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Selection Principles in Function Spaces with the Compact-Open Topology

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Abstract. For a Tychonoff space X, we denote by $C_k(X)$ the space of all real-valued continuous functions on X with the compact-open topology. A subset $A \subset X$ is said to be sequentially dense in X if every point of X is the limit of a convergent sequence in A. In this paper, the following properties for $C_k(X)$ are considered.

$$\begin{array}{ccccccc} S_1(\mathcal{S},\mathcal{S}) & \Rightarrow & S_{fin}(\mathcal{S},\mathcal{S}) & \Rightarrow & S_{fin}(\mathcal{S},\mathcal{D}) & \Leftarrow & S_1(\mathcal{S},\mathcal{D}) \\ \Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\ S_1(\mathcal{D},\mathcal{S}) & \Rightarrow & S_{fin}(\mathcal{D},\mathcal{S}) & \Rightarrow & S_{fin}(\mathcal{D},\mathcal{D}) & \Leftarrow & S_1(\mathcal{D},\mathcal{D}) \end{array}$$

For example, a space $C_k(X)$ satisfies $S_1(\mathcal{S},\mathcal{D})$ (resp., $S_{fin}(\mathcal{S},\mathcal{D})$) if whenever $(S_n : n \in \mathbb{N})$ is a sequence of sequentially dense subsets of $C_k(X)$, one can take points $f_n \in S_n$ (resp., finite $F_n \subset S_n$) such that $\{f_n : n \in \mathbb{N}\}$ (resp., $\bigcup\{F_n : n \in \mathbb{N}\}$) is dense in $C_k(X)$. Other properties are defined similarly.

In [22], we obtained characterizations these selection properties for $C_p(X)$. In this paper, we give characterizations for $C_k(X)$.

1. Introduction

For a Tychonoff space X, we denote by $C_k(X)$ the space of all real-valued continuous functions on X with the compact-open topology. Subbase open sets of $C_k(X)$ are of the form $[A, U] = \{f \in C(X) : f(A) \subset U\}$, where A is a compact subset of X and U is a non-empty open subset of $\mathbb{R}$. Since the compact-open topology coincides with the topology of uniform convergence on compact subsets of X, we can represent a basic neighborhood of the point $f \in C(X)$ as $\langle f, A, \epsilon\rangle$ where $\langle f, A, \epsilon\rangle := \{g \in C(X) : |f(x) - g(x)| < \epsilon \ \forall\, x \in A\}$, A is a compact subset of X and $\epsilon > 0$.

Many topological properties are defined or characterized in terms of the following classical selection principles. Let $\mathcal{A}$ and $\mathcal{B}$ be sets consisting of families of subsets of an infinite set X. Then:

$S_1(\mathcal{A},\mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $\{b_n\}_{n\in\mathbb{N}}$ such that for each n, $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$S_{fin}(\mathcal{A},\mathcal{B})$ is the selection hypothesis: for each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $\{B_n\}_{n\in\mathbb{N}}$ of finite sets such that for each n, $B_n \subseteq A_n$, and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

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$U_{fin}(\mathcal{A},\mathcal{B})$ is the selection hypothesis: whenever $\mathcal{U}_1, \mathcal{U}_2, ... \in \mathcal{A}$ and none contains a finite subcover, there are finite sets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\{\bigcup \mathcal{F}_n : n \in \mathbb{N}\} \in \mathcal{B}$.

The following prototype of many classical properties is called "$\mathcal{A}$ choose $\mathcal{B}$" in [29].

$\binom{\mathcal{A}}{\mathcal{B}}$: For each $\mathcal{U} \in \mathcal{A}$ there exists $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{V} \in \mathcal{B}$.

Then $S_{fin}(\mathcal{A},\mathcal{B})$ implies $\binom{\mathcal{A}}{\mathcal{B}}$.

In this paper, by a cover we mean a nontrivial one, that is, $\mathcal{U}$ is a cover of X if $X = \bigcup \mathcal{U}$ and $X \notin \mathcal{U}$.

A cover $\mathcal{U}$ of a space X is called:

- an ω-cover (a k-cover) if each finite (compact) subset C of X is contained in an element of $\mathcal{U}$;
- a γ-cover (a γ_k-cover) if $\mathcal{U}$ is infinite and for each finite (compact) subset C of X the set $\{U \in \mathcal{U} : C \nsubseteq U\}$ is finite.

Note that a γ_k-cover is a k-cover, and a k-cover is infinite. A compact space has no k-covers.

A space X is said to be a γ_k-set if each open k-cover $\mathcal{U}$ of X contains a countable set $\{U_n : n \in \mathbb{N}\}$ which is a γ_k-cover of X [9].

In a series of papers it was demonstrated that γ-covers and k-covers play a key role in function spaces [8–10, 13, 16, 22–26, 28] and many others. We continue to investigate applications of k-covers in function spaces with the compact-open topology.

2. Main Definitions and Notation

If X is a topological space and $A \subseteq X$, then the sequential closure of A, denoted by $[A]_{seq}$, is the set of all limits of sequences from A. A set $D \subseteq X$ is said to be sequentially dense if $X = [D]_{seq}$. A space X is called sequentially separable if it has a countable sequentially dense set. Call X strongly sequentially separable, if X is separable and every countable dense subset of X is sequentially dense. Clearly, every strongly sequentially separable space is sequentially separable, and every sequentially separable space is separable.

For a topological space X we denote:

- $\mathcal{O}$ — the family of open covers of X;
- Γ — the family of open γ-covers of X;
- Γ_k — the family of open γ_k-covers of X;
- Ω — the family of open ω-covers of X;
- $\mathcal{K}$ — the family of open k-covers of X;
- $\mathcal{K}^{\omega}_{cz}$ — the family of countable co-zero k-covers of X;
- $\mathcal{D}$ — the family of dense subsets of $C_k(X)$;
- $\mathcal{D}^{\omega}$ — the family of countable dense subsets of $C_k(X)$;
- $\mathcal{S}$ — the family of sequentially dense subsets of $C_k(X)$;
- $\mathbb{K}(X)$ — the family of all non-empty compact subsets of X.
- A space X is R-separable, if X satisfies $S_1(\mathcal{D},\mathcal{D})$ (Def. 47, [2]).
- A space X is M-separable (selective separability), if X satisfies $S_{fin}(\mathcal{D},\mathcal{D})$.
- A space X is selectively sequentially separable, if X satisfies $S_{fin}(\mathcal{S},\mathcal{S})$ (Def. 1.2, [3]).

