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# Separation of variables for the eight-vertex SOS model with antiperiodic boundary conditions

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To my parents

**Abstract**
This work deals on the one hand with how to use the representation theory of a quantum group ( [14]) to investigate a statistical mechanical model and on the other hand with how to solve the statistical mechanical model by Sklyanin's method of separation of variables. [47]. More concretely, we use the elliptic quantum group $E_{\tau,\eta}(sl_2)$ established by Felder in [29, 25] to investigate the SOS eight-vertex model established by Date, Jimbo, Miwa and Okado [10] with antiperiodic boundary conditions which are the reason that Bethe ansatz fails and we have to use Sklyanin's method of separation of variables [47].
The SOS eight-vertex model is a face-model version of Baxter's original eight–vertex model [3]. It is related to the elliptic quantum group $E_{\tau,\eta}(sl_2)$, since we rediscover its Boltzmann weights $W_e(c,b,a,d|z)$ by a suitably discretized version of the R-matrix $R_e(z,\lambda)$ defining $E_{\tau,\eta}(sl_2)$. This relation reads

$$R_e(z,\lambda = -2\eta d)e[c-d] \otimes e[b-c] = \sum_a W_e(c,b,a,d|z)\, e[b-a] \otimes e[a-d].$$

The antiperiodic boundary conditions of the model are fixed by considering a special family of transfer matrices $T_{SOS,e}(z,\lambda_0)$ depending on the parameter $z \in \mathbb{C}$ which are twisted traces of the defining L-operator of the model $(M(\mathbb{C},V^{2\otimes n}),\bar{L}_{SOS,e}(z,\lambda))$ (cf. Definition 4.21) over the auxiliary space of the quantum group, twisted by the matrix $K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The L-operator is built out of a representation of $E_{\tau,\eta}(sl_2)$ which is an $n$-fold shifted tensor product of fundamental representations of $E_{\tau,\eta}(sl_2)$.
The (finite) partition function of the model is then to be computed out of this transfer matrix to yield $\mathcal{Z}_M = \mathrm{Tr}_{2^M}(T_{SOS,e}(z,\lambda_0))^M$.
To find common eigenvalues and eigenvectors of the family of antiperiodic transfer matrices of the SOS eight-vertex model we use Sklyanin's method of separation of variables [45]. This method reduces the problem of finding those entities, which involves solving a non-linear multidimensional difference equation, to solving $n$ so-called separated equations which are one-dimensional linear difference equations (cf. Definition 4.52). The system of separated equations emerges by evaluating the family of transfer matrices $T_{aux,e}(z,\lambda_0)$ of Sklyanin's [46, 44] auxiliary representation, generalized to the elliptic case, called $(M(\mathbb{C},V^{2\otimes n}),\bar{L}^{\mathbb{C}}_{aux,e}(z,\lambda))$ (Definition 4.33) of $E_{\tau,\eta}(sl_2)$ at $n$ points. The equivalence of solving this eigenvalue problem instead of the original one is due to the fact that the auxiliary representation and the representation of $E_{\tau,\eta}(sl_2)$ which defines the SOS eight-vertex model are isomorphic (Theorem 4.44).
Let us now briefly state the content of this work: In the second chapter – after the introduction –, we briefly present the results on the elliptic Gaudin model by Enriquez, Feigin and Rubtsov [16], i.e. we state the solutions of this model obtained by separation of variables. (This serves as an insight into the separation of variables method as well as describing a model that can be seen as a limiting case of the eight-vertex SOS model.)
In the chapter 4, we deal with the eight-vertex SOS model. We first define the basic notions of the eight-vertex SOS model more heuristically. Then, we describe the basic representation theory of $E_{\tau,\eta}(sl_2)$. In the next section, we describe the eight–vertex SOS model in terms of the representation theory of $E_{\tau,\eta}(sl_2)$ involving the definition of $(M(\mathbb{C},V^{2\otimes n}),\bar{L}_{SOS,e}(z,\lambda))$ and the commutative family of antiperiodic transfer matrices. We propose the auxiliary representation $(M(\mathbb{C},V^{2\otimes n}),\bar{L}^{\mathbb{C}}_{aux,e}(z,\lambda))$ and the emerging

family of auxiliary transfer matrices in the fourth section. In the fifth section, we describe the isomorphism between $(M(\mathbb{C}, V^{2\otimes n}), \bar{L}_{SOS,e}(z,\lambda))$ and $(M(\mathbb{C}, V^{2\otimes n}), \bar{L}^{\mathbb{C}}_{aux,e}(z,\lambda))$. In the final section, we state our main results on the description of the common eigenvalues and eigenvectors of the family of transfer matrices with antiperiodic boundary conditions of the eight-vertex SOS model in terms of the common eigenvalues and eigenvectors of the auxiliary transfer matrices (Proposition 4.54 and Theorem 4.55).

In chapter 5, we treat the simplest non–trivial example of the SOS eight-vertex model, namely $n = 3$, to clarify the notions defined in the preceding chapter.

We deal with another problem in Appendix1, a problem frequently treated in Sklyanin's papers on separation of variables [47, 46]. There he discusses separation of variables of the XXX model [20, 21, 37], which is related to the representation theory of the Yangian [52] $\mathcal{Y}(sl_2)$. Solving this model involves a procedure analogous to the one for the SOS eight-vertex model: a main problem consists in finding an auxiliary representation $(\mathbb{C}^{2^n}, \bar{L}_{aux,r}(z))$ which is isomorphic to the representation $(\mathbb{C}^{2^n}, L_{XXX}(z))$ which comes along with the XXX model. Here, we propose a version of obtaining the isomorphism which differs from what Sklyanin did in [47, 46] and is in analogy to the isomorphism we proposed in the SOS eight-vertex case. Of course, the results agree with what Sklyanin states in [46, 44].

**Zusammenfassung**

Diese Arbeit befasst sich zum einen damit, wie man Darstellungstheorie von Quantengruppen ([13]) auf Modelle der statistischen Mechanik anwenden kann, zum anderen damit, wie man ein entsprechendes Modell aus der statistischen Mechanik mit Sklyanins Methode der Separation der Variablen [47] löst. Konkreter benutzen wir die elliptische Quantengruppe $E_{\tau,\eta}(sl_2)$, wie sie von Felder [29, 25] konstruiert wurde, um das SOS Acht-Vertex-Modell in der Gestalt von Date, Jimbo, Miwa und Okade [10] mit antiperiodischen Randbedingungen zu betrachten, die die Ursache für das Versagen des Bethe-Ansatzes sind und uns dazu anleiten, Sklyanins Methode der Separation der Variablen zu benutzen [47].

Das SOS Acht-Vertex-Modell ist eine Version von Baxters ursprünglichem Acht-Vertex-Modell [3] als "face"-Modell. Es steht mit der elliptischen Quantengruppe in Zusammenhang, was wir an der folgenden Relation erkennen, die die Boltzmann-Gewichte des Modells $W_e(c, b, a, d|z)$ mit der geeignet diskretisierten R-Matrix der elliptischen Quantengruppe $R_e(z, \lambda)$ verbindet:

$$R_e(z, \lambda = -2\eta d)e[c-d] \otimes e[b-c] = \sum_a W_e(c, b, a, d|z)\, e[b-a] \otimes e[a-d].$$

Die antiperiodischen Randbedingungen des Modells werden dadurch fixiert, dass man eine spezielle Familie von Transfermatrizen $T_{SOS,e}(z, \lambda_0)$ betrachtet, die von einem Parameter $z \in \mathbb{C}$ abhängen, und getwistete Spuren über den auxiliären Raum der Quantengruppe des L-Operators zum SOS Acht-Vertex-Modell $(M(\mathbb{C}, V^{2\otimes n}), \bar{L}_{SOS,e}(z, \lambda))$ sind (s. Definition 4.21), mit Twistmatrix $K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Der L-Operator besteht aus dem $n$-fachen verschobenen Tensorprodukt von fundamentalen Darstellungen von $E_{\tau,\eta}(sl_2)$.

Die (endliche) Partitionsfunktion des SOS Acht-Vertex-Modells mit antiperiodischen Randbedingungen ist durch die Transfermatrix gegeben als $\mathcal{Z}_M = \mathrm{Tr}_{2^M}(T_{SOS,e}(z, \lambda_0))^M$. Um Eigenwerte und Eigenvektoren der beschriebenen Transfermatrix zu finden, benutzen wir Sklyanins Methode der Separation der Variablen [47]. Mit Hilfe dieser Methode gelingt es uns, das Problem, das ursprünglich das Lösen einer nichtlinearen multidimensionalen Differenzengleichung beinhaltet, auf das Lösen eines Systems von $n$ eindimensionalen Differenzengleichungen, den sogenannten separierten Gleichungen (s. Definition 4.52), zurückzuführen. Dieses Gleichungssystem kommt durch das Auswerten der Familie von auxiliären Transfermatrizen $T_{aux,e}(z, \lambda_0)$ an $n$ generischen Punkten zustande. Diese sind Transfermatrizen von Sklyanins auxiliärer Darstellung [46, 44], genannt $(M(\mathbb{C}, V^{2\otimes n}), \bar{L}^{\mathbb{C}}_{aux,e}(z, \lambda))$ (Definition 4.33) , die hier auf den elliptischen Fall $E_{\tau,\eta}(sl_2)$ erweitert wird. Die Äquivalenz, die es uns erlaubt, statt des ursprünglichen Problems das System separierter Gleichungen zu lösen, beruht darauf, dass die Darstellung von $E_{\tau,\eta}(sl_2)$ zum SOS Acht-Vertex-Modell und die auxiliäre Darstellung isomorph sind (Theorem 4.44).

Wir möchten nun kurz eine Inhaltsübersicht geben: Das zweite Kapitel, nach der Einführung, enthält ein Resumé der Resultate von Enriquez, Feigin und Rubtsov zum elliptischen Gaudin-Modell [16], d.h. wir geben die Lösungen dieses Modells an, die die Autoren durch Separation der Variablen erhalten. (Dies dient der Einsicht in diese Methode wie auch der Entwicklung eines Modells, das als Grenzfall des SOS Acht-Vertex-Modells aufgefasst werden kann.)

In Kapitel 4 behandeln wir das SOS Acht-Vertex-Modell. Zunächst beschreiben wir die Grundbegriffe des Modells auf heuristische Art. Dann folgt eine kurze Einführung in die Darstellungstheorie von $E_{\tau,\eta}(sl_2)$, soweit wir sie benötigen. Im nächsten Abschnitt wird das SOS Acht-Vertex-Modell dann darstellungstheoretisch formuliert, was die Definition von $(M(\mathbb{C}, V^{2\otimes n}), \bar{L}_{SOS,e}(z,\lambda))$ und der kommutativen Familie antiperiodischer Transfermatrizen umfasst. Wir stellen die auxiliäre Darstellung $(M(\mathbb{C}, V^{2\otimes n}), \bar{L}^{\mathbb{C}}_{aux,e}(z,\lambda))$ und die kommutative Familie auxiliärer Transfermatrizen, die daraus hervorgeht, in Abschnitt 4 vor. Im fünften Abschnitt konstruieren wir den Isomorphismus zwischen $(M(\mathbb{C}, V^{2\otimes n}), \bar{L}_{SOS,e}(z,\lambda))$ und $(M(\mathbb{C}, V^{2\otimes n}), \bar{L}_{SOS,e}(z,\lambda))$. Im letzten Abschnitt formulieren wir unsere Hauptresultate zur Beschreibung der gemeinsamen Eigenwerte und Eigenvektoren der Familie von Transfermatrizen der SOS Acht-Vertex-Modells mit antiperiodischen Randbedingungen mit Hilfe der gemeinsamen Eigenwerte und Eigenvektoren der Familie von auxiliären Transfermatrizen mit antiperiodischen Randbedingungen (Proposition 4.54 und Proposition 4.55).

In Kapitel 5 behandeln wir das einfachste nicht-triviale Beispiel, $n = 3$, um das im vorhergehenden Kapitel Hergeleitete zu illustrieren.

In Appendix 1 streifen wir ein weiteres Problem, das in Sklyanins Artikeln [46, 44] über die Separation der Variablen oft behandelt wird. Dort erklärt er die Separation der Variablen für die XXX-Kette [20, 21, 37], ein Problem, das mit der Darstellungstheorie des Yangian [52] $\mathcal{Y}(sl_2)$ verbunden werden kann. Die Lösung dieses Modells erfordert ein Vorgehen, das in Analogie zu demjenigen beim SOS Acht-Vertex-Modell betrachtet werden kann: Ein Hauptproblem ist es, eine auxiliäre Darstellung des Yangian $(\mathbb{C}^{2^n}, \bar{L}_{aux,r}(z))$ [46, 44] zu finden, die isomorph au derjenigen zur XXX-Kette $(\mathbb{C}^{2^n}, L_{XXX}(z))$ ist. Hier konstruieren wir den dazugehörigen Isomorphismus auf eine Art, die sich von der Herleitung Sklyanins in [46, 44] unterscheidet und in Analogie zu unserem Vorgehen beim SOS Acht-Vertex-Modell steht. Die erhaltenen Resultate stimmen selbstverständlich mit denen Sklyanins in [46, 44] überein.

