A new construction of the Lie algebra $\mathfrak{e}_8$

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1. Introduction

We describe a new construction of the compact real form of the Lie algebra $\mathfrak{e}_8$. The octonion algebra comes to play an important role in this construction.

2. What are Lie algebras?

A Lie algebra is a vector space with Lie multiplication $[x,y]$ that satisfies the following properties

- $[x,y]$ is bilinear
- $[x,x]=0$
- $[[x,y],z]+[[y,z],x]+[[z,x],y]=0$

Lie algebras are endowed with a scalar product called the Killing form. Their nilpotent subalgebras that are equal to their own normalisers are called Cartan subalgebras.

Example

The vector space

$$\mathfrak{sl}_n = \{n \times n \text{ matrices with zero trace}\}$$

is a Lie algebra. The Lie multiplication of matrices A and B is defined by $[A,B]=AB-BA$, and the diagonal matrices form a Cartan subalgebra.

Acknowledgements

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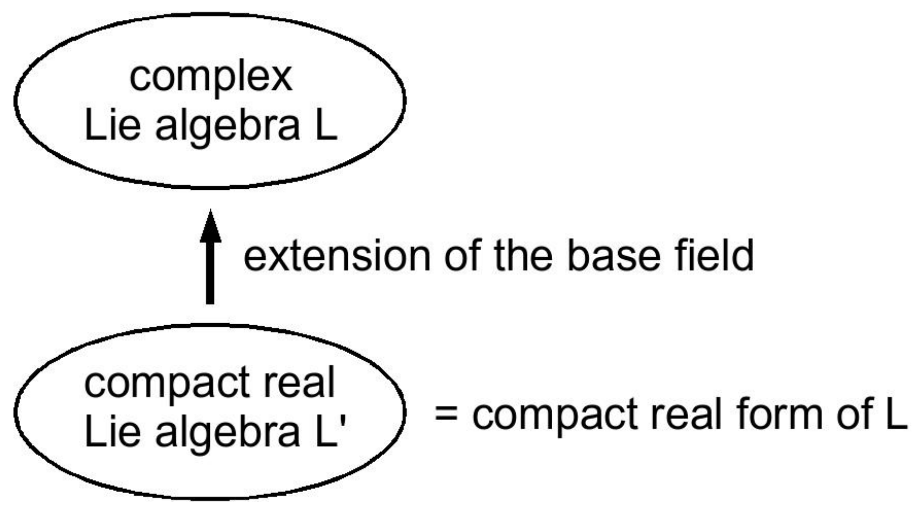
3. Compact real forms

A Lie algebra over $\mathbb{R}$ is compact if its Killing form is negative definite. Compact Lie algebras are interesting because they are associated with compact Lie groups. Every semi-simple Lie algebra over $\mathbb{C}$ has a compact real form.

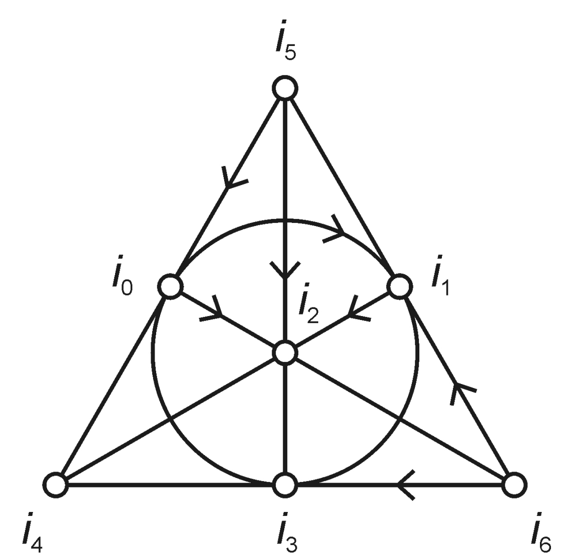


4. What are octonions?

The real octonion algebra $\mathbb{O}$ consists of all the real linear combinations of the unit element 1 and seven square roots of -1:

$$\mathbb{O} = \langle 1,\ i_0,\ i_1,\ \ldots,\ i_6 \rangle.$$

Octonion multiplication is defined as follows: We have $i_r i_s = i_t$, when i_r, i_s and i_t are on the same line in the below diagram. The arrow indicates the sign of the product. For example, $i_1 i_2 = i_4$ and $i_2 i_1 = -i_4$.



5. New construction

The Lie algebra $\mathfrak{e}_8$ is a 248-dimensional simple exceptional Lie algebra. Below we construct a Lie algebra $\mathfrak{L}$ which is the compact real from of $\mathfrak{e}_8$.

Let $\mathfrak{L}$ be the vector space

$$\mathfrak{L} = H_0 \oplus H_1 \oplus \cdots \oplus H_{30}, \qquad (1)$$

where $H_k \cong \mathbb{O}$ for all k. The subspaces H_k will be Cartan subalgebras of $\mathfrak{L}$.

Arrange the index set $S=\{0,\ldots,30\}$ into blocks $\{r,s,t\}$ such that $x^r+x^s+x^t=0$ in $\mathbb{F}_2[x]/(x^5+x^2+1)$. Every pair of elements of S lies in exactly one block.

Define

$$[H_r,H_r]=0 \ \text{ and } \ [H_r,H_s] \subseteq H_t,$$

where $\{r,s,t\}$ is a block in S. Now one needs to define the Lie multiplication in these blocks. The automorphism group $2^{5+10}{\cdot}GL_5(2)$ of the decomposition (1) will be used in this task.

Every block $\{H_r,H_s,H_t\}$ can be mapped to $\{H_0,H_2,H_5\}$. Therefore, it is enough to give the multiplication table for $\{H_0,H_2,H_5\}$, and describe the automorphisms that are needed in mapping the other blocks to this block. This defines the Lie multiplication of $\mathfrak{L}$.