NATIONAL ACADEMY OF SCIENCES

# GEORGE PÓLYA

*1887—1985*

*A Biographical Memoir by*

R. P. BOAS

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# GEORGE PÓLYA

*December 13, 1887–September 7, 1985*

BY R. P. BOAS

GEORGE (GYÖRGY) PÓLYA made many significant contributions to mathematics and at the same time—rather unusually for a distinguished research mathematician—was an effective advocate of improved methods for teaching mathematics. His research publications extend from 1912 to 1976; his publications about teaching began in 1919 and continued throughout his life. For several decades, he was steadily initiating new topics and making decisive contributions to more established ones. Although his main mathematical interest was in analysis, at the peak of his career he was contributing not only to real and complex analysis, but also to probability, combinatorics, occasionally to algebra and number theory, and to the theory of proportional representation and voting. His work typically combined great power and great lucidity of exposition. Although much of his work was so technical that it can be fully appreciated only by specialists, a substantial number of his theorems can be stated simply enough to be appreciated by anyone who has a moderate knowledge of mathematics.

As a whole, Pólya's work is notable for its fruitfulness. All his major contributions have been elaborated on by other mathematicians and have become the foundations of important branches of mathematics. In addition to his more substantial contributions, Pólya made many brief communications, ranging from the many problems that he proposed to brief remarks—a considerable number of which became the germs of substantial theories in the hands of other mathematicians. A student who needs a topic for research could do worse than look through Pólya's short papers.

Pólya's papers were published in four volumes: the first two devoted to complex analysis, the third to other branches of analysis including mathematical physics, the fourth to probability, combinatorics, and teaching and learning in mathematics.[1]

## ORIGINS AND CAREER

Pólya was born in Budapest on December 13, 1887, and died in Palo Alto, California, September 7, 1985. In 1918 he married Stella Vera Weber, who survived him; they had no children. He received his doctorate in mathematics—first having studied law, language, and literature—from the University of Budapest in 1912. After two years at Göttingen and a short period in Paris, he accepted a position as Privatdocent at the Eidgenössische Technische Hochschule (Swiss Federal Institute of Technology) in Zürich in 1914 and rose to full professor there in 1928. In 1924 he was the first International Rockefeller Fellow and spent the year in England. In 1933 he was again a Rockefeller Fellow at Princeton. He emigrated to the United States in 1940, held a position at Brown for two years, spent a short time at Smith College, and in 1952 became a professor at Stanford. He retired in 1954 but continued to teach until 1978. He was elected to the National Academy of Sciences in 1976.

[1] George Pólya, *Collected Papers*, 4 vols. (Cambridge, Massachusetts, and London, England: MIT Press, 1974–1984).

## PROBABILITY

Pólya's first paper was in this field, and during his career he contributed perhaps thirty papers to various problems in probability theory. These papers contain many results that have now become textbook material, or even exercises, so that every student of probability encounters Pólya's work. One of Pólya's best known results is typical of his style, being unexpected but simple enough to prove once it was thought of. The Fourier transform

$$f(t) = \int_{-\infty}^{\infty} e^{itx}\mu\,(dx)$$

of a one-dimensional probability measure is known as the characteristic function. Pólya discovered (1918,3; 1923,2) that a sufficient condition for a real-valued function to be a characteristic function is that $f(0) = 1$, $f(\infty) = 0$, $f(t) = f(-t)$, and f is convex, $t > 0$. This is the only useful general test for characteristic functions, even though the most famous characteristic function, $\exp(-t^2)$, is not covered by it.

In 1921, Pólya initiated the study of random walks (which he named), proving the striking and completely unintuitive theorem that a randomly moving point returns to its initial position with probability 1 in one or two dimensions, but not in three or more dimensions (1921,4). His other contributions to the subject are less easily explained informally but, like those just mentioned, have served as starting points for extensive theories. These include limit laws (Pólya also named the central limit theorem), the continuity theorem for moments, stable distributions, the theory of contagion and exchangeable sequences of random variables, and the roots of random polynomials.