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# Contents

# 1 Introduction

In the introduction we will briefly present the main themes pursued throughout the thesis. First, we will discuss the main notions of statistical mechanics we will be using in the sequel. These include for example the transfer matrix, vertex models and interaction-round-a-face models, the Bethe ansatz – in its appearance before quantum groups were discovered –, the star-triangle-relation and the Yang-Baxter-equation. The main points presented here can of course be found in Baxter's book [3], where we always cite the corresponding pages, as well as in an abbreviated version also in some newer talks, e.g. [40]. At the end of the section, we will also present the model treated in the thesis: the eight-vertex model and its interaction-round-a-face (IRF) or – what is an equivalent notion – its solid-on-solid (SOS) version.
In the second section, we will briefly point out some general facts about quantum groups or Drinfeld-Jimbo quantum affine algebras and the quantum inverse scattering method (cf. e.g. [14, 23, 22, 47]). Since the main object of the thesis, with regard to quantum groups, is the elliptic quantum group $E_{\tau,\eta}(sl_2)$ ([29], [25]), we will also discuss its structure.
After this, we will briefly expose the connection between quantum groups and the solving of the aforementioned models of statistical mechanics, i.e. how some notions that we discussed there can be suitably translated into the language of representation theory of quantum groups (to achieve a certain unification in treatment). This topic relies on the so-called Quantum Inverse Scattering Method (QISM) developed by the Faddeev school. The last part will be devoted to two different realizations of the Bethe ansatz, especially the separation of variables, also an integral part of the QISM.

## 1.1 The SOS eight-vertex model

### 1.1.1 Basic notions

We first have to ask what a model of statistical mechanics is. As we understand it here, a model of statistical mechanics is a description of a system consisting of infinitely many atoms at the sites of an infinite lattice $a\mathbb{Z} + ib\mathbb{Z} \subset \mathbb{C}$ in a two-dimensional plane that interact via their spins or a finite lattice $L \subset \mathbb{C}$, with some boundary conditions concerning the rows and columns of the lattice. For simplicity, let us suppose that to each edge of the lattice we attach a variable $\sigma$, e.g. denoting a spin, which can only take values $\sigma \in \{-1, 1\}$. Solving the model usually implies the computation of (some of) the following quantities:

a) The partition function (infinite or finite respectively) of the system

$$\mathcal{Z} = \sum_s \exp(-E(s)/kT) \text{ or } \mathcal{Z}_M = \sum_s \exp(-E(s)/kT)$$

where the sum is taken over all possible states $s$ of the spins on the infinite lattice or of a finite lattice with $M$ rows and $N$ columns with some boundary conditions w.r.t. the columns (cf. figure below). $E(s)$ is the energy of the system depending on the lattice configuration (cf. below) and a possible external field, $k$ is Boltzmann's constant and $T$ the temperature.

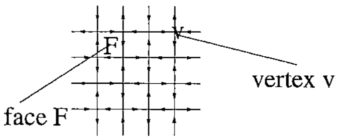
b) The free energy of the system which is given by

$$F = -kT \ln \mathcal{Z} \text{ or } F = \lim_{M \to \infty} \left( -kT \frac{1}{M} \ln(\mathcal{Z}_M) \right),$$

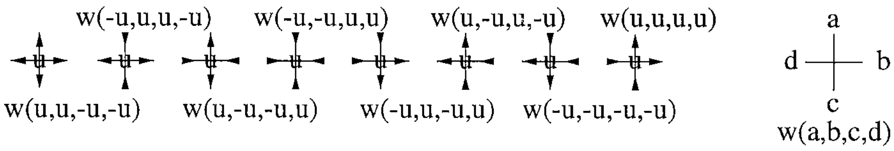
corresponding to whether we start with the infinite lattice or a finite lattice with some boundary conditions.

c) Other physically interesting quantities such as the specific heat and the magnetization.

In order to understand how one usually calculates these quantities, let us start with a two-dimensional finite lattice $L \subset \mathbb{C}$, a part of which is given in the figure below. One way of approaching this lattice is by means of a vertex model.




We state that the lattice consists of horizontal and vertical edges and an arrow is attached to each edge. An intersection of a horizontal and a vertical line is indicated by a vertex $v$. A possible physical interpretation is given in [3], p. 127 (though the physical interpretations of course differ by what combinations of arrows are allowed at each vertex), for the six-vertex model describing the hydrogen bonding of ice: At each vertex there is an oxygen atom surrounded by four hydrogen ions which are placed at the edges. The atom and each ion are attached by a hydrogen bonding. Thus, of every four ions surrounding an atom two are near the corresponding atom, signified by an arrow pointing towards the atom, and two are farther away from the atom, denoted by an arrow pointing away from the atom. (In this case, there also exists a 'non-physical' interpetation of the arrangement of arrows – cf. [3] p.165 –, namely the following problem: In how many different ways can the lattice be colored by three different colors if the colors of two adjacent faces are to be different?) An assignment of arrows to a (finite) lattice is called a configuration of the lattice. If we look at the finite lattice drawn above, we can see that by imposing periodical toroidal boundary conditions we can think of the part as representing an infinite lattice $a\mathbb{Z} + ib\mathbb{Z} \subset \mathbb{C}$. Note that we could also impose antiperiodical boundary conditions w.r.t. rows as we will always do later on.
Let us now turn to the interaction of the edges – with arrows – on the lattice. We admit only nearest-neighbour interactions and interactions of any edge with an external field $H$. How can we describe the interactions? All types of interactions occurring between nearest-neighbour edges can be specified by looking at a vertex with some values of the surrounding edges attached to it. To every combination of arrows around a vertex a corresponding weight $w(a, b, c, d)$ exists that describes the statistical occurrence of the vertex configuration in question, where $a, b, c, d$ are the variables attaches to the surrounding edges of a given vertex. If we classify all allowed combinations of spins at a vertex with their corresponding weights, we obtain the interactions. Depending on the model in question, there are a different number of allowed vertex interactions. For the eight-vertex model, the allowed combinations of the arrows are drawn below, according to the rule that an even number of arrows has to be pointing in and out of the vertex.



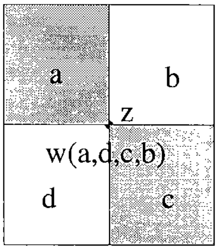

Let us now turn to the quantities we want to calculate: First, the partition function, which is the sum indicated above, where we sum over all possible lattice configurations. To simplify calculations, one usually introduces the transfer matrix $T$. If we have $N$ columns drawn in our finite lattice and impose periodic or antiperiodic boundary conditions concerning the rows in this lattice, this is a $2^N \times 2^N$ matrix, i.e. in general $T \in \text{End}\,((\mathbb{C}^2)^{\otimes N})$. Its entries are the – not yet normalized – probabilities with which one (of the $2^N$ possible) configuration of a row change into any of the $2^N$ possible configurations on the next row. If our finite lattice has $M$ rows, the finite partition function is then given by

$$\mathcal{Z}_M = \text{Tr}_{2^N} T^M,$$

If we can suitably perform the limit $M \to \infty$, we may do so obtaining the partition function. As we perceive by the above formula of the partition function, it proves useful to diagonalize the transfer matrix, since by the cyclicity of the trace, we get in the easiest, periodic, case that $\mathcal{Z}_M = \sum_{i=1}^{2^N} \Lambda_i^M$, where the $\Lambda_i$ for $i = 1, \dots, 2^N$ are the eigenvalues of the transfer matrix. In this case, we also find the free energy

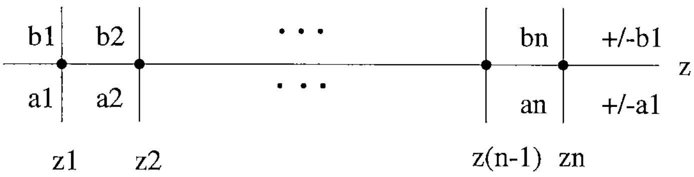
$$f = \lim_{M\to\infty} \left(-\frac{1}{kT}\frac{1}{M} \ln \mathcal{Z}_M\right) = -\frac{1}{kT} \ln \Lambda_1,$$

where $\Lambda_1$ is the largest eigenvalue of the matrix. (For a neat explanation of this and the appearance of the transfer matrix, cf. e.g. [3], pp. 32.)
The second way of approaching the lattice is by means of the SOS or face [34] or IRF model. Here, instead of on the vertices the dynamical variable is put onto a face of the lattice, face $F$ in the first figure. We call this variable height and adjacent heights are to differ by plus or minus one. The weights describing the interaction, commonly denoted as Boltzmann weights, then indicate an interaction between faces in the manner shown in the figure below.




Assignment of a Boltzmann weight to four faces

If we want to calculate the partition function we can do so by using the row-to-row-transfer matrix (cf. [3], pp. 370), visualized in the figure below.




Row-to-row transfer matrix for an SOS model

The row-to-row transfer matrix of a face-model is still a $2^N \times 2^N$ matrix, provided a height to each face that differs from its neighbouring height by plus or minus one to each face. Each entry is of the form

$$\prod_{j=1}^{n} w(b_i, a_i, a_{i+1}, b_{i+1}),$$

where $a_{n+1} = \pm a_1$ and $b_{n+1} = \pm b_1$ according to whether we chose periodic or antiperiodic boundary conditions on the rows. An element of the above form tells how a given assignment of faces $(a_1, a_2, \dots, a_n, a_{n+1} = \pm a_1)$ in the lower row changes into the assignment $(b_1, \dots, b_n, b_{n+1} = \pm b_1)$ in the upper row. The statements on the diagonalization of the transfer matrix remain the same as for the vertex models. Note that for SOS models the common notion of the corner transfer matrix also exists which, as the name suggests, provides a method of calculating a transfer matrix for quadrants of the lattice (cf. [3] pp. 363-401). Since we will not need it here, we will not pursue it further.

### 1.1.2 Two different approaches solving models of statistical mechanics

In chapter 8 of his book on statistical mechanics, Baxter proposed a method of finding common eigenvalues and eigenvectors of a family of transfer matrices of the six-vertex model (cf. [3], pp. 133 – 140), which involves an ansatz using a vector parametrized by a set of parameters $(w_1, \dots, w_m)$. The vector has to obey certain recursive relations. In order to yield an eigenvector, the set of parameters has to obey a set of equations, corresponding to the cancellation of some "unwanted" terms which, if they do not cancel, prevent the vector emerging from the recursion relations from being linear dependent on the vector one started with. This idea goes back to Bethe and is known as the Bethe ansatz [8], the set of equations the parameters are to obey is known as the Bethe ansatz equations.

In the chapter 9 of the same book [3], Baxter suggested a different, fairly general treatment as to when a family of transfer matrices of a model of statistical mechanics can be diagonalized - and hence the solving of the model seems feasible (cf. [3], pp. 180 – 200). This "program", formulated on p. 184 of [3], involves the following: The first step is to find a commuting family $T(u)$ of transfer matrices, where the variable $u \in \mathbb{C}$ is obtained by a reparametrization of the original weights, cf. [3], p. 184 or p. 212, for the eight-vertex model. The transfer matrix, i.e. all of its entries, is an entire function of $u$. The next step is finding a matrix $Q(u)$ with non-zero determinant, also an entire function of $u$, commuting with $T(u)$ for all values of $u$, obeying the so-called Baxter equation (cf. [3], p. 183) for all $u$

$$\Lambda(u) = \left(\Phi(\eta - u)Q(u + 2\eta') + \Phi(\eta + u)Q(u - 2\eta')\right) / Q(u),$$

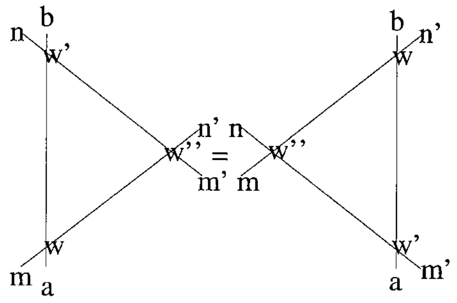
where every appearing parameter is obtained as a consequence of the reparametrization of the Boltzmann weights and $\eta' = \eta - \pi i$ in Baxter's notation. The matrix equation can be understood for every diagonal entry of the matrix $Q(u)$, cf. pp. 182 in [3]. $\Phi(u)$ is an entire scalar function and $\Lambda(u)$ stands for the (diagonal) matrix of eigenvalues of the transfer matrix.