For a topological space X we have the next relations of selectors for sequences of dense sets of X.

$$\begin{array}{ccccccc}
S_1(\mathcal{S},\mathcal{S}) & \Rightarrow & S_{fin}(\mathcal{S},\mathcal{S}) & \Rightarrow & S_{fin}(\mathcal{S},\mathcal{D}) & \Leftarrow & S_1(\mathcal{S},\mathcal{D}) \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
S_1(\mathcal{D},\mathcal{S}) & \Rightarrow & S_{fin}(\mathcal{D},\mathcal{S}) & \Rightarrow & S_{fin}(\mathcal{D},\mathcal{D}) & \Leftarrow & S_1(\mathcal{D},\mathcal{D})
\end{array}$$

Let X be a topological space, and $x \in X$. A subset A of X *converges* to x, $x = \lim A$, if A is infinite, $x \notin A$, and for each neighborhood U of x, $A \setminus U$ is finite. Consider the following collection:

- $\Omega_x = \{A \subseteq X : x \in \overline{A} \setminus A\}$;
- $\Gamma_x = \{A \subseteq X : x = \lim A\}$.

Note that if $A \in \Gamma_x$, then there exists $\{a_n\} \subset A$ converging to x. So, simply Γ_x may be the set of non-trivial convergent sequences to x.

We write $\Pi(\mathcal{A}_x, \mathcal{B}_x)$ without specifying x, we mean $(\forall x)\Pi(\mathcal{A}_x, \mathcal{B}_x)$.

So, we have three types of topological properties described through the selection principles:

• local properties of the form $S_*(\Phi_x, \Psi_x)$;

• global properties of the form $S_*(\Phi, \Psi)$;

• semi-local properties of the form $S_*(\Phi, \Psi_x)$.

Our main goal is to describe the topological properties for sequences of dense sets of $C_k(X)$ in terms of selection principles of X.

3. $S_1(\mathcal{D}, \mathcal{S})$

Recall that X a γ'_k-set if it satisfies the selection hypothesis $S_1(\mathcal{K}, \Gamma_k)$ [9].

Theorem 3.1. ([11]) *For a Tychonoff space X the following statements are equivalent:*

1. $C_k(X)$ *satisfies* $S_1(\Omega_{\mathbf{0}}, \Gamma_{\mathbf{0}})$ *(i.e., $C_k(X)$ is strongly Fréchet-Urysohn);*
2. *X is a γ'_k-set.*

Recall that the *i*-weight $iw(X)$ of a space X is the smallest infinite cardinal number τ such that X can be mapped by a one-to-one continuous mapping onto a Tychonoff space of the weight not greater than τ.

Theorem 3.2. (Noble [19]) *A space $C_k(X)$ is separable if and only if $iw(X) = \aleph_0$.*

Theorem 3.3. *For a Tychonoff space X with $iw(X) = \aleph_0$ the following statements are equivalent:*

1. *$C_k(X)$ satisfies $S_1(\mathcal{D}, \mathcal{S})$;*
2. *Every dense subset of $C_k(X)$ is sequentially dense;*
3. *X satisfies $S_1(\mathcal{K}, \Gamma_k)$ (X is a γ'_k-set);*
4. *X is a γ_k-set;*
5. *$C_k(X)$ is Fréchet-Urysohn;*
6. *$C_k(X)$ satisfies $S_{fin}(\mathcal{D}, \mathcal{S})$;*
7. *X satisfies $S_{fin}(\mathcal{K}, \Gamma_k)$;*
8. *Each finite power of X satisfies $S_{fin}(\mathcal{K}, \Gamma_k)$;*
9. *$C_k(X)$ satisfies $S_1(\Omega_{\mathbf{0}}, \Gamma_{\mathbf{0}})$;*
10. *$C_k(X)$ satisfies $S_1(\mathcal{D}, \Gamma_{\mathbf{0}})$.*

Proof. (1) ⇒ (6) is immediate.

(4) ⇔ (5) By Theorem 4.7.4 in [17].

(3) ⇔ (4) By Theorem 18 in [4].

(3) ⇔ (7) By Theorem 5 in [9].

(3) ⇔ (8) By Theorem 7 in [9].

(3) ⇔ (9) By Theorem 3.1.

(9) ⇒ (10) is immediate.

(6) ⇒ (2) Let D be a dense subset of $C_k(X)$. By $S_{fin}(\mathcal{D}, \mathcal{S})$, for sequence $(D_i : D_i = D$ and $i \in \mathbb{N})$ there is a sequence $(K_i : i \in \mathbb{N})$ such that for each i, K_i is finite, $K_i \subset D_i$, and $\bigcup_{i\in\mathbb{N}} K_i$ is a countable sequentially dense subset of $C_k(X)$. It follows that D is a sequentially dense subset of $C_k(X)$.

(2) ⇒ (4) Let $\mathcal{U}$ be an open k-cover of X. Note that the set $\mathcal{D} := \{f \in C(X) : f \upharpoonright (X \setminus U) \equiv 1$ for some $U \in \mathcal{U}\}$ is dense in $C_k(X)$, hence, it is sequentially dense. Take $f_n \in \mathcal{D}$ such that $f_n \mapsto \mathbf{0}$. Let $f_n \upharpoonright (X \setminus U_n) \equiv 1$ for some $U_n \in \mathcal{U}$. Then $\{U_n : n \in \mathbb{N}\}$ is a γ_k-subcover of $\mathcal{U}$, because of $f_n \mapsto \mathbf{0}$. Hence, X is a γ_k-set.

(3) ⇒ (1) Let $(D_{i,j} : i, j \in \mathbb{N})$ be a sequence of dense subsets of $C_k(X)$ and let $D = \{f_i : i \in \mathbb{N}\}$ be a countable dense subset of $C_k(X)$.

