CHARACTERISTICS OF AIRPLANE ANTENNAS FOR RADIO RANGE-BEACON RECEPTION

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ABSTRACT

This paper gives the results of an investigation on the characteristics of airplane receiving antennas to determine whether an antenna arrangement could be devised which would have all the desirable electrical properties of the vertical pole antenna and yet be free from the mechanical difficulties encountered in the use of the pole antenna. The antennas studied include the inclined antenna with both forward and backward inclination, the horizontal dipole antenna, the horizontal L antenna, the horizontal V antenna, the inclined V antenna, the symmetrical transverse T antenna, and the symmetrical longitudinal T antenna. A theoretical treatment is given which enables the voltage induced by a radio range beacon transmitting station to be calculated for any receiving antenna in space. This theoretical analysis is used to determine the received voltage, course error, and localizing effect for each of the antenna types studied. An experimental study was also made to check the theoretical analysis. The results obtained by experiment check very well with the theoretical predictions for each type of antenna. The symmetrical transverse T antenna and the symmetrical longitudinal T antenna, with vertical lead-in portions, are both found to fulfill the desired requirements. Neither of these antennas shows any course errors and give the same received voltage as the vertical pole antenna having much greater actual height, thus reducing the mechanical troubles caused by vibration and ice formation. An appendix gives the mathematical derivation of the equation used as a starting point for the theoretical analysis.

CONTENTS

I. INTRODUCTION

Since the advent of the radio range beacon, the vertical pole antenna has been widely used because of its nondirectional properties and consequent freedom from course errors in radio range-beacon reception. The vertical pole antenna, however, is subject to considerable mechanical trouble because of vibration and ice formation, and for this reason, an antenna free from these mechanical difficulties and yet having all the desirable electrical properties of the simple pole antenna would be preferable.

Accordingly, a study of various types of airplane antennas was begun, a theoretical analysis being checked by experimental observa-

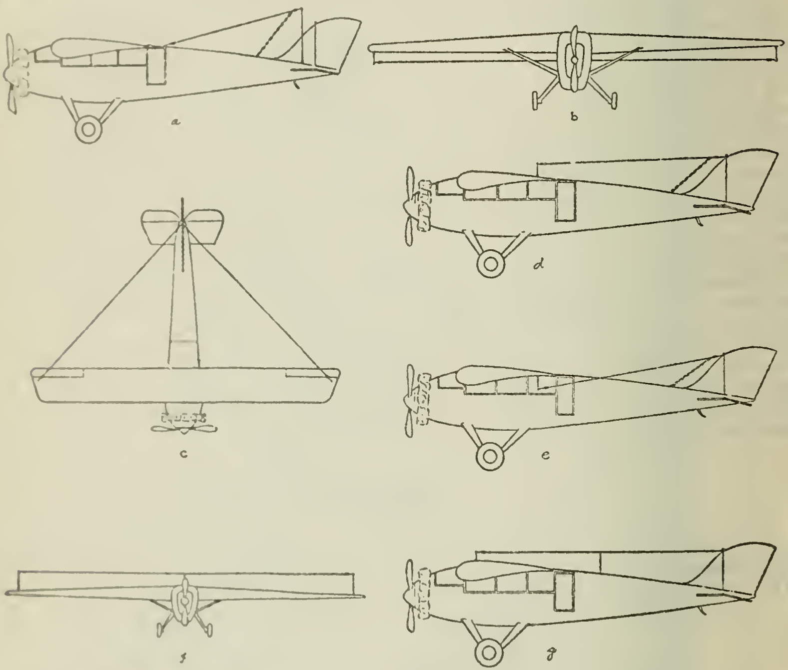

Figure 1.—*Experimental installations of antennas studied*

a, Inclined antenna; *b*, horizontal dipole antenna (half of this was used for a horizontal L antenna); *c*, top view, V antenna; *d*, side view, horizontal V antenna; *e*, side view, inclined V antenna; *f*, symmetrical transverse T antenna; *g*, symmetrical longitudinal T antenna

tions in the air and on the ground. A number of antenna arrangements were included in this study. For each type of antenna, the tests in the air included observation of the received voltage, the localizing effect or variation of the received voltage in the immediate vicinity of the beacon tower, and the course error as observed by circling the beacon. These were compared directly with the results obtained using the vertical pole antenna. The types studied included the inclined antenna, with both forward and backward inclination (one example of the latter being the trailing wire antenna), the horizontal dipole antenna, the horizontal L antenna, the horizontal V antenna, the inclined V antenna, the symmetrical transverse T antenna, and the symmetrical longitudinal T antenna. Figure 1 shows experimental installations of these antennas on an airplane. The results of this experimental work check the theoretical analysis with sufficient accuracy for the desired purpose, although exact quantitative data are almost impossible to obtain in the air because of the difficulties incurred in any attempt to measure the geometrical quantities involved.