If we consider Baxter's equation as an equation of matrix elements, and hence functions, every appearing function is an entire function of $u$. Thus, if we consider $Q(u)$ with zeroes $(w_1, \dots, w_m)$, we obtain $m$ conditions forcing the residues of the above equation to vanish. They read

$$-\frac{\Phi(\eta - w_i)}{\Phi(\eta + w_i)} = \frac{Q(w_i - 2\eta')}{Q(w_i + 2\eta')}$$

for all $i = 1, \dots, m$ and are precisely the Bethe ansatz equations, though deduced in quite another context.
In Baxter's treatment the matrices $Q(u)$ and $V(u)$ have to commute with still another matrix $S$, but since we will not need this operator here we will not go into details.
After the presentation of this program, Baxter proposes some conditions that must be satisfied in order to achieve certain of the above-mentioned steps. For the family of local transfer matrices that later on appeared as R-matrices of a vertex model, $U_i$ in the language of Baxter (cf. [3], p.188), to be commutative it has to obey what later became known as the Yang-Baxter-relation (here formulated in weights according to [3], p.187), which are illustrated in the figure below

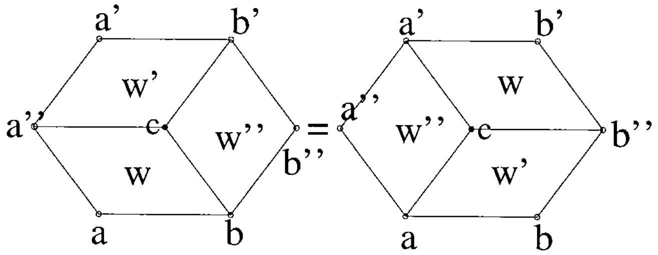
$$\sum_{c,m'',n''} w(m,a,c,m')w'(n,c,b,n'')w''(n'',m'',n',m') =$$
$$\sum_{c,m'',n''} w''(n,m,n'',m'')w'(m'',a,m,n')w(n'',c,b,n')$$




For the row-to-row transfer matrices of an SOS-model to be commutative, the Boltzmann weights have to obey the (generalized) star-triangle-relation

$$\sum_{c} w(a,b,c,a'')w'(a'',c,b',a')w''(c,b,b'',b')$$
$$= \sum_{c} w''(a'',a,c,a')w'(a,b,b'',c)w(c,b'',b',a'),$$

where $w, w', w''$ are different Boltzmann weights. It is visualized by the figure below.



In order to have the operator $Q(u)$ obey Baxter's equation, Baxter also formulated a condition on the columns of this operator, named "Propagation through a vertex" (cf. [3], pp. 192 – 194). But since so far there has not been a consistent treatment of the operator $Q(u)$, cf. [46] and [41], but only explicit examples – e.g. [3],pp. 215 – 222 – and since we will not need it in the sequel, we consider it sufficient to state the condition.

### 1.1.3 The eight-vertex-model and the SOS eight-vertex model

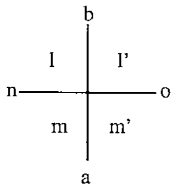
The eight-vertex model can be described by its weights, hence its $R$-matrix. The $R$-matrix was given by Baxter in [3] and reads (cf. [27] in comparison to [3], p. 213)

$$R_{8V}(z) = \begin{pmatrix} a_{8V}(z) & 0 & 0 & b_{8V}(z) \\ 0 & d_{8V}(z) & c_{8V}(z) & 0 \\ 0 & c_{8V}(z) & d_{8V}(z) & 0 \\ b_{8V}(z) & 0 & 0 & a_{8V}(z) \end{pmatrix},$$

with

$$\begin{aligned} a_{8V}(z) &= \frac{\theta_0(z)\theta_0(2\eta)}{\theta_0(z-2\eta)\theta_0(0)}, \\ b_{8V}(z) &= \frac{\theta_1(z)\theta_0(2\eta)}{\theta_1(z-2\eta)\theta_0(0)}, \\ c_{8V}(z) &= -\frac{\theta_0(z)\theta_1(2\eta)}{\theta_1(z-2\eta)\theta_0(0)}, \\ d_{8V}(z) &= -\frac{\theta_1(z)\theta_1(2\eta)}{\theta_0(z-2\eta)\theta_0(0)}. \end{aligned}$$

It is an element of End $(\mathbb{C}^{2^{\otimes 2}})$, where we identified the following basis elements of the standard tensor product basis of $\mathbb{C}^{2^{\otimes 2}}$ $(1, 0, 0, 0)^T = e[1] \otimes e[1], (0, 1, 0, 0)^T = e[1] \otimes e[-1], (0, 0, 1, 0)^T = e[-1] \otimes e[1], (0, 0, 0, 1)^T = e[-1] \otimes e[-1]$. $\theta_1(z)$ is the odd Jacobi theta function and $\theta_0(z) = ie^{-\pi i(z+\frac{\tau}{2})}\theta_1(z)$. Compared to Baxter's notation, the notation in [27] was changed according to: $\Theta(iu) = \theta_0(u)$, $H(iu) = \theta_1(u)$ were fixed, the variables of Baxter's were changed to $\lambda = 2\eta$ and $\frac{v}{2} = z + \eta$ and there was a division of Baxter's weights by $i\rho\,\Theta(\frac{i}{2}(\lambda - v))H(\frac{i}{2}(\lambda - v))$.
This model can be transformed to the $R$-matrix of the elliptic quantum group $E_{\tau,\eta}(sl_2)$ (cf. Definition 4.6) by a so-called vertex-IRF transformation (cf. [4], [27]).
Before we go into details, let us briefly mention the origin of the vertex–IRF transfer matrix as given in [4], section 3. There, Baxter relates the eight-vertex model to a generalized ice-type model, here generalized refers to the transformed model being a face-model and ice-type refers to the fact that there are only six allowed vertices after the transformation, by the following transformation, which we can explain by the following figure attaching labels to edges and faces around a vertex




(b,o,a,n): arrows of eight-vertex model
(l,l',m,m'):heights of eight-vertex SOS model

and by the following equation relating weights w.r.t. the vertices to Boltzmann weights w.r.t. to the faces:

$$\sum_{b,d} R_{8V}(a,b,c,d)\Phi_{l,l'}(b)z_{m',l'}(d) = \sum_{m} W(m,m',l,l')\Phi_{m,m'}(a)z_{m,l}(c),$$

where the values of the labels attached to the faces are to differ by one, i.e. $|l-l'| = |m'-l'| = |m-m'| = |m-l| = 1$ and $a,b,c,d \in \{-1,1\}$. This equation is to be understood in the following manner: The entities $\Phi_{l,l\pm1}(\pm1), z_{l,l\pm1}(\pm1)$ and $W(m,m',l,l')$ for all allowed combinations of these values are unknowns which, if they can be determined, determine the vertex-IRF transformation (cf. [4], pp. 27). A possible solution in terms of Boltzmann weights can be found in [11].
In the case treated in the thesis, we may formulate the vertex-IRF transformation as follows.

**Proposition [27]:**
*Let $S(z,\lambda)$ be the matrix*

$$S(z,\lambda) = \frac{1}{\theta(\lambda)} \begin{pmatrix} \theta_0(z-\lambda+\frac{1}{2}) & -\theta_0(-z-\lambda+\frac{1}{2}) \\ -\theta_1(z-\lambda+\frac{1}{2}) & \theta_1(-z-\lambda+\frac{1}{2}) \end{pmatrix} \in \text{End}\,(\mathbb{C}^2).$$

*Then*

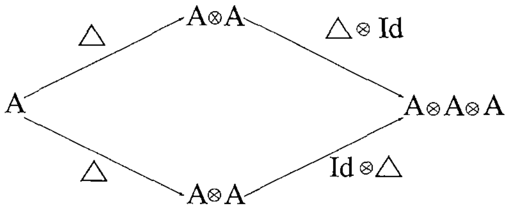
$$(\mathbb{I}\otimes S(w,\lambda))(S(z,\lambda-2\eta h^{(2)})\otimes\mathbb{I})R_e(z-w,\lambda) = R_{8V}(z-w)(S(z,\lambda)\otimes\mathbb{I})(\mathbb{I}\otimes S(w,\lambda-2\eta h^{(1)})),$$

*where $\mathbb{I}$ is the identity matrix on $\mathbb{C}^2$.*
For the notation, we refer to chapter 3 of the thesis.
In paper [27], it is shown how this transformation corresponds to a vertex-IRF transformation.
This decribes that, while working with the elliptic $R$-matrix we in fact describe an SOS-version of the eight-vertex model, where we get to the actual eight-vertex model by the transformation given above.
There is also another way of conceiving that we work with a SOS-model. If we consider the Boltzmann weights given in Definition 4.21 and rewrite the Yang–Baxter–relation 4.7 of the elliptic $R$-matrix in terms of the thus defined Boltzmann weights, this exactly yields the star-triangle- relation as was shown in [27], p. 8.
In the periodic case – i.e. $a_1 = a_{n+1}$ and $b_1 = b_{n+1}$ in the row-to-row transfer matrix –, Felder and Varchenko in [27] explicitly showed that by means of the above IRF-vertex-transformation one also can map the periodic transfer matrix of the SOS eight-vertex model to the periodic transfer matrix of Baxter's eight-vertex model.
This is not done here, in the antiperiodic case. Thus, we are really dealing with the row-to-row transfer matrix of the SOS eight-vertex model with antiperiodic boundary conditions.
The paragraph above merely points out how this model has been obtained from Baxter's original model.

## 1.2 Quantum groups, the QISM and different forms of the Bethe Ansatz

The notion of a quantum group was developed by Drinfeld [14] under the influence of the quantum inverse scattering method (QISM) developed by Faddeev, Sklyanin, [21, 22, 23,

37] to which we will turn later on.
In what follows we will restrict ourselves to simple simply connected complex Lie algebras and Lie groups.
The name of a quantum group came up by the following construction. If we take the category of (Lie) groups $G$ and, by means of a functor, map it to the category of (smooth) functions on $G$, $\mathfrak{F} : G \to$ Fun $(G) = A$, we obtain the category of associative commutative unital algebras which has the property of being a category of Hopf algebras, which will be explained below.
Now, if we take the right hand side of the functorial mapping to be the category of associative unital, not necessarily commutative, Hopf algebras, then the left-hand side is called the category of quantum groups, i.e. every element of it is a quantum group, [14], p.800.
Let us now briefly turn to the attributes of a Hopf algebra. A noncommutative Hopf algebra is a septuple $(A, m, \Delta, i, \epsilon, S, S')$. Here, $A$ is the actual algebra, $m : A \otimes A \to A$ is the multiplication, $\Delta : A \to A \otimes A$ is the comultiplication, $\epsilon : A \to \mathbb{C}$ is the counit, $S : A \to A$ is the antipode and $S' : A \to A$ is the skew antipode with $SS' = Id$, $Id$ being the identity function on $A$, and $i : \mathbb{C} \to A, c \to c \cdot Id$. (Actually, this picture can be enriched by introducing the notion of a quasi-Hopf algebra, where we add an eighth term $\phi : H \otimes H \otimes H \longrightarrow H \otimes H \otimes H$ to the septuple, with $((id \circ \Delta) \circ \Delta) = \phi^{-1}((\Delta \circ Id) \circ \Delta)\phi$, which means that coassociativity – or the consistency diagram shown below – is obeyed up to conjugation only.) These notions and the properties that they are to fulfill are introduced and discussed in every standard volume on quantum groups and Hopf algebras (e.g. [39]), so we will not go into further detail. Let us just write down the commutative diagram the comultiplication $\Delta$ has to satisfy in order to ensure the coassociativity of the algebra as an example.




