y_2=0} 2s\beta(\mathbf{y}^*)\frac{\partial \widetilde{w}}{\partial y_2}\overline{\left\{(\beta(\mathbf{y}^*)\frac{\partial \widetilde{w}}{\partial y_1}\varphi_{y_1}(\mathbf{y}^*) + \beta(\mathbf{y}^*)\frac{\partial \widetilde{w}}{\partial y_2}\varphi_{y_2}(\mathbf{y}^*) - \frac{\partial \widetilde{w}}{\partial y_0}\varphi_{y_0}(\mathbf{y}^*))\right\}}dy_0dy_1$$
$$+\int_{y_2=0} s\beta(\mathbf{y}^*)\varphi_{y_2}(\mathbf{y}^*)\left\{\left|\frac{\partial \widetilde{w}}{\partial y_0}\right|^2 - \beta(\mathbf{y}^*)\left(\left|\frac{\partial \widetilde{w}}{\partial y_1}\right|^2 + \left|\frac{\partial \widetilde{w}}{\partial y_2}\right|^2\right)\right.$$
$$\left. -s^2(\varphi_{y_0}^2(\mathbf{y}^*) - \beta(\mathbf{y}^*)(\varphi_{y_1}^2(\mathbf{y}^*) + \varphi_{y_2}^2(\mathbf{y}^*)))|\widetilde{w}|^2\right\}dy_0dy_1.$$

Then

$$|\Sigma_\beta^{(2)}| \le \epsilon(\delta)s\left\|\left(\frac{\partial \widetilde{w}}{\partial y_2}, \widetilde{w}\right)\right\|^2_{L^2(\mathcal{G})\times H^{1,s}(\mathcal{G})}, \tag{1.60}$$

where $\epsilon(\delta) \to 0$ as $\delta \to +0$. It is known (see e.g., [Im2]) that there exists a parameter $\widehat{\tau} > 1$ such that for any $\tau > \widehat{\tau}$ there exists $s_0(\tau)$ such that

$$\frac{1}{4}\|L_{-,\beta}\widetilde{w}\|^2_{L^2(\mathcal{G})} + \frac{1}{4}\|L_{+,\beta}\widetilde{w}\|^2_{L^2(\mathcal{G})} + ([L_{+,\beta}, L_{-,\beta}]\widetilde{w}, \widetilde{w})_{L^2(\mathcal{G})}$$
$$+C_8 s\|\widetilde{w}\|_{L^2(\partial\mathcal{G})}\|\partial_{y_2}\widetilde{w}\|_{L^2(\partial\mathcal{G})} \ge C_9 s\|\widetilde{w}\|^2_{H^{1,s}(\mathcal{G})} \quad \forall s \ge s_0(\tau), \tag{1.61}$$

where $C_9 > 0$ is independent of s. Combining (1.58) and (1.61), we arrive at

$$\frac{1}{4}\|L_{-,\beta}\chi_\nu\widetilde{w}\|^2_{L^2(\mathcal{G})} + \frac{1}{4}\|L_{+,\beta}\chi_\nu\widetilde{w}\|^2_{L^2(\mathcal{G})} + C_9 s\|\chi_\nu\widetilde{w}\|^2_{H^{1,s}(\mathcal{G})} + \Sigma_\beta$$
$$\le C_{10}(\|q\|^2_{L^2(\mathcal{G})} + s\|\widetilde{w}\|_{L^2(\partial\mathcal{G})}\|\partial_{y_2}\widetilde{w}\|_{L^2(\partial\mathcal{G})} + \|\widetilde{w}\|^2_{H^{1,s}(\partial\mathcal{G})}), \quad \forall s \ge s_0(\tau). \tag{1.62}$$

We argue microlocally to obtain the Carleman estimate for the function $\chi_\nu(\mathbf{u}e^{s\varphi})$. In Section 2, we consider the case where the support of the function χ_ν is in a neighbourhood of ζ^* such that $r_\mu(\mathbf{y}^*, \zeta^*) = 0$. The case $r_{\lambda+2\mu}(\mathbf{y}^*, \zeta^*) = 0$ is discussed in Section 3. In Section 4, we consider the case of $r_\mu(\mathbf{y}^*, \zeta^*) \ne 0$ and

$r_{\lambda+2\mu}(\mathbf{y}^*,\zeta^*) \ne 0$. Hence all the possible cases are covered. Finally, for completing the proof of Theorem 1.1, we combine all these microlocal estimates.

The rest part of this paper is organized as follows.

Section 2. Case $r_\mu = 0$

Section 3. Case $r_{\lambda+2\mu} = 0$

Section 4. Case $r_\mu \ne 0$ and $r_{\lambda+2\mu} \ne 0$

Section 5. Application to an inverse problem.

§2. Case $r_\mu = 0$.

In this section we treat the case where $\operatorname{supp}\chi_\nu \subset \mathcal{O}(\delta_1)$ and $r_\mu(\gamma) = 0$ for $\gamma = (\mathbf{y}^*,\zeta^*) \in \partial\mathcal{G} \times S^3$. Throughout this paper, we use the following notations:

$$\mathbf{v} = \mathbf{u}e^{s\varphi}, \quad \mathbf{w} = \mathbf{z}e^{s\varphi},$$

$$\mathbf{z} = (z_1, z_2) = (\operatorname{rot}\mathbf{u}, \operatorname{div}\mathbf{u}), \quad \mathbf{u} = (u_1, u_2), \quad \mathbf{v} = (v_1, v_2).$$

Henceforth $\widehat{v}(s,\xi',y_2)$ is the Fourier transform of $v(s,y_0,y_1,y_2)$ with respect to y_0, y_1, and we set $\mathbf{w}_\nu \equiv (w_{1,\nu}, w_{2,\nu}) = \chi_\nu(s,D')\mathbf{w}$.

This section is devoted to the proof of the following lemma.

Lemma 2.1. *Let $\gamma = (\mathbf{y}^*,\zeta^*) \in \partial\mathcal{G} \times S^3$ be a point such that $r_\mu(\gamma) = 0$ and $\operatorname{supp}\chi_\nu \subset \mathcal{O}(\delta_1)$. Then for all sufficiently small $\delta_1 > 0$, we have*

$$N\sum_{|\alpha|=0}^{2} s^{4-2|\alpha|}\|\partial_y^\alpha \mathbf{v}_\nu\|^2_{\mathbf{L}^2(\mathcal{G})} + s\left\|\left(\mathbf{v}_\nu, \frac{\partial \mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2 \mathbf{v}_\nu}{\partial y_2^2}\right)\right\|^2_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})}$$
$$\le C(\|\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{v}\|^2_{\mathbf{H}^{2,s}(\mathcal{G})} + \|\mathbf{f}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + s\|\mathbf{g}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}). \quad (2.1)$$

Proof. There exists a constant $C_1 > 0$ such that

$$|\xi_0^2 - s^2\varphi_{y_0}^2(\mathbf{y}^*) - \mu(\mathbf{y}^*)\xi_1^2 + \mu(\mathbf{y}^*)s^2\varphi_{y_1}^2(\mathbf{y}^*)|$$
$$\le C_1\delta_1(|\xi_0|^2 + |\xi_1|^2 + s^2), \quad \forall\zeta \in \mathcal{O}(\delta_1). \quad (2.2)$$

We remind that by (1.58)-(1.61) there exist constants $C_2, C_3 > 0$ such that

$$C_2 s\|w_{1,\nu}\|^2_{H^{1,s}(\mathcal{G})} + \Sigma^{(1)}_\mu \le C_3\|P_{\mu,s}w_1\|^2_{L^2(\mathcal{G})} + \epsilon(\delta)s\left\|\left(\frac{\partial w_{1,\nu}}{\partial y_2}, w_{1,\nu}\right)\right\|^2_{L^2(\partial\mathcal{G})\times H^{1,s}(\partial\mathcal{G})}, \tag{2.3}$$

where $\epsilon(\delta)\to 0$ as $\delta\to +0$. Note that $\Sigma^{(1)}_\mu$ can be written in the form

$$\begin{aligned}\Sigma^{(1)}_\mu = &\int_{\partial\mathcal{G}}\left(s\mu^2(\mathbf{y}^*)\varphi_{y_2}(\mathbf{y}^*)\left|\frac{\partial w_{1,\nu}}{\partial y_2}\right|^2 + s^3\mu^2(\mathbf{y}^*)\varphi^3_{y_2}(\mathbf{y}^*)|w_{1,\nu}|^2\right)d\Sigma\\
&+\mathrm{Re}\int_{\partial\mathcal{G}} 2s\mu(\mathbf{y}^*)\frac{\partial w_{1,\nu}}{\partial y_2}\overline{\left(\mu(\mathbf{y}^*)\varphi_{y_1}(\mathbf{y}^*)\frac{\partial w_{1,\nu}}{\partial y_1} - \varphi_{y_0}(\mathbf{y}^*)\frac{\partial w_{1,\nu}}{\partial y_0}\right)}d\Sigma\\
&+\int_{\partial\mathcal{G}} s\mu(\mathbf{y}^*)\varphi_{y_2}(\mathbf{y}^*)(\xi_0^2 - \mu(\mathbf{y}^*)\xi_1^2 - s^2\varphi^2_{y_0}(\mathbf{y}^*) + s^2\mu(\mathbf{y}^*)\varphi^2_{y_1}(\mathbf{y}^*))|\widehat{w}_{1,\nu}|^2 d\Sigma\\
=&J_1 + J_2 + J_3.\end{aligned} \tag{2.4}$$

If $r_{\lambda+2\mu}(\gamma) = 0$, then

$$\varphi_{y_0}(\mathbf{y}^*) = 0, \quad \varphi_{y_1}(\mathbf{y}^*) = 0, \quad \xi_0^* = \xi_1^* = 0, \ \ s^* = 1.$$

By (1.6) this is impossible.

Therefore $r_{\lambda+2\mu}(\gamma) \ne 0$ and the factorization (1.55) holds true. We set $V^+_{\lambda+2\mu} = (D_{y_2} - \Gamma^+_{\lambda+2\mu}(y,s,D'))w_{2,\nu}$. Then

$$P_{\lambda+2\mu}(y,s,D)w_{2,\nu} = (\lambda+2\mu)|G|(D_{y_2} - \Gamma^-_{\lambda+2\mu}(y,s,D'))V^+_{\lambda+2\mu} + T_{\lambda+2\mu}w_{2,\nu},$$

where $T_{\lambda+2\mu}\in\mathcal{L}(H^{1,s}(\mathcal{G}), L^2(\mathcal{G}))$. Therefore Proposition 1.5 immediately yields

$$\|(D_{y_2} - \Gamma^+_{\lambda+2\mu}(y,s,D'))w_{2,\nu}|_{y_2=0}\|_{L^2(\partial\mathcal{G})} \le C_4(\|P_{\lambda+2\mu,s}w_{2,\nu}\|_{L^2(\mathcal{G})} + \|\mathbf{w}\|_{(H^{1,s}(\mathcal{G}))^2}). \tag{2.5}$$

Now we have to estimate $\Sigma^{(1)}_\mu$. First we note that

$$\begin{aligned}\frac{\partial z_1}{\partial y_2}|_{y_2=0} = &\frac{\partial^2 u_2}{\partial y_1\partial y_2} - \frac{\partial^2 u_1}{\partial y_2^2} - \frac{\partial^2 u_2}{\partial y_2^2}\ell'(y_1) = \left(A_2(y_1), \frac{\partial^2\mathbf{u}}{\partial y_1^2}\right) + \left(A_2'(y_1), \frac{\partial\mathbf{u}}{\partial y_1}\right)\\
&+\frac{\partial}{\partial y_1}(\widetilde{A}_2(y_1), \mathbf{g}) - \frac{\partial^2 u_1}{\partial y_2^2} - \frac{\partial^2 u_2}{\partial y_2^2}\ell'(y_1),\end{aligned} \tag{2.6}$$

$$\frac{\partial z_2}{\partial y_2}|_{y_2=0} = \frac{\partial^2 u_1}{\partial y_1 \partial y_2} + \frac{\partial^2 u_2}{\partial y_2^2} - \frac{\partial^2 u_1}{\partial y_2^2}\ell'(y_1) = \left(A_1(y_1), \frac{\partial^2 \mathbf{u}}{\partial y_1^2}\right) + \left(A_1'(y_1), \frac{\partial \mathbf{u}}{\partial y_1}\right)$$
$$+\frac{\partial}{\partial y_1}(\widetilde{A}_1(y_1), \mathbf{g}) + \frac{\partial^2 u_2}{\partial y_2^2} - \frac{\partial^2 u_1}{\partial y_2^2}\ell'(y_1). \tag{2.7}$$

We may rewrite (2.6) and (2.7) as

$$e^{s\varphi}\begin{pmatrix}\frac{\partial z_1}{\partial y_2}\\ \frac{\partial z_2}{\partial y_2}\end{pmatrix} = -\mathcal{A}(y_1)\mathbf{D}^2_{y_1^2}\mathbf{v} - \widetilde{A}'(y_1)\mathbf{D}_{y_1}\mathbf{v} + K(y', s, D')\mathbf{g}e^{s\varphi} - I(y_1)\mathbf{D}^2_{y_2}\mathbf{v}.$$

where we used the notations

$$\mathcal{A}(y_1) = \begin{pmatrix} a_{21} & a_{22}\\ a_{11} & a_{12}\end{pmatrix}, \quad I(y_1) = \begin{pmatrix} -1 & -\ell'(y_1)\\ -\ell'(y_1) & 1\end{pmatrix}, \quad \mathcal{A}(\mathbf{y}^*) = \begin{pmatrix} -\frac{\lambda}{\lambda+2\mu} & 0\\ 0 & -1\end{pmatrix},$$

a_{ij} are the elements of the matrix A introduced in (1.12) and $K(y', s, D')$ stands for some first order differential operator. Therefore

$$I^{-1}e^{s\varphi}\begin{pmatrix}\frac{\partial z_1}{\partial y_2}\\ \frac{\partial z_2}{\partial y_2}\end{pmatrix} = -I^{-1}\mathcal{A}(y_1)\mathbf{D}^2_{y_1^2}\mathbf{v} - I^{-1}\widetilde{A}'(y_1)\mathbf{D}_{y_1}\mathbf{v} + I^{-1}\widetilde{K}(y', s, D')\mathbf{g}e^{s\varphi} - \mathbf{D}^2_{y_2}\mathbf{v}. \tag{2.8}$$

Using the definition of the operator $\Gamma^+_{\lambda+2\mu}(y, s, D')$, we have

$$\chi_\nu(s, D')\left(\frac{\partial z_2}{\partial y_2}e^{s\varphi}\right) = iV_\mu(y', 0) - [\chi_\nu, s\varphi_{y_2}]w_2$$
$$+i\alpha^+_{\lambda+2\mu}(y', 0, s, D')(b_{21}(y_1, \mathbf{D}_{y_1})v_{1,\nu} + b_{22}(y_1, \mathbf{D}_{y_1})v_{2,\nu} + [\chi_\nu, b_{21}+b_{22}]\mathbf{v} + \chi_\nu\widetilde{C}_2(y_1)\mathbf{g}e^{s\varphi}). \tag{2.9}$$

Here we recall that b_{21} and b_{22} are defined by (1.16). Substituting (2.9) into (2.8), we can obtain

$$\mathbf{D}^2_{y_2}\mathbf{v}_\nu = \widetilde{\mathbf{f}} - iI^{-1}\mathbf{D}_{y_2}w_{1,\nu}\vec{e}_1 + R(y', s, D')\mathbf{v}, \tag{2.10}$$

where we recall that $\vec{e_1} = (1, 0)$, and we set

$$\widetilde{\mathbf{f}} = I^{-1}\begin{pmatrix} s[\chi_\nu, \varphi_{y_2}]w_1\\ iV_\mu(y', 0) + i\alpha^+_{\lambda+2\mu}(y', 0, s, D')\widetilde{C}_2(y_1)\mathbf{g}e^{s\varphi} - [\chi_\nu, s\varphi_{y_2}]w_2 + [\chi_\nu, b_{21}+b_{22}]\mathbf{v}\end{pmatrix}$$
$$+[\chi_\nu, -I^{-1}\mathcal{A}(y_1)\mathbf{D}^2_{y_1} - I^{-1}\widetilde{A}'(y_1)\mathbf{D}_{y_1}]\mathbf{v} + \chi_\nu I^{-1}K(y', s, D)\mathbf{g}e^{s\varphi},$$

$$R(y',s,D')\mathbf{v}_\nu$$
$$=I^{-1}\begin{pmatrix} 0 & 0\\ -i\alpha^+_{\lambda+2\mu}(y',0,s,D')b_{21}(y_1,\mathbf{D}_{y_1}) & -i\alpha^+_{\lambda+2\mu}(y',0,s,D')b_{22}(y_1,\mathbf{D}_{y_1})\end{pmatrix}\mathbf{v}_\nu$$
$$-I^{-1}\mathcal{A}(y_1)\mathbf{D}^2_{y_1}\mathbf{v}_\nu - I^{-1}\widetilde{A}'(y_1)\mathbf{D}_{y_1}\mathbf{v}_\nu,$$

$$\mathbf{w}_\nu=(w_{1,\nu},w_{2,\nu}),\qquad \mathbf{v}_\nu=(v_{1,\nu},v_{2,\nu}),\qquad \mathbf{v}_\nu=\chi_\nu(s,D')\mathbf{v}.$$

By (1.51) and (2.10), we obtain

$$\chi_\nu(s,D')\widetilde{B}(y',\mathbf{D}')\mathbf{v}=\chi_\nu(s,D')\mathbf{f}e^{s\varphi}+\chi_\nu Q^{-1}K\mathbf{g}e^{s\varphi}$$
$$+(\widetilde{\mathbf{f}}-iI^{-1}\mathbf{D}_{y_2}w_{1,\nu}\vec{e}_1+R(y',s,D')\mathbf{v}_\nu). \tag{2.11}$$

Next we note that

$$\det(\widetilde{B}(\mathbf{y}^*,\xi^*+is^*\nabla\varphi(\mathbf{y}^*))-R(\mathbf{y}^*,s^*,\xi^*))=\mu^2(\xi_1^*+is^*\varphi_{y_1}(\mathbf{y}^*))^4. \tag{2.12}$$

Really

$$\widetilde{B}(\mathbf{y}^*,\xi^*+is^*\nabla\varphi(\mathbf{y}^*))=(\xi_1^*+is^*\varphi_{y_1}(\mathbf{y}^*))^2\begin{pmatrix}\frac{2(\lambda+\mu)}{\lambda+2\mu} & 0\\ 0 & -\frac{(\mu+\lambda)}{(\lambda+2\mu)}\end{pmatrix}. \tag{2.13}$$

Moreover

$$\begin{pmatrix} 0 & 0\\ -i\alpha^+_{\lambda+2\mu}(\mathbf{y}^*,s^*,\xi^*)b_{21}(\mathbf{y}^*,\xi^*+is^*\nabla\varphi(\mathbf{y}^*)) & -i\alpha^+_{\lambda+2\mu}(\mathbf{y}^*,s^*,\xi^*)b_{22}(\mathbf{y}^*,\xi^*+is^*\nabla\varphi(\mathbf{y}^*))\end{pmatrix}$$
$$=\begin{pmatrix} 0 & 0\\ i\mathrm{sign}(\xi_1^*)\sqrt{\frac{\lambda+\mu}{\lambda+2\mu}}\frac{2\mu}{\lambda+2\mu}(\xi^*+is^*\varphi_{y_1}(\mathbf{y}^*))^2 & 0\end{pmatrix}$$

and

$$I^{-1}\mathcal{A}(0)=\begin{pmatrix}\frac{\lambda}{\lambda+2\mu} & 0\\ 0 & -1\end{pmatrix}(\xi_1^*+is^*\varphi_{y_1}(\mathbf{y}^*))^2.$$

Thus

$$R(\mathbf{y}^*,s^*,\xi^*)=\begin{pmatrix}\frac{\lambda}{\lambda+2\mu} & 0\\ i\mathrm{sign}(\xi_1^*)2\mu\sqrt{\frac{\lambda+\mu}{\lambda+2\mu}} & 1\end{pmatrix}(\xi_1^*+is^*\varphi_{y_1}(\mathbf{y}^*))^2. \tag{2.14}$$

In terms of (2.13) and (2.14), we easily obtain (2.12). Set

$$S(y', s, \xi_0, \xi_1) = B(y', \xi' + is\nabla_{y'}\varphi(y)) - R(y', s, \xi').$$

Let $S(y', s, D')$ be the corresponding pseudodifferential operator:

$$\chi_\nu(s, D')S(y', s, D')\mathbf{v} = \mathbf{f}_\nu + \chi_\nu Q^{-1}K\mathbf{g}e^{s\varphi} + (\widetilde{\mathbf{f}} - iI^{-1}\mathbf{D}_{y_2}w_{1,\nu}\vec{e}_1).$$

Then

$$S(y', s, D')\mathbf{v}_\nu + [\chi_\nu, B(y, \mathbf{D})]\mathbf{v} = \mathbf{f}_\nu + \chi_\nu Q^{-1}K\mathbf{g}e^{s\varphi} + (\widetilde{\mathbf{f}} - iI^{-1}\mathbf{D}_{y_2}w_{1,\nu}\vec{e}_1). \quad (2.15)$$

Since we can direcly verify that $S(\mathbf{y}^*, s^*, \xi^*) \neq 0$, we have

$$\begin{aligned} & s\left\|\left(\mathbf{v}_\nu, \frac{\partial \mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2 \mathbf{v}_\nu}{\partial y_2^2}\right)\right\|^2_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})} \\ \le & C_5(s\|\mathbf{f}e^{s\varphi}\|^2_{\mathbf{L}^2(\partial\mathcal{G})} + s\|\mathbf{g}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + s\|\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} \\ & + J_1 + \|V_\mu(\cdot, 0)\|^2_{\mathbf{L}^2(\partial\mathcal{G})}). \end{aligned} \quad (2.16)$$

Note by (2.2) that for any $\epsilon > 0$ there exists $\delta_1(\epsilon) > 0$ such that

$$\begin{aligned} J_3 \le & \epsilon s\left\|\left(\mathbf{v}_\nu, \frac{\partial \mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2 \mathbf{v}_\nu}{\partial y_2^2}\right)\right\|^2_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})} \\ & + C_6 s(\|\mathbf{g}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}). \end{aligned} \quad (2.17)$$

In order to estimate the term J_2, we consider two cases. First we assume that $s^* \neq 0$. Then by (1.18)-(1.20) and $r_\mu(\gamma) = 0$, for given $\epsilon > 0$, there exists δ_0 such that if $\delta \in (0, \delta_0)$, then

$$|\xi_1\varphi_{y_1}(\mathbf{y}^*) - \xi_0\varphi_{y_0}(\mathbf{y}^*)| \le \epsilon|\xi'|, \quad \forall \xi \in \mathcal{O}(\delta_1). \quad (2.18)$$

By this inequality, we obtain

$$\begin{aligned} J_2 \le & \epsilon s\left\|\left(\mathbf{v}_\nu, \frac{\partial \mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2 \mathbf{v}_\nu}{\partial y_2^2}\right)\right\|^2_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})} \\ & + C_7 s(\|\mathbf{g}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}). \end{aligned} \quad (2.19)$$