For every $i, j \in \mathbb{N}$ consider $\mathcal{U}_{i,j} = \{U_{h,i,j} : U_{h,i,j} = (f_i - h)^{-1}(-\frac{1}{j}, \frac{1}{j})$ for $h \in D_{i,j}\}$. Note that $\mathcal{U}_{i,j}$ is an k-cover of X for every $i, j \in \mathbb{N}$. Since X a γ'_k-set, there is a sequence $(U_{h(i,j),i,j} : i, j \in \mathbb{N})$ such that $U_{h(i,j),i,j} \in \mathcal{U}_{i,j}$, and $\{U_{h(i,j),i,j} : i, j \in \mathbb{N}\}$ is an element of Γ_k. Claim that $\{h(i, j) : i, j \in \mathbb{N}\}$ is a dense subset of $C_k(X)$. Fix $g \in C(X)$

and a base neighborhood $W = \langle g, A, \epsilon\rangle$ of g, where A is a compact subset of X and $\epsilon > 0$. There are $f_i \in D$ and $j \in \mathbb{N}$ such that $\langle f_i, A, \frac{1}{j}\rangle \subseteq W$. Since $\{U_{h(i,j),i,j} : i, j \in \mathbb{N}\}$ is an element of Γ_k, there is $j' > j$ such that $A \subset U_{h(i,j'),i,j'}$, hence, $h(i, j') \in \langle f_i, A, \frac{1}{j'}\rangle \subseteq \langle f_i, A, \frac{1}{j}\rangle \subseteq W$.

Since $C_k(X)$ is Fréchet-Urysohn, every dense subset of $C_k(X)$ is sequentially dense. It follows that $\{h(i, j) : i, j \in \mathbb{N}\}$ is sequentially dense.

(10) ⇒ (3) Let $\{\mathcal{U}_i : i \in \mathbb{N}\} \subset \mathcal{K}$ and let $D = \{d_j : j \in \mathbb{N}\}$ be a countable dense subset of $C_k(X)$. Consider $D_i = \{f_{K,U,i,j} \in C(X) :$ such that $f_{K,U,i,j} \upharpoonright K \equiv d_j$, $f_{K,U,i,j} \upharpoonright (X \setminus U) \equiv 1$ where $K \in \mathbb{K}(X)$, $K \subset U \in \mathcal{U}_i\}$ for every $i \in \mathbb{N}$. Since D is a dense subset of $C_k(X)$, then D_i is a dense subset of $C_k(X)$ for every $i \in \mathbb{N}$. By (10), there is a set $\{f_{K(i),U(i),i,j(i)} : i \in \mathbb{N}\}$ such that $f_{K(i),U(i),i,j(i)} \in D_i$ and $\{f_{K(i),U(i),i,j(i)} : i \in \mathbb{N}\} \in \Gamma_{\mathbf{0}}$. Claim that a set $\{U(i) : i \in \mathbb{N}\} \in \Gamma_k$. Let $K \in \mathbb{K}(X)$ and let $W = [K, (-\frac{1}{2}, \frac{1}{2})]$ be a base neighborhood of $\mathbf{0}$. Since $\{f_{K(i),U(i),i,j(i)} : i \in \mathbb{N}\} \in \Gamma_{\mathbf{0}}$, there is $i' \in \mathbb{N}$ such that $f_{K(i),U(i),i,j(i)} \in W$ for every $i > i'$. It follows that $K \subset U(i)$ for every $i > i'$ and, hence, $\{U(i) : i \in \mathbb{N}\} \in \Gamma_k$. □

Let $S \subset \mathbb{K}(X)$. An open cover $\mathcal{U}$ of a space X is called:
- a s-cover if each $C \in S$ is contained in an element of $\mathcal{U}$;
- a γ_s-cover if $\mathcal{U}$ is infinite and for each $C \in S$ the set $\{U \in \mathcal{U} : C \nsubseteq U\}$ is finite.

Definition 3.4. Let $S \subset \mathbb{K}(X)$. A space X is called a γ_s-set if each s-cover of X contains a sequence which is a γ_s-cover of X.

Definition 3.5. A space X is called a γ_k^{ω}-set if each countable cozero k-cover $\mathcal{U}$ of X contains a set $\{U_n : n \in \mathbb{N}\}$ which is a γ_k-cover of X.

For a mapping $f : X \mapsto Y$ we will denote by $f(k) = \{f(K) : K \in \mathbb{K}(X)\}$.

Theorem 3.6. *For a Tychonoff space X with $iw(X) = \aleph_0$, the following statements are equivalent:*

1. *$C_k(X)$ satisfies $S_1(\mathcal{D}^{\omega}, \mathcal{S})$;*
2. *$C_k(X)$ is strongly sequentially separable;*
3. *X is a γ_k^{ω}-set;*
4. *X satisfies $S_1(\mathcal{K}_{cz}^{\omega}, \Gamma_k)$;*
5. *for every a condensation (one-to-one continuous mapping) $f : X \mapsto Y$ from the space X on a separable metric space Y, the space Y is a $\gamma_{f(k)}$-set.*

Proof. (3) ⇒ (5) Let f be a condensation $f : X \mapsto Y$ from the space X on a separable metric space Y. If μ is a $f(k)$-cover of Y, then there is $\mu' \subset \mu$ such that μ' is a $f(k)$-cover of Y and $|\mu'| = \aleph_0$. The family $f^{-1}(\mu') = \{f^{-1}(V) : V \in \mu'\}$ is a countable co-zero k-cover of X. By the argument that X is a γ_k^{ω}-set, we have that Y is $\gamma_{f(k)}$-set.

The remaining implications follow from the proofs of Theorem 3.3 and Theorem 18 in [4]. □

Corollary 3.7. *For a separable metrizable space X, the following statements are equivalent:*

1. *$C_k(X)$ satisfies $S_1(\mathcal{D}, \mathcal{S})$;*
2. *Every dense subset of $C_k(X)$ is sequentially dense;*
3. *$C_k(X)$ is strongly sequentially separable;*
4. *$C_k(X)$ is a Fréchet-Urysohn;*
5. *$C_k(X)$ is metrizable and separable;*
6. *X satisfies $S_1(\mathcal{K}, \Gamma_k)$;*
7. *X satisfies $S_1(\mathcal{K}, \mathcal{K})$;*
8. *X satisfies $S_{fin}(\mathcal{K}, \mathcal{K})$;*
9. *X is a hemicompact.*

Proof. By Theorem 3.3 and Theorem 6 in [4]. □

A space X is called a *k-Lindelöf space* if for each open k-cover $\mathcal{U}$ of X there is a $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{V}$ is countable and $\mathcal{V} \in \mathcal{K}$. Each k-Lindelöf space is Lindelöf, so normal, too.