Now, we will trace the steps towards solutions of the Yang-Baxter-equation in the quantum group setting. Since this is a rather complicated process, we can only present it in a superficial manner. A thorough exposition can be found in [13].
The first notion to mention is the notion of a Poisson-Lie group $G_0$ (plus additional compatibility conditions on the Poisson, Lie, Hopf structures, cf. [14], p.801, 802). It is a Lie group $G_0$ whose algebra of functions Fun $(G_0) = A_0$ is endowed with a Poisson bracket $[\cdot,\cdot] : A_0 \otimes A_0 \to A_0$. Thus, $A_0$ is a Poisson Hopf algebra. There exists the notion of quantizing/deforming the Poisson-Lie group, which means obtaining a deformation $A$ of $A_0$ as a free $\mathbb{C}[[h]]$-module, where $h$ is the parameter of the deformation. (A Poisson-Lie group and a Poisson-Lie Hopf algebra are equivalent by the above mentioned functor.)
The next step is the notion of a Lie-bialgebra, a Lie algebra which is also a Lie coalgebra, i.e. endowed with the structures of comultiplication $\Delta$ and counit $\epsilon$. These strucutures have to satisfy some commutative consistency diagrams, the diagram of coassociativity shown above and another diagram involving the counit.
It can be shown (cf. [14], Theorem 1) that the category of connected and simply-connected

complex Poisson-Lie groups is equivalent to the category of finite dimensional complex Lie bialgebras described by $(g, \delta|_g)$, where $\delta|_g \in g \otimes g$ is a co-Poisson-structure obtained by a canonical procedure (cf. [14], p.802).
There exists a construction (cf. [14], p.803/4) to parametrize Lie bialgebras involving as parameters $X$, a projective curve over $\mathbb{C}$, $\omega$, a rational differential on $X$, and $g$, a simple simply connected Lie algebra.
A special case of Lie bialgebras are quasitriangular Lie-bialgebras, given by $(g, r)$, where $g$ is a Lie bialgebra and $r \in g \otimes g$ satisfies the classical Yang–Baxter–equation

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0.$$

In [7], Belavin and Drinfeld showed that under a certain regularity condition there exist solutions using the parametrized construction of Lie bialgebras mentioned above in terms of rational, trigonometrical and elliptic functions, which correspond to possible choices of $X$ and $\omega$.
From this expression we derive the notion of a quasitriangular Hopf-algebra. A Hopf algebra $(A, R \in A \otimes A)$ is called *coboundary* if it obeys

$$(\sigma \circ \Delta)(a) = R\Delta(a)R^{-1} \text{ for all } a \in A$$

and $R\sigma(R) = 1$, where $\sigma : A \otimes A \to A \otimes A$, $\sigma(x \otimes y) = y \otimes x$. $(A, R)$ is called *quasitriangular*, if additionally $(\Delta \otimes Id)(R) = R^{13}R^{23}$ and $(Id \otimes \Delta)R = R^{13}R^{12}$ on $A \otimes A \otimes A$, where $R = \sum_{ij} a_i \otimes b_j$ and $R^{13} = \sum_{ij} a_i \otimes 1 \otimes b_j$ for example. (The other terms are constructed similarly.) The two conditions imply that a quasitriangular $(A, R)$ satisfies the quantum Yang-Baxter-relation

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}.$$

If $(A, R = 1 + hr + \dots)$ is a quasitriangular quantized universal enveloping algebra of a Lie bialgebra $g$ – this quantization can be understood by means of the correspondence of Lie bialgebras and Poisson-Lie groups -, it can be shown that its classical limit, i.e. the limit $h \to 0$ of the deformation parameter $h$, is a quasitriangular Lie bialgebra $(g, r \in g \otimes g)$.
Well known examples of quantized bialgebras are the Yangians $\mathcal{Y}(a)$, where $a$ is a simple Lie algebra, corresponding to the rational solutions of the quantum Yang-Baxter-relation, and the trigonometric solution to the Yang-Baxter-equation obtained by the quantum double construction [42] (cf. [14], p. 814 and 816).
The Yangian, which we will need in the first appendix, originally constructed by Yang [52], is obtained by quantizing the bialgebra $a[u]$, $a$ being a simple Lie algebra. This bialgebra has a pseudotriangular structure, what means: $r \notin (g \otimes g)$, but can be developed by means of an additional shift operator into a power series whose coefficients are in $(g \otimes g)$. The treatment involving the shift operator can be transferred to the quantum case to yield a shifted Yang-Baxter-equation (cf. Proposition 6.3)

$$R^{(12)}(z-w)R^{(13)}(z)R^{(12)}(w) = R^{(12)}(w)R^{(13)}(z)R^{(12)}(z-w).$$

Concerning pseudotriangular Hopf algebras quantizing elliptic solutions to the classical Yang–Baxter–equation, early attempts have been made by Sklyanin (cf. [43], Sklyanin's elliptic algebra) and Cherednik [9]. So far, we have not yet pointed out the connection between the elliptic quantum group $E_{\tau,\eta}(sl_2)$ and a possible Hopf algebra structure. This work was achieved by Enriquez and Felder in [17] using the work of Babelon, Bernard and Billey [2]: In [18, 19], Enriquez and Rubtsov constructed a quantized universal enveloping algebra with quasi-Hopf properties as the quantization of a higher genus Manin pair [15] – the Manin pair is also a typical method of obtaining a Lie bialgebra – , whereas Felder in [29] developed the notion of $E_{\tau,\eta}(sl_2)$ using dynamic RLL-relations (cf. Definition 4.8). In particular Felder's elliptic R-matrix $R_e(z,\lambda)$ (Definition 4.6) containing an additional complex parameter $\lambda$, obeys the dynamical Yang-Baxter-relation (Proposition 4.7), which heavily depends on the (shifted) parameter $\lambda$. The idea of [2] used by Felder and Enriquez consists introduces a family of twists $F_\lambda \in A \otimes A$ – obeying a condition called "shifted cocycle" condition in order to be admissible ([2], p.2) – such that after conjugating the original R-matrix obeying the dynamical Yang-Baxter-equation one obtains a new R-matrix obeying the Yang-Baxter-equation:

$$R_{DYBE}(\lambda) = F_\lambda^{-1} R_{YBE} F_\lambda.$$

In their paper [17], the authors construct such a family of twists, thus constructing an R-matrix obeying the Yang-Baxter-equation and establishing a correspondence between Felder's construction and the one of Enriquez and Rubtsov.

### 1.2.1 The connection between quantum groups and statistical mechanics

The following two parts of the section are devoted to aspects of the QISM. There are some reviews on this subject which we cited before [21, 22, 23, 47, 37]. In the first part we will briefly explain how the language of QISM is related to the language of Hopf algebras shown in the preceding part. In particular, we will show how to obtain the RLL-relations (cf. Proposition 4.4 and 6.4). We will also show how to obtain and understand commuting families of transfer matrices, a problem which already emerged in the section on statistical mechanics. These also have the property of relating to the consistency condition of the first chapter, the Yang-Baxter-equation. We will conclude with a comment on how different Lie bialgebras as solutions to the classical Yang-Baxter-equation and pseudotriangular Hopf algebras are attached to different models of statistical mechanics. Let us start with the re-interpretation of the coboundary and quasitriangularity condition stated above. To this end, instead of the category of quasi- or pseudotriangular Hopf algebras $A$ we consider the category of representations of quasi- or pseudotriangular Hopf algebras (as an algebra) $\text{Rep}_A$ [14], p. 812. Let us suppose two representations of $A$: $\rho_1 : A \to \text{End}\,(V_1)$, $\rho_2 : A \to \text{End}\,(V_2)$. Then, by the comultiplication property of $A$ we obtain a new representation $\rho = \rho_1 \otimes \rho_2 : A \to \text{End}\,(V_1 \otimes V_2)$ by $A \xrightarrow{\Delta} A \otimes A \xrightarrow{\rho_1 \otimes \rho_2} \text{End}\,(V_1 \otimes V_2)$. If we now suppose that $\rho : A \to \text{Mat}(n, \mathbb{C})$ ( $\text{Mat}(n, \mathbb{C})$ being the $n \times n$ matrices with complex entries) then we obtain for its matrix elements $t_{ij} \in A^*$ – if we use the above property yielded by comultiplication – for the coboundary relation

$$R_\rho T^{(1)} T^{(2)} = T^{(2)} T^{(1)} R_\rho,$$

where $R_\rho = (\rho \otimes \rho)(R)$, $T^{(2)} = \mathbb{I} \otimes T$ and $T^{(1)} = T \otimes \mathbb{I}$ with $\mathbb{I}$ being the identity matrix in $\text{Mat}(n, \mathbb{C})$. (In the case of $n \times n$ matrices, the space $\mathbb{C}^n$ is also sometimes referred to as the auxiliary space.) Now, the L-operator of the RLL-relation, under the condition that $(A, R)$ is pseudotriangular, can be identified with the operator $T$ mentioned above

(cf. [14], p.813).
This is where QISM starts. Let us set $n = 2$ for $\mathrm{Mat}(n, \mathbb{C})$, since this is the case in question for the rest of the thesis, although, in principle, we could take any other integer. Let us first state two basic facts concerning the method: as objects the QISM uses the pseudotriangular (quasi-)Hopf algebras obtained by quantizing the pseudotriangular Lie bialgebras that solved the classical Yang–Baxter equation. One principle of QISM is borrowed from quantum mechanics, cf. [47], p. 68. The commutative algebra of observables of an, as yet unspecified, physical system should be part of a larger algebra $A$ whose (irreducible) representations describe possible states of the system in question, cf. [47]. Now, the larger algebra is to be generated by generators $T_{ij}(u)$, $i, j = 1, 2$, being entries of a $2 \times 2$ matrix $T(u)$, obviously an element of $\mathrm{Mat}(2, \mathbb{C})$, where $u \in \mathbb{C}$ is called the spectral parameter. (Note that there is no restriction concerning the objects $T_{ij}(u)$ or a representations $\tilde{\rho}(T_{ij}(u)) = L_{ij}(u)$ respectively. They can be chosen as elements of some suitable space of functions or as matrices in $\mathrm{Mat}(n, \mathbb{C})$ for some $n \in \mathbb{N}$. Note also that by the comultiplication property, we can inductively build new representations of given representations.) This matrix $T(u)$ is subject to the structure relation of the algebra, which is exactly the RLL-relation we deduced above, by means of identification of $T$ and $L$ as indicated. Thus, we see that the above pattern of a coboundary Hopf algebra is indeed reproduced. Since we are working with a pseudotriangular Hopf algebra, we also know that the Yang-Baxter-equation (with spectral parameters $u, v$) is satisfied. This establishes a connection to the section on statistical mechanics, where the Yang–Baxter–equation emerged as a consistency condition of the local transfer matrices: An R-matrix satisfying the Yang–Baxter–equation is connected to a solvable model of statistical mechanics by means of identifying it with the local transfer matrix called $U_i$ in Baxter's language. The second link to the chapter on statistical mechanics can be seen by the following: By taking the trace of the auxiliary space of the RLL relation, we see that this way we obtain commuting operators $t(u) = \mathrm{Tr}_{aux} T(u)$, where $\mathrm{Tr}_{aux}$ denotes the trace on the auxiliary space, called transfer matrices. (Indeed, as we will see in the fourth and sixth chapter, this is not the only way of obtaining a family of commuting operators, but we should rather call this the case of periodic boundary conditions.) Hence, we get a commutative subalgebra of $A$.
What remains to be done is to find the spectrum of the commuting family of transfer matrices, i.e. the set of common eigenvalues. This also corresponds to what we aimed at in the statistical mechanics setting, namely to find eigenvalues and eigenvectors to the transfer matrix in order to diagonalize it, and thus solve the corresponding model.
The steps mentioned here – take an $R$-matrix, find representations of the generators $T_{ij}(u)$, where $i, j = 1, 2$, of the algebra given by the $R$-matrix, construct the transfer matrix family, find the common spectrum of the family and compute the usual physically interesting quantities – represent the QISM program as formulated in [47].
Let us mention the connection of some models of statistical mechanics to Lie bialgebras being the solutions of the classical Yang–Baxter–equation [7] or to pseudotriangular Hopf algebras being their quantization. The Gaudin model is related [32] to the mentioned Lie bialgebras. The elliptic Gaudin model, presented in chapter 2 of the thesis, is connected to the elliptic solution of the classical Yang–Baxter-relation, what can be seen by studying the structure of the operator $S_e(z)$ as presented in Proposition 2.17, which involves the operators $e_e(z), f_e(z)$ and $H_e(z)$. By looking at the commutation relations of those operators given in Proposition 2.9, we can recover the elliptic classical $r$-matrix which

solves the, modified, classical Yang–Baxter–equation. There also exist rational [31] and trigonometric versions of the Gaudin model. The operators $S_r(z)$ and $S_t(z)$ are constructed in a completely analoguous way.
As for the pseudotriangular Hopf algebras, the XXX magnetic chain was connected to the representation theory of the Yangian $\mathcal{Y}(sl_2)$ in [44, 37, 20, 21]. The representation theory of the elliptic quantum group $E_{\tau,\eta}(sl_2)$ is related to the eight-vertex SOS model, as will be shown in the sequel.