Second let us assume that $s^* = 0$. We solve equation (2.15) with respect to the variable $\mathbf{v}_\nu$:

$$\mathbf{v}_\nu = S(y', s, D')^{-1}(-[\chi_\nu, B(y, \mathbf{D})]\mathbf{v}$$
$$+\chi_\nu Q^{-1}K\mathbf{g}e^{s\varphi} + \mathbf{f}_\nu + (\widetilde{\mathbf{f}} - iI^{-1}\mathbf{D}_{y_2}w_{1,\nu}\vec{e}_1)), \tag{2.20}$$

where the principal symbol of the operator $S(y', s, D')^{-1}$ is the inverse matrix to the matrix $S(y', s, \xi_0, \xi_1)$. Therefore on $\partial\mathcal{G}$ we have

$$\mu(\mathbf{y}^*)\varphi_{y_1}(\mathbf{y}^*)\frac{\partial w_{1,\nu}}{\partial y_1} - \varphi_{y_0}(\mathbf{y}^*)\frac{\partial w_{1,\nu}}{\partial y_0}$$
$$= i(\mu(\mathbf{y}^*)\varphi_{y_1}(\mathbf{y}^*)D_{y_1} - \varphi_{y_0}(\mathbf{y}^*)D_{y_0})(b_{11} + b_{12})(y_1, \mathbf{D}')$$
$$\times S(y', s, D')^{-1}(-[\chi_\nu, B(y, \mathbf{D})]\mathbf{v} + \mathbf{f}_\nu + \chi_\nu Q^{-1}K\mathbf{g}e^{s\varphi}$$
$$+(\widetilde{\mathbf{f}} - iI^{-1}\mathbf{D}_{y_2}w_{1,\nu}\vec{e}_1)). \tag{2.21}$$

By $\mathcal{M}$ we denote the pseudodifferential operator with the symbol

$$\mathcal{M}(y, s, \xi') = i(\mu(\mathbf{y}^*)\varphi_{y_1}(\mathbf{y}^*)\xi_1 - \varphi_{y_0}(\mathbf{y}^*)\xi_0)b_1(y_1, \xi' + is\nabla'\varphi)S(y', s, \xi)^{-1}I^{-1}\vec{ie}_1.$$

Since $b_{11}(\mathbf{y}^*, \xi') = 0$ and $b_{12}(\mathbf{y}^*, \xi') = 2i\xi_1$, we have $\mathrm{Re}\,\mathcal{M}(\mathbf{y}^*, s^*, \xi_0^*, \xi_1^*) = 0$. Therefore, by Gårding's inequality, we see

$$\mathrm{Re}\int_{\partial\mathcal{G}} \frac{\partial w_{1,\mu}}{\partial y_2}\overline{(\mu(\mathbf{y}^*)\varphi_{y_1}(\mathbf{y}^*)\partial_{y_1} - \varphi_{y_0}(\mathbf{y}^*)\partial_{y_0})(b_{11} + b_{12})(y_1, \mathbf{D}')S(y', s, D')^{-1}\vec{e}_1 w_{1,\nu}}d\Sigma$$
$$\ge -\epsilon\left\|\left(\mathbf{v}_\nu, \frac{\partial \mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2 \mathbf{v}_\nu}{\partial y_2^2}\right)\right\|^2_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})}. \tag{2.22}$$

On the other hand

$$s\|\mathrm{Re}(\mu(\mathbf{y}^*)\varphi_{y_1}(\mathbf{y}^*)\partial_{y_1} - \varphi_{y_0}(\mathbf{y}^*)\partial_{y_0})(b_{11} + b_{12})(y_1, \mathbf{D}')$$
$$\times S(y', s, D')^{-1}(-[\chi_\nu, B(y, \mathbf{D}) - Q\mathbf{D}^2_{y_2}]\mathbf{v}$$
$$+\mathbf{f}_\nu + \chi_\nu Q^{-1}K\mathbf{g}e^{s\varphi} + \vec{e}_2 iV^+_{\lambda+2\mu}(y', 0)\|^2_{L^2(\partial\mathcal{G})}$$
$$\le C_8(\|P_{\lambda+2\mu,s}w_{2,\nu}\|^2_{L^2(\mathcal{G})} + s\|\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + s\|\mathbf{f}e^{s\varphi}\|^2_{\mathbf{L}^2(\partial\mathcal{G})} + s\|\mathbf{g}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}). \tag{2.23}$$

Here we recall that $\vec{e}_2=(0,1)$. Inequalities (2.22) and (2.23) imply

$$J_2 \le \epsilon s \left\| \left(\mathbf{v}_\nu, \frac{\partial \mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2 \mathbf{v}_\nu}{\partial y_2^2}\right)\right\|^2_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times \mathbf{H}^{1,s}(\partial\mathcal{G})\times \mathbf{L}^2(\partial\mathcal{G})}$$
$$+C_9(s\|\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{f}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + s\|\mathbf{g}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}). \tag{2.24}$$

By (2.16), (2.17), (2.19) and (2.24), there exist constants $C_{10}>0$ and $C_{11}>0$ such that

$$\Sigma_\mu^{(1)} \ge C_{10} s \left\| \left(\mathbf{v}_\nu, \frac{\partial \mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2 \mathbf{v}_\nu}{\partial y_2^2}\right)\right\|^2_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times \mathbf{H}^{1,s}(\partial\mathcal{G})\times \mathbf{L}^2(\partial\mathcal{G})}$$
$$-C_{11}(s\|\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{f}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + s\|\mathbf{g}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}). \tag{2.25}$$

By (2.3), (2.4) and (2.25), we obtain

$$s\|w_{1,\nu}\|^2_{H^{1,s}(\mathcal{G})} + s \left\| \left(\mathbf{v}_\nu, \frac{\partial \mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2 \mathbf{v}_\nu}{\partial y_2^2}\right)\right\|^2_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times \mathbf{H}^{1,s}(\partial\mathcal{G})\times \mathbf{L}^2(\partial\mathcal{G})}$$
$$\le C_{12}(s\|\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{f}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + s\|\mathbf{g}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}). \tag{2.26}$$

By (1.61) and (1.62) with $\beta=\lambda+2\mu$, we have

$$s\|w_{2,\nu}\|^2_{H^{1,s}(\mathcal{G})} \le C_{13}(s\|\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{f}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + s\|\mathbf{g}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}). \tag{2.27}$$

Therefore combining (2.26) and (2.27), we obtain

$$s\|\mathbf{w}_\nu\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + s \left\| \left(\mathbf{v}_\nu, \frac{\partial \mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2 \mathbf{v}_\nu}{\partial y_2^2}\right)\right\|^2_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times \mathbf{H}^{1,s}(\partial\mathcal{G})\times \mathbf{L}^2(\partial\mathcal{G})}$$
$$\le C_{14}(s\|\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{f}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + s\|\mathbf{g}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}).$$

This inequality and Proposition 1.3 imply (2.1). Thus the proof of Lemma 2.1 is complete. ■

§3. Case $r_{\lambda+2\mu}=0$.

In this section, we wil prove

Lemma 3.1. *Let $\gamma = (\mathbf{y}^*, \zeta^*) \in \partial\mathcal{G} \times S^3$ be a point such that $r_{\lambda+2\mu}(\gamma) = 0$ and $\operatorname{supp}\chi_\nu \subset \mathcal{O}(\delta_1)$. Then for all sufficiently small $\delta_1 > 0$, estimate (2.1) holds true.*

Proof. By (1.19) and (1.20), there exist $\delta_0 > 0$ and $C_1 > 0$ such that for all $\delta_1 \in (0, \delta_0)$ we have

$$\xi_0^2 \le C_1(\xi_1^2 + s^2), \qquad \forall \zeta \in \mathcal{O}(\delta_1). \tag{3.1}$$

We note that if $r_\mu(\gamma) = 0$, then $\xi_0^* = \xi_1^* = 0$, $s^* = 1$ and $\varphi_{y_0}(\mathbf{y}^*) = \varphi_{y_1}(\mathbf{y}^*) = 0$. By (1.6) this is impossible. Therefore $r_\mu(\gamma) \neq 0$ must be true. Then there exists a constant $C_2 > 0$ such that

$$-\operatorname{Im}\Gamma_\mu^\pm(y,\zeta) \ge C_2 s, \qquad \forall (y,\zeta) \in B_\delta(\mathbf{y}^*) \times \mathcal{O}(\delta_1),$$

provided that $|\delta| + |\delta_1|$ is sufficiently small. We set $V_\mu^\pm = (D_{y_2} - \Gamma_\mu^\pm(y, s, D'))w_{1,\nu}$. Then we can represent $P_{\mu,s}$ as

$$P_{\mu,s}(y, s, D)w_{1,\nu} = \mu|G|(D_{y_2} - \Gamma_\mu^-(y, s, D'))V_\mu^+ + T_\mu^\pm w_{1,\nu},$$

where $T_\mu^\pm \in \mathcal{L}(H^{1,s}(\mathcal{G}), L^2(\mathcal{G}))$. This decomposition and Proposition 1.5 immediately imply

$$\begin{aligned}&\|\sqrt{s}(D_{y_2} - \Gamma_\mu^\pm(y, s, D'))w_{1,\nu}|_{y_2=0}\|_{L^2(\partial\mathcal{G})}\\ \le& C_3(\|P_{\mu,s}w_{1,\nu}\|_{L^2(\mathcal{G})} + \|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})}).\end{aligned} \tag{3.2}$$

We consider the following two cases.

Case A. Assume that

$$s^* = 0.$$

Then by decreasing the parameter δ_1, we can assume that for some constant $C_4 > 0$

$$\xi_0^2 + s^2 \le C_4\xi_1^2, \qquad \forall \zeta \in \mathcal{O}(\delta_1). \tag{3.3}$$

We consider two subcases. First we assume that

$$\lim_{\zeta\to\zeta^*} \operatorname{Im} r_\mu(\mathbf{y}^*,\zeta)/|s| \neq 0.$$

Then, since $s^*=0$, we see that $\operatorname{Re} r_\mu(\mathbf{y}^*,\zeta^*)>0$. Set $\mathbf{I} = \text{sign} \lim_{\zeta\to\zeta^*} \operatorname{Im} r_\mu(\mathbf{y}^*,\zeta)/|s|$. For all $(y,\zeta)\in B_\delta(\mathbf{y}^*)\times\mathcal{O}(\delta_1)$, we have

$$\Gamma^+_\mu(\mathbf{y}^*,\zeta^*) = \mathbf{I}\sqrt{\operatorname{Re} r_\mu(\mathbf{y}^*,\zeta^*)}.$$

Therefore

$$\Gamma^+_\mu(\mathbf{y}^*,\zeta^*)(\mu(\mathbf{y}^*)\varphi_{y_1}(\mathbf{y}^*)\xi_1^* - \varphi_{y_0}(\mathbf{y}^*)\xi_0^*)>0.$$

Taking the parameters $\delta>0$ and $\delta_1>0$ sufficiently small, we obtain

$$\operatorname{Re}\Gamma^+_\mu(y,\zeta)(\mu(\mathbf{y}^*)\varphi_{y_1}(\mathbf{y}^*)\xi_1 - \varphi_{y_0}(\mathbf{y}^*)\xi_0)>0, \quad \forall (y,\zeta)\in B_\delta(\mathbf{y}^*)\times\mathcal{O}(\delta_1). \quad (3.4)$$

Let us estimate $\Sigma^{(1)}_\mu$ by (2.4). We have

$$\begin{aligned}
J_2 &= \operatorname{Re}\int_{\partial\mathcal{G}} 2s\mu(\mathbf{y}^*)\frac{\partial w_{1,\nu}}{\partial y_2}\overline{\left(\mu(\mathbf{y}^*)\frac{\partial w_{1,\nu}}{\partial y_1}\varphi_{y_1}(\mathbf{y}^*) - \frac{\partial w_{1,\nu}}{\partial y_0}\varphi_{y_0}(\mathbf{y}^*)\right)}d\Sigma\\
&=\operatorname{Re}\int_{\partial\mathcal{G}} 2s\mu(\mathbf{y}^*)i\Gamma^+_\mu(y,s,D')w_{1,\nu}\overline{\left(\mu(\mathbf{y}^*)\frac{\partial w_{1,\nu}}{\partial y_1}\varphi_{y_1}(\mathbf{y}^*) - \frac{\partial w_{1,\nu}}{\partial y_0}\varphi_{y_0}(\mathbf{y}^*)\right)}d\Sigma\\
&+\operatorname{Re}\int_{\partial\mathcal{G}} 2s\mu(\mathbf{y}^*)iV^+_\mu(\cdot,0)\overline{\left(\mu(\mathbf{y}^*)\frac{\partial w_{1,\nu}}{\partial y_1}\varphi_{y_1}(\mathbf{y}^*) - \frac{\partial w_{1,\nu}}{\partial y_0}\varphi_{y_0}(\mathbf{y}^*)\right)}d\Sigma\\
&=\operatorname{Re}\int_{\partial\mathcal{G}} 2s\mu(\mathbf{y}^*)(D_{y_1}\varphi_{y_1}(\mathbf{y}^*) - D_{y_0}\varphi_{y_0}(\mathbf{y}^*))\Gamma^+_\mu(y,s,D')\widehat{w}_{1,\nu}\overline{\widehat{w}_{1,\nu}}d\Sigma\\
&+\operatorname{Re}\int_{\partial\mathcal{G}} 2s\mu(\mathbf{y}^*)iV^+_\mu(\cdot,0)\overline{\left(\mu(\mathbf{y}^*)\frac{\partial w_{1,\nu}}{\partial y_1}\varphi_{y_1}(\mathbf{y}^*) - \frac{\partial w_{1,\nu}}{\partial y_0}\varphi_{y_0}(\mathbf{y}^*)\right)}d\Sigma. \qquad (3.5)
\end{aligned}$$

By (3.4), (3.5) and Gårding's inequality, we obtain

$$\begin{aligned}
J_2 \ge &-C_5\epsilon(\delta,\delta_1)\left\|\left(\frac{\partial w_{1,\nu}}{\partial y_2}, w_{1,\nu}\right)\right\|^2_{L^2(\partial\mathcal{G})\times H^{1,s}(\partial\mathcal{G})}\\
&-C_6(\delta,\delta_1)(\|P_{\mu,s}w_{1,\nu}\|^2_{L^2(\mathcal{G})} + \|\mathbf{v}\|^2_{\mathbf{H}^{2,s}(\mathcal{G})}). \qquad (3.6)
\end{aligned}$$

Second we assume that

$$\lim_{\zeta\to\zeta^*} \operatorname{Im} r_\mu(\mathbf{y}^*, \zeta)/|s| = 0.$$

Then $\mu\xi_1^*\varphi_{y_1}(\mathbf{y}^*) - \xi_0^*\varphi_{y_0}(\mathbf{y}^*) = 0$. Hence also in this subcase, we have (3.6).

Now we estimate J_3. By (1.18)-(1.20) there exists a constant $C_7 > 0$ such that

$$|\xi_0^2 - s^2\varphi_{y_0}^2(\mathbf{y}^*) - (\lambda+2\mu)(\mathbf{y}^*)\xi_1^2 + (\lambda+2\mu)(\mathbf{y}^*)s^2\varphi_{y_1}^2(\mathbf{y}^*)| \le C_7\delta_1(\xi_0^2+\xi_1^2+s^2), \quad \forall\zeta \in \mathcal{O}(\delta_1),$$

which yields

$$\begin{aligned}
&\xi_0^2 - \mu(\mathbf{y}^*)\xi_1^2 - s^2\varphi_{y_0}^2(\mathbf{y}^*) + s^2\mu(\mathbf{y}^*)\varphi_{y_1}^2(\mathbf{y}^*)\\
=&(\lambda+\mu)(\mathbf{y}^*)(\xi_1^2 - s^2\varphi_{y_1}^2(\mathbf{y}^*)) + (\xi_0^2 - (\lambda+2\mu)(\mathbf{y}^*)\xi_1^2 - s^2\varphi_{y_0}^2(\mathbf{y}^*) + s^2(\lambda+2\mu)(\mathbf{y}^*)\varphi_{y_1}^2(\mathbf{y}^*))\\
\ge&(\lambda+\mu)(\mathbf{y}^*)(\xi_1^2 - s^2\varphi_{y_1}^2(\mathbf{y}^*)) - C_7\delta_1(\xi_0^2+\xi_1^2+s^2).
\end{aligned}$$

Therefore, since $s^* = 0$, for all sufficiently small δ_1, there exists $C_8 > 0$ such that

$$\xi_0^2 - \mu(\mathbf{y}^*)\xi_1^2 - s^2\varphi_{y_0}^2(\mathbf{y}^*) + s^2\mu(\mathbf{y}^*)\varphi_{y_1}^2(\mathbf{y}^*) \ge C_8(\xi_0^2+\xi_1^2+s^2). \tag{3.7}$$

By (3.6), (3.7) and (2.1), there exists a constant $C_9 > 0$ such that

$$\begin{aligned}
\Sigma_\mu^{(1)} \ge C_9 &\left\|\left(\frac{\partial w_{1,\nu}}{\partial y_2}, w_{1,\nu}\right)\right\|^2_{L^2(\partial\mathcal{G})\times H^{1,s}(\partial\mathcal{G})}\\
-C_6(\delta,\delta_1)&(\|P_{\mu,s}w_{1,\nu}\|^2_{L^2(\mathcal{G})} + \|\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}).
\end{aligned} \tag{3.8}$$

Inequalities (3.8) and (1.62) imply

$$s\left\|\left(\frac{\partial w_{1,\nu}}{\partial y_2}, w_{1,\nu}\right)\right\|^2_{L^2(\mathcal{G})\times H^{1,s}(\mathcal{G})} \le C_{10}(\|\mathbf{f}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{v}\|^2_{\mathbf{H}^{2,s}(\mathcal{G})}). \tag{3.9}$$

Next we need the estimate for $\left(\mathbf{v}_\nu, \frac{\partial\mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2\mathbf{v}_\nu}{\partial y_2^2}\right)$. We may rewrite equations (1.45) and (1.46) as

$$-\mathbf{D}_{y_0}^2 v_1 - (\lambda+2\mu)(i\mathbf{D}_{y_1} - i\ell'(y_1)\mathbf{D}_{y_2})w_2 + \mu i\mathbf{D}_{y_2}w_1 + \widetilde{K}_1(y,\mathbf{D})\mathbf{v} = f_1e^{s\varphi}, \tag{3.10}$$

$$-\mathbf{D}_{y_0}^2 v_2 - (\lambda+2\mu)i\mathbf{D}_{y_2}w_2 - \mu(i\mathbf{D}_{y_1} - \ell'(y_1)i\mathbf{D}_{y_2})w_1 + \widetilde{K}_2(y,\mathbf{D})\mathbf{v} = f_2e^{s\varphi}, \quad (3.11)$$

where $\widetilde{K}_1$ and $\widetilde{K}_2$ are first order differential operators. Furthermore, setting

$$q_1 = f_1e^{s\varphi} - \mu i\mathbf{D}_{y_2}w_1 - \widetilde{K}_1(y,\mathbf{D})\mathbf{v},$$

$$q_2 = f_2e^{s\varphi} - \mu(i\mathbf{D}_{y_1} - \ell'(y_1)i\mathbf{D}_{y_2})w_1 - \widetilde{K}_2(y,\mathbf{D})\mathbf{v},$$

we rewrite (3.10) and (3.11) as

$$-\mathbf{D}_{y_0}^2 v_1 - (\lambda+2\mu)(i\mathbf{D}_{y_1} - i\ell'(y_1)\mathbf{D}_{y_2})w_2 = q_1 \quad (3.12)$$

and

$$-\mathbf{D}_{y_0}^2 v_2 - (\lambda+2\mu)i\mathbf{D}_{y_2}w_2 = q_2. \quad (3.13)$$

Using (3.13) we get rid of the term $\mathbf{D}_{y_2}w_2$ in (3.12):

$$\begin{aligned}&-\mathbf{D}_{y_0}^2 v_1 - (\lambda+2\mu)i\mathbf{D}_{y_1}(b_{21}(y_1,\mathbf{D}')v_1 + b_{22}(y_1,\mathbf{D}')v_2) - \ell'(y_1)\mathbf{D}_{y_0}^2 v_2\\ =&q_1 + i\ell'(y_1)q_2 + K_3\mathbf{g}. \end{aligned}\quad (3.14)$$

Here K_3 is a first order differential operator. Using (1.13), we can obtain

$$\mathbf{D}_{y_1}b_{11}(y_1,\mathbf{D}_{y_1})v_1 + \mathbf{D}_{y_1}b_{12}(y_1,\mathbf{D}_{y_1})v_2 = \mathbf{D}_{y_1}w_1|_{y_2=0} + K_3\mathbf{g}. \quad (3.15)$$

Set

$$\mathcal{K}(y',\mathbf{D}') = \begin{pmatrix} -\mathbf{D}_{y_0}^2 - (\lambda+2\mu)i\mathbf{D}_{y_1}b_{21}(y_1,\mathbf{D}') & -(\lambda+2\mu)i\mathbf{D}_{y_1}b_{22}(y_1,\mathbf{D}') - \ell'(y_1)\mathbf{D}_{y_0}^2\\ \mathbf{D}_{y_1}b_{11}(y_1,\mathbf{D}_{y_1}) & \mathbf{D}_{y_1}b_{12}(y_1,\mathbf{D}_{y_1})\end{pmatrix}.$$

By (3.14) and (3.15), we have

$$\mathcal{K}(y',\mathbf{D}')\mathbf{v} = \mathbf{m},$$

where $\mathbf{m} = (q_1 + i\ell'(y_1)q_2, \mathbf{D}_{y_1}w_1|_{y_2=0} + K_3\mathbf{g})$. Therefore

$$\mathcal{K}(y',\mathbf{D}')\mathbf{v}_\nu = \chi_\nu(s,D')\mathbf{m} - [\chi_\nu(s,D'),\mathcal{K}]\mathbf{v},$$

and since $\det \mathcal{K}(y',0,\xi^*) \neq 0$, we have

$$\mathbf{v}_\nu = \mathcal{K}^{-1}(y',\mathbf{D}')(\chi_\nu(s,D')\mathbf{m} - [\chi_\nu(s,D'),\mathcal{K}]\mathbf{v}) + T(y',s,D')\mathbf{v}_\nu.$$

Hence

$$\begin{aligned} & s\left\|\left(\mathbf{v}_\nu, \frac{\partial \mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2 \mathbf{v}_\nu}{\partial y_2^2}\right)\right\|^2_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})} \\ \le & C_{11}\Bigg(s\|\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{f}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + s\|\mathbf{g}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} \\ & + s\left\|\left(\frac{\partial w_{1,\nu}}{\partial y_2}, w_{1,\nu}\right)\right\|^2_{L^2(\partial\mathcal{G})\times H^{1,s}(\partial\mathcal{G})}\Bigg). \end{aligned} \tag{3.16}$$

By (1.62) with $\beta = \lambda + 2\mu$ and $\beta = 2\mu$, we obtain

$$\begin{aligned} & s\|\mathbf{w}_\nu\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + s\left\|\left(\mathbf{v}_\nu, \frac{\partial \mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2 \mathbf{v}_\nu}{\partial y_2^2}\right)\right\|^2_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})} \\ \le & C_{12}(s\|\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{f}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G}))^2} + s\|\mathbf{g}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}). \end{aligned} \tag{3.17}$$

Applying Proposition 1.3 we obtain (2.1).