Lemma 3.8. ([17]) *$C_k(X)$ has countable tightness if and only if X is k-Lindelöf.*

By Theorem 3.3, Theorem 3.6 and Lemma 3.8 we have

Theorem 3.9. *For a Tychonoff space X with $iw(X) = \aleph_0$ the following statements are equivalent:*

1. *$C_k(X)$ is Fréchet-Urysohn;*
2. *$C_k(X)$ is strongly sequentially separable and has countable tightness;*
3. *X satisfies $S_1(\mathcal{K}^\omega, \Gamma_k)$ and is k-Lindelöf;*
4. *Every dense subset of $C_k(X)$ contains a countable sequentially dense subset of $C_k(X)$.*

In Doctoral Dissertation, A.J. March considered the following problem (Problem 117 in [15]): Is it possible to find a space X such that $C_k(X)$ is strongly sequentially separable but $C_k(X)^2$ is not strongly sequentially separable?

We get a negative answer to this question.

Proposition 3.10. *Suppose X has the property $S_1(\mathcal{K}^\omega_{cz}, \Gamma_k)$. Then $X \bigsqcup X$ has the property $S_1(\mathcal{K}^\omega_{cz}, \Gamma_k)$.*

Proof. Let $\mathcal{U} = \{U_i : i \in \mathbb{N}\}$ be a countable k-cover of $X \bigsqcup X$ by cozero sets. Let $X \bigsqcup X = X_1 \bigsqcup X_2$ where $X_i = X$ for $i = 1, 2$. Consider $\mathcal{V}_1 = \{U^1_i = U_i \bigcap X_1 : X_1 \setminus U_i \neq \emptyset, i \in \mathbb{N}\}$ and $\mathcal{V}_2 = \{U^2_i = U_i \bigcap X_2 : X_2 \setminus U_i \neq \emptyset, i \in \mathbb{N}\}$ as families of subsets of the space X. Define $\mathcal{V} := \{U^1_i \bigcap U^2_i : U^1_i \in \mathcal{V}_1$ and $U^2_i \in \mathcal{V}_2\}$. Note that $\mathcal{V}$ is a countable k-cover of X by cozero sets. By Theorem 18 in [4], there is $\{U^1_{i_n} \bigcap U^2_{i_n} : n \in \mathbb{N}\} \subset \mathcal{V}$ such that $\{U^1_{i_n} \bigcap U^2_{i_n} : n \in \mathbb{N}\}$ is a γ_k-cover of X. It follows that $\{U_{i_n} : n \in \mathbb{N}\}$ is a γ_k-cover of $X \bigsqcup X$. □

Theorem 3.11. *For a Tychonoff space X the following statements are equivalent:*

1. *$C_k(X)$ is strongly sequentially separable;*
2. *$(C_k(X))^n$ is strongly sequentially separable for each $n \in \mathbb{N}$.*

Proof. By Theorem 3.6, Proposition 9.1 and the argument that $C_k(X \bigsqcup X) = C_k(X) \times C_k(X)$. □

A.J. March considered the problem (Problem 116 in [15]): Is it possible to find spaces X, Y such that $C_k(X)$ and $C_k(Y)$ are strongly sequentially separable but $C_k(X) \times C_k(Y)$ is not strongly sequentially separable?

A. Miller constructed the following example [18].

Example 3.12. There exist disjoint subsets of the plane X and Y such that both X and Y are γ_k-sets but $X \cup Y$ is not. Let X be the open disk of radius one, i.e., $X = \{(x, y) : x^2 + y^2 < 1\}$, and Y be any singleton on the boundary of X, e.g., $Y = \{(1, 0)\}$.

Thus, we have the example of the subsets of the plane X and Y such that $C_k(X)$ and $C_k(Y)$ are strongly sequentially separable, but $C_k(X \cup Y)$ is not.

Note that (in contrast to the C_p-theory) $C_k(X \cup Y) \neq C_k(X) \times C_k(Y)$.

In [4], the authors considered the next problem (Problem 21 in [4]) : Is the class of γ_k-sets closed for finite unions ?

A particular answer to this problem and March's problem is the following

Theorem 3.13. *Suppose that X and Y are γ_k-sets, $iw(X) = iw(Y) = \aleph_0$ and Y is first-countable. Then $X \bigsqcup Y$ is a γ_k-set.*

Proof. By Theorem 3.6, $C_k(X)$ and $C_k(Y)$ are strongly sequentially separable. Notice that each hemicompact space belong to the class $S_1(\mathcal{K}, \Gamma_k)$, and the converse holds for first countable spaces [16]. It follows that $C_k(Y)$ is a separable metrizable (first countable) space. By Theorem 9 in [6], $C_k(X) \times C_k(Y)$ is strongly sequentially separable. Since $C_k(X) \times C_k(Y) = C_k(X \bigsqcup Y)$ and, by Theorem 3.6, we have that $X \bigsqcup Y$ is a γ_k-set. □

Corollary 3.14. *The product $C_k(X) \times C_k(Y)$ of strongly sequentially separable space $C_k(X)$ and strongly sequentially separable first-countable space $C_k(Y)$ belongs to the class of strongly sequentially separable spaces.*

4. $S_1(\mathcal{D}, \mathcal{D})$

In [10] it was shown that a Tychonoff space X belongs to the class $S_1(\mathcal{K}, \mathcal{K})$ if and only if $C_k(X)$ has countable strong fan tightness (i.e. for each $f \in C_k(X)$, $S_1(\Omega_f, \Omega_f)$ holds [27]).

Lj.D.R. Kočinac proved the next

Theorem 4.1. ([4, Theorem 6]) *For a first countable Tychonoff space X the following statements are equivalent:*

1. $C_k(X)$ *is first countable;*
2. $C_k(X)$ *has countable strong fan tightness;*
3. $C_k(X)$ *has countable fan tightness;*
4. *X is locally compact Lindelöf space;*
5. *X satisfies* $S_1(\mathcal{K}, \mathcal{K})$*;*
6. *X satisfies* $S_{fin}(\mathcal{K}, \mathcal{K})$*;*

We consider the generalizations (Theorem 4.2 and Theorem 5.3) of the Theorem 4.1 to the class of Tychonoff spaces with $iw(X) = \aleph_0$.