II. THEORETICAL ANALYSIS

It is found convenient to treat the subject by the use of vector analysis. A somewhat similar treatment, using a trigonometric method, has been given by Capt. William H. Murphy, of the Signal

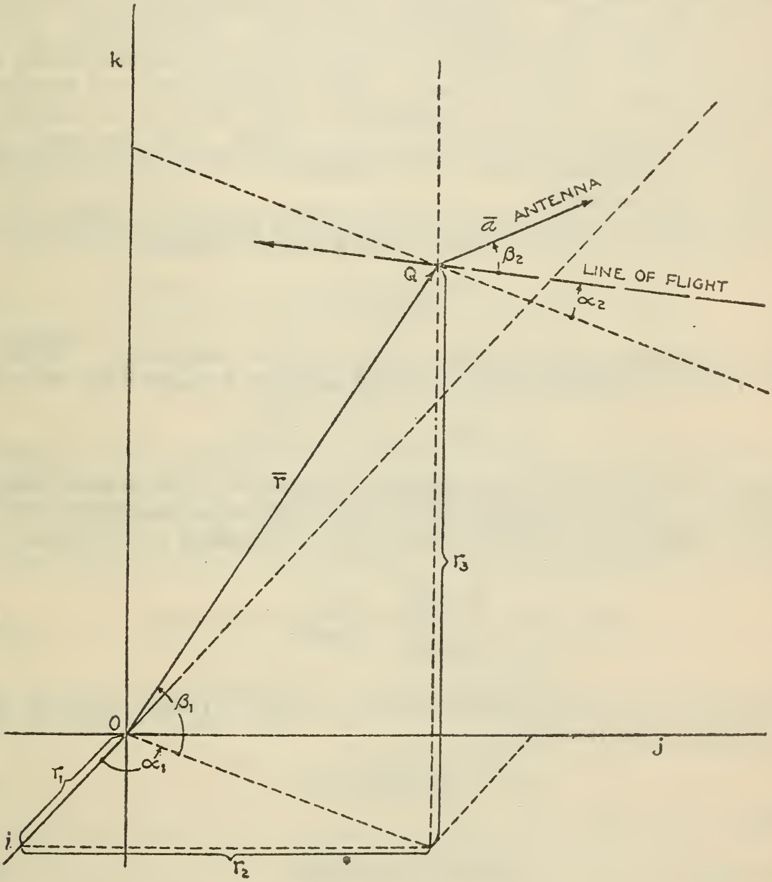

FIGURE 2.—*Diagram showing quantities used in the theoretical analysis*

Corps.[1] In this work, as in Captain Murphy's, relative rather than absolute field intensities and received voltages are desired. Consequently, all constants are considered equal to unity and omitted from the vector equations. The airplane is also considered to be at unit distance from the transmitter.

The transmitting loop antenna is taken in the $\bar{j}$-$\bar{k}$ plane, with its center at *o* (fig. 2), so that the $\bar{i}$-axis coincides with the axis of the

[1] W. H. Murphy, "Space Characteristics of Antennae," J. Frank. Inst., **201**, pp. 411–429; April, 1926; **203**, pp. 289–312; February, 1927.

loop antenna. The airplane is located at the point Q whose position vector is $\bar{r}$, and is flying in the direction indicated by the line marked "Line of flight." The line of flight is assumed to lie in a plane parallel to the $\bar{\imath}$-$\bar{\jmath}$ plane. The vector $\bar{a}$ is a unit vector in the direction of the antenna, pointing away from Q. α_1 is the angle between the axis of the loop antenna ($\bar{\imath}$-axis) and the projection of the line of sight (or the vector $\bar{r}$) on the $\bar{\imath}$-$\bar{\jmath}$ plane. α_2 is the angle between the line of flight and the projection of the line of sight on the horizontal plane containing the line of flight. β_1 is the angle of elevation of the airplane. β_2 is the angle of inclination of the antenna with respect to the horizontal plane. These angles are taken as positive when measured in the directions indicated in the figure. The antenna on the airplane is here considered to be in the vertical plane containing the longitudinal axis of the fuselage. This limitation may be removed, however, and the antenna considered in any position, by replacing α_2 in the following equations by $\alpha_2+\alpha'_2$, where α'_2 is the angle between the longitudinal axis of the airplane and the horizontal projection of the antenna.

The effective value of the electric field at Q, at unit distance from the transmitter, may be written, using Gibb's notation

$$\bar{E}_1=\bar{r}\times\bar{\imath} \tag{1}$$

The derivation of this equation is given in the Appendix. The effective voltage induced in the receiving antenna by this field vector is

$$E_1=\bar{E}_1\cdot\bar{a}=\bar{r}\times\bar{\imath}\cdot\bar{a}=[\bar{r}\bar{\imath}\bar{a}] \tag{2}$$

If the vectors $\bar{r}$ and $\bar{a}$ are evaluated in terms of their scalar components r_1, r_2, r_3, a_1, a_2, and a_3, the value of this expression may be computed. However, equation (2) is readily simplified still further:

$$E_1=[\bar{r}\bar{\imath}\bar{a}]=\begin{vmatrix} r_1 & r_2 & r_3 \\ 1 & 0 & 0 \\ a_1 & a_2 & a_3 \end{vmatrix}=-\begin{vmatrix} r_2 & r_3 \\ a_2 & a_3 \end{vmatrix}=a_2r_3-r_2a_3 \tag{3}$$

Similarly, for a second loop antenna in the $\bar{\imath}$-$\bar{k}$ plane, with its axis coinciding with the $\bar{\jmath}$-axis

$$\bar{E}_2=\bar{\jmath}\times\bar{r} \tag{4}$$

and

$$E_2=\bar{\jmath}\times\bar{r}\cdot\bar{a}=[\bar{\jmath}\bar{r}\bar{a}] \tag{5}$$

$$E_2=\begin{vmatrix} 0 & 1 & 0 \\ r_1 & r_2 & r_3 \\ a_1 & a_2 & a_3 \end{vmatrix}=-\begin{vmatrix} r_1 & r_3 \\ a_1 & a_3 \end{vmatrix}=a_1r_3-r_1a_3 \tag{6}$$