Case B. Assume that $s^* \neq 0$. If $\delta_1 > 0$ is small enough, then there exists a constant $C_{13} > 0$ such that

$$|\xi_0\varphi_{y_1}(\mathbf{y}^*) - (\lambda+2\mu)\xi_1\varphi_{y_1}(\mathbf{y}^*)|^2 \le \delta_1^2 C_{13}(|\xi_1|^2 + s^2). \tag{3.18}$$

By (1.60) and (1.62), there exist constants $C_{14} > 0$ and $C_{15} > 0$ such that

$$\begin{aligned} & \Sigma^{(1)}_{\lambda+2\mu} + C_{14}s\|w_{2,\nu}\|^2_{H^{1,s}(\mathcal{G})} \le C_{15}(\|P_{\lambda+2\mu,s}w_2\|^2_{L^2(\mathcal{G})} \\ & + \|\mathbf{v}\|^2_{\mathbf{H}^{2,s}(\mathcal{G})}) + \epsilon\left\|\left(\frac{\partial w_{2,\nu}}{\partial y_2}, w_{2,\nu}\right)\right\|^2_{L^2(\partial\mathcal{G})\times H^{1,s}(\partial\mathcal{G})}. \end{aligned} \tag{3.19}$$

Note that $\Sigma^{(1)}_{\lambda+2\mu}$ can be written in the form

$$\Sigma^{(1)}_{\lambda+2\mu} = \int_{\partial\mathcal{G}} \left\{ s(\lambda+2\mu)^2(\mathbf{y}^*)\varphi_{y_2}(\mathbf{y}^*)\left|\frac{\partial w_{2,\nu}}{\partial y_2}\right|^2 + s^3(\lambda+2\mu)^2(\mathbf{y}^*)\varphi^3_{y_2}(\mathbf{y}^*)w^2_{2,\nu}\right\} d\Sigma$$

$$+\mathrm{Re}\int_{\partial\mathcal{G}} 2s(\lambda+2\mu)(\mathbf{y}^*)\frac{\partial w_{2,\nu}}{\partial y_2}\overline{\left\{(\lambda+2\mu)(\mathbf{y}^*)\varphi_{y_1}(\mathbf{y}^*)\frac{\partial w_{2,\nu}}{\partial y_1} - \varphi_{y_0}(\mathbf{y}^*)\frac{\partial w_{2,\nu}}{\partial y_0}\right\}} d\Sigma$$

$$+\int_{\partial\mathcal{G}} s(\lambda+2\mu)(\mathbf{y}^*)\varphi_{y_2}(\mathbf{y}^*)\{\xi_0^2 - (\lambda+2\mu)(\mathbf{y}^*)\xi_1^2 - s^2\varphi^2_{y_0}(\mathbf{y}^*)$$

$$+s^2(\lambda+2\mu)(\mathbf{y}^*)\varphi^2_{y_1}(\mathbf{y}^*)\}|\widehat{w}_{2,\nu}|^2 d\Sigma$$

$$=\widetilde{J}_1 + \widetilde{J}_2 + \widetilde{J}_3. \tag{3.20}$$

By (3.15) and (3.18), we have

$$|\widetilde{J}_2 + \widetilde{J}_3| \le C_{16}\epsilon(\delta_1)\left\|\left(\frac{\partial w_{2,\nu}}{\partial y_2}, w_{2,\nu}\right)\right\|^2_{L^2(\partial\mathcal{G})\times H^{1,s}(\partial\mathcal{G})}, \tag{3.21}$$

where $\epsilon(\delta_1) \to 0$ as $\delta_1 \to 0$. By (3.21) we obtain from (3.20) that there exists a constant $C_{17} > 0$ such that

$$\Sigma^{(1)}_{\lambda+2\mu} \ge C_{17}\int_{\partial\mathcal{G}} \left\{ (s(\lambda+2\mu)^2(\mathbf{y}^*)\varphi_{y_2}(\mathbf{y}^*)\left|\frac{\partial w_{2,\nu}}{\partial y_2}\right|^2 + s^3(\lambda+2\mu)^2(\mathbf{y}^*)\varphi^3_{y_2}(\mathbf{y}^*)|w_{2,\nu}|^2\right\} d\Sigma$$

$$-\epsilon\left\|\left(\frac{\partial w_{2,\nu}}{\partial y_2}, w_{2,\nu}\right)\right\|^2_{L^2(\partial\mathcal{G})\times H^{1,s}(\partial\mathcal{G})}. \tag{3.22}$$

In terms of (1.9), we can represent

$$\mathbf{D}^2_{y_0}v_{2,\nu} - 2\mu\mathbf{D}^2_{y_1}v_{2,\nu} + T_5\mathbf{v}_\nu + T_6\mathbf{v} + T_7(\mathbf{f}e^{s\phi}, \mathbf{g}e^{s\phi})$$

$$=(\lambda+2\mu)\mathbf{D}_{y_2}w_{2,\nu} + \chi_\nu(s, D')(f_2 e^{s\varphi}), \tag{3.23}$$

where we have estimates: Note that

$$\|T_5\mathbf{v}_\nu\|_{\mathbf{L}^2(\partial\mathcal{G})} \le \epsilon(\delta)\|\mathbf{v}_\nu\|_{\mathbf{H}^{2,s}(\partial\mathcal{G})}, \qquad \|T_6\mathbf{v}_\nu\|_{\mathbf{L}^2(\partial\mathcal{G})} \le C_{18}\|\mathbf{v}_\nu\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})},$$

$$\|T_7(\mathbf{f}e^{s\phi}, \mathbf{g}e^{s\phi})\|_{\mathbf{L}^2(\partial\mathcal{G})} \le C_{18}\|(\mathbf{f}e^{s\phi}, \mathbf{g}e^{s\phi})\|_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})},$$

where $\epsilon(\delta) \to +0$ as $\delta \to +0$. Hence, since

$$(\xi_0^* + i\varphi_{y_0}(\mathbf{y}^*)s^*)^2 - 2\mu(\mathbf{y}^*)(\xi_1^* + i\varphi_{y_1}(\mathbf{y}^*)s^*)^2 \neq 0,$$

from (3.1) and (3.23) we obtain

$$\|\sqrt{s}v_{2,\nu}\|^2_{H^{2,s}(\partial\mathcal{G})} \le C_{19}\left\{\left\|\sqrt{s}\frac{\partial w_{2,\nu}}{\partial y_2}\right\|^2_{L^2(\partial\mathcal{G})} + \|\sqrt{s}\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\sqrt{s}f_2 e^{s\varphi}\|^2_{L^2(\partial\mathcal{G})}\right\}$$
$$+\epsilon s\left\|\left(\mathbf{v}_\nu, \frac{\partial \mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2 \mathbf{v}_\nu}{\partial y_2^2}\right)\right\|^2_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})}. \tag{3.24}$$

Thanks to (1.12), (1.15) and the fact that $\xi_1^* + is^*\phi_{y_1}(\mathbf{y}^*) \neq 0$, we have

$$s\|v_{1,\nu}\|^2_{H^{2,s}(\partial\mathcal{G})}$$
$$\le C_{19}\left\{s\left\|\frac{\partial w_{2,\nu}}{\partial y_2}\right\|^2_{L^2(\partial\mathcal{G})} + \|\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{f}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + s\|\mathbf{g}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}\right.$$
$$\left.+\epsilon(\delta,\delta_1)s\left\|(\mathbf{v}_\nu, \frac{\partial \mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2 \mathbf{v}_\nu}{\partial y_2^2})\right\|^2_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})}\right\}. \tag{3.25}$$

Consequently (3.24), (3.25), (1.12), (1.8) and (1.9) imply

$$s\left\|\left(\mathbf{v}_\nu, \frac{\partial \mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2 \mathbf{v}_\nu}{\partial y_2^2}\right)\right\|^2_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})} \le C_{20}\left\{s\left\|\frac{\partial w_{1,\nu}}{\partial y_2}\right\|^2_{L^2(\partial\mathcal{G})}\right.$$
$$\left.+s\|\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{f}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + s\|\mathbf{g}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}\right\}. \tag{3.26}$$

By (3.22) and (3.26)

$$\Sigma^{(1)}_{\lambda+2\mu} \ge C_{21}s\left\|\left(\mathbf{v}_\nu, \frac{\partial \mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2 \mathbf{v}_\nu}{\partial y_2^2}\right)\right\|^2_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})}$$
$$-C_{21}(s\|\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{f}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + s\|\mathbf{g}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}).$$

Hence, from this inequality and (3.19) with $\beta = \lambda + 2\mu$, we obtain

$$s\left\|\left(\mathbf{v}_\nu, \frac{\partial \mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2 \mathbf{v}_\nu}{\partial y_2^2}\right)\right\|^2_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})} + s\|w_{2,\nu}\|^2_{H^{1,s}(\mathcal{G})}$$
$$\le C_{22}(s\|\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{f}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + s\|\mathbf{g}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}). \tag{3.27}$$

By (3.27) and (1.62) with $\beta = \mu$, we have

$$s\left\|\left(\mathbf{v}_\nu, \frac{\partial \mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2 \mathbf{v}_\nu}{\partial y_2^2}\right)\right\|^2_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})} + s\|\mathbf{w}_\nu\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}$$
$$\le C_{23}(s\|\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{f}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + s\|\mathbf{g}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}).$$

Finally using Proposition 1.3, we obtain (2.1). The proof of Lemma 3.1 is complete. ■

§4. Case $r_\mu \ne 0$ and $r_{\lambda+2\mu} \ne 0$.

In this section, we consider the case where

$$|r_\mu(\mathbf{y}^*, \zeta^*)| \ne 0 \quad \text{and } |r_{\lambda+2\mu}(\mathbf{y}^*, \zeta^*)| \ne 0. \tag{4.1}$$

We have

Lemma 4.1. *Let (4.1) hold at $\gamma \equiv (\mathbf{y}^*, \zeta^*)$ and let supp $\chi_\nu \subset \mathcal{O}(\delta_1)$ where δ_1 is a sufficiently small positive number. If $\zeta^* \in \Psi_2$, then we have*

$$\|\mathbf{v}_\nu\|_{\mathbf{H}^{2,s}(\mathcal{G})} \le C_1\left\{\frac{1}{s^{\frac14}}(\|\mathbf{f}e^{s\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{v}\|_{\mathbf{H}^{1,s}(\mathcal{G})})\right.$$
$$\left. + s^{\frac14}\|\mathbf{g}e^{s\varphi}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \frac{1}{s^{\frac14}}\|\mathbf{v}\|_{\mathbf{H}^{2,s}(\mathcal{G})}\right\} \tag{4.2}$$

and

$$\left\|\left(\mathbf{v}_\nu, \frac{\partial \mathbf{v}_\nu}{\partial y_2}\right)\right\|_{\mathbf{H}^{\frac32,s}(\partial\mathcal{G})\times\mathbf{H}^{\frac12,s}(\partial\mathcal{G})} \le C_1(\|\mathbf{v}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})}$$
$$+\|\mathbf{f}e^{s\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \sqrt{s}\|\mathbf{g}e^{s\phi}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})}). \tag{4.3}$$

If $\zeta^* \notin \Psi_2$, then estimate (2.1) hold true.

Proof. Thanks to (4.1) and Proposition 1.4, decomposition (1.55) holds true for $\beta = \mu$ and $\beta = \lambda + 2\mu$. Therefore we have

$$(D_{y_2} - \Gamma^+_\mu(y, s, D'))w_{1,\nu}|_{y_2=0} = V^+_\mu(\cdot, 0), \tag{4.4}$$

$$(D_{y_2} - \Gamma^+_{\lambda+2\mu}(y,s,D'))w_{2,\nu}|_{y_2=0} = V^+_{\lambda+2\mu}(\cdot,0). \tag{4.5}$$

By Proposition 1.5 we have the a priori estimate:

$$\begin{aligned}&\sqrt{s}\|V^+_\mu(\cdot,0)\|^2_{L^2(\partial\mathcal{G})} + \sqrt{s}\|V^+_{\lambda+2\mu}(\cdot,0)\|^2_{L^2(\partial\mathcal{G})}\\ \le& C_2(\|P_{\lambda+2\mu,s}w_2\|^2_{L^2(\mathcal{G})} + \|P_{\mu,s}w_1\|^2_{L^2(\mathcal{G})} + \|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}).\end{aligned} \tag{4.6}$$

By (1.13) and (1.15), we rewrite (1.51) in the form:

$$\mathcal{B}(y',s,D')\mathbf{v}_\nu = \mathbf{q}, \quad y' \in \partial\mathcal{G}, \tag{4.7}$$

where we recall that the operator $\mathcal{B}(y',s,D')$ is defined by (1.22) and we set

$$\mathbf{q} = T_1(\mathbf{g}e^{s\varphi}) + T_2(\mathbf{f}(y',0)e^{s\varphi}) + \mathbb{G}(y')(V^+_\mu(\cdot,0), V^+_{\lambda+2\mu}(\cdot,0)), \tag{4.8}$$

$T_1 \in \mathcal{L}(\mathbf{H}^{1,s}(\partial\mathcal{G}), \mathbf{L}^2(\partial\mathcal{G})), T_2 \in \mathcal{L}((L^2(\partial\mathcal{G}))^2, (L^2(\partial\mathcal{G}))^2)$ and $\mathbb{G}(y')$ is a C^1 matrix-valued function.

Now we consider the following three cases.

Case A. $\det \mathcal{B}(\gamma) \ne 0$ and $\zeta^* \notin \Psi$. Here we recall that the sets Ψ, Ψ_1 and Ψ_2 are defined by (1.29) and Lemma 1.1. In that case, there exists a parametrix of the operator $\mathcal{B}(y',s,D')$ which we denote by $\mathcal{B}^{-1}(y',s,D')$, and we have

$$(v_{1,\nu}, v_{2,\nu}) = \mathcal{B}^{-1}(y',s,D')(V^+_\mu(\cdot,0) - q_1, V^+_{\lambda+2\mu}(\cdot,0) - q_2) + K(v_{1,\nu}, v_{2,\nu}), \tag{4.9}$$

where

$$K \in \mathcal{L}(\mathbf{L}^2(\partial\mathcal{G}), \mathbf{H}^{1,s}(\partial\mathcal{G})).$$

By (4.6) and (4.9)

$$|\Sigma_\mu| + |\Sigma_{\lambda+2\mu}| \le C_3(\|\mathbf{f}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + s\|\mathbf{g}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}). \tag{4.10}$$

Inequalities (1.62) and (4.10) yield (2.1).

If $\det B(\gamma)=0$ and $\zeta^*\notin\Psi$ then either $\frac{\Gamma_\mu}{s}(\gamma)<0$ or $\frac{\Gamma_{\lambda+2\mu}}{s}(\gamma)<0$. In the first case we have the estimate

$$\|(b_{11}+b_{12})(y_1,s,D')\mathbf{v}_\nu\|_{\mathbf{H}^{1,s}(\mathcal{G})}\le C_4(\|P_{\lambda+2\mu,s}w_2\|^2_{L^2(\mathcal{G})}+\|P_{\mu,s}w_1\|^2_{L^2(\mathcal{G})}+\|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})})$$

and in the second case we have the additional estimate:

$$\|(b_{21}+b_{22})(y_1,s,D')\mathbf{v}_\nu\|_{\mathbf{H}^{1,s}(\mathcal{G})}\le C_4(\|P_{\lambda+2\mu,s}w_2\|^2_{L^2(\mathcal{G})}+\|P_{\mu,s}w_1\|^2_{L^2(\mathcal{G})}+\|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}).$$

We combine one of these estimates with (4.7) to obtain (4.10). Then, by (4.10) and (1.62), we obtain (2.1). Now we consider the case where $\zeta^*\in\Psi$.

In order to treat this case, we will use the Calderon method. First we introduce the new variables $U=(U_1,U_2,U_3,U_4)$, where $(U_1,U_2)=\Lambda(s,D')\mathbf{v},(U_3,U_4)=\mathbf{D}_{y_2}\mathbf{v}$, and Λ is the pseudodifferential operator with the symbol $(s^2+\xi_1^2+\xi_0^2+1)^{\frac12}$. Then problem (1.8) - (1.9) can be written in the form:

$$D_{y_2}U=M(y,s,D')U+F\quad \text{in } \mathbb{R}^3\times[0,1],\quad \mathbb{B}(y',s,D')U(y)|_{y_2=0}=\mathbf{g}e^{s\phi},\quad (4.11)$$

where $F=(0,P(y,\mathbf{D})\mathbf{v})$ and we set $\mathbf{D}=(\mathbf{D}_{y_0},\mathbf{D}_{y_1},\mathbf{D}_{y_2})$, $\mathbf{D}_{y_j}=\frac1i\frac{\partial}{\partial y_j}+is\partial_{y_j}\phi$ and $M(y,s,D')$ is the matrix pseudodifferential operator whose principal symbol $M_1(y,\zeta)$ is given by formula (see [Y]):

$$M_1(y,\zeta)=\begin{pmatrix}0 & \Lambda_1E_2\\ A^{-1}M_{21}\Lambda_1^{-1} & A^{-1}M_{22}\end{pmatrix}-is\varphi_{y_2}E_4,$$

where $\Lambda_1=|\zeta|$, $M_{21}(y,\xi'+is\nabla_{y'}\varphi)=((\xi_0+is\varphi_{y_0})^2-\mu(\xi'+is\varphi_{y_1})^2)E_2-(\lambda+\mu)\vec\theta^T\vec\theta$, $M_{22}(y,\xi')=-(\lambda+\mu)(\vec\theta^TG+G^T\vec\theta)-2\mu\vec\theta G^TE_2$, $A=(\lambda+\mu)G^TG+\mu|G|^2E_2$, $G(y_1)=(-\ell'(y_1),1)$, $\vec\theta=(\xi_1+is\varphi_{y_1},0)$. Hence and henceforth, $\vec\theta^T$ denotes the transpose of the vector $\vec\theta$. For the stress boundary conditions, we have

$$\mathbb{B}(y',s,D')U=(\mathbb{B}_1(y',s,D'),\mathbb{B}_2(y',s,D'))U=\mathbf{g}e^{s\varphi},$$

where

$$\mathbb{B}_1(y', s, \xi') = \frac{\lambda G^T\vec{\theta} + \mu\vec{\theta}^T G + \mu\vec{\theta}G^T E_2}{\sqrt{s^2 + |\xi_1^2 + \xi_0^2}}, \quad \mathbb{B}_2(y', s, D') = A.$$

Case B. $\det \mathcal{B}(\gamma) = 0$ and $\zeta^* \in \Psi_2$.

We introduce the matrix symbol $\mathcal{K}(y', s, \xi_0, \xi_1)$ by formula

$$\mathcal{K}(y', \zeta) = \frac{1}{1 + s^2 + \xi_1^2 + \xi_2^2} \begin{pmatrix} B_{22}(y', \zeta) & -B_{12}(y', \zeta) \\ -B_{21}(y', \zeta) & B_{22}(y', \zeta) \end{pmatrix}. \tag{4.12}$$

Applying the pseudodifferential operator $\mathcal{K}(y', s, D')$ to equation (4.7), so that

$$\mathcal{K}(y', s, D')\mathcal{B}(y', s, D')\mathbf{v}_\nu = \mathcal{K}(y', s, D')\mathbf{q}. \tag{4.13}$$

The principal symbol of the operator $\mathcal{K}(y', s, D')\mathcal{B}(y', s, D')$ is given by the formula $\mathcal{K}(y', \zeta)\mathcal{B}(y', \zeta) = \det B(y', \zeta)E_4/|\zeta|^2$. Note that if $\zeta^* \in \Psi$ and $s^* \neq 0$, then Im $\alpha_\mu^-(\gamma) < 0$ and Im $\alpha_{\lambda+2\mu}^-(\gamma) < 0$. Hence we may rewrite estimate (4.6) in the form

$$\begin{aligned} &\|V_\mu^+(\cdot, 0)\|^2_{H^{\frac{1}{2},s}(\partial\mathcal{G})} + \|V_{\lambda+2\mu}^+(\cdot, 0)\|^2_{H^{\frac{1}{2},s}(\partial\mathcal{G})} \\ \le & C_5(\|P_{\lambda+2\mu,s}w_2\|^2_{L^2(\mathcal{G})} + \|P_{\mu,s}w_1\|^2_{L^2(\mathcal{G})} + \|\mathbf{w}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}). \end{aligned} \tag{4.14}$$

Thanks to (4.8), (4.14) and Condition 1.2, we have estimate (4.3).

Now we need to show that estimate (4.2) holds true. By $\zeta^* \in \Psi_2$, the matrix $M_1(\gamma)$ has four distinct eigenvalues given by (1.18)-(1.20). Following [T], in terms of the change of variables $W = S^{-1}(y, s, D')U$, we can transform system (4.11) to

$$D_{y_2}W = \widetilde{M}(y, s, D')W + T(y, s, D')W + \widetilde{\mathbf{F}}, \tag{4.15}$$

where

$$\|\widetilde{\mathbf{F}}\|_{L^2(\mathbb{R}^1;\mathbf{H}^{1,s}(\partial\mathcal{G}))} \le C_6(\|P(y, \mathbf{D})\mathbf{v}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{v}\|_{L^2(\mathbb{R}^1;\mathbf{H}^{1,s}(\partial\mathcal{G}))}). \tag{4.16}$$

Here the matrix $\widetilde{M}$ has the form

$$\widetilde{M}(y,\zeta)=\begin{pmatrix} M_+(y,\zeta) & 0\\ 0 & M_-(y,\zeta)\end{pmatrix},\quad M_\pm(y,\zeta)=\begin{pmatrix}\Gamma^\pm_{\lambda+2\mu}(y,\zeta) & 0\\ 0 & \Gamma^\pm_\mu(y,\zeta)\end{pmatrix},$$

and the operator $T(y,s,D')\in L^\infty(0,1;\mathcal{L}(\mathbf{H}^{1,s}(\mathcal{G}),\mathbf{H}^{1,s}(\mathcal{G})))$. We represent the symbol S in the form $S=(s_1^+,s_2^+,s_1^-,s_2^-)$. Here

$$\begin{aligned} s_1^\pm =&((\vec{\theta}+\alpha^\pm_{\lambda+2\mu}G)\Lambda_1^{-1},\alpha^\pm_{\lambda+2\mu}((\vec{\theta}+\alpha^\pm_{\lambda+2\mu}G)\Lambda_1^{-1})),\\ s_2^\pm =&((\alpha^\pm_\mu(\xi_1+is\varphi_{y_1}-\alpha^\pm_\mu\ell'),-(\xi_1+is\varphi_{y_1}-\alpha^\pm_\mu\ell')^2)\Lambda_1^{-2},\\ &\alpha^\pm_\mu\Lambda_1^{-1}(\alpha^\pm_\mu(\xi_1+is\varphi_{y_1}-\alpha^\pm_\mu\ell'),-(\xi_1+is\varphi_{y_1}-\alpha^\pm_\mu\ell')^2)\Lambda_1^{-2}),\end{aligned}$$

are the eigenvectors of the matrix $M_1(y,\zeta)$, $\zeta\in S^2$, which corresponds to the eigenvalues $\Gamma^\pm_{\lambda+2\mu}$ and $\Gamma^\pm_\mu$.

Now using the standard arguments (see e.g., §4 of Chapter 7 in [Ku]), we can estimate the last two components of W as follows

$$\|(W_3,W_4)\|_{\mathbf{H}^{1,s}(\mathcal{G})}\le C_7(\|P(y,\mathbf{D})\mathbf{v}\|_{\mathbf{L}^2(\mathcal{G})}+\|\mathbf{v}\|_{\mathbf{H}^{1,s}(\mathcal{G})}),\tag{4.17}$$

where the constant C_7 is independent of N.

Now we need the estimate for the first two components of vector W. Henceforth we set $j(\beta)=2$ if $\beta=\mu$ and $j(\beta)=1$ if $\beta=\lambda+2\mu$.

There are two possibilities (i) and (ii):

(i) $Im\,\Gamma^+_\beta(\gamma)>0$ *for any* $\beta\in\{\mu,\lambda+2\mu\}$.

Then, by the same argument (see e.g., [Ku], pp. 241-247), we have

$$\|W_{j(\beta)}\|_{\mathbf{H}^{1,s}(\mathcal{G})}\le C_8(\|P(y,\mathbf{D})\mathbf{v}\|_{\mathbf{L}^2(\mathcal{G})}+\|W_{j(\beta)}(\cdot,0)\|_{\mathbf{H}^{1/2,s}(\partial\mathcal{G})}+\|\mathbf{v}\|_{\mathbf{H}^{1,s}(\mathcal{G})}).\tag{4.18}$$

Combining this inequality with a priori estimate (4.3), we have

$$\|W_{j(\beta)}\|_{\mathbf{H}^{1,s}(\mathcal{G})} \le C_9(\|P(y,\mathbf{D})\mathbf{v}\|_{\mathbf{L}^2(\mathcal{G})} + \|\mathbf{v}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \|\mathbf{g}e^{s\varphi}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \frac{1}{\sqrt{s}}(\|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{f}e^{s\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})}) + \|\mathbf{v}\|_{\mathbf{H}^{1,s}(\mathcal{G})}). \quad (4.19)$$

(ii) There exists $\beta \in \{\mu, \lambda+2\mu\}$ such that $Im\Gamma^+_\beta(\gamma) = 0$.