## COMPLEX ANALYSIS

Complex analysis is the study of analytic functions in two dimensions—the field to which Pólya made his most numerous contributions. As every student of the subject learns at an early stage, a function f that is analytic at a point, say 0 (for the sake of simplicity), is represented by a convergent power series

$$\sum_{n=0}^{\infty} a_n z^n;$$

and conversely such a series, if convergent, represents an analytic function. In principle the sequence $\{a_n\}$ of coefficients contains all the properties of the function. The problem is to make the sequence surrender the desired information. The most attractive results are those that connect a simple property of the coefficients with a simple property of the function.

Pólya made many contributions to this subject. He proved that the circle of convergence of a power series is "usually" a natural boundary for the function—that is, a curve past which the sum of the series cannot be continued analytically (1916,1; 1929,1). It is, in fact, always possible to change the signs of the coefficients in such a way that the new series cannot be contained outside the original circle of convergence.

According to Fabry's famous gap theorem, the circle of convergence of a power series is a natural boundary if the density of zero coefficients is 1. Pólya proved that no weaker condition will suffice for the same conclusion. He also extended this theorem in several ways and found analogs of Fabry's theorem for Dirichlet series, which have a more complex theory.

In 1929, Pólya systematized his methods for dealing with problems about power series (1929,1). This very influential paper deals with densities of sequences of numbers, with convex sets and with entire functions of exponential type—that is, with functions analytic in the whole complex plane whose absolute values grow no faster than a constant multiple of some exponential function $e^{A|z|}$. Functions of this kind have proved widely applicable in physics, communication theory, and in other branches of mathematics. The central theorem is Pólya's representation of a function f as a contour integral that resembles a Laplace transform,

$$f(z) = \frac{1}{2\pi i} \int_C F(w)e^{zw}dw,$$

a representation important in contexts far beyond those that Pólya originally envisioned.

Another topic that interested Pólya was how the general character of a function is revealed by the behavior of the function on a set of isolated points. The whole subject originated with Pólya's discovery (1915, 2) that $2^z$ is the "smallest" entire function, not a polynomial, that has integral values at the positive integers. There are many generalizations, on which research continued at least into the 1970s, and the theory is still far from complete. Pólya also contributed to many other topics in complex analysis, including the theory of conformal mapping and its extensions to three dimensions.

One of Pólya's favorite topics was the connections between properties of an entire function and the set of zeros of polynomials that approximate that function. He and I. Schur introduced two classes (now known as Pólya-Schur or Laguerre-Pólya functions) that are limits of polynomials that have either only real zeros or only real positive zeros. There are now many more applications, both in pure and applied mathematics, than Pólya himself envisaged, including, for example, the inversion theory of convolution transforms and the theory of interpolation by spline functions.

Another series of papers starting in 1927 was devoted to zeros of trigonometric integrals,

$$\int f(t)e^{izt}dt.$$

Pólya was much interested in the Riemann hypothesis about the zeros of the zeta function. His work on trigonometric integrals was inspired by the fact that a sufficiently strong theorem about their zeros would establish the hypothesis. That this hope has, so far, proved illusory, does not diminish the importance of Pólya's results in both mathematics and physics.

Pólya devoted a great deal of attention to the question of how the behavior in the large of an analytic or meromorphic function affects the distribution of the zeros of the derivatives of the function. One of the simplest results (simplest to state, that is) is that when a function is meromorphic in the whole plane (has no singular points except for poles), the zeros of its successive derivatives become concentrated near the polygon whose points are equidistant from the two nearest poles. The situation for entire functions is much more complex, and Pólya conjectured a number of theorems that are only now becoming possible to prove.

## REAL ANALYSIS, APPROXIMATION THEORY, NUMERICAL ANALYSIS

Pólya's most important contributions to this area are contained in the book on inequalities he wrote in collaboration with Hardy and Littlewood (1934,2). This was the first systematic study of the inequalities used by all working analysts in their research and has never been fully superseded by any of the more recent books on the subject.