Now the components r_1, r_2, r_3, a_1, a_2, and a_3 of the unit vectors $\bar{r}$ and $\bar{a}$ may be expressed in any system of coordinates, but the most convenient form is that of the spherical coordinates shown in Figure

2. In this system, and since the vectors are unit vectors (that is, $|\bar{r}|=|\bar{a}|=1$)

$$\left.\begin{aligned} r_1&=\cos\alpha_1\cos\beta_1\\ r_2&=\sin\alpha_1\cos\beta_1\\ r_3&=\sin\beta_1\end{aligned}\right\}\qquad(7)$$

and

$$\left.\begin{aligned} a_1&=\cos(\alpha_1+\alpha_2)\cos\beta_2\\ a_2&=\sin(\alpha_1+\alpha_2)\cos\beta_2\\ a_3&=\sin\beta_2\end{aligned}\right\}\qquad(8)$$

On substitution of these values, equations (3) and (6) become

$$E_1=a_2r_3-r_2a_3=\sin(\alpha_1+\alpha_2)\sin\beta_1\cos\beta_2-\sin\alpha_1\cos\beta_1\sin\beta_2 \qquad (9)$$

$$E_2=a_1r_3-r_1a_3=\cos(\alpha_1+\alpha_2)\sin\beta_1\cos\beta_2-\cos\alpha_1\cos\beta_1\sin\beta_2 \qquad (10)$$

These two equations form the basis of the remainder of the theoretical analysis.

A course is obtained whenever $E_1=E_2$. Thus, for the general antenna shown in Figure 2, a course is obtained when

$$\begin{aligned}&\sin(\alpha_1+\alpha_2)\sin\beta_1\cos\beta_2-\sin\alpha_1\cos\beta_1\sin\beta_2\\ &=\cos(\alpha_1+\alpha_2)\sin\beta_1\cos\beta_2-\cos\alpha_1\cos\beta_1\sin\beta_2\end{aligned}\qquad(11)$$

or when

$$\tan\alpha_1=\frac{\tan\beta_1(\cos\alpha_2-\sin\alpha_2)-\tan\beta_2}{\tan\beta_1(\sin\alpha_2+\cos\alpha_2)-\tan\beta_2} \qquad (12)$$

where α_1 is the course angle measured from axis of loop antenna 1.

If the currents in the two loop antennas are equal, the true courses bisect the angles between the transmitting loop antennas. Assuming equal currents, then, the course error for any type of receiving antenna may be found. If the course error is here defined by e,[2] where

$$e=\alpha_1-\pi/4 \qquad (13)$$

then

$$\tan e=\tan(\alpha_1-\pi/4)=\frac{\tan\alpha_1-1}{1+\tan\alpha_1} \qquad (14)$$

Combining (12) and (14)

$$\tan e=\frac{\sin\alpha_2\tan\beta_1}{\tan\beta_2-\cos\alpha_2\tan\beta_1} \qquad (15)$$

Mathematically, it is possible to make this course error zero in several ways: $\beta_1=0$, $\alpha_2=0$, or $\beta_2=\frac{\pi}{2}$. The last is the only practicable method (heretofore accomplished through the use of a vertical antenna). However, a consideration of the physical significance of equation (11) shows that the course error is introduced by the horizontal component of the field vector, for equation (11) can be true for $\alpha_1=45°$ (that is, $e=0$) only if $\alpha_2=0$. Since α_2 can be considered as a generalized direction angle of the vector sum of the horizontal projections of the antenna, it follows that α_2 can be zero for all directions of

[2] This is the usual way of defining course error in connection with flight on a radio range beacon.

flight only if the vector sum of the horizontal projections of the antenna is zero. Obviously, the horizontal projection of the antenna is affected only by the horizontal component of the field vector, so that it is evident that the course error is introduced by this horizontal component. The horizontal component of the field vector can not be eliminated (except by changes in the transmitting antenna, which are not convenient), so that it becomes necessary to construct the receiving antenna so that it will not respond to this horizontal component; that is, as noted above, the vector sum of the horizontal projections of the receiving antenna must be zero. Since the use of flat top portions is thus permissible so long as they fulfill these conditions, it becomes possible to design antennas having smaller actual height than that of the usual pole antenna, and yet having the same effective height. Such an antenna would give the same received voltage as the vertical pole antenna, and would likewise be immune to course errors.

III. APPLICATION OF THEORY TO SPECIFIC ANTENNA STRUCTURES

The theory is readily applied to indicate the results obtainable from several types of simple antennas. In each case, it is desirable to determine the localizing effect (the behavior of the signal in the immediate vicinity of the beacon), the course error, and the relative received voltage. The accompanying graphs (see fig. 3) show the localizing effect for the antennas considered.

1. VERTICAL ANTENNA

$$\beta_2=\frac{\pi}{2} \tag{16}$$

(a) LOCALIZING EFFECT

Substituting this condition in equations (9) and (10) and adding, the total received voltage is

$$E=E_1+E_2=-\sin\alpha_1\cos\beta_1-\cos\alpha_1\cos\beta_1 \tag{17}$$

This equation is strictly correct only when the two loop antennas are energized simultaneously and in the same time phase (for example, in the case of the two-course visual type beacon). However, the results secured in the following analysis are independent of this limitation, although the equations used in securing these results are somewhat different if the loop antenna currents are displaced from each other in time phase, or if the loop antennas are energized separately as in the case of the aural type beacon.