Applying Proposition 1.6, we obtain

$$\|W_{j(\beta)}\|_{\mathbf{H}^{1,s}(\mathcal{G})} \le C_{10}\left(\frac{1}{\sqrt{s}}(\|P(y,\mathbf{D})\mathbf{v}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{v}\|_{\mathbf{H}^{2,s}(\mathcal{G})}) + s^{\frac14}\|W_{j(\beta)}(\cdot,0)\|_{\mathbf{H}^{\frac12,s}(\partial\mathcal{G})}\right).$$

Combining this inequality with a priori estimate (4.3), we have

$$\|W_{j(\beta)}\|_{\mathbf{H}^{1,s}(\mathcal{G})} \le C_{11}\left\{\frac{1}{\sqrt{s}}(\|P(y,\mathbf{D})\mathbf{v}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{v}\|_{\mathbf{H}^{2,s}(\mathcal{G})}) + s^{\frac14}\|\mathbf{g}e^{s\varphi}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + s^{\frac14}(\|\mathbf{v}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \frac{1}{\sqrt{s}}\|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \frac{1}{\sqrt{s}}\|\mathbf{f}e^{s\varphi}\|_{\mathbf{H}^{1,s}(\mathcal{G})})\right\}. \quad (4.20)$$

In view of (4.17), (4.18) and (4.20), we obtain (4.2).

Case C. $\det \mathcal{B}(\gamma) = 0$ and $\zeta^* \in \Psi_1$.

In that case we have

$$\xi_0^* = 0, \;\; s^*\varphi_{y_0}(\mathbf{y}^*) = 0. \quad (4.21)$$

Then we can assume that

$$\operatorname{Im}\Gamma^+_\mu(\gamma) = Im\Gamma^+_{\lambda+2\mu}(\gamma) \ge 0. \quad (4.22)$$

In fact, if

$$\operatorname{Im}\Gamma^+_\mu(\gamma) = Im\Gamma^+_{\lambda+2\mu}(\gamma) < 0, \quad (4.23)$$

then the situation is simple because we have the decomposition

$$P_{\beta,s}(y,s,D)w_{j(\beta),\nu} = \beta|G|(D_{y_2} - \Gamma^{\mp}_\beta(y,s,D'))V^{\pm}_\beta + T^{\pm}_\mu w_{j(\beta),\nu},$$

where $T^\pm_\beta \in \mathcal{L}(H^{1,s}(\mathcal{G}), L^2(\mathcal{G}))$, $\beta \in \{\mu, \lambda+2\mu\}$, $j(\beta)=1$ for $\beta=\mu$ and $j(\beta)=2$ for $\beta=\lambda+2\mu$. This decomposition, (4.23) and Proposition 1.5 imply

$$\|\sqrt{s}(D_{y_2}-\Gamma^\pm_\beta(y,s,D'))w_{j(\beta),\nu}|_{y_2=0}\|_{L^2(\partial\mathcal{G})} \le C_{12}(\|P_{\beta,s}w_{j(\beta),\nu}\|_{L^2(\mathcal{G})}+\|\mathbf{v}\|_{\mathbf{H}^{2,s}(\mathcal{G})}). \tag{4.24}$$

Obviously

$$V^+_\mu(\cdot,0)-V^-_\mu(\cdot,0)=(\alpha^+_\mu(y',0,s,D')$$
$$-\alpha^-_\mu(y',0,s,D'))((b_{11}(y_1,\mathbf{D}_{y_1})+b_{12}(y_1,\mathbf{D}_{y_1}))\mathbf{v}_\nu+\widetilde{C}_1(y_1)\mathbf{g}e^{s\phi}) \quad \text{on } \partial\mathcal{G}.$$

and

$$V^+_{\lambda+2\mu}(\cdot,0)-V^-_{\lambda+2\mu}(\cdot,0)=(\alpha^+_{\lambda+2\mu}(y',0,s,D')$$
$$-\alpha^-_{\lambda+2\mu}(y',0,s,D'))((b_{21}(y_1,\mathbf{D}_{y_1})+b_{22}(y_1,\mathbf{D}_{y_1}))\mathbf{v}_\nu+\widetilde{C}_2(y_1)\mathbf{g}e^{s\phi}) \quad \text{on } \partial\mathcal{G}.$$

Since $\alpha^+_\beta(\mathbf{y}^*,\zeta^*)-\alpha^-_\beta(\mathbf{y}^*,\zeta^*)=2\sqrt{r_\mu(\mathbf{y}^*,\zeta^*)}\ne 0$ and the determinant of the matrix

$$\begin{pmatrix} 2\sqrt{r_\mu(\mathbf{y}^*,\zeta^*)}b_{11}(\mathbf{y}^*,\xi^*+is^*\varphi_{y_1}(\mathbf{y}^*)) & 2\sqrt{r_\mu(\mathbf{y}^*,\zeta^*)}b_{12}(\mathbf{y}^*,\xi^*+is^*\varphi_{y_1}(\mathbf{y}^*)) \\ 2\sqrt{r_\mu(\mathbf{y}^*,\zeta^*)}b_{21}(\mathbf{y}^*,\xi^*+is^*\varphi_{y_1}(\mathbf{y}^*)) & 2\sqrt{r_\mu(\mathbf{y}^*,\zeta^*)}b_{22}(\mathbf{y}^*,\xi^*+is^*\varphi_{y_1}(\mathbf{y}^*)) \end{pmatrix}$$

is not equal to zero, by (4.22) and Gårding's inequality, we obtain

$$s\left\|\left(\mathbf{v}_\nu,\frac{\partial\mathbf{v}_\nu}{\partial y_2},\frac{\partial^2\mathbf{v}_\nu}{\partial y_2^2}\right)\right\|^2_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})}$$
$$\le C_{13}(\|\mathbf{f}e^{s\varphi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}+s\|\mathbf{g}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}+\|\mathbf{v}\|^2_{\mathbf{H}^{2,s}(\mathcal{G})}). \tag{4.25}$$

In terms of (4.25) and (1.62), we obtain (2.1).

The matrix $M_1(\gamma)$ has only two eigenvalues given by (1.18)-(1.20). Moreover it is known that the Jordan form of the matrix $M_1(\gamma)$ has two Jordan blocks of the form:

$$M^\pm=\begin{pmatrix}\Gamma^\pm_\mu(\gamma) & 1\\ 0 & \Gamma^\pm_\mu(\gamma)\end{pmatrix}.$$

Following [T], in terms of the change of variables $W = S^{-1}(y,s,D')U$, we can transform system (4.11) to the form

$$D_{y_2}W = \widetilde{M}(y,s,D')W + T(y,s,D')W + \widetilde{\mathbf{F}}, \tag{4.26}$$

$$\widetilde{\mathbb{B}}(y',s,D')W = \mathbf{g}e^{s\phi}, \tag{4.27}$$

where

$$\|\widetilde{\mathbf{F}}\|_{L^2(\mathbb{R}^1;H^{1,s}(\partial\mathcal{G}))} \le C_{14}(\|P(y,\mathbf{D})\mathbf{v}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{v}\|_{L^2(\mathbb{R}^1;\mathbf{H}^{1,s}(\partial\mathcal{G}))}).$$

and the principal symbol of the operator $\widetilde{\mathbb{B}}$ is defined by formula

$$\widetilde{\mathbb{B}}(y',s,\xi') = \mathbb{B}(y',s,\xi')S(y',0,s,\xi'). \tag{4.28}$$

Here the matrix $\widetilde{M}$ has the form

$$\widetilde{M}(y,\zeta) = \begin{pmatrix} M_+(y,\zeta) & 0 \\ 0 & M_-(y,\zeta) \end{pmatrix}, \quad M_\pm(y,\zeta) = \begin{pmatrix} \Gamma^\pm_{\lambda+2\mu}(y,\zeta) & m^\pm_{12}(y,\zeta) \\ 0 & \Gamma^\pm_\mu(y,\zeta) \end{pmatrix},$$

and the operator $T(y,s,D') \in L^\infty(0,1;\mathcal{L}(\mathbf{H}^{1,s}(\mathcal{G}),\mathbf{H}^{1,s}(\mathcal{G})))$. We describe the construction of the pseudodifferential operator S. We write the symbol S in the form $S = (s_1^+, s_2^+, s_1^-, s_2^-)$. Here $s_1^\pm = ((\vec{\theta} + \alpha^\pm_{\lambda+2\mu}G)\Lambda_1^{-1}, \alpha^\pm_{\lambda+2\mu}((\vec{\theta} + \alpha^\pm_{\lambda+2\mu}G)\Lambda_1^{-1}))$ are the eigenvectors of the matrix $M_1(y,\zeta)$ on the sphere $\zeta \in S^2$ which corresponds to the eigenvalue $\Gamma^\pm_{\lambda+2\mu}$ and the vectors $s_2^\pm$ are given by

$$s_2^\pm = E_\pm s^\pm, \quad E_\pm = \frac{1}{2\pi i}\int_{C^\pm}(z - M_1(y,\zeta))^{-1}dz,$$

where $C^\pm$ are a small circles oriented counterclockwise and centered at $\Gamma^\pm_\mu(\gamma)$, and $s^\pm$ solves the equation $M_1(\gamma)s^\pm - \Gamma^\pm_\mu(\gamma)s^\pm = s_1^\pm$. By (4.22) the circles $C^\pm$ can be taken such that the disks bounded by these circles do not intersect. Note that the vectors $s_j^{\pm} \in C^2(B_\delta \times \mathcal{O}_{\delta_1})$ are homogeneous functions of the order zero in (s, ξ_0, ξ_1). Now, similarly to (4.18), we can estimate the last two components of W as follows

$$\|(W_3, W_4)\|_{\mathbf{H}^{\frac{3}{2},s}(\partial\mathcal{G})} \le C_{15}(\|P(y, \mathbf{D})\mathbf{v}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \|\mathbf{v}\|_{\mathbf{H}^{2,s}(\mathcal{G})}), \tag{4.29}$$

where the constant C_{15} is independent of N. Now we need to estimate the first two components of the vector W on $\partial\mathcal{G}$. We can decompose the boundary operator $\widetilde{\mathbb{B}}(y', s, D') = (\widetilde{\mathbb{B}}^+(y', s, D'), \widetilde{\mathbb{B}}^-(y', s, D'))$ such that

$$\widetilde{\mathbb{B}}^+(y', s, D')(W_1, W_2) = -\widetilde{\mathbb{B}}^-(y', s, D')(W_3, W_4) + \mathbf{g}e^{s\phi}, \tag{4.30}$$

where $\widetilde{\mathbb{B}}^+(y, \zeta) = (\widetilde{\mathbb{B}}_1(y, \zeta), A)S_+(y, \zeta)$, $\widetilde{\mathbb{B}}_1$ is the principal symbol of the boundary operator $\widetilde{\mathbb{B}}$ and $S_+ = (s_1^+, s_2^+)$.

At the point γ the vectors s_1, s_2 are given explicitly by

$$\widetilde{\eta} = (\xi_1^* + is^*\varphi_{y_1}(\mathbf{y}^*), i\mathrm{sign}(\xi_1^*)(\xi_1^* + is^*\varphi_{y_1}(\mathbf{y}^*)))$$

$$s_1^+(\gamma) = \left(\widetilde{\eta}, i\frac{\mathrm{sign}(\xi_1^*)(\xi_1^* + is^*\varphi_{y_1}(\mathbf{y}^*))}{\sqrt{(\xi_1^*)^2 + (s^*)^2}}\widetilde{\eta}\right),$$

$$\vec{\varsigma} = -\frac{(\lambda + 3\mu)(\mathbf{y}^*)}{2\sqrt{(\xi_1^*)^2 + (s^*)^2}(\lambda + \mu)(\mathbf{y}^*)}(i\mathrm{sign}(\xi_1^*), 1),$$

$$s_2^+(\gamma) = \left(\vec{\varsigma}, \frac{1}{\sqrt{(\xi_1^*)^2 + (s^*)^2}}\{i\mathrm{sign}(\xi_1^*)(\xi_1^* + is^*\varphi_{y_1}(\mathbf{y}^*))\vec{\varsigma} + \widetilde{\eta}\}\right).$$

Therefore

$$\det \widetilde{\mathbb{B}}^+(\gamma) = \begin{pmatrix} 2\mu i(\xi_1^* + is^*\varphi_{y_1}(\mathbf{y}^*))^2 & \mu(\xi_1^* + is^*\varphi_{y_1}(\mathbf{y}^*)) \\ 2\mu(\xi_1^* + is^*\varphi_{y_1}(\mathbf{y}^*))^2 & -\mu i\frac{\lambda+2\mu}{\lambda+\mu}(\xi_1^* + is^*\varphi_{y_1}(\mathbf{y}^*)) \end{pmatrix}$$

$$= (\xi_1^* + is^*\varphi_{y_1}(\mathbf{y}^*))^3\mu^2\frac{2\lambda + 4\mu}{\lambda + \mu} \ne 0. \tag{4.31}$$

By (4.28)-(4.31) and Gårding's inequality, we obtain

$$\sqrt{s}\left\|\left(\mathbf{v}_\nu, \frac{\partial \mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2 \mathbf{v}_\nu}{\partial y_2^2}\right)\right\|_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})} \le C_{16}(\|P(y,\mathbf{D})\mathbf{v}\|_{\mathbf{H}^{1,s}(\mathcal{G})}+\|\mathbf{v}\|_{\mathbf{H}^{2,s}(\mathcal{G})}). \tag{4.32}$$

By (4.10), (1.62) and Proposition 1.3 we obtain (2.1). ■

Now we will proceed

Proof of Theorem 1.1. Microlocally we obtain two types of estimates: If $\zeta^* \in \Psi_2$, then we have estimate (4.2), while if $\zeta^* \notin \Psi_2$, then we have estimate (2.1). By $\mathcal{O}\Psi_2(\delta_2)$, we denote the δ_2-neighbourhood of the set Ψ_2 in S^2. We take the parameter δ_2 sufficiently small. From the covering of the set $\mathcal{O}\Psi_2(\frac{15}{4}\delta_2)$ by balls of radius $4\delta_2$, we take the finite subcovering $\{B_{4\delta_2}(\zeta_j)\}_{j\in\Upsilon_1}$, $\zeta_j \in \Psi_2$. Let $\{\chi_\nu\}_{\nu\in\Upsilon_1}$ be a partition of unity associated with this subcovering. For the set $\overline{S^2\setminus\mathcal{O}\Psi_2(3\delta_2)}$, we take the finite covering by balls of radius δ_2. Let $\{\chi_\nu\}_{\nu\in\Upsilon_2}$ be a partition of unity associated with this subcovering. We extend the functions χ_ν as a homogeneous functions of order zero to a function in $C^\infty(\mathbb{R}^3)$. Since $\Psi_2 \subset \mathcal{O}\Psi_2(\frac{15}{4}\delta_2)\cup\overline{S^2\setminus\mathcal{O}\Psi_2(3\delta_2)}$, it follows from (4.2) and (2.1) that

$$\begin{aligned}\|\mathbf{v}\|^2_{\mathbf{H}^{2,s}(\mathcal{G})} &\le C_{17}\sum_{\nu\in\Upsilon_1\cup\Upsilon_2}\|\chi_\nu\mathbf{v}\|^2_{\mathbf{H}^{2,s}(\mathcal{G})}\\ &\le C_{17}\Bigg\{\left(\frac{1}{N}+\frac{1}{s^{\frac14}}\right)\|P(y,\mathbf{D})\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}+s\|\mathbf{g}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}\\ &+\left(\frac{1}{N}+\frac{1}{s^{\frac14}}\right)\sum_{|\alpha|=0}^{2}s^{4-2|\alpha|}\|\partial_y^\alpha\mathbf{v}\|^2_{\mathbf{L}^2(\mathcal{G})}\Bigg\}, \quad \forall N\ge\widehat{N}, s\ge s_0(\widehat{N}).\end{aligned} \tag{4.33}$$

Fixing the parameters $\widehat{N}$ and $s_0(N)$ sufficiently large, we obtain

$$\|\mathbf{v}\|^2_{\mathbf{H}^{2,s}(\mathcal{G})} \le C_{18}(\|P(y,\mathbf{D})\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})}+s\|\mathbf{g}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}), \quad \forall s\ge s_0. \tag{4.34}$$

Combination of (4.34) with estimates (4.3) and (2.1), yields

$$s\left\|\left(\mathbf{v}_\nu, \frac{\partial \mathbf{v}_\nu}{\partial y_2}, \frac{\partial^2 \mathbf{v}_\nu}{\partial y_2^2}\right)\right\|^2_{\mathbf{H}^{2,s}(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})\times\mathbf{L}^2(\partial\mathcal{G})}$$
$$\le C_{19}(\|P(y,\mathbf{D})\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + s\|\mathbf{g}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}), \quad \forall \nu \in \Upsilon_2 \tag{4.35}$$

and

$$s\|\mathbf{v}_\nu\|^2_{\mathbf{H}^{\frac{3}{2},s}(\partial\mathcal{G})} \le C_{19}(\|P(y,\mathbf{D})\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + s\|\mathbf{g}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}), \quad \forall \nu \in \Upsilon_1. \tag{4.36}$$

Estimates (4.35) and (4.36) yield

$$s\|\mathbf{u}e^{s\phi}\|^2_{\mathbf{H}^{\frac{3}{2},s}(\partial\mathcal{G})} \le C_{20}(P(y,\mathbf{D})\mathbf{v}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + s\|\mathbf{g}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}), \tag{4.37}$$

where we used $\phi = \varphi$ on $\partial\Omega$.

We note that estimate (4.37) is obtained under additional assumption (1.35). Now we will get rid of (1.35). For this, we consider the function $\theta\mathbf{u}$ instead of the function $\mathbf{u}$, where θ is a smooth cut-off function such that $\theta|_{\partial\Omega} = 1$ and $\theta|_{\Omega_N\setminus\Omega_{\frac{N}{2}}} = 1$. Then it suffices to modify (4.37) and prove

$$s\|\mathbf{u}e^{s\phi}\|^2_{\mathbf{H}^{\frac{3}{2},s}(\partial\mathcal{G})} \le C_{21}\left(\|\mathbf{f}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + s\|\mathbf{g}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})} + \frac{1}{s}\|\mathbf{u}e^{s\phi}\|^2_{\mathbf{H}^{2,s}(\mathcal{G})}\right), \quad \forall s \ge s_0, \tag{4.38}$$

where we used the fact that $\varphi(x) \le \psi(x)$ in $\Omega_N \setminus \Omega_{N/2}$. Next by Lemma 4.1 we note that the analogue of the estimate (4.3) and (4.2) holds true for the weight function ϕ instead of φ:

$$s\|\chi_\nu(\mathbf{u}e^{s\phi})\|^2_{\mathbf{H}^{2,s}(\mathcal{G})} + s\|\chi_\nu(\mathbf{u}e^{s\phi})\|^2_{\mathbf{H}^{\frac{3}{2},s}(\partial\mathcal{G})}$$
$$\le C_{22}(\|\mathbf{u}e^{s\phi}\|^2_{\mathbf{H}^{2,s}(\mathcal{G})} + \|\mathbf{f}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + s\|\mathbf{g}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}), \quad \forall \nu \in \Upsilon_1. \tag{4.39}$$

Let χ_{-1} and χ_{-2} be C^∞-functions on the sphere S^2 such that $\chi_{-1} \in C_0^\infty(\mathcal{O}\Psi_2(\frac{15}{4}\delta_2))$, $\chi_{-2} \in C_0^\infty(S^2 \setminus \mathcal{O}\Psi_2(3\delta_2))$ and $\chi_{-1}|_{\mathcal{O}\Psi_2(\frac{15}{16}\delta_2)} = 1$, $\chi_{-2}|_{S^2\setminus\mathcal{O}\Psi_2(\frac{8}{9}\delta_2)} = 1$. Hence we have $|\chi_{-1}(\zeta)| + |\chi_{-2}(\zeta)| \geq 1$ for all $\zeta \in S^2$. We extend the functions χ_{-1} and χ_{-2} as homogeneous functions of order zero to functions in $C^\infty(\mathbb{R}^3)$. By (1.8) and (1.9), we have

$$A^{-1}P(y,\mathbf{D})\chi_{-1}\mathbf{v} + [\chi_{-1}, A^{-1}P(y,\mathbf{D})]\mathbf{v} = \chi_{-1}A^{-1}\mathbf{f}e^{s\phi}. \tag{4.40}$$

Note that we can estimate

$$\|[\chi_{-1}, A^{-1}P(y,\mathbf{D})]\mathbf{v}\|_{\mathbf{H}^{1,s}(\mathcal{G})} \leq C_{23}(\|\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{2,s}(\mathcal{G})} + \|\mathbf{f}e^{s\phi}\|_{\mathbf{H}^{1,s}(\mathcal{G})}). \tag{4.41}$$

Hence applying estimate (4.39) and (4.41) to (4.40), we obtain.

$$\begin{aligned} &s\|\chi_{-1}(s,D')(\mathbf{u}e^{s\phi})\|^2_{\mathbf{H}^{2,s}(\mathcal{G})} \\ \leq &C_{24}(\|\mathbf{u}e^{s\phi}\|^2_{\mathbf{H}^{2,s}(\mathcal{G})} + \|\mathbf{f}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(\mathcal{G})} + s\|\mathbf{g}e^{s\phi}\|^2_{\mathbf{H}^{1,s}(\partial\mathcal{G})}). \end{aligned} \tag{4.42}$$

Setting $\widetilde{P}_{\beta,s}(y,s,D) = \frac{1}{\beta(1+|\ell'(y_1)|^2)}P_{\beta,s}(y,s,D)$, we have

$$\widetilde{P}_{\mu,s}(y,s,D)\chi_{-2}w_1 + [\chi_{-2}, \widetilde{P}_{\mu,s}]w_1 = \chi_{-2}\left(\frac{1}{\mu(1+|\ell'(y_1)|^2)}q_1\right)$$

and

$$\widetilde{P}_{\lambda+2\mu,s}(y,s,D)\chi_{-2}w_2 + [\chi_{-2}, \widetilde{P}_{\lambda+2\mu,s}]w_2 = \chi_{-2}\left(\frac{1}{(\lambda+2\mu)(1+|\ell'(y_1)|^2)}q_2\right).$$

By (1.62) we have

$$\begin{aligned} \sqrt{s}\|\chi_{-2}\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})} \leq C_{25}\Bigg\{ &\|\mathbf{f}e^{s\phi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \sqrt{s}\|\mathbf{g}e^{s\phi}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})} \\ + &\|[\chi_{-2}, \widetilde{P}_{\mu,s}]w_1\|_{\mathbf{L}^2(\mathcal{G})} + \|[\chi_{-2}, \widetilde{P}_{\lambda+2\mu,s}]w_2\|_{L^2(\mathcal{G})} \\ + &\sqrt{s}\left\|\chi_{-2}\left(\frac{\partial\mathbf{w}}{\partial y_2}, \mathbf{w}\right)\right\|_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}\Bigg\}. \end{aligned} \tag{4.43}$$

Thanks to the estimates

$$\|[\chi_{-2},\widetilde{P}_{\mu,s}]w_1\|_{L^2(\mathcal{G})}+\|[\chi_{-2},\widetilde{P}_{\lambda+2\mu,s}]w_2\|_{L^2(\mathcal{G})}\le C_{26}(\|\mathbf{f}e^{s\phi}\|_{\mathbf{H}^{1,s}(\mathcal{G})}+\|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})})$$

and

$$\left\|\chi_{-2}\left(\frac{\partial \mathbf{w}}{\partial y_2},\mathbf{w}\right)\right\|_{\mathbf{L}^2(\partial\mathcal{G})\times\mathbf{H}^{1,s}(\partial\mathcal{G})}$$
$$\le C_{26}\left\{\frac{1}{\sqrt{s}}\|\mathbf{f}e^{s\phi}\|_{\mathbf{H}^{1,s}(\mathcal{G})}+\|\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})}+\|\chi_{-2}\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{2,s}(\partial\mathcal{G})}\right\},$$

we obtain

$$\sqrt{s}\|\chi_{-2}\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})}\le C_{27}(\|\mathbf{f}e^{s\phi}\|_{\mathbf{H}^{1,s}(\mathcal{G})}+\sqrt{s}\|\mathbf{g}e^{s\phi}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})}+\|(1-\chi_{-2})\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})}$$
$$+\sqrt{s}(\|\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})}+\|\chi_{-2}\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{2,s}(\partial\mathcal{G})})). \tag{4.44}$$