Theorem 4.2. *For a Tychonoff space X with* $iw(X) = \aleph_0$ *the following statements are equivalent:*

1. $C_k(X)$ *satisfies* $S_1(\mathcal{D}, \mathcal{D})$*;*
2. *X satisfies* $S_1(\mathcal{K}, \mathcal{K})$*;*
3. *Each finite power of X satisfies* $S_1(\mathcal{K}, \mathcal{K})$*;*
4. $C_k(X)$ *satisfies* $S_1(\Omega_\mathbf{0}, \Omega_\mathbf{0})$ *[countable strong fan tightness];*
5. $C_k(X)$ *satisfies* $S_1(\mathcal{D}, \Omega_\mathbf{0})$.

Proof. (2) ⇔ (3) By Theorem 5 in [14].

(2) ⇔ (4) By Theorem 2.2 in [10].

(1) ⇒ (2) Let $\mathcal{K}_i \in \mathcal{K}$ for every $i \in \mathbb{N}$ and let D be a countable dense subset of $C_k(X)$. Consider $D_i = \{f_{K,U,d} \in C(X) : f|(X \setminus U) \equiv 1$ and $f|K = d$ where K is a compact subset of X, $U \in \mathcal{K}_i$ such that $K \subset U$ and $d \in D\}$. Since D is a dense subset of $C_k(X)$, we have that D_i is a dense subset of $C_k(X)$ for every $i \in \mathbb{N}$. By (1), there is a sequence $\{f_{K_i,U_i,d_i}\}_{i\in\mathbb{N}}$ such that for each i, $f_{K_i,U_i,d_i} \in D_i$, and $\{f_{K_i,U_i,d_i} : i \in \mathbb{N}\}$ is a dense subset of $C_k(X)$. Note that $U_i \in \mathcal{K}_i$ for each $i \in \mathbb{N}$ and $\{U_i : i \in \mathbb{N}\} \in \mathcal{K}$.

(2) ⇒ (1) Let $(D_{i,j} : i, j \in \mathbb{N})$ be a sequence of dense subsets of $C_k(X)$ and let $D = \{d_n : n \in \mathbb{N}\}$ be a countable dense subset of $C_k(X)$. For every couple (i, j), $i, j \in \mathbb{N}$ and $f \in D_{i,j}$ consider $K_{i,j,f} = \{x \in X : |f(x) - d_j(x)| < \frac{1}{i}\}$ and $\mathcal{K}_{i,j} = \{K_{i,j,f} : f \in D_{i,j}\}$. We claim that $\mathcal{K}_{i,j} \in \mathcal{K}$ for every couple (i, j), $i, j \in \mathbb{N}$. Let $K \in \mathbb{K}(X)$ and $\langle d_j, K, \frac{1}{i}\rangle$ a base neighborhood of d_j. Since $D_{i,j}$ is a dense subset of $C_k(X)$, there is $f \in D_{i,j}$ such that $f \in \langle d_j, K, \frac{1}{i}\rangle$, hence, $K \subset K_{i,j,f}$. Fix $j \in \mathbb{N}$, by (2), there is a family $\{K_{i,j,f(i,j)} : i \in \mathbb{N}\}$ such that $K_{i,j,f(i,j)} \in \mathcal{K}_{i,j}$ and $\{K_{i,j,f(i,j)} : i \in \mathbb{N}\} \in \mathcal{K}$. So $f(i, j) \in D_{i,j}$ for $i, j \in \mathbb{N}$. Claim that $\{f(i, j) : i, j \in \mathbb{N}\}$ is dense in $C_k(X)$. Let $p \in C(X)$, $K \in \mathbb{K}(X)$, $\epsilon > 0$ and let $\langle p, K, \epsilon\rangle$ be a base neighborhood of p. There is $j' \in \mathbb{N}$ such that $d_{j'} \in \langle p, K, \frac{\epsilon}{2}\rangle$. Since $\{K_{i,j',f(i,j')} : i \in \mathbb{N}\} \in \mathcal{K}$, there is $i' \in \mathbb{N}$ such that $K \subset K_{i',j',f(i',j')}$ and $\frac{1}{i'} < \frac{\epsilon}{2}$. It follows that $|f(i', j')(x) - p(x)| < |f(i', j')(x) - d_{j'}(x)| + |d_{j'}(x) - p(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ for every $x \in K$. Hence, $f(i', j') \in \langle p, K, \epsilon\rangle$ and $\{f(i, j) : i, j \in \mathbb{N}\}$ is dense in $C_k(X)$.

(4) ⇒ (5) is immediate.

(5) ⇒ (1) Let $(D_{i,j} : i \in \mathbb{N})$ be a sequence of dense subsets of $C_k(X)$ for each $j \in \mathbb{N}$ and let $D = \{d_j : j \in \mathbb{N}\}$ be a countable dense subset of $C_k(X)$. By (5), for every $j \in \mathbb{N}$ there is a family $\{d^i_j : i \in \mathbb{N}\}$ such that $d^i_j \in D_{i,j}$ and $\{d^i_j : i \in \mathbb{N}\} \in \Omega_{d_j}$. Note that $\{d^i_j : i, j \in \mathbb{N}\} \in \mathcal{D}$. □

5. $S_{fin}(\mathcal{D},\mathcal{D})$

According to [11] X belongs to $S_{fin}(\mathcal{K},\mathcal{K})$ if and only if $C_k(X)$ has countable fan tightness (i.e., for each $f \in C_k(X)$, $S_{fin}(\Omega_f,\Omega_f)$ holds [1]).

Theorem 5.1. *For a Tychonoff space X with $iw(X) = \aleph_0$ the following statements are equivalent:*

1. *$C_k(X)$ satisfies $S_{fin}(\mathcal{D},\mathcal{D})$;*
2. *X satisfies $S_{fin}(\mathcal{K},\mathcal{K})$;*
3. *Each finite power of X satisfies $S_{fin}(\mathcal{K},\mathcal{K})$.*
4. *$C_k(X)$ satisfies $S_{fin}(\Omega_{\mathbf{0}},\Omega_{\mathbf{0}})$ [countable fan tightness];*
5. *$C_k(X)$ satisfies $S_{fin}(\mathcal{D},\Omega_{\mathbf{0}})$.*

Proof. (2) ⇔ (3) By Theorem 6 in [14].

(2) ⇔ (4) see in [11].