Peano's space-filling curve passes through every point of a plane area but passes through some points four times. In 1913, Pólya produced a construction for a similar curve that has, at most, triple points, the smallest possible number. In keeping with Pólya's principle of drawing pictures whenever possible, the construction is quite geometrical (1913,1).

Pólya devoted two papers more than fifty years apart to Graeffe's method for numerical solution of polynomial equations (1914,2; 1968,1). Although this method is useful for functions other than polynomials as well, it was not highly regarded originally because of the large amount of computation it requires. With the availability of modern high-speed computers, however, the method is becoming more useful. His pioneering investigation of the theory of numerical integration (1933,1) is still important today in numerical analysis.

## COMBINATORICS

Combinatorics addresses questions about the number of ways there are to do something that is too complicated to be analyzed intuitively. Pólya's chief discovery was the enumeration of the isomers of a chemical compound, that is, the chemical compounds with different properties but the same numbers of each of their constituent elements. The problem had baffled chemists. Pólya treated it abstractly as a problem in group theory and was able to obtain formulas that made the solution of specific problems relatively routine. With the abstract theory in hand, Pólya could solve many concrete problems in chemistry, logic, and graph theory. His ideas and methods have been still further developed by his successors.

A related problem is the study of the symmetry of geometric figures, for example, tilings of the plane by tiles of particular shapes. Pólya's paper (1937,2) came to the attention of the artist M. C. Escher, who used it in constructing his famous pictures of interlocked figures.

The theory of symmetries also plays an important role in Pólya's work in mathematical physics.

## MATHEMATICAL PHYSICS

Physical problems in two or three dimensions usually depend in essential ways on the shape of the domain in which the problems are considered. For example, the shape of a drumhead affects the sound of the drum; the electrostatic capacitance of an object depends on its shape. Except for very simple shapes, such as circles or spheres, the mathematical equations that describe the properties are too difficult to solve exactly; the solutions must be approximated in some way. Pólya's contributions to mathematical physics consisted of developing methods for such approximations. These methods, like his work in other fields, were subsequently developed further by others.

Pólya was interested in estimating quantities of physical interest connected with particular domains, as, for example, electrostatic capacitance, torsional rigidity, and the lowest vibration frequency. Usually one wants an estimate for some property of a domain in terms of another. The simplest problem of this kind (and the oldest—it goes back to antiquity) is the isoperimetric problem, in which the area inside a curve is compared with the perimeter, or the volume of a solid is compared with its surface area. Problems of this kind, consequently, go by the generic name of isoperimetric problems. One method to which Pólya devoted a great deal of work, including the production of a widely read book (1951,1), is to replace a given domain by a more symmetric one with one property (say, the area inside a curve) the same, and for which the other property is more easily discussed. If we know that symmetrization increases or decreases the quantity in which we are interested, the result is an inequality for the other property.

One of the earliest successes of this technique was a simple proof of Rayleigh's conjecture that a circular membrane has the lowest vibration frequency (that is, the smallest eigenvalue of the corresponding differential equation) among all membranes of a specified area. For different problems, different kinds of symmetrization are needed.

Many physical quantities that are determined as the solutions of extremal problems can be estimated by making appropriate changes of variable, a technique known as transplantation. Pólya exploited this technique in a long paper in collaboration with M. Schiffer (1954,1). He also contributed several refinements to the standard technique of approximating solutions of partial differential equations by solving difference equations (1952,1; 1954,2).

## TEACHING AND LEARNING MATHEMATICS

Pólya believed that one should learn mathematics by solving problems. This led him to write, with G. Szegő, *Problems and Theorems in Analysis* (1925,1 [2 vols., in German]; 1972,1 [vol. 1], and 1976,1 [vol. 2] revised and enlarged English translation) in which topics are developed through series of problems. Besides their use for systematic instruction, these volumes are a convenient reference for special topics and methods. Pólya thought a great deal about how people solve problems and how they can learn to do so more effectively. His first book on this subject (1945,2) was very popular and has been translated into many languages. He wrote two additional books (1954,3; 1962,1) and many articles on the same general theme.