Estimating the last term at the right hand side of (4.44), we obtain

$$\sqrt{s}\|\chi_{-2}\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})}\le C_{28}(\|\mathbf{f}e^{s\phi}\|_{\mathbf{H}^{1,s}(\mathcal{G})}+\sqrt{s}\|\mathbf{g}e^{s\phi}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})}+\|(1-\chi_{-2})\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})}). \tag{4.45}$$

Since $\chi_{-1}|_{\mathrm{supp}(1-\chi_{-2})}=1$ by (4.39), we have

$$\|(1-\chi_{-2})\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})}\le\|\chi_{-1}\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})}\le C_{29}(\|\chi_{-1}(\mathbf{u}e^{s\phi})\|_{\mathbf{H}^{2,s}(\mathcal{G})}+\|\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{1,s}(\mathcal{G})})$$
$$\le C_{30}\left(\frac{1}{\sqrt{s}}\|\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{2,s}(\mathcal{G})}+\|\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{1,s}(\mathcal{G})}\right).$$

Hence, using this estimate, we obtain from (4.45)

$$\sqrt{s}\|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})}\le C_{31}\Bigg(\|\mathbf{f}e^{s\phi}\|_{\mathbf{H}^{1,s}(\mathcal{G})}+\sqrt{s}\|\mathbf{g}e^{s\phi}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})}$$
$$+\frac{1}{\sqrt{s}}\|\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{2,s}(\mathcal{G})}+\|\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{1,s}(\mathcal{G})}\Bigg). \tag{4.46}$$

Applying Proposition 1.2, we obtain

$$\sum_{|\alpha|=0,\alpha=(0,\alpha_1,\alpha_2)}^{2}s^{2-|\alpha|}\|\partial^\alpha(\mathbf{u}e^{s\phi})\|_{\mathbf{L}^2(\mathcal{G})}\le C_{32}\Bigg(\sqrt{s}\|\mathbf{w}\|_{\mathbf{H}^{1,s}(\mathcal{G})}$$
$$+\sqrt{s}\left\|\left(\mathbf{u},\frac{\partial \mathbf{u}}{\partial\vec{n}}\right)e^{s\phi}\right\|^2_{\mathbf{H}^{\frac{3}{2},s}(\partial\mathcal{G})\times\mathbf{H}^{\frac{1}{2},s}(\partial\mathcal{G})}+\|\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{1,s}(\mathcal{G})}\Bigg). \tag{4.47}$$

In view of equations (1.8) and (1.9), we can estimate $\partial_{y_0}^2(\mathbf{u}e^{s\phi})$:

$$\|\partial_{y_0}^2(\mathbf{u}e^{s\phi})\|_{\mathbf{L}^2(\mathcal{G})} \le C_{33}\left(\sum_{|\alpha|=0,\alpha=(0,\alpha_1,\alpha_2)}^{2} s^{2-|\alpha|}\|\partial^\alpha(\mathbf{u}e^{s\phi})\|_{\mathbf{L}^2(\mathcal{G})} + \|\mathbf{f}e^{s\phi}\|_{\mathbf{L}^2(\mathcal{G})}\right). \tag{4.48}$$

Hence (4.38) and (4.46) - (4.48) yield

$$\|\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{2,s}(\mathcal{G})} \le C_{34}\Bigg\{\|\mathbf{f}e^{s\phi}\|_{\mathbf{H}^{1,s}(\mathcal{G})} + \sqrt{s}\|\mathbf{g}e^{s\phi}\|_{\mathbf{H}^{1,s}(\partial\mathcal{G})}$$
$$+\frac{1}{\sqrt{s}}\|\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{2,s}(\mathcal{G})} + \|\mathbf{u}e^{s\phi}\|_{\mathbf{H}^{1,s}(\mathcal{G})}\Bigg\}. \tag{4.49}$$

Estimate (4.49) implies (1.24). Thus the proof of Theorem 1.1 is complete. ■

§5. Determination of the density and the Lamé coefficients by a single measurement.

As one important application of our Carleman estimate, in this section, we will solve an inverse problem of determining ρ, λ, μ with a single measurement, which is an open problem. We start with the introduction of notations. By $L_{\lambda,\mu}$, we denote the stationary part of the Lamé operator P:

$$(L_{\lambda,\mu}\mathbf{v})(\widetilde{x}) = \mu(\widetilde{x})\Delta\mathbf{v}(\widetilde{x}) + (\mu(\widetilde{x}) + \lambda(\widetilde{x}))\nabla_{\widetilde{x}}(\operatorname{div}\mathbf{v}(\widetilde{x}))$$
$$+(\operatorname{div}\mathbf{v}(\widetilde{x}))\nabla_{\widetilde{x}}\lambda(\widetilde{x}) + (\nabla_{\widetilde{x}}\mathbf{v} + (\nabla_{\widetilde{x}}\mathbf{v})^T)\nabla_{\widetilde{x}}\mu(\widetilde{x}), \quad \widetilde{x} \in \Omega.$$

We assume (1.3) for ρ, λ, μ. By $\mathbf{u} = \mathbf{u}(\lambda, \mu, \rho, \mathbf{p}, \mathbf{q}, \eta)(x)$, we denote the sufficiently smooth solution to

$$\rho(\widetilde{x})(\partial_{x_0}^2\mathbf{u})(x) = (L_{\lambda,\mu}\mathbf{u})(x), \quad x \in Q, \tag{5.1}$$

$$(\mathbb{B}(x, D)\mathbf{u})(x) = \eta(x), \qquad x \in (0,T) \times \partial\Omega, \tag{5.2}$$

$$\mathbf{u}(T/2, \widetilde{x}) = \mathbf{p}(\widetilde{x}), \quad (\partial_{x_0}\mathbf{u})(T/2, \widetilde{x}) = \mathbf{q}(\widetilde{x}), \quad \widetilde{x} \in \Omega, \tag{5.3}$$

where the functions η, $\mathbf{p}$ and $\mathbf{q}$ are suitably given functions in order that we can prove the unique existence of a sufficiently smooth solution to (5.1) - (5.3). For concise exposition, we do not describe conditions on $\eta, \mathbf{p}, \mathbf{q}$ for the well-posedness of the initial/boundary value problem (5.1) - (5.3), which can be executed by Lions and Magenes [LM].

Let $\omega \subset \Omega$ be a suitably given subdomain. In this section, we consider

Inverse Problem. Let $\mathbf{p}_j, \mathbf{q}_j, \eta_j$, $1 \le j \le \mathcal{N}$, be appropriately given. One have to determine $\lambda(\widetilde{x})$, $\mu(\widetilde{x})$, $\rho(\widetilde{x})$, $\widetilde{x} \in \Omega$, by

$$\mathbf{u}(\lambda, \mu, \rho, \mathbf{p}_j, \mathbf{q}_j, \eta_j)(x), \quad x \in Q_\omega \equiv (0,T) \times \omega. \tag{5.4}$$

In our formulation of the inverse problem, we exclusively take a finite number of observations (i.e., $\mathcal{N} < \infty$), especially $\mathcal{N} = 1$, and ours is different from the formulation by infinitely many boundary observations, that is, the Dirichlet-to-Neumann map (e.g., Rachele [Ra]).

For the formulation with a finite number of observations, we refer to Bukhgeim and Klibanov [BuK] as the first paper which used a Carleman estimate and established the uniqueness for similar inverse problems for scalar partial differential equations. See also Baudouin and Puel [BP], Bukhgeim [Bu], Bukhgeim, Cheng, Isakov and Yamamoto [BCIY], Imanuvilov and Yamamoto [IY1], [IY2], [IY3], Isakov [Is1], [Is2], [Is3], Isakov and Yamamoto [IsY], Khaĭdarov [Kh1], [Kh2], Klibanov [K], Klibanov and Timonov [KT], Puel and Yamamoto [PY1], [PY2], Yamamoto [Ya].

For inverse problems for the Lamé system with the Dirichlet boundary condition, we refer to Ikehata, Nakamura and Yamamoto [INY], Imanuvilov, Isakov and Yamamoto [IIY], Imanuvilov and Yamamoto [IY7], Isakov [Is1] where for determination of some (or all) of ρ, λ, μ, one measures the boundary stress on the whole boundary $\partial\Omega$ over a time interval $(0,T)$ ([INY], [Is1]) or $\mathbf{u}$ in $(0,T)\times\omega$ where ω is a suitable subdomain ([IIY]). In particular, as long as the Dirichlet case is concerned, in [IY7], the conditional stability in determining ρ, λ, μ in the case of $\mathcal{N}=1$, is proved.

On the other hand, the stress boundary condition is important also from the physical viewpoint and is more difficult mathematically: when we can observe solutions only on a subboundary or in a subdomain (which is quite practical), the previously know Carleman estimates are not applicable, since they require the Dirichlet boundary conditions on $(0,T)\times\partial\Omega$. Thus there are no results on the inverse problems for the Lamé system equipped by the free stress boundary condition with data on a subboundary or in a subdomain, although such a formulation is important.

The purpose of this section is prove the uniqueness and the conditional stability for this inverse problem with a single choice of initial data (i.e., $\mathcal{N}=1$.) Our argument here is similar to [IY3] and the key machinery is the Carleman estimate given in Corollary 1.1.

In order to formulate our main result, we will introduce notations and an admissible set of unknown parameters λ, μ, ρ. Henceforth we set $(\widetilde{x},\widetilde{y}) = \sum_{j=1}^2 x_j y_j$ for $\widetilde{x}=(x_1,x_2)$ and $\widetilde{y}=(y_1,y_2)$, and $\vec{t}(\widetilde{x})$ denotes the tangential vector on $\partial\Omega$ which is oriented counterclockwise. Let a subdomain $\omega\subset\Omega$ satisfy

$$\begin{aligned}\partial\omega \supset &\{\widetilde{x}\in\partial\Omega; ((\widetilde{x}-\widetilde{y}),\vec{n}(\widetilde{x}))\ge 0\}\\ &\cup\{\widetilde{x}\in\partial\Omega; |((\widetilde{x}-\widetilde{y}),\vec{t}(\widetilde{x}))|\le\varepsilon_0\}\equiv\Gamma \qquad (5.5)\end{aligned}$$

with some $\varepsilon_0>0$ and some $\widetilde{y}\notin\overline{\Omega}$.

Remark 1. In comparison with the condition on ω in the Dirichlet case ([IY7]), we have to choose a larger set ω in general. In fact, in the Dirichlet case for the Lamé system, it is sufficient that ω satisfies

$$\partial\omega \supset \{\widetilde{x} \in \partial\Omega; ((\widetilde{x}-\widetilde{y}), \vec{n}(\widetilde{x})) \ge 0\}, \tag{5.5'}$$

while $\partial\omega$ must include the second component of the right hand side of (5.5) in the free-stress case. As a consequence, for example, in the case where Ω is not simply connected whose boundary $\partial\Omega$ is divided into disjoint closed curves $\Gamma_1, ..., \Gamma_\ell$, condition (5.5) requires that $\partial\omega \cap \Gamma_j$ should contain relatively open subsets for all $j \in \{1, ..., \ell\}$. Accordingly for the inverse problem, we have to choose larger T than in the Dirichlet case, as is seen from (5.12) where θ is smaller in view of the extra third constraint in (5.9). We note that condition (5.5) is one sufficient condition for Condition 1.2, but it is very difficult to sharpen the geometric condition on ω and replace (5.5) by (5.5)'.

Remark 2. Under condition (5.5) on ω, we can prove the observability inequality for the wave equation with constant coefficients for sufficiently large T (e.g., [BLR], [Li]). Moreover, if $\frac{2\mu(\lambda+\mu)}{\rho(3\lambda+4\mu)} > 0$ on $\overline{\Omega}$ and $\frac{\lambda+2\mu}{\rho}$, $\frac{\mu}{\rho}$ satisfy (5.7) below, and $T > 0$ is sufficiently large, then by means of Corollary 1.1, we can prove the observability inequality for (5.1) with $\mathbb{B}(x, D)\mathbf{u} = 0$ on $(0, T) \times \partial\Omega$.

Denote

$$d = (\sup_{\widetilde{x}\in\Omega} |\widetilde{x}-\widetilde{y}|^2 - \inf_{\widetilde{x}\in\Omega} |\widetilde{x}-\widetilde{y}|^2)^{\frac{1}{2}}. \tag{5.6}$$

Next we define an admissible set of unknown coefficients λ, μ, ρ. We introduce the conditions:

$$\beta(\widetilde{x}) \ge \theta_1 > 0, \quad \widetilde{x} \in \overline{\Omega},$$
$$\|\beta\|_{C^3(\overline{\Omega})} \le M_0, \quad \frac{(\nabla_{\widetilde{x}}\beta(\widetilde{x}), (\widetilde{x}-\widetilde{y}))}{2\beta(\widetilde{x})} \le 1-\theta_0, \quad \widetilde{x} \in \overline{\Omega \setminus \omega} \tag{5.7}$$

for any fixed constants $M_0 \ge 0$ and $0 < \theta_0 \le 1$, $\theta_1 > 0$. For real-valued functions a_0, b_0, $\mathbb{R}^2$-valued functions $\mathbf{a}$, $\mathbf{b}$, η on $\partial\Omega$ and $\mathbf{p}$, $\mathbf{q}$ in Ω, we set

$$\mathcal{W} = \mathcal{W}(M_0, M_1, \theta_0, \theta_1, \theta_2, a_0, b_0, \mathbf{a}, \mathbf{b}) = \Big\{(\lambda, \mu, \rho) \in (C^3(\overline{\Omega}))^3;$$
$$\lambda = a_0, \mu = b_0, \nabla\lambda = \mathbf{a}, \nabla\mu = \mathbf{b} \quad \text{on } \partial\Omega,$$
$$\frac{2\mu(\lambda+\mu)}{\rho(3\lambda+4\mu)} \ge \theta_2 > 0 \quad \text{on } \overline{\Omega},$$
$$\frac{\lambda+2\mu}{\rho}, \frac{\mu}{\rho} \text{ satisfy (5.7)}, \|\mathbf{u}(\lambda, \mu, \rho, \mathbf{p}, \mathbf{q}, \eta)\|_{W^{7,\infty}(Q)} \le M_1\Big\}, \tag{5.8}$$

where the constants M_1 and $\theta_2 > 0$ are given. We choose $\theta > 0$ such that

$$\theta + \frac{M_0 d}{\sqrt{\theta_1}}\sqrt{\theta} < \theta_0\theta_1, \quad \theta_1 \inf_{\widetilde{x}\in\Omega\setminus\omega} |\widetilde{x}-\widetilde{y}|^2 - \theta d^2 > 0, \quad d\sqrt{\theta} < \varepsilon_0\sqrt{\theta_2}. \tag{5.9}$$

Here we note that since $\widetilde{y} \notin \overline{\Omega}$, such $\theta > 0$ exists.

Let (λ, μ, ρ) be an arbitrary element of $\mathcal{W}$.

Now we are ready to state

Theorem 5.1. *We assume that the functions* $\mathbf{p} = (p_1, p_2)^T$ *and* $\mathbf{q} = (q_1, q_2)^T$ *satisfy*

$$det \begin{pmatrix} (L_{\lambda,\mu}\mathbf{p})(\widetilde{x}) & (div\,\mathbf{p}(\widetilde{x}))E_2 & (\nabla_{\widetilde{x}}\mathbf{p}(\widetilde{x}) + (\nabla_{\widetilde{x}}\mathbf{p}(\widetilde{x}))^T)(\widetilde{x}-\widetilde{y}) \\ (L_{\lambda,\mu}\mathbf{q})(\widetilde{x}) & (div\,\mathbf{q}(\widetilde{x}))E_2 & (\nabla_{\widetilde{x}}\mathbf{q}(\widetilde{x}) + (\nabla_{\widetilde{x}}\mathbf{q}(\widetilde{x}))^T)(\widetilde{x}-\widetilde{y}) \end{pmatrix} \ne 0, \forall \widetilde{x} \in \overline{\Omega}, \tag{5.10}$$

$$det \begin{pmatrix} (L_{\lambda,\mu}\mathbf{p})(\widetilde{x}) & \nabla_{\widetilde{x}}\mathbf{p}(\widetilde{x}) + (\nabla_{\widetilde{x}}\mathbf{p}(\widetilde{x}))^T & (div\,\mathbf{p})(\widetilde{x}-\widetilde{y}) \\ (L_{\lambda,\mu}\mathbf{q})(\widetilde{x}) & \nabla_{\widetilde{x}}\mathbf{q}(\widetilde{x}) + (\nabla_{\widetilde{x}}\mathbf{q}(\widetilde{x}))^T & (div\,\mathbf{q})(\widetilde{x}-\widetilde{y}) \end{pmatrix} \ne 0, \forall \widetilde{x} \in \overline{\Omega}, \tag{5.11}$$

and

$$T > \frac{2}{\sqrt{\theta}} d. \tag{5.12}$$

Then there exist constants $\kappa = \kappa(\mathcal{W}, \omega, \Omega, T, \lambda, \mu, \rho) \in (0,1)$ *and* $C = C(\mathcal{W}, \omega, \Omega, T, \lambda, \mu, \rho) > 0$ *such that*

$$\begin{aligned} &\|\widetilde{\lambda} - \lambda\|_{H^1(\Omega)} + \|\widetilde{\mu} - \mu\|_{H^1(\Omega)} + \|\widetilde{\rho} - \rho\|_{L^2(\Omega)} \\ \le & C \sum_{k=1}^{4} \|\partial_{x_0}^k (\mathbf{u}(\lambda, \mu, \rho, \mathbf{p}, \mathbf{q}, \eta) - \mathbf{u}(\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\rho}, \mathbf{p}, \mathbf{q}, \eta))\|^{\kappa}_{\mathbf{H}^1(Q_\omega)} \end{aligned}$$

for any $(\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\rho}) \in \mathcal{W}$.

Our stability and uniqueness result requires only one measurement: $\mathcal{N} = 1$, but the conditions on the initial values $\mathbf{p}$, $\mathbf{q}$ are restrictive and we have to choose $\mathbf{p}$ and $\mathbf{q}$ satisfying (5.10) and (5.11).

Example of Ω, p, q meeting (5.10) - (5.11). We assume that λ, μ are positive constants. If we take

$$\mathbf{p}(\widetilde{x}) = \begin{pmatrix} 0 \\ (x_1 - y_1)(x_2 - y_2) \end{pmatrix} \quad \text{and} \quad \mathbf{q}(\widetilde{x}) = \begin{pmatrix} (x_2 - y_2)^2 \\ 0 \end{pmatrix},$$

then (5.10) and (5.11) are satisfied.

We set

$$\psi(x) = |\widetilde{x} - \widetilde{y}|^2 - \theta \left(x_0 - \frac{T}{2} \right)^2, \quad \phi(x) = e^{\tau \psi(x)}, \quad x = (x_0, \widetilde{x}) \in Q \tag{5.13}$$

with parameter $\tau > 0$.

First we show

Lemma 5.1. *Let* $(\lambda, \mu, \rho) \in \mathcal{W}$ *and let us assume (5.9). Then there exists* $\varepsilon_1 > 0$, *depending on* θ, θ_0, θ_1, θ_2, M_0, d, *such that the following property holds: For any*

$T \in \left(\frac{2d}{\sqrt{\theta}}, \frac{2d}{\sqrt{\theta}} + \varepsilon_1\right)$, *if we choose* $\tau > 0$ *sufficiently large, then the function* ψ *given by (5.13) satisfies Conditions 1.1 and 1.2 in* $Q \equiv (0,T) \times \Omega$ *with* Q_ω. *Therefore the conclusion of Corollaries 1.1 and 1.2 hold true and the constants* $C(\tau)$, $\widehat{\tau}$ *and* $s_0(\tau)$ *in the Carleman estimates can be taken uniformly in* $(\lambda, \mu, \rho) \in \mathcal{W}$.

Proof. The second and the third conditions in (1.6) are directly verified by means of (5.5). Conditions (1.4) and (1.5) can be verified by the same way as in Imanuvilov and Yamamoto [IY3], for example. Next we have to verify the first condition in (1.6). Without loss of generality, we may assume that $T = \frac{2d}{\sqrt{\theta}} + \widetilde{\varepsilon}$, where $\widetilde{\varepsilon} > 0$ is sufficiently small. Let $\beta = \frac{\lambda+2\mu}{\rho}$ or $= \frac{\mu}{\rho}$. Then it suffices to verify

$$-(\theta(x_0 - T/2))^2 + \beta(\widetilde{x})|\widetilde{x} - \widetilde{y}|^2 > 0$$

for $x \in [0,T] \times \overline{(\partial\Omega \setminus \partial\omega)}$. In fact, by means of (5.7) and (5.9), we have

$$\begin{aligned} &4\beta(\widetilde{x})|\widetilde{x} - \widetilde{y}|^2 - 4\theta^2\left(x_0 - \frac{T}{2}\right)^2 \ge 4\theta_1 \inf_{\widetilde{x}\in\Omega\setminus\omega} |\widetilde{x} - \widetilde{y}|^2 - \theta(\theta T^2) \\ \ge &4\theta_1 \inf_{\widetilde{x}\in\Omega\setminus\omega} |\widetilde{x} - \widetilde{y}|^2 - \theta(2d + \widetilde{\varepsilon}\sqrt{\theta})^2 > 0 \end{aligned}$$

because $\widetilde{\varepsilon} > 0$ is sufficiently small.

Finally we have to verify (1.27), which is written by means of the original $\widetilde{x}$-coordinate by

$$\frac{\partial\psi}{\partial x_0}(x) \pm \sqrt{\mathcal{C}}\frac{\partial\psi}{\partial\vec{t}}(x) \ne 0, \quad x \in [0,T] \times \overline{(\partial\Omega \setminus \partial\omega)}. \tag{5.14}$$

Here $\mathcal{C} \in \left(0, \frac{\mu}{\rho}\right)$ is the root of a cubic equation in t:

$$\begin{aligned} &\widetilde{h}(t) = t^3 - t^2\left(\frac{8\mu}{\rho}\right)(\widetilde{x}) + t\left(\frac{24\mu^2}{\rho^2} - \frac{16\mu^3}{\rho^2(\lambda+2\mu)}\right)(\widetilde{x}) \\ +&\left(\frac{16\mu^3(\lambda+3\mu)}{\rho^3(\lambda+2\mu)} - \frac{32\mu^3}{\rho^3}\right)(\widetilde{x}). \end{aligned}$$

We will give a lower bound of $\mathcal{C}$. Henceforth we fix $\widetilde{x} \in \overline{\Omega}$ arbitrarily. We have

$$\widetilde{h}'(t) = 3t^2 - \frac{16\mu}{\rho}t + \left(\frac{24\mu^2}{\rho^2} - \frac{16\mu^3}{\rho^2(\lambda+2\mu)}\right),$$

and, noting that $\left(\frac{8\mu}{3\rho}, \widetilde{h}\left(\frac{8\mu}{3\rho}\right)\right)$ is the apex of the parabola defined by $\widetilde{h}'(t)$, by (1.3), we can directly see that

$$\max_{0\le t\le \frac{\mu}{\rho}} |\widetilde{h}'(t)| = \max\left\{|\widetilde{h}'(0)|, \left|\widetilde{h}'\left(\frac{\mu}{\rho}\right)\right|\right\}$$
$$= \frac{8\mu^2}{\rho^2}\frac{3\lambda+4\mu}{\lambda+2\mu}.$$

Since $\widetilde{h}(\mathcal{C}) = 0$, in terms of (1.3) and the mean value theorem, we can choose $\widehat{t} \in \left(0, \frac{\mu}{\rho}\right)$ such that

$$\frac{16\mu^3(\lambda+\mu)}{\rho^3(\lambda+2\mu)} = |\widetilde{h}(0)| = |\widetilde{h}(0) - \widetilde{h}(\mathcal{C})| = |\widetilde{h}'(\widehat{t})|\mathcal{C} \le \frac{8\mu^2(3\lambda+4\mu)}{\rho^2(\lambda+2\mu)}\mathcal{C}.$$

Hence

$$\mathcal{C} \ge \frac{2\mu(\lambda+\mu)}{\rho(3\lambda+4\mu)} \ge \theta_2 \quad \text{on } \overline{\Omega} \tag{5.15}$$

by the condition in (5.8). Therefore, applying (5.15) in (5.14), by (5.5) we have

$$\left|\frac{\partial\psi}{\partial x_0}(x) \pm \sqrt{\mathcal{C}}\frac{\partial\psi}{\partial\vec{t}}(x)\right| \ge 2\sqrt{\mathcal{C}}|((\widetilde{x}-\widetilde{y}), \vec{t}(\widetilde{x}))| - 2\theta\left|t - \frac{T}{2}\right|$$
$$\ge 2\sqrt{\theta_2}\varepsilon_0 - \theta T = 2(\sqrt{\theta_2}\varepsilon_0 - \sqrt{\theta}d) - \theta\varepsilon_1$$

for $x \in [0,T] \times \overline{(\partial\Omega \setminus \partial\omega)}$. Consequently, for sufficiently small $\varepsilon_1 > 0$, we have

$$\left|\frac{\partial\psi}{\partial x_0}(x) \pm \sqrt{\mathcal{C}}\frac{\partial\psi}{\partial\vec{t}}(x)\right| > 0$$

for $x \in [0,T] \times \overline{(\partial\Omega \setminus \partial\omega)}$. Thus (1.27) holds, so that Condition 1.2 is satisfied by Proposition 1.1.