The value of E in equation (17) becomes zero when

$$\beta_1=\frac{\pi}{2} \tag{18}$$

Thus, this antenna gives a zero signal zone directly over the beacon. (Fig. 3(*A*).)

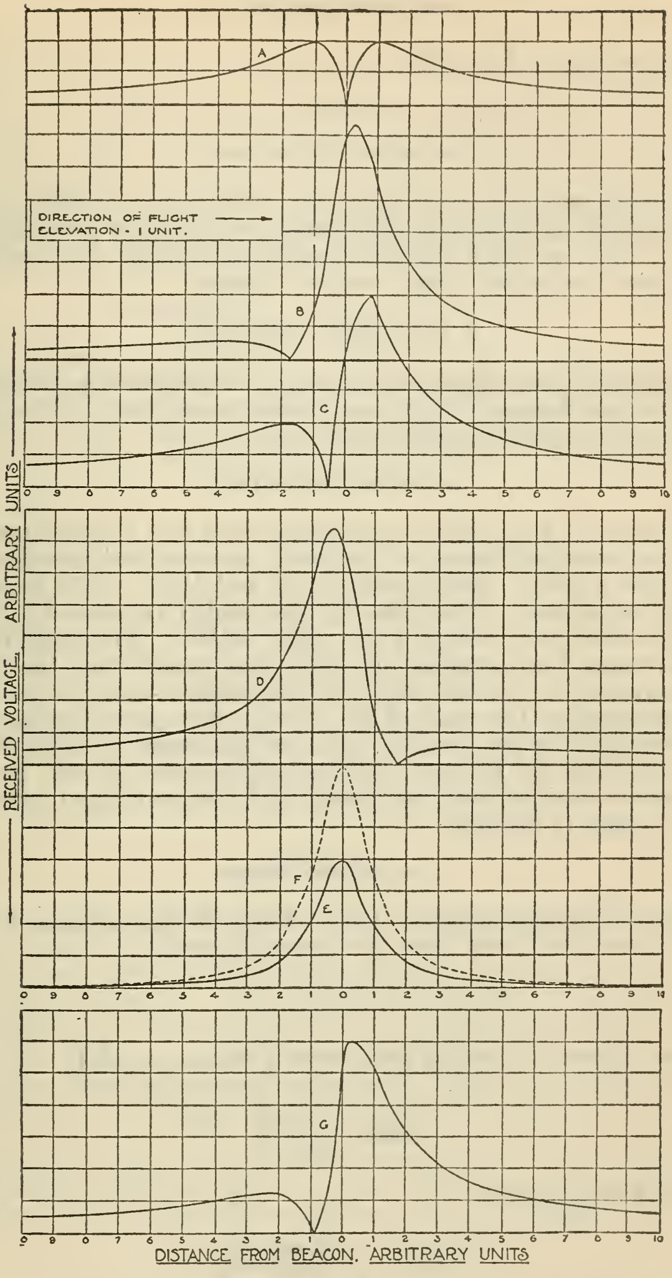

FIGURE 3.—*Graphs of localizing effects for antennas studied*

A, vertical antenna; *B*, inclined antenna ($\beta_2=30°$); *C*, inclined antenna ($\beta_2=60°$); *D*, trailing wire antenna ($\beta_2=-30°$); *E*, horizontal L antenna; *F*, horizontal V antenna (angle between each wire and fuselage center line=30°); *G*, inclined V antenna, angle between each wire and fuselage center line=30°, angle of inclination=45°.

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(b) COURSE ERROR

Substituting $\beta_2 = \frac{\pi}{2}$ in (15)

$$e = 0 \tag{19}$$

(c) RECEIVED VOLTAGE

In equation (17), α_1 may have any value, but is constant for any particular line of flight over the beacon, so that E is proportional to $\cos \beta_1$. Thus, at great distances, where β_1 is small, the vertical antenna gives a relatively large received voltage.

2. INCLINED ANTENNA

The antenna is considered in the same vertical plane as the fuselage, either inclined forward ($\beta_2 > 0$), or trailing wire ($\beta_2 < 0$). The general equations given above apply to this case.

(a) LOCALIZING EFFECT

This antenna gives both a zero signal zone and a maximum signal zone, one occurring before the beacon is reached and the other after it has been passed. (See equations (9) and (10).) If β_2 is positive, the zero signal zone occurs first (fig. 3(B) and (C)), while if β_2 is negative, the maximum occurs first. (Fig. 3(D).) The magnitude of β_2 determines the distances of these two zones from the beacon. The sharpness of the zero and the magnitude of the maximum are also dependent on the value of β_2. As β_2 is decreased, the zero signal zone recedes from the beacon, and the maximum increases in value and approaches the beacon. In the limiting case ($\beta_2 = 0$), the maximum occurs directly over the beacon and the zero signal zone disappears (recedes to infinity).

(b) COURSE ERROR

This is given by equation (15). Thus, if the airplane is flying directly toward or away from the beacon ($\alpha_2 = 0$)

$$e = 0 \tag{20}$$

If the airplane is circling the beacon $\left(\text{that is, } \alpha_2 = \frac{\pi}{2}\right)$,

$$\tan e = \frac{\tan \beta_1}{\tan \beta_2} \tag{21}$$

If $\alpha_2 = \pi/4$ and $\beta_2 = \pi/4$

$$\tan e = \frac{\sqrt{2} \tan \beta_1}{2 - \sqrt{2} \tan \beta_1} \tag{22}$$

In equation (15), when $\tan \beta_2 - \cos \alpha_2 \tan \beta_1$ becomes very small, $\tan e$ becomes very large and e becomes nearly equal to $\frac{\pi}{2}$.