Pólya also stressed the importance of heuristics (essentially, intelligent guessing) in teaching mathematics and in mathematical research. In the preface to *Problems and Theorems*, for example, he and Szegő remarked that—since a straight line is determined by a point and a parallel—geometry suggests, by analogy, ways of approaching problems that have nothing to do with geometry. One can hope both to generate new problems and to guess methods for solving them by generalizing a well-understood problem, by interpolating between two problems or by thinking of a parallel situation. One can see these principles at work in some of Pólya's research, and many other mathematicians have found them helpful.

Whether heuristics can really be successful on a large scale as a teaching technique has not yet been established. Some researchers in artificial intelligence have not found it effective for teaching mathematics. It is not clear, however, whether these results reflect more unfavorably on Pólya or artificial intelligence. It does seem clear that putting Pólya's ideas into practice on a large scale would entail major changes both in the mathematics curriculum and in the training of teachers of mathematics.

Pólya also stressed geometric visualization of mathematics wherever possible, and "Draw a figure!" was one of his favorite adages.

IN PREPARING THIS MEMOIR I have drawn on the introductions and notes in the *Collected Papers* and, to a large extent, on the more detailed memoir prepared by G. L. Alexanderson and L. H. Lange for the *Bulletin of the London Mathematical Society*, which I had the opportunity of seeing in manuscript and to which I also contributed.

## SELECTED BIBLIOGRAPHY


1913

Über eine Peanosche Kurve. Bull. Acad. Sci. Cracovie, A, 305–13.

Sur un algorithme toujours convergent pour obtenir les polynomes de meilleure approximation de Tchebychef pour une fonction continue quelconque. C. R. Acad. Sci. (Paris), 1957:840–43.

Über Annäherung durch Polynome mit lauter reelen Wurzeln. Rend. Circ. Mat. Palermo, 36:279–95.

Über Annäherung durch Polynome deren sämtliche Wurzeln in einen Winkelraum fallen. Nachr. Ges. Wiss. Göttingen, 1913:326–30.

1914

With G. Lindwart. Über einen Zusammenhang zwischen der Konvergenz von Polynomfolgen und der Verteilung ihrer Wurzeln. Rend. Circ. Mat. Palermo, 37:297–304.

Über das Graeffesche Verfahren. Z. Mat. Phys., 63:275–90.

With I. Schur. Über zwei Arten von Faktorenfolgen in der Theorie der algebraischen Gleichungen. J. Reine Angew. Math., 144:89–113.

Sur une question concernant les fonctions entières. C. R. Acad. Sci. (Paris), 158:330–33.

1915

Algebraische Untersuchungen über ganze Functionen vom Geschlechte Null und Eins. J. Reine Angew. Math., 145:224–49.

Über ganzwertige ganze Funktionen. Rend. Circ. Mat. Palermo, 40:1–16.

1916

With A. Hurwitz. Zwei Beweise eines von Herrn Fatou vermuteten Satzes. Acta Math., 40:179–83.

Über den Zusammenhang zwischen dem Maximalbetrage einer analytischen Funktion und dem grössten Gliede der zugehörigen Taylorschen Reihe. Acta Math., 40:311-19.

Über Potenzreihen mit ganzzahligen Koeffizienten. Math. Ann., 77:497–513.

1917

Über geometrische Wahrscheinlichkeiten. S.-B. Akad. Wiss. Denksch. Philos. Hist. Kl., 126:319–28.

Über die Potenzreihen, deren Konvergenzkreis natürliche Grenze ist. Acta Math., 41:99–118.

1918

Über Potenzreihen mit endlich vielen verscheidenen Koeffizienten. Math. Ann., 78:286–93.

Über die Verteilung der quadratischen Reste und Nichtreste. Nachr. Ges. Wiss. Göttingen, 1918:21–29.

Über die Nullstellen gewisser ganzer Funktionen. Math. Z., 2:352–83.

1919

Über das Gauss'sche Fehlergesetz. Astronom. Nachr., 208:186–91; 209:111.

Proportionalwahl und Wahrscheinlichkeitsrechnung. Z. Gesamte Staatswiss., 74:297–322.