### 1.2.2 The method of separation of variables

So far, we have not yet studied the problem of how to find common eigenvectors and eigenvalues to the commuting family of transfer matrices from the point of view of quantum groups.
If we recall the section on statistical mechanics, we see that there were two different settings of how to obtain eigenvectors of the transfer matrix depending on a set of parameters that had to obey the Bethe ansatz equations: one recursive approach and one that consisted of solving a system of difference equations that had to obey some regularity conditions.
In some sense in the quantum group setting the two different forms of Bethe ansatz are reproduced: the first is known as the algebraic Bethe ansatz, the second approach is known as the functional Bethe ansatz or the method of separation of variables and will be used in the following.
The algebraic Bethe ansatz, cf. e.g. [23], relies on the notion of a (finite-dimensional, irreducible) highest weight representation of a quantum group – for the elliptic quantum group $E_{\tau,\eta}(sl_2)$ (cf. Definition 4.16) and for the Yangian $\mathcal{Y}(sl_2)$ (cf. [47], p. 76) – which is in fact quite reminiscent of a highest weight representation of a Lie algebra (cf. Chapter 2). We can use the algebraic Bethe ansatz to obtain the common spectrum of the transfer matrices, if the highest weight vector $v_{h.w.} \in V$, if the representation of the generators of the quantum group are mappings $A \to \text{End}\,(V)$, is a common eigenvector of the family of commuting transfer matrices, which is the case for the quantum models studied here with periodic boundary conditions, e.g. [27]. Out of the highest weight vector, we construct the possible Bethe eigenvector $B_m(z, w_1, \dots, w_m) = \prod_{i=1}^m L_{12}(z - w_i)v_{h.w.}$ and act on it by an element of the family of transfer matrices. We can then show inductively that the Bethe vector is indeed an eigenvector if the Bethe ansatz equations are satisfied, which leads to a cancellation of those terms not linearly dependent on the Bethe eigenvector. The Bethe ansatz equations for the parameters $w_1, \dots, w_m$ thus obtained show the same structure as the ones obtained in the section on Baxter's approaches.
This type of Bethe ansatz obviously fails if the highest weight vector is not an eigenvector of the commuting family of transfer matrices in question; in other words, if we cannot suitably build up a highest weight representation for the case in question. Then, we can use the functional Bethe ansatz, also known as the method of separation of variables. Separation of variables in general reduces a possibly non-linear multidimensional problem, here a difference or differential equation, to a system of some number of one-dimensional, thus easier to study, problems: the system of separated equations. Thus, if we want to use separation of variables, we first have to show that solving the original problem – an eigenvalue problem of a family of transfer matrices – can be suitably related to another eigenvalue problem of a related family of transfer matrices which is reducible

to solving $n$ one-dimensional difference or differential equations. In the cases treated here, this is achieved by finding a representation of the corresponding quantum group, the elliptic quantum group $E_{\tau,\eta}(sl_2)$ in chapter 4, and the Yangian $\mathcal{Y}(sl_2)$ [46, 44] in chapter 6, which is isomorphic (cf. Theorem 4.44 and Proposition 6.20) to the representation which we obtained the original commuting transfer matrices from. These new representations yield families of transfer matrices that can be shown to generically split up into a certain number separated difference equations which have to be solved. As it turns out, the separated difference equations in both cases (cf. p.72, and [44], p. 23) show the same structure as Baxter's equation and, due to conditions posed on the eigenvalues and eigenfunctions, the condition of vanishing residues also holds true. Thus, the Bethe ansatz equations also emerge in this case (cf. p. 72).

Let us briefly discuss the problem of finding a complete set of common eigenvectors and eigenvalues of the transfer matrix. This is easier to study by the method of separation of variables, since once we have established that solving the system of separated equations is equivalent to studying the original problem, it is easier to describe what a complete set of solutions to these differential or difference equations is, than to deduce that the set of solutions found by algebraic Bethe ansatz is complete.

# 2 The elliptic Gaudin Hamiltonian

## 2.1 Introduction

The work on the elliptic Gaudin Hamiltonian described in this section, has been achieved by Enriquez, Feigin and Rubtsov [16]. It was carried out by them also in the context of the geometric Langlands correspondence [6] (for the general Langlands correspondence, cf. [38]) connected to the Lie group $GL_2$ and a rational curve $X$ over $\mathbb{C}$. More specifically, Frenkel [31] showed by a very intricate discussion that the separation of variables established by Sklyanin [45] can be interpreted as constructing an equivalence between two different approaches realizing the geometric Langland's correspondence [6, 12]. Frenkel showed how to obtain the equivalence in the case of genus zero, i.e. involving a rational curve, using separation of variables for the rational Gaudin model [44]. Enriquez, Feigin and Frenkel established this correspondence in the case of genus one, hence an elliptic curve, using the elliptic version of the Gaudin model.
We will not pursue the Langlands program further here – though there exist attempts of a quantized version [49], a problem already raised in [31] –, but rather use [16] as a model where we can see how the separation of variables works and what can furthermore be seen as a limit, by Proposition 4.51, of the eight–vertex SOS model that will be discussed afterwards.

## 2.2 The setting corresponding to $sl_2(\mathbb{C})$

**Synopsis:**
Here, we develop the basic representation theoretical notions concerning $sl_2$ which we will need in the sequel. We first define a Verma module of the Lie algebra $sl_2$ (Proposition 2.2) and show that it contains a finite dimensional irreducible quotient (Corollary 2.3). Then, we define the finite-dimensional irreducible quotient of a tensor product of Verma modules of $sl_2$ (Proposition 2.4). Finally, we define the space $V[0]$ (Definition 2.6) which is a subspace of the finite-dimensional irreducible quotient of a tensor product of Verma modules of $sl_2$ defined in Proposition 2.4. This space will be the space which the solutions of the elliptic Gaudin model will be elements of.

**Definition 2.1** *Let $e, f, g$ be the generators of the Lie algebra $sl_2(\mathbb{C})$. They obey the commutation relations*

$$[e, f] = h, \tag{1}$$
$$[h, e] = 2e, \tag{2}$$
$$[h, f] = -2f. \tag{3}$$

**Proposition 2.2** *A representation of the Lie algebra $sl_2(\mathbb{C})$ on $\mathbb{C}[t]$ is defined by*

$$e \to -t\frac{d^2}{dt^2} + \Lambda\frac{d}{dt} \tag{4}$$
$$f \to t \tag{5}$$
$$h \to -2t\frac{d}{dt} + \Lambda, \tag{6}$$

*where $\Lambda \in \mathbb{C}$. This representation is called a Verma module $V_\Lambda$ of highest weight $\Lambda$.*

**Proof:**
The proof is straightforward by checking the defining commutation relations.

**Corollary 2.3** *If $\Lambda \in \mathbb{N}$, the quotient $\mathbb{C}[t]/t^{\Lambda+1}\mathbb{C}[t]$ is an irreducible finite dimensional sub-module with highest weight vector $v_\Lambda = 1 \in Ker(e)$.*

**Proof:**
We have to show that $t^{\Lambda+1}\mathbb{C}[t]$ is an invariant subspace of $e, f, h$. If $\Lambda \in \mathbb{N}$, we find a term $t^{\Lambda+1} \in \mathbb{C}[t]$. The only thing to check is that $et^{\Lambda+1} = 0$, since by applying $f$ and $h$ to an element of this subspace or by applying $e$ to an element $t^l$, $l > \Lambda+1$ we stay in it.

$$et^{\Lambda+1} = -\Lambda(\Lambda+1) + \Lambda(\Lambda+1)t^{\Lambda} = 0\,.$$

A highest weight vector also has to obey $ev_\Lambda = 0$ which is obvious here, since $v_\Lambda$ is a constant. The weight of $v_\Lambda$ is by $hv_\Lambda = \Lambda v_\Lambda$ equal to $\Lambda$.

**Proposition 2.4** *The tensor product of $n$ Verma modules of highest weight $\sum_{i=1}^n \Lambda_i$ - denoted by $V = \otimes_{i=1}^n V_{\Lambda_i}$ is given by the operators*

$$\begin{aligned} e^{(i)} &\rightarrow -t_i \frac{d^2}{dt_i^2} + \Lambda_i \frac{d}{dt_i}, && (7)\\ f^{(i)} &\rightarrow t_i, && (8)\\ h^{(i)} &\rightarrow -2t_i \frac{d}{dt_i} + \Lambda_i\,. && (9) \end{aligned}$$

*acting on $\mathbb{C}[t_1, \dots, t_n]/(\sum_{i=1}^n t_i^{\Lambda_i+1}\mathbb{C}[t_1, \dots, t_n])$. The generators $e^{(i)}, f^{(i)}, h^{(i)}$ for $i = 1, \dots, n$ obey the commutation relations*

$$\begin{aligned} [e^{(i)}, f^{(j)}] &= \delta_{ij} h^{(i)}, && (10)\\ [h^{(i)}, e^{(j)}] &= \delta_{ij} 2e^{(i)}, && (11)\\ [h^{(i)}, f^{(j)}] &= -\delta_{ij} 2f^{(i)}\,. && (12) \end{aligned}$$

**Definition 2.5** ($H^0$) *For the Verma module $\otimes_{i=1}^n V_{\Lambda_i}$, we may define the following operator $H^0 = \sum_{i=1}^n h^{(i)}$.*

**Definition 2.6** ($V[0]$) *Let $\otimes_{i=1}^n V_{\Lambda_i}$ be the Verma module defined above. Then we define the space $V[0] \subset \mathbb{C}[t_1, \dots, t_n]/(\sum_{i=1}^n t_i^{\Lambda_i+1}\mathbb{C}[t_1, \dots, t_n])$ as*

$$\begin{aligned} V[0] = \{f(t_1, \dots, t_n) \in \mathbb{C}[t_1, \dots, t_n]/(\textstyle\sum_{i=1}^n t_i^{\Lambda_i+1}\mathbb{C}[t_1, \dots, t_n]) \mid \\ H^0 f(t_1, \dots, t_n) = 0\}. \end{aligned}$$

*Due to the action of $h^{(i)} = -2t_i \frac{d}{dt_i} + \Lambda_i$, the space is equivalently described by*

$$\begin{aligned} V[0] = \{f(t_1, \dots, t_n) \in \mathbb{C}[t_1, \dots, t_n]/(\sum_{i=1}^n t_i^{\Lambda_i+1}\mathbb{C}[t_1, \dots, t_n]) \mid \\ f(ct_1, \dots ct_i, \dots, ct_n) = c^{\sum_{i=1}^n \frac{\Lambda_i}{2}} f(t_1, \dots, t_n)\}, \end{aligned}$$

*i.e. it consists of complex polynomials that are homogeneous of degree $m = \sum_{i=1}^n \frac{\Lambda_i}{2}$ in the variables $t_1, \dots, t_n$.*

## 2.3 The setting corresponding to the elliptic Gaudin Hamiltonian

**Synopsis:**
Here, we first define the needed elliptic functions which are basic for this chapter and also the chapter on the elliptic quantum group $E_{\tau,\eta}(sl_2)$ (Definition 2.7). With the help of these functions, we write down the operators $e_e(z), f_e(z), h_e(z)$ and its commutation relations (Definition 2.8 and Proposition 2.9) which are the ones of a generalized elliptic $r$-matrix algebra (cf. Defintition 2.10). We will need these operators in the following section to formulate the Gaudin eigenvalue problem.

**Definition 2.7 (Basic notions)**

*a) Let $\tau \in \mathbb{C}$, $\mathcal{I}m(\tau) > 0$. If we define the lattice $\Gamma = \mathbb{Z} + \tau\mathbb{Z}$, the elliptic curve $E_\tau$ is correspondingly defined by $E_\tau = \mathbb{C}/\Gamma$.*

*b) Let $\theta(z) \equiv \theta(z,\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i(n+\frac{1}{2})^2\tau} e^{2\pi i(n+\frac{1}{2})(z+\frac{1}{2})}$ be the odd Jacobi Theta function.*
*Its transformation properties are given by*

$$\theta(z+1) = -\theta(z),\, \theta(z+\tau) = e^{-2\pi i z}\theta(z)\,.$$

*We also need two other functions to be defined by means of Theta functions: $\frac{\theta'(z)}{\theta(z)}$ transforming like*

$$\frac{\theta'(z+1)}{\theta(z+1)} = -\frac{\theta'(z)}{\theta(z)},\, \frac{\theta'(z+\tau)}{\theta(z+\tau)} = -\frac{\theta'(z)}{\theta(z)} - 2\pi i\,,$$

*and its derivative $\wp(z) = (\frac{\theta'(z)}{\theta(z)})'$ which transforms as*

$$\wp(z+1) = \wp(z+\tau) = \wp(z)\,.$$

In what folllows, we always consider the tensor product $V \equiv \otimes_{i=1}^n V_{\Lambda_i}$, $\Lambda_i \in \mathbb{N}$, $i = 1,\ldots,n$.