The uniformity of the constants $C_1(\tau)$, $\widehat{\tau}$ and $s_0(\tau)$ follows similarly to [IIY]. Thus the proof of Lemma 5.1 is complete. ■

Next we prove a Carleman estimate for a first order partial differential operator

$$(P_0 g)(\widetilde{x}) = \sum_{j=1}^{2} p_{0,j}(\widetilde{x})\partial_{x_j} g(\widetilde{x}),$$

where $p_{0,j} \in C^1(\overline{\Omega})$, $j = 1, 2$.

Lemma 5.2. *We assume*

$$\sum_{j=1}^{2} p_{0,j}(\widetilde{x})\partial_{x_j}\phi(T/2, \widetilde{x}) \neq 0, \quad \widetilde{x} \in \overline{\Omega}. \tag{5.16}$$

Then there exists a constant $\tau_0 > 0$ such that for all $\tau > \tau_0$, there exist $s_0 = s_0(\tau) > 0$ and $C_1 = C_1(s_0, \tau_0, \Omega, \omega) > 0$ such that

$$s^2 \|g e^{s\phi(T/2,\cdot)}\|^2_{H^{1,s}(\Omega)} \le C_1 \|(P_0 g) e^{s\phi(T/2,\cdot)}\|^2_{H^{1,s}(\Omega)}$$

for all $s > s_0$ and $g \in H^2_0(\Omega)$.

Proof. For simplicity, we set $P_0 g = h$, $\phi_0(\widetilde{x}) = \phi(T/2, \widetilde{x})$ and $w = e^{s\phi_0} g$, $Q_0 w = e^{s\phi_0} P_0(e^{-s\phi_0} w)$. By Lemma 3.3 (see [IY7]) we have

$$s^4 \int_\Omega |g|^2 e^{2s\phi_0} d\widetilde{x} \le C_2'' s^2 \int_\Omega |P_0 g|^2 e^{2s\phi_0} d\widetilde{x} \tag{5.17}$$

for all sufficiently large $s > 0$. Next we observe

$$P_0(\partial_{x_k} g) = \partial_{x_k} h - \sum_{j=1}^{2} (\partial_{x_k} p_{0,j})(\widetilde{x})(\partial_{x_j} g)(\widetilde{x}), \quad k = 1, 2.$$

By $\partial_{x_k} g \in H^1_0(\Omega)$, we can apply Lemma 3.3 from [IY7] again, so that we have

$$\begin{aligned} & s^2 \int_\Omega |\nabla' g|^2 e^{2s\phi_0} d\widetilde{x} \\ \le & C_3 \int_\Omega |\nabla' h|^2 e^{2s\phi_0} d\widetilde{x} + C_3 \int_\Omega \left| \sum_{j=1}^{2} (\partial_{x_k} p_{0,j})(\widetilde{x})(\partial_{x_j} g)(\widetilde{x}) \right|^2 e^{2s\phi_0} d\widetilde{x} \\ \le & C_3' \int_\Omega |\nabla'(P_0 g)|^2 e^{2s\phi_0} d\widetilde{x} + C_3' \int_\Omega |\nabla' g|^2 e^{2s\phi_0} d\widetilde{x}. \end{aligned}$$

Consequently we can absorb the second term at the right hand side into the left hand side, so that we can obtain

$$s^2\int_\Omega |\nabla' g|^2 e^{2s\phi_0} d\widetilde{x} \le C_3''\int_\Omega |\nabla'(P_0 g)|^2 e^{2s\phi_0} d\widetilde{x}. \tag{5.18}$$

Since $\partial_{x_k}(ge^{\phi_0}) = (\partial_{x_k} g)e^{s\phi_0} + s(\partial_{x_k}\phi_0)ge^{s\phi_0}$, there exist constants $C_4, C_4' > 0$ such that

$$\begin{aligned} &C_4\int_\Omega (s^2|g|^2 + |\nabla' g|^2)e^{2s\phi_0} d\widetilde{x} \le \|ge^{s\phi_0}\|^2_{H^{1,s}(\Omega)} \\ \le &C_4'\int_\Omega (s^2|g|^2 + |\nabla' g|^2)e^{2s\phi_0} d\widetilde{x}. \end{aligned}$$

Thus estimates (5.17) and (5.18) are equivalent to the conclusion of the lemma, and the proof is complete. ■

Now we proceed to

Proof of Theorem 5.1. The proof is similar to Imanuvilov and Yamamoto [IY7], and the main difference is the application of the L^2-Carleman estimate (Corollary 1.1). Henceforth, for simplicity, we set

$$\mathbf{u} = \mathbf{u}(\lambda, \mu, \rho, \mathbf{p}, \mathbf{q}, \eta), \quad \mathbf{v} = \mathbf{u}(\widetilde{\lambda}, \widetilde{\mu}, \widetilde{\rho}, \mathbf{p}, \mathbf{q}, \eta)$$

and

$$\mathbf{y} = \mathbf{u} - \mathbf{v}, \quad f = \rho - \widetilde{\rho}, \quad g = \lambda - \widetilde{\lambda}, \quad h = \mu - \widetilde{\mu}.$$

Then

$$\widetilde{\rho}\partial_{x_0}^2 \mathbf{y} = L_{\widetilde{\lambda},\widetilde{\mu}}\mathbf{y} + G\mathbf{u} \quad \text{in } Q \tag{5.19}$$

and

$$\mathbf{y}\left(\frac{T}{2}, \widetilde{x}\right) = \partial_{x_0}\mathbf{y}\left(\frac{T}{2}, \widetilde{x}\right) = 0, \qquad \widetilde{x} \in \Omega \tag{5.20}$$

and

$$\mathbb{B}(x, D)\mathbf{y} = 0 \qquad \text{in } (0, T) \times \partial\Omega. \tag{5.21}$$

Here we set

$$G\mathbf{u}(x) = -f\partial_{x_0}^2\mathbf{u}(x) + (g+h)(\widetilde{x})\nabla_{\widetilde{x}}(\operatorname{div}\mathbf{u})(x) + h(\widetilde{x})\Delta\mathbf{u}(x)$$

$$+(\operatorname{div}\mathbf{u})(x)\nabla_{\widetilde{x}}g(\widetilde{x}) + (\nabla_{\widetilde{x}}\mathbf{u}(x) + (\nabla_{\widetilde{x}}\mathbf{u}(x))^T)\nabla h(\widetilde{x}). \tag{5.22}$$

By (5.12), we have the inequality $\frac{\theta T^2}{4} > d^2$. Therefore, by (5.6) and definition (5.13) of the function ϕ, we have

$$\phi(T/2, \widetilde{x}) \geq d_1, \quad \phi(0, \widetilde{x}) = \phi(T, \widetilde{x}) < d_1, \qquad \widetilde{x} \in \overline{\Omega}$$

with $d_1 = \exp(\tau \inf_{\widetilde{x}\in\Omega} |\widetilde{x} - \widetilde{y}|^2)$. Thus, for given $\varepsilon > 0$, we can choose a sufficiently small $\delta = \delta(\varepsilon) > 0$ such that

$$\phi(x) \geq d_1 - \varepsilon, \quad x \in \left[\frac{T}{2} - \delta, \frac{T}{2} + \delta\right] \times \overline{\Omega} \tag{5.23}$$

and

$$\phi(x) \leq d_1 - 2\varepsilon, \quad x \in ([0, 2\delta] \cup [T - 2\delta, T]) \times \overline{\Omega}. \tag{5.24}$$

In order to apply Lemma 5.1, it is necessary to introduce a cut-off function χ satisfying $0 \leq \chi \leq 1$, $\chi \in C^\infty(\mathbb{R})$ and

$$\chi(x_0) = \begin{cases} 0, & x_0 \in [0, \delta] \cup [T - \delta, T], \\ 1, & x_0 \in [2\delta, T - 2\delta]. \end{cases} \tag{5.25}$$

Henceforth $C_j > 0$ denotes generic constants depending on s_0, τ, M_0, M_1, θ_0, θ_1, θ_2, η, Ω, T, $\widetilde{y}$, ω, χ and $\mathbf{p}$, $\mathbf{q}$, ε, δ, but independent of $s > s_0$.

Setting $\mathbf{z}_1 = \chi\partial_{x_0}^2\mathbf{y}$, $\mathbf{z}_2 = \chi\partial_{x_0}^3\mathbf{y}$ and $\mathbf{z}_3 = \chi\partial_{x_0}^4\mathbf{y}$, we have

$$\begin{cases} \widetilde{\rho}\partial_{x_0}^2\mathbf{z}_1 = L_{\widetilde{\lambda},\widetilde{\mu}}\mathbf{z}_1 + \chi G(\partial_{x_0}^2\mathbf{u}) + 2\widetilde{\rho}(\partial_{x_0}\chi)\partial_{x_0}^3\mathbf{y} + \widetilde{\rho}(\partial_{x_0}^2\chi)\partial_{x_0}^2\mathbf{y}, \\ \widetilde{\rho}\partial_{x_0}^2\mathbf{z}_2 = L_{\widetilde{\lambda},\widetilde{\mu}}\mathbf{z}_2 + \chi G(\partial_{x_0}^3\mathbf{u}) + 2\widetilde{\rho}(\partial_{x_0}\chi)\partial_{x_0}^4\mathbf{y} + \widetilde{\rho}(\partial_{x_0}^2\chi)\partial_{x_0}^3\mathbf{y}, \\ \widetilde{\rho}\partial_{x_0}^2\mathbf{z}_3 = L_{\widetilde{\lambda},\widetilde{\mu}}\mathbf{z}_3 + \chi G(\partial_{x_0}^4\mathbf{u}) + 2\widetilde{\rho}(\partial_{x_0}\chi)\partial_{x_0}^5\mathbf{y} + \widetilde{\rho}(\partial_{x_0}^2\chi)\partial_{x_0}^4\mathbf{y} \quad \text{in } Q. \end{cases} \tag{5.26}$$

Henceforth we set

$$\mathcal{E} = \sum_{j=0}^{4} \|(\partial_{x_0}^j \mathbf{y})e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q_\omega)}.$$

Noting $\mathbf{u} \in W^{7,\infty}(Q)$, in view of (5.25) and Lemma 5.1, we can apply Corollary 1.1 to (5.26), so that

$$\begin{aligned}
&\sum_{j=2}^{4} \|(\partial_{x_0}^j \mathbf{y})\chi e^{s\phi}\|^2_{\mathbf{H}^{1,s}(Q)} \\
\le& C_5 \left\{ \|fe^{s\phi}\|^2_{L^2(Q)} + \|ge^{s\phi}\|^2_{L^2(Q)} + \|(\nabla g)e^{s\phi}\|^2_{L^2(Q)} + \|he^{s\phi}\|^2_{L^2(Q)} + \|(\nabla h)e^{s\phi}\|^2_{L^2(Q)} \right\} \\
+& C_5 \sum_{j=3}^{5} \|(\partial_{x_0}\chi)(\partial_{x_0}^j \mathbf{y})e^{s\phi}\|^2_{L^2(Q)} + C_5 \sum_{j=2}^{4} \|(\partial_{x_0}^2\chi)(\partial_{x_0}^j \mathbf{y})e^{s\phi}\|^2_{L^2(Q)} + C_5\mathcal{E} \\
\le& C_6 \left\{ \|fe^{s\phi}\|^2_{L^2(Q)} + \|ge^{s\phi}\|^2_{L^2(Q)} + \|(\nabla g)e^{s\phi}\|^2_{L^2(Q)} + \|he^{s\phi}\|^2_{L^2(Q)} + \|(\nabla h)e^{s\phi}\|^2_{L^2(Q)} \right\} \\
+& C_6 e^{2s(d_1 - 2\varepsilon)} + C_6\mathcal{E}
\end{aligned} \tag{5.27}$$

for all large $s > 0$.

On the other hand, it follows from (5.24) and (5.25) that

$$\begin{aligned}
&\|(\partial_{x_0}^2 \mathbf{y})(T/2, \widetilde{x})e^{s\phi(T/2,\widetilde{x})}\|^2_{\mathbf{H}^{1,s}(\Omega)} \\
=& \int_0^{T/2} \frac{\partial}{\partial x_0} \left(\chi^2(x_0) \|(\partial_{x_0}^2 \mathbf{y})(x_0, \widetilde{x})e^{s\phi(x_0,\widetilde{x})}\|^2_{\mathbf{H}^{1,s}(\Omega)} \right) dx_0 \\
=& \int_0^{T/2} 2((\partial_{x_0}^3 \mathbf{y})e^{s\phi}, (\partial_{x_0}^2 \mathbf{y})e^{s\phi})_{\mathbf{H}^{1,s}(\Omega)} \chi^2 dx \\
+& 2s \int_0^{T/2} \chi^2 \|(\partial_{x_0}^2 \mathbf{y})(\partial_{x_0}\phi)e^{s\phi}\|^2_{\mathbf{H}^{1,s}(\Omega)} dx_0 + \int_0^{T/2} \partial_{x_0}(\chi^2) \|(\partial_{x_0}^2 \mathbf{y})e^{s\phi}\|^2_{\mathbf{H}^{1,s}(\Omega)} dx_0 \\
\le& C_7 \int_0^T s\chi^2 (\|(\partial_{x_0}^3 \mathbf{y})e^{s\phi}\|^2_{H^{1,s}(\Omega)} + \|(\partial_{x_0}^2 \mathbf{y})e^{s\phi}\|^2_{\mathbf{H}^{1,s}(\Omega)}) dx_0 + C_7 e^{2s(d_1 - 2\varepsilon)}.
\end{aligned}$$

Therefore (5.27) yields

$$\begin{aligned}
&\|(\partial_{x_0}^2 \mathbf{y})(T/2, \widetilde{x})e^{s\phi(T/2,\widetilde{x})}\|^2_{\mathbf{H}^{1,s}(\Omega)} \\
\le& C_8 s \int_Q (|f|^2 + |g|^2 + |\nabla g|^2 + |h|^2 + |\nabla h|^2) e^{2s\phi} dx + C_8 s e^{2s(d_1 - 2\varepsilon)} + C_8 s\mathcal{E}
\end{aligned} \tag{5.28}$$

for all large $s>0$. Similarly we can estimate $\|(\partial_{x_0}^3\mathbf{y})(T/2,\widetilde{x})e^{s\phi(T/2,\widetilde{x})}\|^2_{\mathbf{H}^{1,s}(\Omega)}$ to obtain

$$\begin{aligned}
&\|(\partial_{x_0}^2\mathbf{y})(T/2,\widetilde{x})e^{s\phi(T/2,\widetilde{x})}\|^2_{\mathbf{H}^{1,s}(\Omega)}+\|(\partial_{x_0}^3\mathbf{y})(T/2,\widetilde{x})e^{s\phi(T/2,\widetilde{x})}\|^2_{\mathbf{H}^{1,s}(\Omega)}\\
\le& C_9 s\int_Q(|f|^2+|g|^2+|h|^2+|\nabla g|^2+|\nabla h|^2)e^{2s\phi}dx\\
&+C_9 s e^{2s(d_1-2\varepsilon)}+C_9 s\mathcal{E} \qquad (5.29)
\end{aligned}$$

for all large $s>0$.

On the other hand, by (5.19), (5.20) and $\mathbf{u},\mathbf{v}\in W^{7,\infty}(Q)$, we have

$$\widetilde{\rho}\partial_{x_0}^2\mathbf{y}\left(\frac{T}{2},\widetilde{x}\right)=G\mathbf{u}\left(\frac{T}{2},\widetilde{x}\right),\quad \widetilde{\rho}\partial_{x_0}^3\mathbf{y}\left(\frac{T}{2},\widetilde{x}\right)=G\partial_{x_0}\mathbf{u}\left(\frac{T}{2},\widetilde{x}\right). \qquad (5.30)$$

Then, setting

$$\begin{cases}
\begin{pmatrix}a_{11}\\a_{21}\end{pmatrix}=-\dfrac{1}{\rho}L_{\lambda,\mu}\mathbf{p}, \quad \begin{pmatrix}a_{12}\\a_{22}\end{pmatrix}=-\dfrac{1}{\rho}L_{\lambda,\mu}\mathbf{q},\\
b_1=\operatorname{div}\mathbf{p}, \qquad b_2=\operatorname{div}\mathbf{q},\\
\begin{pmatrix}c_1 & d_1\\ d_1 & e_1\end{pmatrix}=\nabla_{\widetilde{x}}\mathbf{p}+(\nabla_{\widetilde{x}}\mathbf{p})^T, \quad \begin{pmatrix}c_2 & d_2\\ d_2 & e_2\end{pmatrix}=\nabla_{\widetilde{x}}\mathbf{q}+(\nabla_{\widetilde{x}}\mathbf{q})^T,\\
\begin{pmatrix}G_{11}\\G_{21}\end{pmatrix}=\widetilde{\rho}(\partial_{x_0}^2\mathbf{y})\left(\dfrac{T}{2},\widetilde{x}\right)-(g+h)\nabla_{\widetilde{x}}(\operatorname{div}\mathbf{p})-h\Delta\mathbf{p},\\
\begin{pmatrix}G_{12}\\G_{22}\end{pmatrix}=\widetilde{\rho}(\partial_{x_0}^3\mathbf{y})\left(\dfrac{T}{2},\widetilde{x}\right)-(g+h)\nabla_{\widetilde{x}}(\operatorname{div}\mathbf{q})-h\Delta\mathbf{q},
\end{cases} \qquad (5.31)$$

we rewrite (5.30) as

$$\begin{pmatrix}a_{11} & b_1 & 0\\ a_{21} & 0 & b_1\\ a_{12} & b_2 & 0\\ a_{22} & 0 & b_2\end{pmatrix}\begin{pmatrix}f\\ \partial_{x_1}g\\ \partial_{x_2}g\end{pmatrix}=\begin{pmatrix}G_{11}-c_1\partial_{x_1}h-d_1\partial_{x_2}h\\ G_{21}-d_1\partial_{x_1}h-e_1\partial_{x_2}h\\ G_{12}-c_2\partial_{x_1}h-d_2\partial_{x_2}h\\ G_{22}-d_2\partial_{x_1}h-e_2\partial_{x_2}h\end{pmatrix}. \qquad (5.32)$$

Because linear system (5.32) possesses a solution $(f,\partial_{x_1}g,\partial_{x_2}g)$, the coefficient matrix must satisfy

$$\det\begin{pmatrix}a_{11} & b_1 & 0 & G_{11}-c_1\partial_{x_1}h-d_1\partial_{x_2}h\\ a_{21} & 0 & b_1 & G_{21}-d_1\partial_{x_1}h-e_1\partial_{x_2}h\\ a_{12} & b_2 & 0 & G_{12}-c_2\partial_{x_1}h-d_2\partial_{x_2}h\\ a_{22} & 0 & b_2 & G_{22}-d_2\partial_{x_1}h-e_2\partial_{x_2}h\end{pmatrix}=0,$$

that is,

$$(\partial_{x_1} h)\det\begin{pmatrix} a_{11} & b_1 & 0 & c_1 \\ a_{21} & 0 & b_1 & d_1 \\ a_{12} & b_2 & 0 & c_2 \\ a_{22} & 0 & b_2 & d_2 \end{pmatrix} + (\partial_{x_2} h)\det\begin{pmatrix} a_{11} & b_1 & 0 & d_1 \\ a_{21} & 0 & b_1 & e_1 \\ a_{12} & b_2 & 0 & d_2 \\ a_{22} & 0 & b_2 & e_2 \end{pmatrix}$$
$$= \det\begin{pmatrix} a_{11} & b_1 & 0 & G_{11} \\ a_{21} & 0 & b_1 & G_{21} \\ a_{12} & b_2 & 0 & G_{12} \\ a_{22} & 0 & b_2 & G_{22} \end{pmatrix}, \tag{5.33}$$

by the linearity of the determinant. Under condition (5.11), taking into consideration that $h \equiv \mu - \widetilde{\mu} \in H_0^2(\Omega)$ by the definition of $\mathcal{W}$ and considering (5.35) as a first order partial differential operator in h, we apply Lemma 5.2, so that

$$s^2 \| h e^{s\phi(T/2,\cdot)} \|^2_{H^{1,s}(\Omega)} \le C_{10} \left\| \det\begin{pmatrix} a_{11} & b_1 & 0 & G_{11} \\ a_{21} & 0 & b_1 & G_{21} \\ a_{12} & b_2 & 0 & G_{12} \\ a_{22} & 0 & b_2 & G_{22} \end{pmatrix} e^{s\phi(T/2,\cdot)} \right\|^2_{H^{1,s}(\Omega)}$$
$$\le C_{11}(\|(\partial^2_{x_0}\mathbf{y})(T/2,\cdot)e^{s\phi(T/2,\cdot)}\|^2_{\mathbf{H}^{1,s}(\Omega)} + \|(\partial^3_{x_0}\mathbf{y})(T/2,\cdot)e^{s\phi(T/2,\cdot)}\|^2_{\mathbf{H}^{1,s}(\Omega)})$$
$$+ C_{11}(\|g e^{s\phi(T/2,\cdot)}\|^2_{H^{1,s}(\Omega)} + \|h e^{s\phi(T/2,\cdot)}\|^2_{H^{1,s}(\Omega)}), \tag{5.34}$$

in view of (5.31). We rewrite (5.30) as

$$\begin{pmatrix} a_{11} & c_1 & d_1 \\ a_{21} & d_1 & e_1 \\ a_{12} & c_2 & d_2 \\ a_{22} & d_2 & e_2 \end{pmatrix} \begin{pmatrix} f \\ \partial_{x_1} h \\ \partial_{x_2} h \end{pmatrix} = \begin{pmatrix} G_{11} - b_1\partial_{x_1} g \\ G_{21} - b_1\partial_{x_2} g \\ G_{12} - b_2\partial_{x_1} g \\ G_{22} - b_2\partial_{x_2} g \end{pmatrix}$$

and, using (5.12), we can similarly deduce

$$s^2 \| g e^{s\phi(T/2,\cdot)} \|^2_{H^{1,s}(\Omega)}$$
$$\le C_{12}(\|(\partial^2_{x_0}\mathbf{y})(T/2,\cdot)e^{s\phi(T/2,\cdot)}\|^2_{\mathbf{H}^{1,s}(\Omega)} + \|(\partial^3_{x_0}\mathbf{y})(T/2,\cdot)e^{s\phi(T/2,\cdot)}\|^2_{\mathbf{H}^{1,s}(\Omega)})$$
$$+ C_{12}(\|g e^{s\phi(T/2,\cdot)}\|^2_{H^{1,s}(\Omega)} + C_{12}\|h e^{s\phi(T/2,\cdot)}\|^2_{H^{1,s}(\Omega)}) \tag{5.35}$$

for all large $s > 0$. In (5.34) and (5.35), for sufficiently large $s > 0$, we can absorb the third and the fourth terms at the right hand sides into the left hand sides, so that we have

$$s^2(\|ge^{s\phi(T/2,\cdot)}\|^2_{H^{1,s}(\Omega)} + \|he^{s\phi(T/2,\cdot)}\|^2_{H^{1,s}(\Omega)})$$
$$\le C_{13}(\|(\partial^2_{x_0}\mathbf{y})(T/2,\cdot)e^{s\phi(T/2,\cdot)}\|^2_{\mathbf{H}^{1,s}(\Omega)} + \|(\partial^3_{x_0}\mathbf{y})(T/2,\cdot)e^{s\phi(T/2,\cdot)}\|^2_{\mathbf{H}^{1,s}(\Omega)}). \tag{5.36}$$