(c) RECEIVED VOLTAGE

If β_2 is large $\left(\text{greater than } \frac{\pi}{3}\right)$, the received voltage in an inclined antenna is very nearly the same as that in a vertical antenna of the same length. However, if β_2 is small, the inclined antenna is markedly inferior to the vertical antenna at great distances from the beacon, although there is a region, in the vicinity of the maximum signal zone mentioned above, where the inclined antenna is superior to the vertical antenna. This is a small region close to the beacon, and of little value practically, except for localizing effects.

3. HORIZONTAL DIPOLE ANTENNA

This antenna is in the horizontal plane, and perpendicular to the line of flight. The results apply also to an L-type antenna from the wing tip to the fuselage. In this case, the antenna does not lie in the vertical plane containing the longitudinal axis of the fuselage, and consequently α_2 in the preceding equations must be replaced by $\alpha_2+\frac{\pi}{2}$. Also

$$\beta_2=0$$

Now

$$\left.\begin{aligned}\sin\left(\alpha_2+\frac{\pi}{2}\right)&=\cos\alpha_2\\ \cos\left(\alpha_2+\frac{\pi}{2}\right)&=-\sin\alpha_2\end{aligned}\right\}\qquad(23)$$

By substitution of these values in equation (15), it is found that

$$\tan e=\cot\alpha_2\qquad(24)$$

and

$$e=\frac{\pi}{2}-\alpha_2\qquad(25)$$

Thus the course error is the complement of the angle at which the airplane crosses the beacon course.

(1) *Localizing effect.*—With this antenna, a maximum signal zone occurs over the beacon. As mentioned before, this is the localizing effect obtained in the limiting case of the inclined antenna, when $\beta_2=0$. (Fig. 3(*E*).)

(2) *Course error.*—As noted immediately above, the course error is the complement of the angle at which the airplane crosses the beacon course.

(3) *Received voltage.*—From equation (24), it is seen that the received voltage varies as $\sin\beta_1$, so that it becomes very small when the airplane is any appreciable distance from the beacon.

4. HORIZONTAL V ANTENNA

One variation of the horizontal antenna is the V antenna from wing tips to vertical fin. The performance of this type of antenna is essentially the same as that of the horizontal dipole, except that the received voltage is somewhat greater, since both horizontal components of the transmitted signal are received. The localizing effect is shown in Figure 3(*F*).

5. SYMMETRICAL T ANTENNA

This can be either longitudinal or transverse. The received voltage of this type of antenna is due entirely to the vertical portion, the flat top merely increasing the charging current at the top of the vertical portion, thus increasing the effective height. Consequently, the antenna will have the characteristics of its vertical or lead-in portion. If this is truly vertical, the arrangement will act the same as the simple vertical antenna, while if the lead-in portion is inclined, the characteristics will be those of the inclined antenna. The T antenna gives greater effective height than either type simple antenna having the same actual height because of the more uniform current distribution resulting from the additional capacity at the top of the antenna.

6. OTHER ANTENNA SYSTEMS

The general theory outlined in Section II is also applicable to other and more complicated antennas. For example, consider the inclined V antenna from the wing tips to the vertical fin, with the lead-in running from the junction of the two wires at the vertical fin. This is essentially a form of T antenna, the effects of the horizontal components perpendicular to the axis of the fuselage cancelling. The antenna, therefore, performs as an inclined antenna having a vertical portion and also having a horizontal portion in the axis of the fuselage. The localizing effect for this type of antenna is shown in Figure 3(*G*).

Similarly, consider any antenna having a vertical lead-in and a flat-top arrangement symmetrically disposed about the vertical lead-in so that the vector sum of the horizontal projections of the individual elements of the flat top is equal to zero. This antenna will behave in every way as a vertical antenna, but giving greater received voltage than a vertical antenna of the same actual height. It is this type of antenna which offers advantages of reduced mechanical vibration over the conventional vertical pole antenna. When the flat-top portions lie along the longitudinal axis of the fuselage, for example, the longitudinal T antenna, the problems due to ice formation are also reduced.

IV. EXPERIMENTAL RESULTS

Antennas of each of the types analyzed above were installed on the National Bureau of Standards' airplane and compared (in flight on the visual radio range beacon) with the vertical pole antenna as a standard of comparison. Relative effective heights were determined by switching the antenna under test and the pole antenna alternately to the same receiving set and recording the receiving set output voltage. Course error effects were obtained by using two receiving sets, one connected to the antenna under test and the other to the pole antenna, and comparing the course indications (as obtained on two reed indicators connected in the output circuits of the two receiving sets), while circling the beacon at various altitudes and various distances from the beacon. The localizing effects were obtained by flying at a constant altitude on a beacon course directly over the beacon tower.

The results secured in the air were approximate only, exact quantitative measurements being very difficult because of the difficulty in measuring the geometric quantities involved. However, the results were sufficiently accurate to corroborate the theoretical analysis.

1. RELATIVE RECEIVED VOLTAGES

The data obtained showed that, of the antennas studied, the symmetrical T antenna of either type and the wing tip to vertical fin inclined V antenna both gave effective heights equivalent to that obtained with the vertical pole antenna without requiring nearly the same actual height as the pole antenna. For example, a symmetrical transverse T antenna having a 12-inch vertical lead-in and a flat top extending 15 feet on each side of the fuselage, parallel to and 12 inches above the wing surfaces, gave slightly better effective height than a 5-foot vertical pole.