**Definition 2.8** *Let the $n$ points $(z_1,\ldots,z_n) \in (E_\tau)^n -$ diag be the projections of $(\tilde{z}_1,\ldots,\tilde{z}_n) \in \mathbb{C}^n -$ diag , $i = 1,\ldots,n$ on the elliptic curve $E_\tau$. Let $\lambda \in \mathbb{C}$ be some parameter and $z \in \mathbb{C}$ a complex coordinate. Let*

$$e_e(z) = \sum_{i=1}^n \frac{\theta(\lambda+z-z_i)\theta'(0)}{\theta(\lambda)\theta(z-z_i)} e^{(i)}\,, \tag{13}$$

$$f_e(z) = \sum_{i=1}^n \frac{\theta(\lambda-z+z_i)\theta'(0)}{\theta(\lambda)\theta(z-z_i)} f^{(i)}\,, \tag{14}$$

$$h_e(z) = \sum_{i=1}^n \frac{\theta'(z-z_i)}{\theta(z-z_i)} h^{(i)}\,. \tag{15}$$

**Remark:**
We may define $\frac{\theta(\lambda-z+z_i)\theta'(0)}{\theta(\lambda)\theta(z-z_i)} \equiv \sigma_\lambda(z-z_i)$. Note that $\sigma_\lambda(z)$ has the transformation properties:

$$\sigma_\lambda(z+1) = \sigma_\lambda(z),\, \sigma_\lambda(z+\tau) = e^{2\pi i\lambda}\,.$$

**Proposition 2.9** *The operators defined above obey the following commutation relations: if $z \neq w$*

$$[e_e(z), f_e(w)] = \frac{\theta(\lambda+z-w)\theta'(0)}{\theta(\lambda)\theta(z-w)}(-h_e(z)+h_e(w)) + (\frac{\partial}{\partial\lambda}(\frac{\theta(\lambda+z-w)\theta'(0)}{\theta(\lambda)\theta(z-w)}))\sum_{i=1}^{n} h^{(i)}, \tag{16}$$

$$[h_e(z), e_e(w)] = -2\frac{\theta(\lambda-z+w)\theta'(0)}{\theta(z-w)\theta(\lambda)}e_e(z) +2(\frac{\theta'(z-w)}{\theta(z-w)}+\frac{\partial}{\partial\lambda})e_e(w), \tag{17}$$

$$[h_e(z), f_e(w)] = +2\frac{\theta(\lambda+z-w)\theta'(0)}{\theta(\lambda)\theta(z-w)}f_e(z) \tag{18}$$

$$-2(\frac{\theta'(z-w)}{\theta(z-w)}-\frac{\partial}{\partial\lambda})f_e(w)\,. \tag{19}$$

*If $z = w$ , the relations are given by*

$$[e_e(z), f_e(z)] = -h'_e(z) - \wp(\lambda)\sum_{i=1}^{n} h^{(i)}, \tag{20}$$

$$[h_e(z), e_e(z)] = -2e'_e(z) + 2(\frac{\theta'(\lambda)}{\theta(\lambda)}+\frac{\partial}{\partial\lambda})e_e(z), \tag{21}$$

$$[h_e(z), f_e(z)] = 2f'_e(z) + 2(\frac{\theta'(\lambda)}{\theta(\lambda)}+\frac{\partial}{\partial\lambda})f_e(z)\,. \tag{22}$$

**Proof:**
The proof is straightforward and uses the fact that two theta functions are equal if their residues, zeroes and transformation properties under $z \to z+1$, $z \to z+\lambda$ coincide.
**Remark:**
Note that by defining

$$H_e(z) = H_e(z,\lambda) = \frac{1}{2}h_e(z) - \frac{\partial}{\partial\lambda}, \tag{23}$$

we may rewrite the above commutation relations in order to deal more effectively with the Bethe ansatz (cf. the Appendix). This yields

$$[e_e(z), f_e(z)] = \frac{\theta(\lambda+z-w)\theta'(0)}{\theta(\lambda)\theta(z-w)}(-2H_e(z)+2H_e(w)) +(\frac{\partial}{\partial\lambda}(\frac{\theta(\lambda+z-w)\theta'(0)}{\theta(\lambda)\theta(z-w)}))\sum_{i=1}^{n} h^{(i)}, \tag{24}$$

$$[H_e(z), e_e(w)] = -\frac{\theta(\lambda-z+w)\theta'(0)}{\theta(\lambda)\theta(z-w)}e_e(z) + \frac{\theta'(z-w)}{\theta(z-w)}e_e(w), \tag{25}$$

$$[H_e(z), f_e(w)] = \frac{\theta(\lambda+z-w)\theta'(0)}{\theta(\lambda)\theta(z-w)}f_e(z) - \frac{\theta'(z-w)}{\theta(z-w)}f_e(w)\,. \tag{26}$$

**Definition 2.10 (Elliptic r-matrix)** *The elliptic r-matrix is given by*

$$r(z-w,\lambda) = \begin{pmatrix} \frac{\theta'}{\theta}(z-w) & 0 & 0 & 0 \\ 0 & \frac{\theta'}{\theta}(z-w) & \frac{\theta(\lambda+z-w)\theta'(0)}{\theta(z-w)\theta(\lambda)} & 0 \\ 0 & \frac{\theta(\lambda-z+w)\theta'(0)}{\theta(z-w)\theta(\lambda)} & \frac{\theta'}{\theta}(z-w) & 0 \\ 0 & 0 & 0 & \frac{\theta'}{\theta}(z-w) \end{pmatrix}.$$

**Remark:**
Note that analogously to the quantum case, Proposition 4.3, a classical elliptic r-matrix can be defined that satisfies the modified classical Yang–Baxter relation, cf. [28]. There, the structure of the elliptic r-matrix is also described in more detail.

## 2.4 The elliptic Gaudin eigenvalue problem

**Synopsis:**
First, we define the Gaudin Hamiltonians $H_e^i$ in Definition 2.11.
Consider $n$ atoms on the elliptic curve $E_\tau = \mathbb{C}/\Gamma$, each at a site $z_i$. To each atom we attach a representation of $sl_2(\mathbb{C})$, a Verma module $V_{\Lambda_i}$, with highest weight $\Lambda_i \in \mathbb{N}$ corresponding to its spin. The $i$th atom is interacting with the other ones by the Hamiltonian $H_e^i$.
These operators commute (Proposition 2.12). Thus it is feasible to treat the problem of finding common eigenvectors as noted below Proposition 2.12.
Out of these Hamiltonians, we then develop the operator $S_e(z)$ (Definition 2.15) which also commutes with the Hamiltonians (Lemma 2.16). Thus, it is sensible to study its eigenvalue problem. Here, we restrict ourselves onto the space $V[0]$, thus imposing a condition on possible eigenvalues (Corollary 2.14).
We reformulate the operator $S_e(z)$ in terms of the operators $e_e(z), f_e(z), h_e(z)$ in Proposition 2.17. This reformulation allows us to perform the separation of variables in the next section, since this method is formulated for these operators (cf. the first lines of the next section).

**Definition 2.11** ($H_e^i$)

$$H_e^i = -h^{(i)}\frac{\partial}{\partial\lambda} + \sum_{j\neq i,j=1}^{n}\left(\frac{1}{2}\frac{\theta'(z_i-z_j)}{\theta(z_i-z_j)}h^{(i)}h^{(j)} + \frac{\theta(\lambda-z_i+z_k)\theta'(0)}{\theta(\lambda)}e^{(i)}f^{(j)}\right.$$
$$\left. + \frac{\theta(\lambda+z_i-z_j)\theta'(0)}{\theta(\lambda)\theta(z_i-z_j)}f^{(i)}e^{(j)}\right). \tag{27}$$

*The n Hamiltonians thus obtained are called the elliptic Gaudin Hamiltonians.*

**Proposition 2.12** *The elliptic Gaudin Hamiltonians commute with each other and with* $H^0$.

$$[H_e^i, H_e^j] = 0, \text{ for all } i,j = 1,\ldots,n, \tag{28}$$
$$[H_e^i, H^0] = 0. \tag{29}$$

**Proof:**
The proof of this proposition is straightforward.
**Remark:**

The fact that the elliptic Gaudin commute allows for their simultaneous diagonalization, since an eigenvector of one Hamiltonian is an eigenvector of every Hamiltonian.
We now turn to the problem of how to find eigenvalues - corresponding to possible energy levels of the atoms on the elliptic curve - and eigenvectors of the Hamiltonians, hence possible solutions $(\mu_i, \psi)$ of the equations

$$H_e^i \psi(\lambda, t_1, \dots, t_n) = \mu_i \psi(\lambda, t_1, \dots, t_n), \text{ for } i = 1, \dots, n. \tag{30}$$

To find a complete set of eigenvectors for a certain given set $(\mu_1, \dots, \mu_n)$ will be our aim in this chapter.

**Lemma 2.13** *If restricted on the space $V[0]$,*

$$\sum_{i=1}^{n} H_e^i \psi(\lambda, t_1, \dots, t_n) = 0. \tag{31}$$

**Proof:**
This is done by a straightforward calculation.

**Corollary 2.14** *While working on $V[0]$,*

$$\sum_{i=1}^{n} \mu_i = 0. \tag{32}$$

**Remark:**
Note that since we made the restriction of working on $V[0]$, possible eigenfunctions $\psi(\lambda, t_1, \dots, t_n)$ are to be homogeneous of degree $m = \sum_{i=1}^{n} \frac{\Lambda_i}{2}$ in the variables $(t_1, \dots, t_n)$ and polynomial in the variables $(t_1, \dots, t_n)$. The dependence on $\lambda$ is dictated by the transformation behaviour of the operators $H_e^i$ with respect to this variable.
Since the elliptic Gaudin Hamiltonians can be simultaneously diagonalized, we may instead of them investigate the following operator.

**Definition 2.15** *Let $z \in E_\tau$ be a coordinate on the elliptic curve $E_\tau$.*

$$S_e(z) = \sum_{i=1}^{n} H_e^i \frac{\theta'(z - z_i)}{\theta(z - z_i)} + \sum_{i=1}^{n} \wp(z - z_i) c^{(i)} + H_e^c, \tag{33}$$

*where $H_e^c$ is given by*

$$H_e^c = \frac{\partial^2}{\partial \lambda^2} + \frac{1}{2} \sum_{i,j=1}^{n} \Big( \frac{1}{2} \frac{\theta''(z_i - z_j)}{\theta(z_i - z_j)} h^{(i)} h^{(j)} \\ -\big(\frac{\partial}{\partial \lambda} \frac{\theta(\lambda - z_i + z_j)\theta'(0)}{\theta(\lambda)\theta(z_i - z_j)}\big)(e^{(i)} f^{(j)} + f^{(i)} e^{(j)}) \Big).$$

**Remark:**
The eigenvalue problem now reads as follows: We want to determine a complete set of eigenvectors $\psi(\lambda, t_1, \dots, t_n)$ to a given set $(\mu_c, \mu_1, \dots, \mu_n)$ of

$$S_e(z)\psi = q_e(z)\psi, \text{ with} \tag{34}$$

$$q_e(z) = \sum_{i=1}^{n} (\mu_i \frac{\theta'(z - z_i)}{\theta(z - z_i)} + \frac{\Lambda_i(\Lambda_i + 2)}{4} \wp(z - z_i)) + \mu_c, \tag{35}$$

where $\mu_c$ corresponds to the value of $H_e^c$ on $\psi(\lambda, t_1, \dots, t_n)$.

**Lemma 2.16** *The following identities holde true:*

*a)* $[S_e(z), S_e(w)] = 0,$

*b)* $[S_e(z), H^0] = 0,$

*c)* $[S_e(z), \frac{\partial}{\partial\lambda}] = 0.$

*d) If* $\psi(\lambda, t_1, \ldots, t_n)$ *solves the elliptic Gaudin eigenvalue problem , it can be written*

$$\psi(\lambda, t_1, \ldots, t_n) = e^{c\lambda} f(t_1, \ldots, t_n). \tag{36}$$

**Proof:**

a) The proof is given in [26].

b) $H^0$ and $H_e^i$ commute by Lemma 2.12, $H^0$ and $c^{(i)}$ by a short calculation. Proving the commutativity of $H^0$ and $H_e^c$ is straightforward.

c) The calculation is straightforward.

d) This is a corollary of c). Since the operators $S_e(z)$ and $\frac{\partial}{\partial\lambda}$ commute, they can be digonalized simultaneously. The eigenfunctions of the $\frac{\partial}{\partial\lambda}$ to the eigenvalue $c \in \mathbb{C}$ correspond to $e^{c\lambda}$ .

**Remark:**
Due to the first part of Lemma 2.16 possible eigenvectors $\psi$ are indeed independent of the variable $z \in E_\tau$ .
In order to use the operators and relations developed in the first part of this chapter to solve the eigenvalue problem, we need the following

**Proposition 2.17**

$$S_e(z) = \frac{1}{2}(e_e(z)f_e(z) + f_e(z)e_e(z)) + \left(\frac{\partial}{\partial\lambda} - \frac{1}{2}h_e(z)\right)^2 . \tag{37}$$

**Proof:**
$S_e(z)$ is a meromorphic doubly periodic function with at most double poles at the points $z = z_j$. Expanding $S_e(z)$ into a Laurent series at $z = z_i$ for all $i = 1, \ldots, n$ up to the constant term, we see that the difference of the left and the right hand side vanishes. This is due to the fact that the difference yields a differential operator whose coefficients are regular elliptic functions vanishing at least at one point, thus vanishing everywhere by Liouville's Theorem.