On the other hand, by (5.32), we have

$$\begin{pmatrix} -\frac{1}{\rho}L_{\lambda,\mu}\mathbf{p} \\ -\frac{1}{\rho}L_{\lambda,\mu}\mathbf{q} \end{pmatrix} f = \begin{pmatrix} -b_1\partial_{x_1}g + G_{11} - c_1\partial_{x_1}h - d_1\partial_{x_2}h \\ -b_1\partial_{x_2}g + G_{21} - d_1\partial_{x_1}h - e_1\partial_{x_2}h \\ -b_2\partial_{x_1}g + G_{12} - c_2\partial_{x_1}h - d_2\partial_{x_2}h \\ -b_2\partial_{x_2}g + G_{22} - d_2\partial_{x_1}h - e_2\partial_{x_2}h \end{pmatrix}$$

on $\overline{\Omega}$. Since, in view of (5.11), we have

$$\left|\left(\frac{1}{\rho}L_{\lambda,\mu}\mathbf{p}\right)(\widetilde{x})\right| + \left|\left(\frac{1}{\rho}L_{\lambda,\mu}\mathbf{q}\right)(\widetilde{x})\right| > 0, \quad \widetilde{x} \in \overline{\Omega},$$

we can solve the above equation in f, and

$$|f(\widetilde{x})| \le C_{14}\left(|\nabla' g(\widetilde{x})| + |\nabla' h(\widetilde{x})| + \sum_{j,k=1}^{2}|G_{jk}(\widetilde{x})|\right), \quad \widetilde{x} \in \overline{\Omega}.$$

Therefore (5.36) yields

$$s^2\int_\Omega |f(\widetilde{x})|^2 e^{2s\phi(T/2,\widetilde{x})}d\widetilde{x}$$
$$\le C_{15}s^2(\|ge^{s\phi(T/2,\cdot)}\|^2_{H^{1,s}(\Omega)} + \|he^{s\phi(T/2,\cdot)}\|^2_{H^{1,s}(\Omega)})$$
$$+C_{15}(\|(\partial^2_{x_0}\mathbf{y})(T/2,\cdot)e^{s\phi(T/2,\cdot)}\|^2_{\mathbf{H}^{1,s}(\Omega)} + \|(\partial^3_{x_0}\mathbf{y})(T/2,\cdot)e^{s\phi(T/2,\cdot)}\|^2_{\mathbf{H}^{1,s}(\Omega)})$$
$$\le C_{16}(\|(\partial^2_{x_0}\mathbf{y})(T/2,\cdot)e^{s\phi(T/2,\cdot)}\|^2_{\mathbf{H}^{1,s}(\Omega)} + \|(\partial^3_{x_0}\mathbf{y})(T/2,\cdot)e^{s\phi(T/2,\cdot)}\|^2_{\mathbf{H}^{1,s}(\Omega)}) \tag{5.37}$$

for all large $s > 0$.

Consequently, substituting (5.36) and (5.37) into (5.29) and using $\phi(T/2,\widetilde{x}) \ge \phi(x_0,\widetilde{x})$ for $(x_0,\widetilde{x}) \in Q$, we obtain

$$\int_\Omega (|f|^2 + |g|^2 + |\nabla g|^2 + |h|^2 + |\nabla h|^2)e^{2s\phi(T/2,\widetilde{x})}d\widetilde{x}$$
$$\le \frac{C_{17}T}{s}\int_\Omega (|f|^2 + |g|^2 + |\nabla g|^2 + |h|^2 + |\nabla h|^2)e^{2s\phi(T/2,\widetilde{x})}d\widetilde{x} + \frac{C_{17}}{s}e^{2s(d_1-2\varepsilon)} + \frac{C_{17}}{s}\mathcal{E}$$

for all large $s > 0$. Taking $s > 0$ sufficiently large and noting $e^{2s\phi(T/2,\widetilde{x})} \ge e^{2sd_1}$ for $\widetilde{x} \in \overline{\Omega}$, we obtain

$$\int_\Omega (|f|^2 + |g|^2 + |\nabla g|^2 + |h|^2 + |\nabla h|^2) d\widetilde{x} \le C_{18} e^{-4s\varepsilon} + C_{19} e^{2sC_{20}} \mathcal{E}$$

for all large $s > s_0$: a constant which is dependent on τ, but independent of s. Therefore, setting $C_{21} = C_{19} e^{2s_0 C_{20}}$, we have

$$\int_\Omega (|f|^2 + |g|^2 + |\nabla g|^2 + |h|^2 + |\nabla h|^2) d\widetilde{x} \le C_{18} e^{-4s\varepsilon} + C_{21} e^{2sC_{20}} \mathcal{E} \tag{5.38}$$

for all $s > s_0$.

Now we choose $s > 0$ such that

$$e^{2sC_{20}} \mathcal{E} = e^{-4s\varepsilon},$$

that is,

$$s = -\frac{1}{4\varepsilon + 2C_{20}} \ln \mathcal{E}.$$

Here we may assume that $\mathcal{E} < 1$ and so $s > 0$. Then it follows from (5.38) that

$$\begin{aligned} &\int_\Omega (|f|^2 + |g|^2 + |h|^2 + |\nabla g|^2 + |\nabla h|^2) d\widetilde{x} \\ \le & 2C \mathcal{E}^{\frac{4\varepsilon}{4\varepsilon + 2C}}. \end{aligned}$$

The proof of Theorem 5.1 is complete. ■

Appendix I. Proof of Proposition 1.3.

Proof of Proposition 1.3. It suffices to prove the estimate for an arbitrary but fixed $x_0 \in [0, T]$. That is, we should establish the following estimate: There exist

$\widehat{\tau} > 1$ and $N_0 > 1$ such that for any $\tau > \widehat{\tau}$ and $N > N_0$, there exists $s_0(\tau, N) > 0$ such that

$$N \int_{\Omega_N} \left(\frac{1}{s\varphi} \sum_{j,k=1}^{2} |\partial^2_{x_j x_k} \mathbf{u}|^2 + s\varphi |\nabla_{\widetilde{x}} \mathbf{u}|^2 + s^3 \varphi^3 |\mathbf{u}|^2 \right) e^{2s\varphi} d\widetilde{x}$$
$$\leq C_1 (\|(\mathrm{rot}\mathbf{u})\, e^{s\varphi}\|^2_{H^1(\Omega_N)} + \|(\mathrm{div}\mathbf{u}) e^{s\varphi}\|^2_{H^1(\Omega_N)}),$$
$$\forall \mathbf{u} \in (H^1_0(\Omega_N))^2, \forall s \geq s_0(\tau, N), \quad \mathrm{supp}\, \mathbf{u} \subset B_\delta \cap \Omega_N, \tag{1}$$

where the constant C_1 is independent of N.

First we choose $N_0 > 0$ sufficiently large such that

$$\nabla_{\widetilde{x}} \psi(x) \neq 0, \quad \forall \widetilde{x} \in \Omega_N,\ x_0 \in (0, T).$$

The existence of such N_0 follows from condition (1.6).

Denote $\mathrm{rot}\, \mathbf{u} = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} = \mathbf{y}$, and $\mathrm{div}\, \mathbf{u} = \mathbf{w}$. Let $\mathrm{rot}^* v = (\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1})$. Using the well-known formula $\mathrm{rot}^*\mathrm{rot} = -\Delta + \nabla_{\widetilde{x}}\mathrm{div}$, we obtain

$$-\Delta_{\widetilde{x}} \mathbf{u} = -\mathrm{rot}^*\, \mathbf{y} - \nabla_{\widetilde{x}} \mathbf{w} \quad \text{in } \Omega_N, \quad \mathbf{u}|_{\partial\Omega_N} = 0.$$

The function $\widetilde{\mathbf{u}} = \mathbf{u} e^{s\varphi}$ satisfies the equation

$$L_1 \widetilde{\mathbf{u}} + L_2 \widetilde{\mathbf{u}} = q_s \quad \text{in } \Omega_N,\ \widetilde{\mathbf{u}}|_{\partial\Omega_N} = 0, \tag{2}$$

where $L_1 \widetilde{\mathbf{u}} = -\Delta_{\widetilde{x}} \widetilde{\mathbf{u}} - s^2 |\nabla_{\widetilde{x}} \varphi|^2 \widetilde{\mathbf{u}}$, $L_2 \widetilde{\mathbf{u}} = 2s \sum_{k=1}^{2} \widetilde{\mathbf{u}}_{x_k} \varphi_{x_k} + s(\Delta_{\widetilde{x}} \varphi) \widetilde{\mathbf{u}}$ and $q_s = (-\mathrm{rot}^*\, \mathbf{y} - \nabla_{\widetilde{x}} \mathbf{w}) e^{s\varphi}$. Taking the L^2 norms of the right and the left hand sides of equation (2), we obtain

$$\|L_1 \widetilde{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega_N)} + \|L_2 \widetilde{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega_N)} + 2(L_1 \widetilde{\mathbf{u}}, L_2 \widetilde{\mathbf{u}})_{\mathbf{L}^2(\Omega_N)} = \|q_s\|^2_{\mathbf{L}^2(\Omega_N)}.$$

After integrations, we will arrive at the formula:

$$(L_1\widetilde{\mathbf{u}}, L_2\widetilde{\mathbf{u}})_{\mathbf{L}^2(\Omega_N)} = \int_{\Omega_N}\Bigg\{2s\sum_{k,j=1}^{2}\widetilde{\mathbf{u}}_{x_j}\widetilde{\mathbf{u}}_{x_k}\varphi_{x_jx_k} + s^3(\mathrm{div}(|\nabla_{\widetilde{x}}\varphi|^2\nabla_{\widetilde{x}}\varphi) - |\nabla_{\widetilde{x}}\varphi|^2\Delta_{\widetilde{x}}\varphi)|\widetilde{\mathbf{u}}|^2$$

$$-\frac{s}{2}\sum_{j=1}^{2}\frac{\partial^2\Delta_{\widetilde{x}}\varphi}{\partial x_j^2}|\widetilde{\mathbf{u}}|^2\Bigg\}d\widetilde{x} - \int_{\partial\Omega}\left(\frac{\partial\widetilde{\mathbf{u}}}{\partial\vec{n}}, L_2\widetilde{\mathbf{u}}\right)d\sigma + s\int_{\partial\Omega}\left|\frac{\partial\widetilde{\mathbf{u}}}{\partial\vec{n}}\right|^2(\nabla_{\widetilde{x}}\varphi,\vec{n})d\sigma$$

$$+s\int_{\partial\Omega}(\nabla_{\widetilde{x}}\varphi,\vec{n})s^3|\nabla_{\widetilde{x}}\varphi|^2|\widetilde{\mathbf{u}}|^2 d\sigma. \tag{3}$$

Denote $\psi_1(x) = \psi(x) - \widehat{\epsilon}\ell_1(x)$. Then

$$\mathrm{div}(|\nabla_{\widetilde{x}}\varphi|^2\nabla_{\widetilde{x}}\varphi) - |\nabla_{\widetilde{x}}\varphi|^2\Delta_{\widetilde{x}}\varphi = 2\sum_{k,j=1}^{2}\varphi_{x_k}\varphi_{x_j}\varphi_{x_kx_j}$$

$$=2\varphi^3\sum_{k,j=1}^{2}\tau^4(\partial_{x_k}\psi_1 + 2N\ell_1\partial_{x_k}\ell_1)^2(\partial_{x_j}\psi_1 + 2N\ell_1\partial_{x_j}\ell_1)^2$$

$$+\tau^3(\partial_{x_k}\psi_1 + 2N\ell_1\partial_{x_k}\ell_1)(\partial_{x_j}\psi_1 + 2N\ell_1\partial_{x_j}\ell_1)(\partial^2_{x_kx_j}\psi_1 + 2N(\partial_{x_k}\ell_1)\partial_{x_j}\ell_1 + 2N\ell_1\partial^2_{x_kx_j}\ell_1).$$

Since $(\nabla_{\widetilde{x}}\psi_1, \nabla_{\widetilde{x}}\ell_1) > 0$ on $\partial\Omega$, there exists a constant $C_2 > 0$ independent of $N, \widetilde{\tau}, s$, such that

$$\mathrm{div}(|\nabla_{\widetilde{x}}\varphi|^2\nabla_{\widetilde{x}}\varphi) - |\nabla_{\widetilde{x}}\varphi|^2\Delta_{\widetilde{x}}\varphi \ge 2\varphi^3\tau^4|\nabla_{\widetilde{x}}\psi_1|^4 + C_2N\tau^3\varphi^3 + \varphi^2 O(\tau^3). \tag{4}$$

On the other hand

$$\sum_{k,j=1}^{2}\widetilde{\mathbf{u}}_{x_j}\widetilde{\mathbf{u}}_{x_k}\varphi_{x_jx_k} = \tau^2(\nabla_{\widetilde{x}}\widetilde{\mathbf{u}}, \nabla_{\widetilde{x}}\widetilde{\psi})^2\varphi + \tau\sum_{k,j=1}^{2}\widetilde{\mathbf{u}}_{x_j}\widetilde{\mathbf{u}}_{x_k}(\psi_{x_jx_k}$$

$$+N\ell_1\partial^2_{x_jx_k}\ell_1)\varphi + N\tau(\nabla_{\widetilde{x}}\widetilde{\mathbf{u}}, \nabla_{\widetilde{x}}\ell_1)^2\varphi. \tag{5}$$

Note that there exists a constant $C_3 > 0$, independent of N, such that

$$\|N\ell_1\partial^2_{x_jx_k}\ell_1\|_{C^0(\overline{\Omega_N})} \le \frac{C_3}{N}. \tag{6}$$

By (3)-(6) we obtain

$$\|L_1\widetilde{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega_N)} + \|L_2\widetilde{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega_N)} + \int_{\Omega_N}(2\varphi^3\tau^4|\nabla_{\widetilde{x}}\psi_1|^4 + C_1N\tau^3\varphi^3)|\widetilde{\mathbf{u}}|^2 d\widetilde{x}$$

$$-s\tau C_4\int_{\Omega_N}\varphi|\nabla_{\widetilde{x}}\widetilde{\mathbf{u}}|^2 d\widetilde{x} \le \|q_s\|^2_{\mathbf{L}^2(\Omega_N)} + C_5\left(s\|\widetilde{\mathbf{u}}\|^2_{\mathbf{H}^{1,s}(\partial\Omega)} + s\left\|\frac{\partial\widetilde{\mathbf{u}}}{\partial\vec{n}}\right\|^2_{\mathbf{L}^2(\partial\Omega)}\right). \tag{7}$$

Multiplying equation (2) by $sN\varphi\widetilde{u}$ and integrating by parts, we obtain

$$\int_{\Omega_N}\left\{sN\varphi|\nabla_{\widetilde{x}}\widetilde{\mathbf{u}}|^2+s^2N\Delta_{\widetilde{x}}\varphi\varphi|\widetilde{\mathbf{u}}|^2-s^3\varphi^3|\nabla_{\widetilde{x}}\varphi|^2|\widetilde{\mathbf{u}}|^2-\frac{sN}{2}\mathrm{div}\varphi|\widetilde{\mathbf{u}}|^2\right\}d\widetilde{x}$$
$$+\int_{\partial\Omega}\left\{-\left(\frac{\partial\widetilde{\mathbf{u}}}{\partial\vec{n}},sN\varphi\widetilde{\mathbf{u}}\right)+\left(s^2\varphi N+\frac{sN}{2}\right)(\nabla_{\widetilde{x}}\varphi,\vec{n})|\widetilde{\mathbf{u}}|^2\right\}d\sigma=\int_{\Omega_N}q_s sN\varphi\widetilde{\mathbf{u}}d\widetilde{x}. \tag{8}$$

Next we note that

$$\Delta_{\widetilde{x}}\varphi=(|\nabla_{\widetilde{x}}\widetilde{\psi}|^2\tau^2+\tau\Delta_{\widetilde{x}}\psi_1+2\tau N|\nabla_{\widetilde{x}}\ell_1|^2+2\tau N\ell_1\Delta_{\widetilde{x}}\ell_1)\varphi\ge C_6\tau N\varphi.$$

This inequality and (8) imply

$$\int_{\Omega_N}\left(sN\varphi|\nabla_{\widetilde{x}}\widetilde{\mathbf{u}}|^2+\frac{1}{2}s^2N(\Delta_{\widetilde{x}}\varphi)\varphi|\widetilde{\mathbf{u}}|^2-s^3\varphi^3|\nabla_{\widetilde{x}}\varphi|^2|\widetilde{\mathbf{u}}|^2\right)d\widetilde{x}$$
$$\le C_6\|q_s\|^2_{L^2(\Omega_N)}+C_6\left(s\|\widetilde{\mathbf{u}}\|^2_{\mathbf{H}^{1,s}(\partial\Omega)}+s\left\|\frac{\partial\widetilde{\mathbf{u}}}{\partial\vec{n}}\right\|^2_{\mathbf{L}^2(\partial\Omega)}\right). \tag{9}$$

In view of (7) and (9), we obtain

$$\|L_1\widetilde{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega_N)}+\|L_2\widetilde{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega_N)}+\int_{\Omega_N}\left(\frac{1}{2}\varphi^3\tau^4|\nabla_{\widetilde{x}}\psi_1|^4+C_2N\tau^3\varphi^3\right)|\widetilde{\mathbf{u}}|^2d\widetilde{x}$$
$$+sN\int_{\Omega_N}\varphi|\nabla_{\widetilde{x}}\widetilde{\mathbf{u}}|^2d\widetilde{x}\le C_7\|q_s\|^2_{L^2(\Omega_N)}+C_8\left(s\|\widetilde{\mathbf{u}}\|^2_{\mathbf{H}^{1,s}(\partial\Omega)}+s\left\|\frac{\partial\widetilde{\mathbf{u}}}{\partial\vec{n}}\right\|^2_{\mathbf{L}^2(\partial\Omega)}\right). \tag{10}$$

Let $\widetilde{\mathbf{u}}=\widetilde{\mathbf{u}}_1+\widetilde{\mathbf{u}}_2$ where

$$-\Delta_{\widetilde{x}}\widetilde{\mathbf{u}}_1=L_1\widetilde{\mathbf{u}}\text{ in }\Omega_{N_0}\ \widetilde{\mathbf{u}}_1|_{\partial\Omega_{N_0}}=\widetilde{\mathbf{u}},\quad -\Delta_{\widetilde{x}}\widetilde{\mathbf{u}}_2=s^2|\nabla_{\widetilde{x}}\varphi|^2\widetilde{\mathbf{u}}\quad\text{in}\quad\Omega_{N_0}\ \widetilde{\mathbf{u}}_2|_{\partial\Omega_{N_0}}=0.$$

By standard a priori estimates for the Laplace operator we have

$$\|\widetilde{\mathbf{u}}_1\|_{\mathbf{H}^2(\Omega_N)}\le C_9(\|L_1\widetilde{\mathbf{u}}\|_{\mathbf{L}^2(\Omega_N)}+\|\widetilde{\mathbf{u}}\|_{\mathbf{H}^{\frac{3}{2}}(\partial\Omega)}), \tag{11}$$

$$\frac{\sqrt{N}}{\sqrt{s}}\|\widetilde{\mathbf{u}}_2\|_{\mathbf{H}^2(\Omega_N)}\le C_{10}\sqrt{N}\|s^{\frac{3}{2}}|\nabla_{\widetilde{x}}\varphi|^2\widetilde{\mathbf{u}}\|_{\mathbf{L}^2(\Omega_N)}, \tag{12}$$

where $C_9, C_{10} > 0$ are independent of N. Taking $s_0(\tau, N) \ge N$, we obtain (1) from (9)-(12):

$$N \sum_{|\alpha|=0,\alpha=(\alpha_0,\alpha_1,0)}^{2} s^{4-2|\alpha|}\|(\partial^\alpha \mathbf{u})e^{s\phi}\|^2_{\mathbf{L}^2(Q)}$$
$$\le C_{11}\left\{\|\mathbf{w}\|^2_{\mathbf{L}^2(Q)} + s^2\|\widetilde{\mathbf{u}}\|^2_{\mathbf{H}^{1,s}(\partial Q)} + s^2\left\|\frac{\partial \widetilde{\mathbf{u}}}{\partial \vec{n}}\right\|^2_{\mathbf{L}^2(\partial Q)} + N\|\widetilde{\mathbf{u}}\|^2_{L^2(0,T;\mathbf{H}^{\frac{3}{2}}(\partial\Omega))}\right\}. \tag{13}$$

By (1.8) and (1.9), we estimate the norm of $\partial^2_{x_0}\mathbf{u}$:

$$N\|(\partial^2_{x_0}\mathbf{u})e^{s\phi}\|^2_{\mathbf{L}^2(Q)} \le C_{12}\left\{\|\mathbf{w}\|^2_{\mathbf{L}^2(Q)} + s^2\|\widetilde{\mathbf{u}}\|^2_{\mathbf{H}^{1,s}(\partial Q)}\right.$$
$$\left. + s^2\left\|\frac{\partial \widetilde{\mathbf{u}}}{\partial \vec{n}}\right\|^2_{\mathbf{L}^2(\partial Q)} + N\|\widetilde{\mathbf{u}}\|^2_{L^2(0,T;\mathbf{H}^{\frac{3}{2}}(\partial\Omega))} + N\|\mathbf{f}e^{s\varphi}\|^2_{\mathbf{L}^2(Q)}\right\}. \tag{14}$$

Finally we note

$$Ns^2\|(\partial_{x_0}\mathbf{u})e^{s\phi}\|^2_{\mathbf{L}^2(Q)} \le C_{13}(N\|(\partial^2_{x_0}\mathbf{u})e^{s\phi}\|^2_{\mathbf{L}^2(Q)} + s^4 N\|\mathbf{u}e^{s\phi}\|^2_{\mathbf{L}^2(Q)}). \tag{15}$$

and

$$N\|\partial_{x_k x_j}\widetilde{\mathbf{u}}\|^2_{\mathbf{L}^2(Q)} \le C_{14}\left(N\|\partial^2_{x_k}\widetilde{\mathbf{u}}\|^2_{\mathbf{L}^2(Q)} + \|\partial^2_{x_j}\widetilde{\mathbf{u}}\|^2_{\mathbf{L}^2(Q)} + \left\|\left(\widetilde{\mathbf{u}}, \frac{\partial \widetilde{\mathbf{u}}}{\partial \vec{n}}\right)\right\|^2_{\mathbf{H}^{\frac{3}{2}}(\partial Q)\times \mathbf{H}^{\frac{1}{2}}(\partial Q)}\right). \tag{16}$$

Thus the proof of Proposition 1.3 is complete. ■

Appendix II. Proof of Proposition 1.6.