2. COURSE ERROR EFFECTS

(a) INCLINED ANTENNA ($\beta_2=20°$, INCLINATION FORWARD)

When circling the beacon at constant distance from the beacon tower, the on-course indications of the reed indicator operated from the receiving set connected to the inclined antenna were in advance of the on-course indications of the reed indicator in the output circuit of the receiving set connected to the pole antenna. The angle of lead increased with decreasing distance from the beacon and also increased with the altitude of the airplane for a given distance from the beacon. This angle of lead is equal to the course error. The results are in accordance with equation (15), placing $\alpha_2=\frac{\pi}{2}$, since, when circling the beacon, the line of flight is perpendicular to the line of sight. Equation (15) then resolves into equation (21), namely

$$\tan e=\frac{\tan \beta_1}{\tan \beta_2}$$

where β_1 is the angle of elevation of the line connecting the airplane with the beacon (the line of sight) and β_2 the angle of inclination of the antenna (20° in this case). Some idea of the magnitude of the course errors involved may be obtained from one specific case. Flying at an elevation of 3,000 feet along a circle of three miles radius, the course error is 27.5°.

When flying along a beacon course directly to or from the beacon, no course error was obtained. This is also in accordance with equation (15), since, when $\alpha_2=0$, equation (15) resolves into equation (20), namely

$$\tan e=0$$

(b) INCLINED ANTENNA ($\beta_2=-20°$, INCLINATION BACKWARD)

The results obtained were exactly the same as for the antenna with forward inclination except that the on-course indications of the reed indicator operated from the receiving set connected to the inclined antenna lagged the on-course indications obtained on the reed indicator in the output circuit of the set connected to the pole antenna.

(c) INCLINED V ANTENNA

The results secured were the same as for the inclined antenna with forward inclination.

(d) SYMMETRICAL T ANTENNA (EITHER TYPE)

No course errors were obtained under any conditions. Both the transverse T and longitudinal T antennas were tested.

3. LOCALIZING EFFECTS

The localizing effect for each of the antennas studied was found to be very nearly the same as shown in the graphs of Figure 3. This was, perhaps, the most satisfactory test of all, checking, within the limits of error, the exact theoretical analysis. At 3,000 feet altitude, by changing the angle of inclination of the inclined antenna from 20° to 90°, the zero signal zone could be moved from approximately 1 mile from the beacon to directly over the beacon tower. When setting up the longitudinal T antenna, the zero signal effect directly over the beacon tower was employed as a check on the electrical symmetry of the flat-top arrangement.

V. CONCLUSIONS

For reception of signals from radio range beacons, an antenna arrangement of which the symmetrical T antenna is a typical example was found to give good electrical performance and to be free from trouble due to vibration. In addition, the performance of this type of antenna under conditions favoring ice formation should prove superior to that of the vertical pole antenna. To reduce the effects of mechanical vibration, this antenna employs vertical poles considerably shorter than the conventional pole antenna (10 to 18 inches instead of 5 to 6 feet). Equivalent effective height is secured through the addition of a flat top. In order to prevent directional effects, which introduce course errors, the flat top is made up of two, three, or more elements symmetrically disposed about the vertical lead-in, so that the vector sum of the horizontal projections of the individual elements is equal to zero. When but two flat-top elements are employed, the antenna takes the form of either a transverse or longitudinal T. The latter arrangement is preferable from the viewpoint of freedom from trouble due to ice formation. Furthermore, such a longitudinal T antenna has an aerodynamical resistance very much smaller than that of the transverse T antenna, and probably as small, if not smaller, than that of the vertical pole antenna.

VI. APPENDIX

This calculation of the field due to a loop antenna is based on the Lorentz solution of the electromagnetic field equations for a known distribution of currents and charges.

The notation used (Gibbs-"Vector analysis") is as follows:

$\underline{H}$ = magnetic field intensity.

$\underline{E}$ = electric field intensity.

ρ = density of charge.

v = velocity of motion of charge.

μ, c, K = constants depending upon system of units used
$\nabla\times$ = curl.
$\nabla\cdot$ = divergence.

Pot $\bar{w} = \iiint \frac{\bar{w}}{r} d\tau$ where the integration is extended over all space, $d\tau$ represents the element of volume, and r the distance from $d\tau$ to the point at which the potential is computed.

I_0 = Maximum value of current in loop antenna.
ω = angular velocity of current in loop antenna.
t = time
V = velocity of electromagnetic waves in medium surrounding loop.
$[u]$ = value of u at time $t-\frac{r}{V}$, where r is the distance from source of disturbance to point at which $[u]$ is considered.

The loop is assumed to be square and the current to be the same throughout the loop.

In this notation, Maxwell's equations of the electromagnetic field are

$$\left.\begin{aligned} \nabla\times\bar{H} &= \frac{4\pi}{c}\left(\rho\bar{v}+\frac{K}{4\pi}\frac{\partial\bar{E}}{\partial t}\right) \\ \nabla\times\bar{E} &= -\frac{\mu}{c}\frac{\partial\bar{H}}{\partial t} \\ \nabla\cdot\bar{E} &= \frac{4\pi\rho}{K} \\ \nabla\cdot\bar{H} &= 0 \end{aligned}\right\} \tag{1}$$