The remaining implications are proved similarly to the proof of Theorem 4.2. □

Remark 5.2. It is easy to see that every hemicompact space is in the class $S_1(\mathcal{K},\mathcal{K})$ and, thus, in $S_{fin}(\mathcal{K},\mathcal{K})$. By Proposition 5 in [4], the converse is also true in the class of first countable spaces.

Corollary 5.3. *For a first countable Tychonoff space X with $iw(X) = \aleph_0$ the following statements are equivalent:*

1. *$C_k(X)$ satisfies $S_1(\mathcal{D},\mathcal{D})$;*
2. *$C_k(X)$ satisfies $S_{fin}(\mathcal{D},\mathcal{D})$;*
3. *X satisfies $S_1(\mathcal{K},\mathcal{K})$.*

6. $S_1(\mathcal{S},\mathcal{D})$

Definition 6.1. A γ_k-cover $\mathcal{U}$ of co-zero sets of X is **γ_k-shrinkable** if there exists a γ_k-cover $\{F(U) : U \in \mathcal{U}\}$ of zero-sets of X with $F(U) \subset U$ for every $U \in \mathcal{U}$.

For a topological space X we denote:

• Γ_k^{sh} — the family of γ_k-shrinkable covers of X.

Theorem 6.2. *For a Tychonoff space X the following statements are equivalent:*

1. *$C_k(X)$ satisfies $S_1(\Gamma_{\mathbf{0}},\Omega_{\mathbf{0}})$;*
2. *X satisfies $S_1(\Gamma_k^{sh},\mathcal{K})$.*

Proof. (1) ⇒ (2) Let $C_k(X)$ satisfies $S_1(\Gamma_{\mathbf{0}},\Omega_{\mathbf{0}})$ and $\{\mathcal{F}_i : i \in \mathbb{N}\} \subset \Gamma_k^{sh}$.

For each $i \in \mathbb{N}$ we consider a set $D_i = \{f_{F(U),U,i} \in C(X) : f_{F(U),U,i} \upharpoonright F(U) = 0$ and $f_{F(U),U,i} \upharpoonright (X \setminus U) = 1$ for $U \in \mathcal{F}_i\}$.

Since $\{F(U) : U \in \mathcal{F}_i\}$ is a γ_k-cover of X, we have that D_i converges to $f \equiv \mathbf{0}$ for each $i \in \mathbb{N}$.

Since $C_k(X)$ satisfies $S_1(\Gamma_{\mathbf{0}},\Omega_{\mathbf{0}})$, there is a sequence $(f_{F(U_i),U_i,i} : i \in \mathbb{N})$ such that for each i, $f_{F(U_i),U_i,i} \in D_i$, and $\{f_{F(U_i),U_i,i} : i \in \mathbb{N}\}$ is an element of $\Omega_{\mathbf{0}}$.

Consider $\{U_i : i \in \mathbb{N}\}$.

(a) $U_i \in \mathcal{F}_i$.

(b) $\{U_i : i \in \mathbb{N}\}$ is a k-cover of X.

Let K be a non-empty compact subset of X and $U = \langle f, K, \frac{1}{2}\rangle$ be a base neighborhood of f, then there is $f_{F(U_{i'}),U_{i'},i'} \in U$. It follows that $K \subset U_{i'}$. We thus get X satisfies $S_1(\Gamma_k^{sh},\mathcal{K})$.

(2) ⇒ (1) Let $(f_{k,i} : k \in \mathbb{N})$ be a sequence converging to f for each $i \in \mathbb{N}$. Without loss of generality we can assume that $f = \mathbf{0}$, a set $W_k^i = \{x \in X : -\frac{1}{i} < f_{k,i}(x) < \frac{1}{i}\} \neq X$ for any $i \in \mathbb{N}$ and $S_k^i = \{x \in X : -\frac{1}{i} \leq f_{k,i}(x) \leq \frac{1}{i}\} \neq X$ for any $i \in \mathbb{N}$.

Consider $\mathcal{V}_i = \{W^i_k : k \in \mathbb{N}\}$ and $\mathcal{S}_i = \{S^i_k : k \in \mathbb{N}\}$ for each $i \in \mathbb{N}$. We claim that $\mathcal{V}_i$ is a γ_k-cover of X. Since $\{f_{k,i}\}_{k\in\mathbb{N}}$ converges to f, for each compact subset $K \subset X$ there is $k_0 \in \mathbb{N}$ such that $f_{k,i} \in \langle f, K, \frac{1}{i}\rangle$ for $k > k_0$. It follows that $K \subset W^i_k$ for any $k > k_0$. Since $\mathcal{V}_{i+1}$ is a γ_k-cover, $\mathcal{S}_{i+1}$ is a γ_k-cover, too. $\mathcal{S}_{i+1}$ is a refinement of the family $\mathcal{V}_i$, hence, $\mathcal{V}_i \in \Gamma^{sh}_k$.

By X satisfies $S_1(\Gamma^{sh}_k, \mathcal{K})$, there is a sequence $(W^i_{k(i)} : i \in \mathbb{N})$ such that $W^i_{k(i)} \in \mathcal{V}_i$ for each i, and $\{W^i_{k(i)} : i \in \mathbb{N}\}$ is an element of $\mathcal{K}$.

We claim that $f \in \overline{\{f_{k(i),i} : i \in \mathbb{N}\}}$. Let $U = \langle f, K, \epsilon\rangle$ be a base neighborhood of f where $\epsilon > 0$ and $K \in \mathbb{K}(X)$, then there is $i_1 \in \mathbb{N}$ such that $\frac{1}{i_1} < \epsilon$ and $W^{i_1}_{k(i_1)} \supset K$. It follows that $f_{k(i_1),i_1} \in \langle f, K, \epsilon\rangle$ and, hence, $f \in \overline{\{f_{k(i),i} : i \in \mathbb{N}\}}$. □

Lemma 6.3. *Let $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ be a γ_k-shrinkable cover of a space X. Then the set $S = \{f \in C(X) : f \upharpoonright (X \setminus U_n) \equiv 1$ for some $n \in \mathbb{N}\}$ is sequentially dense in $C_k(X)$.*

Proof. Let $h \in C(X)$. For each $n \in \mathbb{N}$, take $f_n \in C(X)$ such that $f_n \upharpoonright F(U_n) = h \upharpoonright F(U_n)$ and $f_n \upharpoonright (X \setminus U_n) \equiv 1$. Then obviously $f_n \in S$, and $f_n \mapsto h$, because $\{F(U_n) : n \in \mathbb{N}\}$ is a γ_k-cover. □