## 2.5 Separation of variables for the elliptic Gaudin Hamiltonian

**Synopsis:**
Here, we first write down how to obtain out of the separated variables the variables we used for the tensor product of Verma modules of $sl_2$, i.e. the "old" variables (Proposition 2.18).
To show that this transformation of variables is indeed useful, we reformulate the operator

$S_e(z)$ in the separated variables (Proposition 2.23) and especially this operator evaluated at the $n$ points – sites of the atoms – $(z_1, \ldots, z_n)$. By this evaluation, we obtain the system of separated equations (Proposition 2.21, Definition 2.22) which all show the same structure of a second order differential equation solvable by Lamé's method (cf. [51]). That the solution of this system of differential equations is equivalent to solving the eigenvalue problem for $S_e(z)$ while restricted on $V[0]$ is shown in Proposition 2.24.
The main idea of this paragraph relies on the following identity (cf. [16], [44]):

$$f_e(z) = \sum_{i=1}^{n} \frac{\theta(\lambda - z + z_i)\theta'(0)}{\theta(\lambda)\theta(z - z_i)} t_i = C \cdot \frac{\prod_{j=1}^{n} \theta(z - y_j)}{\prod_{i=1}^{n} \theta(z - z_i)}. \tag{38}$$

The variables $(C, y_1, \ldots, y_n)$ are called separated variables.

**Proposition 2.18** *Let* $(z_1, \ldots, z_n) \in (E_\tau)^n -$ *diag . The mapping*

$$t_i = C \frac{\prod_{j=1}^{n} \theta(z_i - y_j)}{\prod_{j \neq i, j=1}^{n} \theta(z_i - z_j)\theta'(0)}, \tag{39}$$

$$\lambda = -\sum_{i=1}^{n} (z_i - y_i) \tag{40}$$

*defines a bijection between*

$$\{(t_1, \ldots, t_n, \lambda) | \sum_{i=1}^{n} t_i \neq 0\} \text{ and}$$
$$\{(y_1, \ldots, y_n, C) \in \mathcal{S}^n(E_\tau) \times \mathbb{C}/0\}.$$

**Proof:**
Note that the identity for $\lambda$ is obtained by looking at the transformation properties $z \to z + \tau$ of

$$\sum_{i=1}^{n} \frac{\theta(\lambda - z + z_i)\theta'(0)}{\theta(\lambda)\theta(z - z_i)} t_i = C \cdot \frac{\prod_{j=1}^{n} \theta(z - y_j)}{\prod_{i=1}^{n} \theta(z - z_i)}.$$

The transformation of the left hand side yields a factor $e^{2\pi i \lambda}$ whereas the right hand side yields $e^{-2\pi i \sum_{j=1}^{n}(-y_j + z_j)}$. The $y_j$ for $j = 1, \ldots, n$ are in $\mathcal{S}^n(E_\tau)$.

**Corollary 2.19** *Note the following properties of the transformation:*

$$C = \sum_{i=1}^{n} t_i \frac{\prod_{j \neq i, j=1}^{n} \theta(\sum_{j=1}^{n} y_j - \sum_{i \neq j, i=1}^{n} z_i)\theta'(0)}{\theta(\sum_{j=1}^{n}(-z_j + y_j)) \prod_{j=1}^{n} \theta(y_j)}, \tag{41}$$

$$C \frac{\partial}{\partial C} = \sum_{i=1}^{n} t_i \frac{\partial}{\partial t_i}, \tag{42}$$

$$\frac{\partial}{\partial y_j} = \sum_{i=1}^{n} \frac{\theta'(y_j - z_i)}{\theta(y_j - z_i)} t_i \frac{\partial}{\partial t_i} + \frac{\partial}{\partial \lambda} \tag{43}$$

*They are obtained by studying the map of Proposition 2.18.*

**Lemma 2.20** *On $V[0]$ we get*

$$S_e(z) = \left( f_e(z) e_e(z) - \frac{1}{2} h'_e(z) + (\frac{\partial}{\partial \lambda} - \frac{1}{2} h_e(z))^2 \right) . \tag{44}$$

**Proof:**
Note that if on $V[0]$, the last commutation relation of Proposition 2.9 reduces to $[e_e(z), f_e(z)] = -h'_e(z)$. If we put this into the definition of $S_e(z)$ of Proposition 2.15, we obtain the desired result.

**Proposition 2.21 ([16])**

$$S_e(z)|_{z=y_j} = \left( \frac{\partial}{\partial y_j} - \sum_{i=1}^{n} \frac{\Lambda_i}{2} \frac{\theta'(y_j - z_i)}{\theta(y_j - z_i)} \right)^2 . \tag{45}$$

**Proof:**
First note that

$$\begin{aligned} \frac{\partial^2}{\partial y_j^2} &= \frac{\partial}{\partial y_j} (\frac{\partial}{\partial \lambda} + \sum_{i=1}^{n} \frac{\theta'(y_j - z_i)}{\theta(y_j - z_i)}) t_i \frac{\partial}{\partial t_i} \\ = -\sum_{i=1}^{n} \wp(y_j - z_i) t_i \frac{\partial}{\partial t_i} &+ \frac{\partial^2}{\partial \lambda^2} + 2 \sum_{i=1}^{n} \frac{\theta'(y_j - z_i)}{\theta(y_j - z_i)} t_i \frac{\partial}{\partial t_i} \frac{\partial}{\partial \lambda} \\ + \sum_{i,l=1}^{n} (\frac{\theta'(y_j - z_i)\theta'(y_j - z_l)}{\theta(y_j - z_i)\theta(y_j - z_l)} t_i t_l \frac{\partial}{\partial t_i} \frac{\partial}{\partial t_l}) &+ \sum_{i=1}^{n} (\frac{\theta'(y_j - z_i)}{\theta(y_j - z_i)})^2 t_i \frac{\partial}{\partial t_i} . \end{aligned}$$

Then, we have by our representation of $sl_2(\mathbb{C})$

$$-\frac{1}{2} h'(y_j) = -\sum_{i=1}^{n} \wp(y_j - z_i) t_i \frac{\partial}{\partial t_i} + \sum_{i=1}^{n} \wp(y_j - z_i) \frac{\Lambda_i}{2} .$$

Thus, we get

$$\begin{aligned} (H_e(z))^2|_{z=y_j} = (\frac{\partial^2}{\partial \lambda^2} + 2 \sum_{i=1}^{n} \frac{\theta'(z - z_i)}{\theta(z - z_i)} t_i \frac{\partial}{\partial t_i} \frac{\partial}{\partial \lambda} &- \sum_{i=1}^{n} \Lambda_i \frac{\theta'(z - z_i)}{\theta(z - z_i)} \frac{\partial}{\partial \lambda} \\ &+ \sum_{i,l=1, l \neq i}^{n} \frac{\Lambda_i \Lambda_l}{4} \frac{\theta'(z - z_i)\theta'(z - z_l)}{\theta(z - z_i)\theta(z - z_l)} \\ -2 \sum_{i=1}^{n} \frac{\Lambda_i}{2} \frac{\theta'(z - z_i)}{\theta(z - z_i)} \sum_{i=1}^{n} \frac{\theta'(z - z_i)}{\theta(z - z_i)} t_i \frac{\partial}{\partial t_i} &+ \sum_{i,l=1}^{n} \frac{\theta'(z - z_i)\theta'(z - z_l)}{\theta(z - z_i)\theta(z - z_l)} t_i t_l \frac{\partial}{\partial t_i} \frac{\partial}{\partial t_l} \\ &+ \sum_{i=1}^{n} (\frac{\theta'(z - z_i)}{\theta(z - z_i)})^2 t_i \frac{\partial}{\partial t_i} |_{z=y_j} \\ = \frac{\partial^2}{\partial y_j^2} + \sum_{i=1}^{n} \wp(y_j - z_i) t_i \frac{\partial}{\partial t_i} &- \sum_{=1}^{n} \Lambda_i \frac{\theta'(y_j - z_i)}{\theta(y_j - z_l)} \frac{\partial}{\partial y_j} \\ &+ \sum_{i,l=1}^{4} \frac{\Lambda_i \Lambda_l}{4} \frac{\theta'(y_j - z_i)\theta'(y_j - z_l)}{\theta(y_j - z_i)\theta(y_j - z_l)} \\ = \frac{\partial^2}{\partial y_j^2} - \sum_{=1}^{n} \Lambda_i \frac{\theta'(y_j - z_i)}{\theta(y_j - z_l)} \frac{\partial}{\partial y_j} &+ \sum_{i=1}^{n} \wp(y_j - z_i) \frac{\Lambda_i}{2} \end{aligned}$$

$$+\sum_{i,l=1}^{4}\frac{\Lambda_i\Lambda_l}{4}\frac{\theta'(y_j-z_i)\theta'(y_j-z_l)}{\theta(y_j-z_i)\theta(y_j-z_l)}+\frac{1}{2}h'(y_j)\,.$$

This yields the desired result.

**Definition 2.22 (Separated differential equations)** *The n differential operators thus obtained all show the same structure and define the separated differential equation*

$$\left(\frac{\partial}{\partial t}-\sum_{i=1}^{n}\frac{\theta'(t-z_i)}{\theta(t-z_i)}\frac{\Lambda_i}{2}\right)^2 v(t)=q_e(t)v(t)\,.$$

**Proposition 2.23**

$$S_e(z)=\sum_{i=1}^n c^{(i)}\wp(z-z_i)+\sum_{i=1}^n\frac{\prod_{j=1}^n\theta(y_i-z_j)}{\prod_{j=1}^n\theta(z-z_j)}\times\frac{\theta(z+\sum_{j=1,j\neq i}y_j-\sum_{i=1}^n z_i)}{\theta(\sum_{j=1}^n y_j-\sum_{i=1}^n z_i)}$$
$$\times\prod_{j=1,j\neq i}^n\frac{\theta(z-y_i)}{\theta(y_j-y_i)}\left(S_e(y_i)-\sum_{j=1}^n c^{(j)}\wp(y_i-z_j)\right), \qquad (46)$$

*where $S_e(y_j)$ for $j=1,\dots,n$ is calculated in Proposition 2.21.*

**Proof:**

First, we notice that $S_e^D(z)=S_e(z)-\sum_{i=1}^{n}c^{(i)}\wp(z-z_i)$ has only simple poles on the fundamental domain $F=\{x+y\tau|\,x,y\in[0,1)\}$ at the points $z=z_i, i=1,\dots,n$. Thus, $S_e^D(z)\prod_{i=1}^{n}\theta(z-z_i)$ is an elliptic polynomial in $\Theta_n(e^{\sum_{i=1}^n z_i})$ (cf. Appendix B) which can be calculated by interpolating the $n$ known values of it at the points $z=y_j, j=1,\dots,n$ to yield

$$\sum_{i=1}^{n}\prod_{j=1}^{n}\theta(y_i-z_j)\frac{\theta(z+\sum_{j=1,j\neq i}y_j-\sum_{i=1}^n z_i)}{\theta(\sum_{j=1}^n y_j-\sum_{i=1}^n z_i)}\prod_{j=1,j\neq i}^{n}\frac{\theta(z-y_i)}{\theta(y_j-y_i)}\left(S_e(y_i)-\sum_{j=1}^{n}c^{(j)}\wp(y_i-z_j)\right),$$

which expression is in turn useful to calculate $S_e^D(z)$ and $S_e(z)$. $S_e(z)$ as defined in Definition 2.15 transforms doubly periodic on $\Gamma$ - due to $\sum_{i=1}^{n}H_e^i=0$ - and so does its expression in this Proposition. Both expressions also coincide at the residues and the $n$ points $z=y_j, j=1,\dots,n$.

**Remark:**

We will need this expression in the sequel, cf. Chapter 4.

Let us now proceed in formulating a proposition on the strucuture of possible eigenfunctions in the terms of the new variables $(C,y_1,\dots,y_n)$.