Let us consider the following problem

$$L^* p = \left(-\frac{\partial}{\partial y_2} - \Gamma^{+,*}_\beta(y, s, D')\right) p = \chi_\nu w \quad \text{in } \mathcal{G}, \tag{1}$$

where $\beta \in \{\mu, \lambda + 2\mu\}$ and $\Gamma^{+,*}_\beta$ is the operator which is formally adjoint to Γ^+_β.

We have

Lemma 1. *There exist constants $C_1 > 0$ and $s_0 > 0$ such that for every $s \geq s_0$, there exists p satisfying (1) and*

$$s\int_{\mathcal{G}} |p|^2 dy + \sqrt{s}\int_{\mathcal{G}\cap\{y_2=0\}} |p(y',0)|^2 dy' \leq C\int_{\mathcal{G}} |\chi_\nu w|^2 dy. \tag{2}$$

Proof. For $\epsilon > 0$, let us consider the functional:

$$J_\epsilon(p) = \frac{1}{2}\|p_\epsilon\|^2_{L^2(\mathcal{G})} + \frac{1}{2\epsilon}\left\| \frac{\partial p}{\partial y_2} + \Gamma^{+,*}_\beta(y,s,D')p + \chi_\nu w \right\|^2_{L^2(\mathcal{G})}. \tag{3}$$

Notice that there exists p such that $J_\epsilon(p)$ is finite and, for example, we can set $p = 0$. We consider the minimization problem

$$\min_{p\in U} J_\epsilon(p), \tag{4}$$

where

$$U = \left\{ p \in L^2(\mathcal{G});\ \frac{\partial p}{\partial y_2} + \Gamma^{+,*}_\beta(y,s,D')p + \chi_\nu w \in L^2(\mathcal{G}) \right\}. \tag{5}$$

There exists a minimizing sequence $\{p_n\}_{n=1}^\infty$ such that $p_n \in U$ and

$$\lim_{n\to\infty} J_\epsilon(p_n) = \inf_{p\in U} J_\epsilon(p). \tag{6}$$

Then $\|p_n\|^2_{L^2(\mathcal{G})}$ and $\|\frac{\partial p_n}{\partial y_2} + \Gamma^{+,*}_\beta(y,s,D')p_n + \chi_\nu w\|^2_{L^2(\mathcal{G})}$ are bounded. Therefore $\Gamma^{+,*}_\beta(y,s,D')p_n$ is bounded in $L^2(0,1;H^{-1}(\mathbb{R}^2))$ and $\frac{\partial p_n}{\partial y_2}$ is bounded in $L^2(0,1;H^{-1}(\mathbb{R}^2))$. Therefore We can extract a subsequence, still denoted by $\{p_n\}_{n=1}^\infty$ such that

$$\begin{aligned} &p_n \rightharpoonup p_\epsilon \text{ in } L^2(\mathcal{G}) \quad \text{weakly,} \\ &\frac{\partial p_n}{\partial y_2} \rightharpoonup \frac{\partial p_\epsilon}{\partial y_2} \quad \text{in} \quad L^2(0,1;H^{-1}(\mathbb{R}^2)) \quad \text{weakly,} \\ &\frac{\partial p_n}{\partial y_2} + \Gamma^{+,*}_\beta(y,s,D')p_n + \chi_\nu w \rightharpoonup \frac{\partial p_\epsilon}{\partial y_2} + \Gamma^{+,*}_\beta(y,s,D')p_\epsilon + \chi_\nu w \\ &\qquad \text{in } L^2(0,1;H^{-1}(\mathbb{R}^2)) \text{ weakly.} \end{aligned} \tag{7}$$

On the other hand, as $\left\|\frac{\partial p_n}{\partial y_2}+\Gamma^{+,*}_\beta(y,s,D')p_n+\chi_\nu w\right\|_{L^2(\mathcal{G})}$ remains bounded, we have

$$\frac{\partial p_n}{\partial y_2}+\Gamma^{+,*}_\beta(y,s,D')p_n+\chi_\nu w \rightharpoonup \frac{\partial p_\epsilon}{\partial y_2}+\Gamma^{+,*}_\beta(y,s,D')p_\epsilon+\chi_\nu w$$

weakly in $L^2(\mathcal{G})$. Then p_ϵ is a minimizer of J_ϵ, that is, $p_\epsilon \in U$ and

$$J_\epsilon(p_\epsilon)=\min_{p\in U} J_\epsilon(p). \tag{8}$$

Writing the first order optimality conditions, we have for every $r\in H^1(\mathcal{G})$:

$$< J'_\epsilon(p_\epsilon), r >= 0. \tag{9}$$

Let us define q_ϵ by

$$q_\epsilon=\frac{1}{\epsilon}\left(\frac{\partial p_\epsilon}{\partial y_2}+\Gamma^{+,*}_\beta(y,s,D')p_\epsilon+w\right). \tag{10}$$

In view of (9), for every $r\in H^1(\mathcal{G})$, we see:

$$\int_{\mathcal{G}} p_\epsilon \overline{r} dy+\int_{\mathcal{G}} q_\epsilon \overline{\left(\frac{\partial r}{\partial y_2}+\Gamma^{+,*}_\beta(y,s,D')r\right)} dy=0. \tag{11}$$

Then q_ϵ satisfies the following problem

$$\frac{\partial q_\epsilon}{\partial y_2}-\Gamma^{+}_\beta(y,s,D')q_\epsilon=p_\epsilon \quad \text{in} \ \ \mathcal{G}, \tag{12}$$

$$q_\epsilon(y',0)=0,\ q_\epsilon(y',1)=0,\ y'\in\mathbb{R}^2. \tag{13}$$

Denote $L_1=\frac{1}{2}(-\Gamma^{+}_\beta-\Gamma^{+,*}_\beta)$ and $L_2=\frac{\partial}{\partial y_2}+\frac{1}{2}(\Gamma^{+,*}_\beta-\Gamma^{+}_\beta)$. Then we can rewrite (12) and (13) as follows.

$$Lq_\epsilon\equiv(L_1+L_2)q_\epsilon=p_\epsilon \text{ in } \mathcal{G},\ q_\epsilon(y',0)=0,\ q_\epsilon(y',1)=0,\ y'\in\mathbb{R}^2. \tag{14}$$

There exist constants $C_2>0$ and $s_0>0$ such that

$$\|L_1q_\epsilon\|^2_{L^2(\mathcal{G})}+\|L_2q_\epsilon\|^2_{L^2(\mathcal{G})}+s\int_{\mathcal{G}}|q_\epsilon|^2dy\le C_2\|p_\epsilon\|^2_{L^2(\mathcal{G})}, \quad \forall s\ge s_0. \tag{15}$$

Notice that $L_1 q_\epsilon \in L^2(\mathcal{G})$ implies $q_\epsilon \in L^2(0,1;H^1(\mathbb{R}^2))$, which implies $\frac{\partial q_\epsilon}{\partial x_0} \in L^2(\mathcal{G})$ from (12). Now it follows from definition (10) of q_ϵ that p_ϵ satisfies

$$\frac{\partial p_\epsilon}{\partial y_2} + \Gamma_\beta^{+,*}(y,s,D')p_\epsilon = \epsilon q_\epsilon - \chi_\nu w, \tag{16}$$

which can be written as

$$(L_2 - L_1)p_\epsilon = \frac{\partial p_\epsilon}{\partial y_2} + \Gamma_\beta^{+,*}(y,s,D')p_\epsilon = \epsilon q_\epsilon - \chi_\nu w. \tag{17}$$

Multiplying (17) by q_ϵ in $L^2(\mathcal{G})$ and using the boundary conditions for q_ϵ, we obtain

$$-\int_{\mathcal{G}} p_\epsilon \overline{(L_1 + L_2)q_\epsilon} dy = \epsilon \int_{\mathcal{G}} |q_\epsilon|^2 dy - \int_{\mathcal{G}} \chi_\nu w \overline{q_\epsilon} dy, \tag{18}$$

so that

$$\int_{\mathcal{G}} |p_\epsilon|^2 dy + \epsilon \int_{\mathcal{G}} |q_\epsilon|^2 dy = \int_{\mathcal{G}} \chi_\nu w \overline{q_\epsilon} dy$$

and

$$\int_{\mathcal{G}} |p_\epsilon|^2 dy \le \left(\int_{\mathcal{G}} |\chi_\nu w|^2 dy\right)^{\frac{1}{2}} \left(\int_{\mathcal{G}} |q_\epsilon|^2 dy\right)^{\frac{1}{2}} \le \frac{C_3}{\sqrt{s}} \left(\int_{\mathcal{G}} |\chi_\nu w|^2 dy\right)^{\frac{1}{2}} \left(\int_{\mathcal{G}} |p_\epsilon|^2 dy\right)^{\frac{1}{2}}. \tag{19}$$

Therefore we obtain the first estimate on p_ϵ

$$s \int_{\mathcal{G}} |p_\epsilon|^2 dy \le C_4 \int_{\mathcal{G}} |\chi_\nu w|^2 dy. \tag{20}$$

Let us now notice from (12) and (13) that we have

$$p_\epsilon(y',0) = \frac{\partial q_\epsilon}{\partial y_2}(y',0), \ \ y' \in \mathbb{R}^2.$$

Let $\theta = \theta(y_2) \in C^\infty[0,1]$ such that $0 \le \theta \le 1$, $\theta(0) = 1$ and $\theta(1) = 0$. We have

$$(L_1 + L_2)(\theta q_\epsilon) = \theta(L_1 + L_2)(q_\epsilon) + \frac{\partial \theta}{\partial y_2} q_\epsilon = \theta p_\epsilon + \frac{\partial \theta}{\partial y_2} q_\epsilon \ \text{ in } \mathcal{G}, \tag{21}$$

$$(\theta q_\epsilon)(y',0)=0,\ (\theta q_\epsilon)(y',1)=0,\ \frac{\partial(\theta q_\epsilon)}{\partial y_2}(y',1)=0,\quad y'\in\mathbb{R}^2. \tag{22}$$

Now we apply the operator (L_2-L_1) to the first equation, so that

$$\begin{aligned}&(L_2-L_1)(L_2+L_1)(\theta q_\epsilon)=(L_2-L_1)(\theta p_\epsilon)+(L_2-L_1)\left(\frac{\partial\theta}{\partial y_2}q_\epsilon\right)\\ =&\theta(L_2-L_1)(p_\epsilon)+\frac{\partial\theta}{\partial y_2}p_\epsilon+\frac{\partial\theta}{\partial y_2}(L_2-L_1)q_\epsilon+\frac{\partial^2\theta}{\partial y_2^2}q_\epsilon.\end{aligned} \tag{23}$$

Then we have

$$\begin{aligned}&L_2^2(\theta q_\epsilon)-L_1^2(\theta q_\epsilon)+[L_2,L_1](\theta q_\epsilon)=\epsilon\theta q_\epsilon-\theta w+\frac{\partial\theta}{\partial y_2}p_\epsilon\\ &+\frac{\partial\theta}{\partial y_2}(L_2-L_1)q_\epsilon+\frac{\partial^2\theta}{\partial y_2^2}q_\epsilon.\end{aligned} \tag{24}$$

We now multiply this equation by $L_2(\theta q_\epsilon)$ scalary in $L^2(\mathcal{G})$ and afterwards take the real part. Henceforth we give the computations of the successive terms. We can calculate the first term as follows.

$$\begin{aligned}&(L_2^2(\theta q_\epsilon),L_2(\theta q_\epsilon))_{L^2(\mathcal{G})}=\left(\frac{\partial}{\partial y_2}(L_2(\theta q_\epsilon)),L_2(\theta q_\epsilon)\right)_{L^2(\mathcal{G})}\\ &-\frac{1}{2}((\Gamma_\beta^+(y,s,D')-\Gamma_\beta^{+,*}(y,s,D'))L_2(\theta q_\epsilon),L_2(\theta q_\epsilon))_{L^2(\mathcal{G})}\\ &=-\|L_2(\theta q_\epsilon)(0)\|^2_{L^2(\mathbb{R}^2)}-(L_2(\theta q_\epsilon),L_2^2(\theta q_\epsilon))_{L^2(\mathcal{G})}\\ &=-\left\|\frac{\partial q_\epsilon}{\partial y_2}(0)\right\|^2_{L^2(\mathbb{R}^2)}-(L_2(\theta q_\epsilon),L_2^2(\theta q_\epsilon))_{L^2(\mathcal{G})}.\end{aligned} \tag{25}$$

Therefore we have

$$\mathrm{Re}\,(L_2^2(\theta q_\epsilon),L_2(\theta q_\epsilon))_{L^2(\mathcal{G})}=-\frac{1}{2}\left\|\frac{\partial q_\epsilon}{\partial y_2}(0)\right\|^2_{L^2(\mathbb{R}^2)}. \tag{26}$$

Now for the second term

$$\begin{aligned}&2\mathrm{Re}\,(L_1^2(\theta q_\epsilon),L_2(\theta q_\epsilon))_{L^2(\mathcal{G})}=(L_1^2(\theta q_\epsilon),L_2(\theta q_\epsilon))_{L^2(\mathcal{G})}+(L_2(\theta q_\epsilon),L_1^2\theta q_\epsilon)_{L^2(\mathcal{G})}\\ =&-(L_2L_1^2(\theta q_\epsilon),(\theta q_\epsilon))_{L^2(\mathcal{G})}+(L_1^2L_2(\theta q_\epsilon),(\theta q_\epsilon))_{L^2(\mathcal{G})}+\frac{1}{2}(L_1^2(\theta q_\epsilon),(\theta q_\epsilon))_{L^2(\mathcal{G})}\\ =&(L_1[L_1,L_2](\theta q_\epsilon),(\theta q_\epsilon))_{L^2(\mathcal{G})}+([L_1,L_2]L_1(\theta q_\epsilon),(\theta q_\epsilon))_{L^2(\mathcal{G})}\\ =&2\mathrm{Re}\,([L_1,L_2](\theta q_\epsilon),L_1(\theta q_\epsilon))_{L^2(\mathcal{G})}.\end{aligned} \tag{27}$$

(Notice that $([L_1, L_2]u, v)_{L^2(\mathcal{G})} = (u, [L_1, L_2]v)_{L^2(\mathcal{G})}$.)

We have already seen that

$$[L_1, L_2] = K(y_2), \tag{28}$$

where $K \in C([0,1]; \mathcal{L}(H^1(\mathbb{R}^2), L^2(\mathbb{R}^2)))$ is an operator which is independent of s, τ.

Therefore

$$|\mathrm{Re}\,(L_1^2(\theta q_\epsilon), L_2(\theta q_\epsilon))_{L^2(\mathcal{G})}| \leq \|s(\theta q_\epsilon)\|_{L^2(\mathcal{G})}\|L_1(\theta q_\epsilon)\|_{L^2(\mathcal{G})}$$
$$+C_5\|\theta q_\epsilon\|_{L^2(0,1;H^1(\mathbb{R}^2))}\|L_1(\theta q_\epsilon)\|_{L^2(\mathcal{G})} + \frac{1}{4}\|L_1(\theta q_\epsilon)\|^2_{L^2(\mathcal{G})}. \tag{29}$$

We already know from (15) that

$$\|L_1(\theta q_\epsilon)\|_{L^2(\mathcal{G})} \leq \|L_1 q_\epsilon\|_{L^2(\mathcal{G})} \leq C_6\|p_\epsilon\|_{L^2(\mathcal{G})} \tag{30}$$

and from the definition of L_1

$$\|\theta q_\epsilon\|_{L^2(0,1;H^1(\mathbb{R}^2))} \leq \|q_\epsilon\|_{L^2(0,1;H^1(\mathbb{R}^2))} \leq C_7\|L_1 q_\epsilon\|_{L^2(\mathcal{G})} + s\|q_\epsilon\|_{L^2(\mathcal{G})}. \tag{31}$$

Using again (15), we obtain

$$|\mathrm{Re}\,(L_1^2(\theta q_\epsilon), L_2(\theta q_\epsilon))_{L^2(\mathcal{G})}| \leq C_8\sqrt{s}\|p_\epsilon\|^2_{L^2(\mathcal{G})} + C_8\|p_\epsilon\|^2_{L^2(\mathcal{G})}, \tag{32}$$

so that we have

$$|\mathrm{Re}\,(L_1^2(\theta q_\epsilon), L_2(\theta q_\epsilon))_{L^2(\mathcal{G})}| \leq C\sqrt{s}\|p_\epsilon\|_{L^2(\mathcal{G})} \tag{33}$$

for $s \geq s_0$. Concerning the third term, we have

$$|\mathrm{Re}\,([L_2, L_1](\theta q_\epsilon), L_2(\theta q_\epsilon))_{L^2(\mathcal{G})}| \leq \|[L_2, L_1](\theta q_\epsilon)\|_{L^2(\mathcal{G})}\|L_2(\theta_\epsilon q_\epsilon)\|_{L^2(\mathcal{G})}. \tag{34}$$

Using the form of $[L_2, L_1]$ we obtain

$$|\mathrm{Re}\,([L_2, L_1](\theta q_\epsilon), L_2(\theta q_\epsilon))_{L^2(\mathcal{G})}|$$
$$\leq \|s(\theta q_\epsilon)\|_{L^2(\mathcal{G})}\|L_2(\theta q_\epsilon)\|_{L^2(\mathcal{G})} + C_9\|\theta q_\epsilon\|_{L^2(0,1;H^1(\mathbb{R}^2))}\|L_2(\theta q_\epsilon)\|_{L^2(\mathcal{G})} \tag{35}$$

and, since $L_2(\theta q_\epsilon) = \theta L_2 q_\epsilon + \frac{\partial \theta}{\partial y_2} q_\epsilon$, from (15) and (31) we have

$$|\mathrm{Re}\,([L_2, L_1](\theta q_\epsilon), L_2(\theta q_\epsilon))_{L^2(\mathcal{G})}| \le C_{10}\sqrt{s}\|p_\epsilon\|^2_{L^2(\mathcal{G})} + C_{10}\|p_\epsilon\|^2_{L^2(\mathcal{G})}$$
$$\le C_{11}\sqrt{s}\|p_\epsilon\|^2_{L^2(\mathcal{G})} \tag{36}$$

for $s \ge s_0$.

On the other hand, for the right hand side of (24), we have

$$\left|\mathrm{Re}\,\left(\epsilon\theta q_\epsilon - \theta\varphi w + \frac{\partial\theta}{\partial y_2} p_\epsilon + \frac{\partial\theta}{\partial y_2}(L_2 - L_1) q_\epsilon + \frac{\partial^2\theta}{\partial y_2^2} q_\epsilon, L_2(\theta q_\epsilon)\right)_{L^2(\mathcal{G})}\right|$$
$$\le C_{12}\epsilon\|q_\epsilon\|_{L^2(\mathcal{G})}\|p_\epsilon\|_{L^2(\mathcal{G})} + C_{12}\|\chi_\nu w\|_{L^2(\mathcal{G})}\|p_\epsilon\|_{L^2(\mathcal{G})} + C_{12}\|p_\epsilon\|^2_{L^2(\mathcal{G})}.$$

Therefore we obtain

$$\left|\mathrm{Re}\,\left(\epsilon\theta q_\epsilon - \theta\varphi w + \frac{\partial\theta}{\partial y_2} p_\epsilon + \frac{\partial\theta}{\partial y_2}(L_2 - L_1) q_\epsilon + \frac{\partial^2\theta}{\partial y_2^2} q_\epsilon, L_2(\theta q_\epsilon)\right)_{L^2(\mathcal{G})}\right|$$
$$\le C_{13}\|p_\epsilon\|^2_{L^2(\mathcal{G})} + C_{13}\|\chi_\nu w\|_{L^2(\mathcal{G})}\|p_\epsilon\|_{L^2(\mathcal{G})}. \tag{37}$$

Putting together (28), (33), (36) and (37), we obtain the following estimate:

$$\left\|\frac{\partial q_\epsilon}{\partial y_2}(0)\right\|^2_{L^2(\mathcal{G}\cap\{y_2=0\})} \le C_{14}\sqrt{s}\|p_\epsilon\|^2_{L^2(\mathcal{G})} + C_{14}\|\chi_\nu w\|_{L^2(\mathcal{G})}\|p_\epsilon\|_{L^2(\mathcal{G})}. \tag{38}$$

Now combining (15), (20) and (38), we can easily obtain

$$s\int_{\mathcal{G}} |p_\epsilon|^2 dy + \sqrt{s}\int_{\mathcal{G}\cap\{y_2=0\}} |p_\epsilon(0, y')|^2 dy' \le C_{15}\int_{\mathcal{G}} |\chi_\nu w|^2 dy, \tag{39}$$

which is estimate (2) for p_ϵ.

Now p_ϵ and $\frac{\partial p_\epsilon}{\partial y_2} + \Gamma^{+,*}_\beta(y, s, D')p_\epsilon$ are bounded in $L^2(\mathcal{G})$ uniformly in ϵ. After extraction of a subsequence (still denoted by p_ϵ), we can assume that

$$p_\epsilon \rightharpoonup p \quad \text{in } L^2(\mathcal{G}) \quad \text{weakly},$$
$$\frac{\partial p_\epsilon}{\partial y_2} \rightharpoonup \frac{\partial p}{\partial y_2} \quad \text{in } L^2(0,1; H^{-1}(\mathbb{R}^2)) \quad \text{weakly}, \tag{40}$$

so that

$$p_\epsilon(0) \rightharpoonup p(0) \quad \text{in} \quad H^{-\frac{1}{2}}(\mathbb{R}^2) \quad \text{weakly.}$$

Since $p_\epsilon(0)$ remains bounded in $L^2(\mathbb{R}^2)$, we also have $p_\epsilon(0) \rightharpoonup p(0)$ weakly in $L^2(\mathcal{G} \cap \{y_2 = 0\})$. By (15) and (40), we easily see that p satisfies

$$L^* p = (-\frac{\partial}{\partial y_2} - \Gamma_\beta^{+,*}(y, s, D'))p = \chi_\nu w \quad \text{in } \mathcal{G},$$

which is (1), and from (39) we see that

$$s \int_{\mathcal{G}} |p|^2 dy + \sqrt{s} \int_{\mathcal{G} \cap \{y_2 = 0\}} |p(y', 0)|^2 dy' \le C \int_{\mathcal{G}} |\chi_\nu w|^2 dy$$

which is (2). The proof of Lemma 1 is now complete. ■

We take the scalar product of equation (1) and the function $\chi_\nu w$ in $L^2(\mathcal{G})$

$$\|\chi_\nu w\|^2_{L^2(\mathcal{G})} = (g, p)_{L^2(\mathcal{G})} + (p(\cdot, 0), \chi_\nu w(\cdot, 0))_{L^2(\mathbb{R}^2)}$$

Applying estimate (2) to this equality, we have

$$\|\chi_\nu w\|^2_{L^2(\mathcal{G})} \le C_{16} \left(\frac{1}{\sqrt{s}} \|g\|_{L^2(\mathcal{G})} \|\chi_\nu w\|_{L^2(\mathcal{G})} + \frac{1}{s^{\frac{1}{4}}} \|\chi_\nu w(\cdot, 0)\|_{L^2(\mathbb{R}^2)} \|\chi_\nu w\|_{L^2(\mathcal{G})} \right).$$

Therefore

$$\sqrt{s} \|\chi_\nu w\|_{L^2(\mathcal{G})} \le C_{17}(\|g\|_{L^2(\mathcal{G})} + s^{\frac{1}{4}} \|\chi_\nu w(\cdot, 0)\|_{L^2(\mathbb{R}^2)}), \quad \forall s \ge s_0. \tag{41}$$

Since $|s - |(\xi_0, \xi_1)||$ is small on $\operatorname{supp} \chi_\nu$, we obtain

$$\|\chi_\nu w\|_{L^2(0,1;H^1(\mathbb{R}^2))} \le C_{18} \left(\frac{1}{\sqrt{s}} \|g\|_{H^{1,s}(\mathcal{G})} + s^{\frac{3}{4}} \|\chi_\nu w(\cdot, 0)\|_{L^2(\mathbb{R}^2)} \right).$$

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