Theorem 6.4. *For a Tychonoff space X with $iw(X) = \aleph_0$ the following statements are equivalent:*

1. *$C_k(X)$ satisfies $S_1(\mathcal{S}, \mathcal{D})$;*
2. *$C_k(X)$ satisfies $S_1(\mathcal{S}, \Omega_{\mathbf{0}})$;*
3. *$C_k(X)$ satisfies $S_1(\Gamma_{\mathbf{0}}, \Omega_{\mathbf{0}})$;*
4. *X satisfies $S_1(\Gamma^{sh}_k, \mathcal{K})$.*

Proof. (1) ⇒ (4) Let $\{\mathcal{F}_i : i \in \mathbb{N}\} \subset \Gamma^{sh}_k$. By Lemma 6.3, $S_i = \{f \in C(X) : f \upharpoonright (X \setminus F^i_n) \equiv 1$ for some $F^i_n \in \mathcal{F}_i\}$ is a sequentially dense subset of $C_k(X)$ for each $i \in \mathbb{N}$.

By (1), there is $\{f_i : i \in \mathbb{N}\}$ such that $f_i \in S_i$ and $\{f_i : i \in \mathbb{N}\} \in \mathcal{D}$.

Consider the sequence $\{F^i_{n(i)} : i \in \mathbb{N}\}$.

(a) $F^i_{n(i)} \in \mathcal{F}_i$ for $i \in \mathbb{N}$.

(b) $\{F^i_{n(i)} : i \in \mathbb{N}\}$ is a k-cover of X.

Let $K \in \mathbb{K}(X)$ and let $U = \langle \mathbf{0}, K, \frac{1}{2}\rangle$ be a base neighborhood of $\mathbf{0}$, then there is $f_{i'} \in \{f_i : i \in \mathbb{N}\}$ such that $f_{i'} \in U$. It follows that $K \subset F^{i'}_{n(i')}$.

(4) ⇒ (3) Let X satisfies $S_1(\Gamma^{sh}_k, \mathcal{K})$ and let $\{f_{i,m}\}_{m\in\mathbb{N}}$ converges to $\mathbf{0}$ for each $i \in \mathbb{N}$.

Consider $\mathcal{F}_i = \{F_{i,m} : m \in \mathbb{N}\} = \{f^{-1}_{i,m}(-\frac{1}{i}, \frac{1}{i}) : m \in \mathbb{N}\}$ for each $i \in \mathbb{N}$. Without loss of generality we can assume that a set $F_{i,m} \neq X$ for any $i, m \in \mathbb{N}$. Otherwise there is a sequence $(f_{i_k,m_k} : k \in \mathbb{N})$ such that $\{f_{i_k,m_k}\}_{k\in\mathbb{N}}$ uniformly converges to $\mathbf{0}$ and $\{f_{i_k,m_k} : k \in \mathbb{N}\} \in \Omega_{\mathbf{0}}$.

Note that $\mathcal{F}_i$ is a γ_k-shrinkable cover of X for each $i \in \mathbb{N}$.

By (4), there is a sequence $(F_{i,m(i)} : i \in \mathbb{N})$ such that for each i, $F_{i,m(i)} \in \mathcal{F}_i$, and $\{F_{i,m(i)} : i \in \mathbb{N}\}$ is an element of $\mathcal{K}$.

We claim that $\mathbf{0} \in \overline{\{f_{i,m(i)} : i \in \mathbb{N}\}}$. Let $W = \langle \mathbf{0}, K, \epsilon\rangle$ be a base neighborhood of $\mathbf{0}$ where $\epsilon > 0$ and $K \in \mathbb{K}(X)$, then there is $i_1 \in \mathbb{N}$ such that $\frac{1}{i_1} < \epsilon$ and $F_{i_1,m(i_1)} \supset K$. It follows that $f_{i_1,m(i_1)} \in \langle \mathbf{0}, K, \epsilon\rangle$ and, hence, $\mathbf{0} \in \overline{\{f_{i,m(i)} : i \in \mathbb{N}\}}$ and $C_k(X)$ satisfies $S_1(\Gamma_{\mathbf{0}}, \Omega_{\mathbf{0}})$.

(3) ⇒ (2) is immediate.

(2) ⇒ (1) Suppose that $C_k(X)$ satisfies $S_1(\mathcal{S}, \Omega_{\mathbf{0}})$. Let $D = \{d_n : n \in \mathbb{N}\}$ be a dense subspace of $C_k(X)$. Given a sequence of sequentially dense subspaces of $C_k(X)$, enumerate it as $\{S_{n,m} : n, m \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, pick $d_{n,m} \in S_{n,m}$ so that $d_n \in \overline{\{d_{n,m} : m \in \mathbb{N}\}}$. Then $\{d_{n,m} : m, n \in \mathbb{N}\}$ is dense in $C_k(X)$. □

7. $S_{fin}(\mathcal{S}, \mathcal{D})$

The following theorems are proved similarly to Theorems 6.2 and 6.4.

Theorem 7.1. *For a Tychonoff space X the following statements are equivalent:*

1. $C_k(X)$ *satisfies* $S_{fin}(\Gamma_{\mathbf{0}}, \Omega_{\mathbf{0}})$;
2. X *satisfies* $S_{fin}(\Gamma_k^{sh}, \mathcal{K})$.

Theorem 7.2. *For a Tychonoff space X with* $iw(X) = \aleph_0$ *the following statements are equivalent:*

1. $C_k(X)$ *satisfies* $S_{fin}(\mathcal{S}, \mathcal{D})$;
2. $C_k(X)$ *satisfies* $S_{fin}(\mathcal{S}, \Omega_{\mathbf{0}})$;
3. $C_k(X)$ *satisfies* $S_{fin}(\Gamma_{\mathbf{0}}, \Omega_{\mathbf{0}})$;
4. X *satisfies* $S_{fin}(\Gamma_k^{sh}, \mathcal{K})$.

8. $S_1(\mathcal{S}, \mathcal{S})$

In [22], we proved the following theorems.

Theorem 8.1. ([22, Theorem 3.3]) *For a Tychonoff space X the following statements are equivalent:*

1. $C_k(X)$ *satisfies* $S_1(\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$;
2. X *satisfies* $S_1(\Gamma_k^{sh}, \Gamma_k)$.