A solution of this system sometimes used (neglecting a term in the value of $\bar{E}$ which is immaterial in this discussion) may be written

$$\left.\begin{aligned} \bar{H} &= \frac{1}{c}\nabla\times\text{Pot }[\rho\bar{v}] = \frac{1}{c}\text{Lap }[\rho\bar{v}] \\ \bar{E} &= -\frac{\mu}{c^2}\frac{\partial}{\partial t}\text{Pot }[\rho\bar{v}] = -\frac{\mu}{c^2}\text{Pot}\left[\frac{\partial}{\partial t}(\rho\bar{v})\right] \end{aligned}\right\} \tag{2}$$

Now

$$\rho\bar{v} = \bar{I} = I\bar{b} = I_0\bar{b}\sin\omega t \tag{3}$$

where $\bar{b}$ is the unit vector in the direction of the current.

$$\therefore \bar{E} = -\frac{\mu}{c^2}\text{Pot}\left[\frac{\partial\bar{I}}{\partial t}\right] \tag{4}$$

$$\frac{\partial\bar{I}}{\partial t} = \omega I_0\bar{b}\cos\omega t \tag{5}$$

$$\bar{E} = -\frac{\mu\omega I_0}{c^2}\text{Pot }[\bar{b}\cos\omega t] = -\frac{\mu\omega I_0}{c^2}\int_{(c)}\frac{\bar{b}\cos\omega\left(t-\frac{h}{V}\right)}{h}ds \tag{6}$$

where h represents the distance from current element to P of Figure 4, which shows the position of the loop antenna and the distances and vectors used in the following treatment:

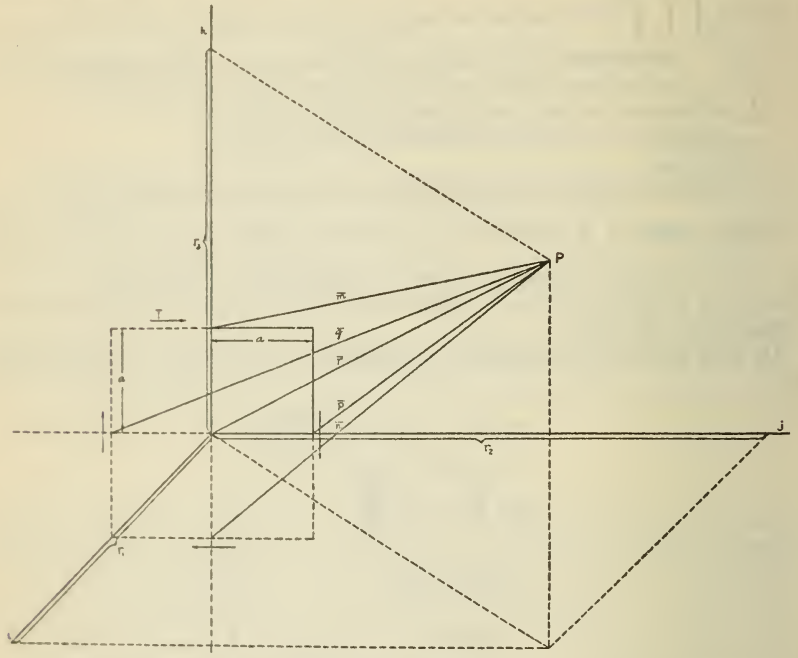

FIGURE 4.—*Diagram showing quantities used in appendix*

$$\bar{E}=-\frac{\mu\omega I_o}{c^2}\left[-\bar{k}\frac{\cos\omega\left(t-\frac{p}{V}\right)}{p}-\bar{j}\frac{\cos\omega\left(t-\frac{n}{V}\right)}{n}+\bar{k}\frac{\cos\omega\left(t-\frac{q}{V}\right)}{q}+\bar{j}\frac{\cos\omega\left(t-\frac{m}{V}\right)}{m}\right]\int_{-a}^{a}ds \tag{7}$$

But

$$\int_{-a}^{a}ds=2a \tag{8}$$

$$\bar{E}=\frac{2\mu\omega a I_o}{c^2}\left[\left\{\frac{\cos\omega\left(t-\frac{p}{V}\right)}{p}-\frac{\cos\omega\left(t-\frac{q}{V}\right)}{q}\right\}\bar{k}+\left\{\frac{\cos\omega\left(t-\frac{n}{V}\right)}{n}-\frac{\cos\omega\left(t-\frac{m}{V}\right)}{m}\right\}\bar{j}\right] \tag{9}$$

Now

$$\left.\begin{aligned}\bar{p}&=\bar{r}-a\bar{j}=r_1\bar{i}+(r_2-a)\bar{j}+r_3\bar{k}\\ \bar{q}&=\bar{r}+a\bar{j}=r_1\bar{i}+(r_2+a)\bar{j}+r_3\bar{k}\\ \bar{m}&=\bar{r}-a\bar{k}=r_1\bar{i}+r_2\bar{j}+(r_3-a)\bar{k}\\ \bar{n}&=\bar{r}+a\bar{k}=r_1\bar{i}+r_2\bar{j}+(r_3+a)\bar{k}\end{aligned}\right\} \tag{10}$$

r_1, r_2, and r_3 are the components of $\bar{r}$ along the $\bar{i}$, $\bar{j}$, and $\bar{k}$ axes, respectively.

$$p=[r_1^2+(r_2-a)^2+r_3^2]^{1/2}=(r^2-2r_2a+a^2)^{1/2}=r-\frac{ar_2}{r} \tag{11}$$

This is an approximation obtained by expanding the value of p by the binomial theorem and neglecting terms involving a in powers higher than the first. Thus

$$\left.\begin{aligned} p&=r-\frac{ar_2}{r}\\ q&=r+\frac{ar_2}{r}\\ m&=r-\frac{ar_3}{r}\\ n&=r+\frac{ar_3}{r}\end{aligned}\right\} \tag{12}$$