1920

Arithmetische Eigenschaften der Reihenentwicklungen rationaler Funktionen. J. Reine Angew. Math., 151:1–31.

Über den zentralen Grenzwertsatz der Wahrscheinlichkeitsrechnung und das Momentenproblem. Math. Z., 8:171–81.

Über ganze ganzwertige Funktionen. Nachr. Ges. Wiss. Göttingen, 1920:1–10.

1921

Bestimmung einer ganzen Funktion endlichen Geschlechts durch viererlei Stellen. Mat. Tidsskr. B.:16–21.

Ein Mittelwertsatz für Funktionen mehrerer Veränderlichen. Tôhoku Math. J., 19:1–3.

Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Strassennetz. Math. Ann., 84:149–60.

1922

Über die Nullstellen sukzessiver Derivierten. Math. Z., 12:36–60.

1923

Sur les séries entières à coefficients entiers. Proc. London Math. Soc., 21:22–38.

Herleitung des Gauss'schen Fehlergesetzes aus einer Funktionalgleichung. Math. Z., 18:96–108.

Bemerkungen über unendliche Folgen und ganze Funktionen. Math. Ann., 88:169-83.

Über die Existenz unendlich vieler singulärer Punkte auf der Konvergenzgeraden gewisser Dirichletscher Reihen. S.-B. Preuss. Akad. Wiss. Göttingen Math. Phys. Kl. Abh. Folge 3., 1923:45–50.

With F. Eggenberger. Über die Statistik verketteter Vorgänge. Z. Angew. Math. Mech., 3:279–89.

On the zeros of an integral function represented by Fourier's integral. Mess. Math., 52:185–88.

1924

Über die Analogie der Krystallsymmetrie in der Ebene. Z. Kristall., 60:278–82.

On the mean-value theorem corresponding to a given linear homogeneous differential equation. Trans. Am. Math. Soc., 24:312–24.

1925

With G. Szegő. *Aufgaben und Lehrsätze aus der Analysis.* 2 vols. Berlin: Springer-Verlag.

1926

On an integral function of an integral function. J. London Math. Soc., 1:12-15.

On the minimum modulus of integral functions of order less than unity. J. London Math. Soc., 1:78–86.

1927

With G. H. Hardy and A. E. Ingham. Theorems concerning mean values of analytic functions. Proc. R. Soc. A., 113:542–69.

Über trigonometrische Integrale mit nur reelen Nullstellen. J. Reine Angew. Math., 158:6–18.

Über die algebraisch-funktionentheoretischen Untersuchungen von J. L. W. V. Jensen. Kgl. Danske Vidensk. Selsk. Math.-Fys. Medd., 7(17).

Eine Verallgemeinerung des Fabryschen Lückensatzes. Nachr. Ges. Wiss. Göttingen, 1927:187–95.

1928

Über gewisse notwendige Determinantenkriterien für die Fortsetzbarkeit einer Potenzreihe. Math. Ann., 99:687–706.

Beitrag zur Verallgemeinerung des Verzerrungssatzes auf mehrfach zusammenhängende Gebiete. S.-B. Preuss. Akad. Wiss. Göttingen Math. Phys. Kl. Abh. Folge 3., 1928:228–32, 280–82; 1929:55–62.

1929

Untersuchungen über Lücken und Singularitäten von Potenzreihen. Math. Z., 29:549–640.

1931

With G. Szegő. Über den transfiniten Durchmesser (Kapazitätskonstante) von ebenen und räumlichen Punktmengen. J. Reine Angew. Math., 165:4–49.

1932

With A. Bloch. On the roots of certain algebraic equations. Proc. London Math. Soc., 33:102–14.

1933

Über die Konvergenz von Quadraturverfahren. Math. Z., 37:264–86.

Qualitatives über Wärmeausgleich. Z. Angew. Math. Mech., 13:125–28.

Untersuchungen über Lücken und Singularitäten von Potenzreihen. II. Ann. of Math. (2), 34:731–77.

1934

Über die Potenzreihenentwicklung gewisser mehrdeutiger Funktionen. Comment. Math. Helv., 7:201–21.