Theorem 8.2. ([22, Theorem 3.5]) *For a Tychonoff space X such that* $C_k(X)$ *is sequentially separable the following statements are equivalent:*

1. $C_k(X)$ *satisfies* $S_1(\mathcal{S}, \mathcal{S})$;
2. $C_k(X)$ *satisfies* $S_1(\mathcal{S}, \Gamma_{\mathbf{0}})$;
3. $C_k(X)$ *satisfies* $S_1(\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$;
4. X *satisfies* $S_1(\Gamma_k^{sh}, \Gamma_k)$;
5. $C_k(X)$ *satisfies* $S_{fin}(\mathcal{S}, \mathcal{S})$;
6. $C_k(X)$ *satisfies* $S_{fin}(\mathcal{S}, \Gamma_{\mathbf{0}})$;
7. $C_k(X)$ *satisfies* $S_{fin}(\Gamma_{\mathbf{0}}, \Gamma_{\mathbf{0}})$;
8. X *satisfies* $S_{fin}(\Gamma_k^{sh}, \Gamma_k)$.

We can summarize the relationships between considered notions in next diagrams.

$$\begin{array}{ccccccc} S_1(\mathcal{S},\mathcal{S}) & \Leftrightarrow & S_{fin}(\mathcal{S},\mathcal{S}) & \Rightarrow & S_1(\mathcal{S},\mathcal{D}) & \Rightarrow & S_{fin}(\mathcal{S},\mathcal{D}) \\ \Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\ S_1(\mathcal{D},\mathcal{S}) & \Leftrightarrow & S_{fin}(\mathcal{D},\mathcal{S}) & \Rightarrow & S_1(\mathcal{D},\mathcal{D}) & \Rightarrow & S_{fin}(\mathcal{D},\mathcal{D}) \end{array}$$

Diagram 1. The Diagram of selectors for sequences of dense sets of $C_k(X)$.

$$\begin{array}{ccccccc} S_1(\Gamma_k^{sh},\Gamma_k) & \Leftrightarrow & S_{fin}(\Gamma_k^{sh},\Gamma_k) & \Rightarrow & S_1(\Gamma_k^{sh},\mathcal{K}) & \Rightarrow & S_{fin}(\Gamma_k^{sh},\mathcal{K}) \\ \Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\ S_1(\mathcal{K},\Gamma_k) & \Leftrightarrow & S_{fin}(\mathcal{K},\Gamma_k) & \Rightarrow & S_1(\mathcal{K},\mathcal{K}) & \Rightarrow & S_{fin}(\mathcal{K},\mathcal{K}) \end{array}$$

Diagram 2. The Diagram of selection principles for a space X corresponding to selectors for sequences of dense sets of $C_k(X)$.

9. On the Particular Solution to one Problem

Recall that Arens' space S_2 is the set $\{(0,0),(\frac{1}{n},0),(\frac{1}{n},\frac{1}{nm}) : n,m \in \mathbb{N} \setminus \{0\}\} \subset \mathbb{R}^2$ carrying the strongest topology inducing the original planar topology on the convergent sequences $C_0 = \{(0,0),(\frac{1}{n},0) : n > 0\}$ and $C_n = \{(\frac{1}{n},0),(\frac{1}{n},\frac{1}{nm}) : m > 0\}$, $n > 0$. The sequential fan is the quotient space $S_\omega = S_2/C_0$ obtained from the Arens's space by identifying the points of the sequence C_0 [12].

Proposition 9.1. *If $C_k(X)$ satisfies $S_{fin}(\Gamma_{\mathbf{0}},\Omega_{\mathbf{0}})$, then S_ω cannot be embedded into $C_k(X)$.*

The following problem was posed in the paper [4].

Problem 9.2. *Does a first countable (separable metrizable) space belong to the class $S_1(\Gamma_k,\mathcal{K})$ if and only if it is hemicompact?*

A particular answer to this problem is the following

Theorem 9.3. *Suppose that X is first countable stratifiable space and $iw(X) = \aleph_0$. Then following the statements are equivalent:*

1. *X satisfies $S_{fin}(\Gamma_k^{sh},\mathcal{K})$;*
2. *X satisfies $S_{fin}(\Gamma_k,\mathcal{K})$;*
3. *X satisfies $S_1(\mathcal{K},\Gamma_k)$;*
4. *X is hemicompact.*

Proof. (1) ⇒ (4) Since X is first countable stratifiable space and, by Proposition 9.1, S_ω cannot be embedded into $C_k(X)$, then, by Theorem 2.2 (+ Remark) in [7], X is a locally compact. A locally compact stratifiable space is metrizable [5]. It is well-known that a locally compact metrizable space can be represented as $X = \bigsqcup_{\alpha<\tau} X_\alpha$ where X_α is a σ-compact for each $\alpha < \tau$. Since $iw(X) = \aleph_0$, then $\tau \leq \mathfrak{c}$. Claim that $\tau < \omega_1$.

Assume that $\tau \geq \omega_1$. Then there is a continuous mapping $f : X \mapsto D$ ($f(X_\alpha) = d_\alpha$) from X onto a discrete space $D = \{d_\alpha : \alpha < \tau\}$. Note that D satisfies $S_{fin}(\Gamma_k^{sh},\mathcal{K})$ ($S_{fin}(\Gamma,\Omega)$) and, hence, D is Lindelöf, but $|D| > \aleph_0$, a contradiction.

It follows that X is a locally compact and Lindelöf, and, hence, X is a hemicompact.

(4) ⇒ (3) Since X is hemicompact and $iw(X) = \aleph_0$, then $C_k(X)$ is a separable metrizable space [17]. Hence, $C_k(X)$ satisfies $S_1(\mathcal{D},\mathcal{S})$, and, by Theorem 3.3, X satisfies $S_1(\mathcal{K},\Gamma_k)$. □

Corollary 9.4. *Suppose that X is a separable metrizable space. Then X satisfies $S_{fin}(\Gamma_k,\mathcal{K})$ if and only if X is hemicompact.*

Remark 9.5. In the class of first countable stratifiable spaces with $iw(X) = \aleph_0$ (in particular, in the class of separable metrizable spaces) all properties in Diagram 1 (and, hence, Diagram 2) coincide.

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