Also, by the same method

$$\left.\begin{aligned} \frac{1}{p}&=\frac{1}{r}+\frac{ar_2}{r^3}\\ \frac{1}{q}&=\frac{1}{r}-\frac{ar_2}{r^3}\\ \frac{1}{m}&=\frac{1}{r}+\frac{ar_3}{r^3}\\ \frac{1}{n}&=\frac{1}{r}-\frac{ar_3}{r^3}\end{aligned}\right\} \tag{13}$$

By the use of equations (12) and (13) and the trigonometrical formulas

$$\cos(x+y)-\cos(x-y)=-2\sin x\sin y$$

$$\cos(x+y)+\cos(x-y)=2\cos x\cos y$$

the coefficients of $\bar{j}$ and $\bar{k}$ in (9) may be simplified.

$$\frac{\cos\omega\left(t-\frac{p}{V}\right)}{p}-\frac{\cos\omega\left(t-\frac{p}{V}\right)}{q}$$

$$=\cos\omega\left(t-\frac{r}{V}+\frac{ar_2}{rV}\right)\left(\frac{1}{r}+\frac{ar_2}{r^3}\right)-\left(\frac{1}{r}-\frac{ar_2}{r^3}\right)\cos\omega\left(t-\frac{r}{V}-\frac{ar_2}{rV}\right)$$

$$=\frac{1}{r}\left[\cos\omega\left(t-\frac{r}{V}+\frac{ar_2}{rV}\right)-\cos\omega\left(t-\frac{r}{V}-\frac{ar_2}{rV}\right)\right]$$

$$+\frac{ar_2}{r^3}\left[\cos\omega\left(t-\frac{r}{V}+\frac{ar_2}{rV}\right)+\cos\omega\left(t-\frac{r}{V}-\frac{ar_2}{rV}\right)\right]$$

$$=-\frac{2}{r}\sin\omega\left(t-\frac{r}{V}\right)\sin\frac{\omega ar_2}{rV}+\frac{2ar_2}{r^3}\cos\omega\left(t-\frac{r}{V}\right)\cos\frac{\omega ar_2}{rV}$$

$$=-\frac{4\pi ar_2}{\lambda r^2}\sin\omega\left(t-\frac{r}{V}\right)+\frac{2ar_2}{r^3}\cos\omega\left(t-\frac{r}{V}\right) \tag{14}$$

Since $\frac{\omega a}{V}=\frac{2\pi a}{\lambda}$, and a is small in comparison with λ, $\sin \frac{\omega\, ar_2}{Vr}$ may be replaced by $\frac{2\pi ar_2}{\lambda r}$ and $\cos \frac{2\pi ar_2}{\lambda r}$ may be considered equal to one. Similarly

$$\frac{\cos \omega \left(t-\frac{n}{V}\right)}{n}-\frac{\cos \omega \left(t-\frac{m}{V}\right)}{m}$$

$$=\frac{4\pi\ ar_3}{\lambda\, r^2} \sin \omega \left(t-\frac{r}{V}\right)-\frac{2ar_3}{r^3} \cos \omega \left(t-\frac{r}{V}\right) \tag{15}$$

Then

$$\bar{E}=\frac{2\mu\omega aI_0}{c^2}\left\{\bar{k}\left[-\frac{4\pi ar_2}{\lambda r^2}\sin \omega \left(t-\frac{r}{V}\right)+\frac{2ar_2}{r^3}\cos \omega\left(t-\frac{r}{V}\right)\right]\right.$$

$$\left.+\bar{j}\left[\frac{4\pi ar_3}{\lambda r^2}\sin \omega \left(t-\frac{r}{V}\right)-\frac{2ar_3}{r^3}\cos \omega \left(t-\frac{r}{V}\right)\right]\right\}$$

$$=\frac{2\mu\omega aI_0}{c^2}\left(r_3\bar{j}-r_2\bar{k}\right)\left[\frac{4\pi a}{\lambda r^2}\sin \omega \left(t-\frac{r}{V}\right)-\frac{2a}{r^3}\cos \omega \left(t-\frac{r}{V}\right)\right] \tag{16}$$

Now

$$r_3\bar{j}-r_2\bar{k}=\bar{r}\times\bar{i}$$

$$\therefore\ \bar{E}=\left[\frac{8\pi\mu\omega a^2I_0}{\lambda c^2 r}\sin \omega \left(t-\frac{r}{V}\right)-\frac{4\mu\omega a^2I_0}{c^2r^2}\cos \omega \left(t-\frac{r}{V}\right)\right]\frac{\bar{r}\times\bar{i}}{r} \tag{17}$$

The first term represents the radiation field and the second term the induction field. If the radiation term alone is considered, no error is introduced except one of magnitude for small values of r. As interest in this treatment centers chiefly on relative rather than absolute field intensities, the quantity $\frac{8\pi\mu\omega a^2I_0}{\lambda c^2}$ may be considered constant and equal to unity. Furthermore, the airplane is considered to be at unit distance from the transmitter, so that equation (17) may be written

$$\bar{E}_{eff.}=r\times\bar{i} \tag{18}$$

However, for the calculation of curves showing the localizing effect when an airplane flies at constant height over the beacon, it is necessary to take the variation of $\bar{r}$ into account and the equation then used is

$$\bar{E}=\frac{\bar{r}\times\bar{i}}{r^2} \tag{19}$$

WASHINGTON, January 24, 1931.

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