With G. H. Hardy and J. E. Littlewood. *Inequalities*. Cambridge: Cambridge University Press.

1936

Algebraische Berechnung der Anzahl der Isomeren einiger organischer Verbindungen. Z. Kristall. (A), 93:315–43.

1937

With M. Plancherel. Fonctions entières et intégrales de Fourier multiples. Comment. Math. Helv., 9:224–48; 10:110–63.

Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen. Acta Math., 68:145–254.

Über die Realität der Nullstellen fast aller Ableitungen gewisser ganzer Funktionen. Math. Ann., 114:622–34.

1938

Sur la promenade au hasard dans un réseau de rues. Actual. Sci. Ind., 734:25-44.

1942

On converse gap theorems. Trans. Am. Math. Soc., 52:65–71.

With R. P. Boas. Influence of the signs of the derivatives of a function on its analytic character. Duke Math. J., 9:406–24.

1943

On the zeros of the derivatives of a function and its analytic character. Bull. Am. Math. Soc., 49:178–91.

1945

With G. Szegő. Inequalities for the capacity of a condenser. Am. J. Math., 67:1–32.

*How To Solve It: A New Aspect of Mathematical Method.* Princeton: Princeton University Press.

1948

Torsional rigidity, principal frequency, electrostatic capacity and symmetrization. Q. Appl. Math., 6:276–77.

1949

With H. Davenport. On the product of two power series. Can. J. Math., 1:1–5.

1950

With A. Weinstein. On the torsional rigidity of multiply connected cross sections. Ann. Math., 52:154–63.

1951

With G. Szegő. *Isoperimetric Inequalities in Mathematical Physics.* Princeton: Princeton University Press.

1952

Sur une interprétation de la méthode des différences finies qui peut fournir des bornes supérieures ou inférieures. C. R. Acad. Sci. (Paris), 235:1079–81.

1954

With M. Schiffer. Convexity of functionals by transplantation. J. Analyse Math., 3:245–345.

Estimates for eigenvalues. In: *Studies in Mathematics and Mechanics Presented to Richard von Mises,* New York: Academic Press, pp. 200–7.

*Mathematics and Plausible Reasoning.* Vol. 1, *Induction and Analogy in Mathematics.* Vol. 2, *Patterns of Plausible Inference.* Princeton: Princeton University Press.

1956

With L. E. Payne and H. F. Weinberger. On the ratio of consecutive eigenvalues. J. Math. Phys., 35:289–98.

1958

With I. J. Schoenberg. Remarks on de la Vallée-Poussin means and convex conformal maps of the circle. Pacific J. Math., 8:295–334.

1959

With M. Schiffer. Sur la représentation conforme de l'extérieur d'une courbe fermée convexe. C.R. Acad. Sci. (Paris), 248:2837–39.

1961

On the eigenvalues of vibrating membranes, In memoriam Hermann Weyl. Proc. London Math. Soc., 11:419–33.

1962

*Mathematical Discovery: On Understanding, Learning, and Teaching Problem Solving.* 2 vols. New York: John Wiley & Sons.

1968

Graeffe's method for eigenvalues. Numer. Math., 11:315–19.

1972

With G. Szegő. *Problems and Theorems in Analysis,* Vol. 1. New York, Heidelberg, Berlin: Springer-Verlag. Revised and enlarged English language version of 1925,1. (See 1976,1, for Vol. 2).

1974

*Collected Papers.* Vol. 1, *Singularities of Analytic Functions.* Vol. 2, *Location of Zeros,* ed. R. P. Boas. Cambridge: MIT Press.

1976

With G. Szegő. *Problems and Theorems in Analysis,* vol. 2. New York, Heidelberg, Berlin: Springer-Verlag. Revised and enlarged English language version of 1925,1. (See 1972,1, for Vol. 1).

1984

*Collected Papers,* Vol. 3, *Analysis,* eds. J. Hersch and G. C. Rota. Vol. 4, *Probability, Combinatorics, Teaching and Learning Mathematics,* ed. G. C. Rota. Cambridge: MIT Press.