**Proposition 2.24** *A function $\psi(\lambda,t_1,\dots,t_n)$, homogeneous of degree $m=\sum_{i=1}^n\frac{\Lambda_i}{2}$ in the variables $(t_1,\dots,t_n)$, is a solution of the partial differential equation $S_e(z)\psi=q_e(z)\psi$ for $z\in\mathbb{C}$ if and only if*

$$\psi(\lambda,t_1,\dots,t_n)=C^m u(y_1,\dots,y_n) \qquad (47)$$

*and*

$$\left(\frac{\partial}{\partial y_j}-\sum_{k=1}^{n}\frac{\Lambda_k}{2}\frac{\theta'}{\theta}(y_j-z_k)\right)^2 u(y_1,\dots y_n)=q_e(y_j)u(y_1,\dots y_n) \qquad (48)$$

*for* $j = 1, \dots, n$.
*Hence,* $u(y_1, \dots, y_n) = \prod_{i=1}^n v(y_i)$.

**Proof:**
Let us first describe how to obtain $\psi(\lambda, t_1, \dots, t_n) = C^m u(y_1, \dots, y_n)$. By using the fact that $[S_e(z), C\frac{\partial}{\partial C}] = 0$, we see that we can simultaneously diagonalize both operators. The eigenfunction of $C\frac{\partial}{\partial C}$ to the eigenvalue $\alpha \in \mathbb{C}$ is given by $C^\alpha$. Further note that by the homogeneity property of $\psi$ and the definition of $(t_1, \dots, t_n)$ in terms of $(y_1, \dots, y_n)$ we get $\psi(\sum_{i=1}^n (y_i - z_i), Cf_1(y_1, \dots, y_n), \dots, Cf_n(y_1, \dots, y_n)) = C^m \psi(\sum_{i=1}^n (y_i - z_i), f_1(y_1, \dots, y_n), \dots, f_n(y_1, \dots, y_n))$.
That it suffices for a function $u(y_1, \dots, y_n)$ in order to be a solution of $S_e(z)C^m u = q_e(z)C^m u$ to be a solution of $S_e(y_j)u = q_e(y_j)u$ for $j = 1, \dots, n$ is seen by the fact that $\prod_{i=1}^n (S_e(z) - q_e(z))u(y_1, \dots, y_n) \in \Theta_n(\chi)$, cf. the Appendix, vanishing at $n$ generic points $y_j$ for $j = 1, \dots, n$, thus vanishing everywhere. That $u(y_1, \dots, y_n) = \prod_{i=1}^n v(y_i)$ can be seen by the structure of the separated differential equation at $y = y_i$. For all $j = 1, \dots, n$ the corresponding differential operators depend only on one variable $y_j$.

## 2.6 Solutions of the elliptic Gaudin eigenvalue problem

**Synopsis:**
Here, we show how to obtain solutions $\psi \in V[0]$ of $(S_e(z) - q_e(z))\psi = 0$ by studying solutions of the system of separated equations.
We study a non-degenerate (Proposition 2.26) and a degenrate case (Proposition 2.27), where degenerate means that poles of the separated equations can be zeroes of the solutions of these equations.
Proposition 2.28 shows how to construct out of the solutions of the separated equations solutions of the eigenvalue problem of $S_e(z)$ that can also be formulated in terms of the operator $f_e(z)$ which was the main ingredient in introducing separation of variables.
In the last part of this section, we will explain how to understand completeness of the solutions which we gave in Proposition 2.28.

### 2.6.1 The structure of the solutions

**Remark:**
Here, we will look at possible solutions to the elliptic eigenvalue problem as obtained by Proposition 2.24.
First, note that the critical exponents of the separated differential equation of Proposition 2.22 are given by 0 and $\Lambda_i + 1$ at every $z_i$ for all $i = 1, \dots, n$.
In the following two propositions, we will indicate how to write down two different kinds of solutions to the separated equations by Bethe ansatz. Then we will show how to obtain out of these solutions solutions to
$S_e(z)\psi(\lambda, t_1, \dots, t_m) = q_e(z)\psi(\lambda, t_1, \dots, t_m)$.
The last theorem will be on the completeness of the Bethe solutions.

**Definition 2.25 (Bethe solution)** *A solution* $\psi(\lambda, t_1, \dots, t_n)$ *to*

$$S_e(z)\psi(\lambda, t_1, \dots, t_n) = q_e(z)\psi(\lambda, t_1, \dots, t_m)$$

*is called a Bethe solution if it is of the form*

$$\psi(\lambda, t_1, \dots, t_n) = e^{c\lambda} \prod_{i=1}^{\tilde{m}} f_e(w_i) v_I, \tag{49}$$

*where* $v_I$ *is a singular element of the Verma module, i.e. it vanishes by the action of* $\sum_{i=1}^n e^{(i)}$. *The value of* $\tilde{m} \in \mathbb{N}$ *must be chosen as to ensure that* $\psi(\lambda, t_1, \dots, t_n) \in V[0]$.

**Remark :**
We start with the ansatz which Hermite used to solve Lamé's differential equation (cf. [51]). i.e. with a function $v(y) \in \Theta_m(\chi)$, where $m = \sum_{i=1}^n \Lambda_i$. By the Appendix, such a function can be written as

$$v(y) = e^{cy} \prod_{i=1}^{m} \theta(y - w_k).$$

**Proposition 2.26** *Let* $v(y) \in \Theta_m(\chi)$ *for some* $\chi \in \Gamma^*$ *be given such that* $z_i \neq w_k$ *for all* $i = 1, \dots, n$ *and* $k = 1, \dots, m$. *This function is a solution to the elliptic Schrödinger equation*

$$\left( \frac{\partial}{\partial y} - \sum_{i=1}^{n} \frac{\Lambda_i}{2} \frac{\theta'}{\theta}(y - z_i) \right)^2 v(y) = q_e(y)$$

*if and only if its parameters* $w_k$ *for all* $k = 1, \dots, m$ *obey the Bethe Ansatz equations*

$$\sum_{i=1}^{n} \Lambda_i \frac{\theta'}{\theta}(w_k - z_i) - 2 \sum_{j=1, j\neq k}^{m} \frac{\theta'}{\theta}(w_k - w_j) = 2c. \tag{50}$$

**Proof:**
Note first that $w_k \neq w_l$ for all $k \neq l$ for $k, l = 1, \dots, m$, since the only solution of the differential equation vanishing with its derivative at a regular point is the trivial solution. If we write down the first and second derivative of $v(y)$, they read $v'(y) = cv(y) + \sum_{j=1}^m \theta'(y - w_j) \prod_{k=1, k\neq j}^m \theta(y - w_k)$ and $v''(y) = c^2 v(y) + 2c \sum_{k=1}^m \frac{\theta'}{\theta}(y - w_k) v(y) + \sum_{k=1}^m \theta''(y - w_k) \prod_{j=1, j\neq k}^m \theta(y - w_j) e^{cy} + \sum_{k=1}^m \sum_{j=1, j\neq k}^m \theta'(y - w_k)\theta(y - w_j) \prod_{l=1, l\neq k,j} \theta(y - w_l) e^{cy}$. Evaluated at a zero $w_k$, we obtain

$$v''(w_k) = 2cv'(w_k) + 2 \sum_{j=1, j\neq k}^{m} \frac{\theta'}{\theta}(w_k - w_j) v'(w_k).$$

If we instead look at the separated differential equation, we notice that it yields $v''(y) = \sum_{i=1}^n \Lambda_i \frac{\theta'}{\theta}(y - z_i) v'(y) + r(y) v(y)$, where $r(y)$ is a regular function at the $w_k$ for all $k = 1, \dots, m$. Evaluated at $w_k$, this expression yields

$$v''(w_k) = \sum_{i=1}^{n} \Lambda_i \frac{\theta'}{\theta}(w_k - z_i) v'(w_k).$$

Since $v'(w_k) \neq 0$ for all $k = 1, \dots, m$ the comparison of the two identities for $v''(w_k)$ for $k = 1, \dots, m$ yields the Bethe Ansatz equations.

That this is indeed the proof, we perceive by the following two arguments: A solution of the differential equation in $\Theta_m(\chi)$ obeys the Bethe Ansatz equation by construction. On the other hand any set of parameters $(w_1, \ldots, w_n, c)$ gives rise to a function $v(y) = e^{cy}\prod_{k=1}^m \theta(y - w_k) \in \Theta_m(\chi)$, which in turn obeys the differential equation. This can be checked by comparing zeroes of $v(y)$ and residues.

**Proposition 2.27** *Let $I \subseteq \{1, \ldots, n\}$ be given. Let $w_k = z_i$ for all $i \in I$ and $w_k \neq z_i$ if $i \notin I$.*
*Then a solution $v(y) \in \Theta_m(\chi)$ for some $\chi \in \Gamma^*$ of the differential equation*

$$\left(\frac{\partial}{\partial y} - \sum_{i=1}^{n} \frac{\Lambda_i}{2} \frac{\theta'}{\theta}(y - z_i)\right)^2 v(y) = q_e(y)$$

*with the property that it vanishes at $z_i$ for $i \in I$ up to order $\Lambda_i + 1$ is given by*

$$v(y) = e^{cy} \prod_{i=1}^{m'} \theta(y - w_i) \prod_{i \in I} \theta(y - z_i)^{\Lambda_i + 1},$$

*where $m' = m - \sum_{i \in I}(\Lambda_i + 1)$ and the parameters $(w_1, \ldots, w_{m'}, c)$ are to obey the modified Bethe Ansatz equations*

$$\sum_{i=1}^{n} \tilde{\Lambda}_i \frac{\theta'}{\theta}(w_k - z_i) - 2\sum_{j=1}^{m'} \frac{\theta'}{\theta}(w_k - w_j) = 2c$$

*for $k = 1, \ldots, m'$ with*

$$\tilde{\Lambda}_i = \begin{cases} \Lambda_i \text{ for } i \notin I, \\ -\Lambda_i - 2 \text{ otherwise}. \end{cases}$$

**Proof:**
First, we may show that $\tilde{v}(y) = \prod_{i \in I} \theta(y - z_i)^{-\Lambda_i - 1} v(y) \in \Theta_{\tilde{m}}(\tilde{\chi})$, as the characteristic exponents of the differential equation were shown to be 0 and $\Lambda_i + 1$ at $z_i$ for all $i = 1, \ldots, n$, where $v(y)$ is a solution to the above differential equation.
By a straightforward calculation, $\tilde{v}(y)$ obeys the alterated differential equation

$$\left(\frac{\partial}{\partial y} - \sum_{i=1}^{n} \frac{\Lambda_i}{2} \frac{\theta'}{\theta}(y - z_i) + \sum_{i \in I}(\Lambda_i + 1)\frac{\theta'}{\theta}(y - z_i)\right)^2 \tilde{v}(y) = q_e(y)\tilde{v}(y)$$

or, by setting $\tilde{\Lambda}_i = \Lambda_i$ if $i \notin I$ and $\tilde{\Lambda}_i = -\Lambda_i - 2$ if $i \in I$, it obeys

$$\left(\frac{\partial}{\partial y} - \sum_{i=1}^{n} \frac{\tilde{\Lambda}_i}{2} \frac{\theta'}{\theta}(y - z_i)\right)^2 \tilde{v}(y) = q_e(y)\tilde{v}(y).$$

Starting with this differential equation, we get by writing $\tilde{v}(y) = e^{cy}\prod_{i=1}^{m'} \theta(y - w_i)$ an elliptic polynomial solution to the differential equation we started with. This solution reads $v(y) = e^{cy}\prod_{i=1}^{m'} \theta(y - w_i) \prod_{j \in I} \theta(y - z_i)^{\tilde{\Lambda}_i + 1} = e^{cy}\prod_{i=1}^{m'} \theta(y - w_i) \prod_{j \in I} \theta(y - z_i)^{\Lambda_i + 1} \in \Theta_m(\chi)$ which vanishes at the $z_i$ for $i \in I$ up to order $\Lambda_i + 1$.

Since $\tilde{v}(y) = e^{cy}\prod_{i=1}^{m'}\theta(y-w_i)$ contains the unknown parameters $(w_1,\dots,w_{m'},c)$ and obeys the alterated differential equation indicated above, we may - by the same way as with the preceding proposition - obtain the Bethe Ansatz equations these parameters are to obey. They read

$$\sum_{i=1}^{n}\tilde{\Lambda}_i\frac{\theta'}{\theta}(w_k-z_i) - 2\sum_{j=1}^{m'}\frac{\theta'}{\theta}(w_k-w_j) = 2c$$

for $k = 1,\dots,m'$.

**Remark:**

Let us now look at the solution of $S_e(z)$ which are obtained by a product of solutions we found in Proposition 2.26 and 2.27.

**Proposition 2.28** *Let $v_0 = 1$ be the highest vector of the Verma module and let $f_e(z)$ be the operator defined at the beginning.*

*a) The first kind of solutions with $w_k \neq z_i$ for all $i = 1,\dots,n$ and $k = 1,\dots,m$ obtained by Proposition 2.26 yields*

$$\psi(\lambda,t_1,\dots,t_m) = \alpha(z_1,\dots,z_n,w_1,\dots,w_m,c)e^{c\lambda}\prod_{j=1}^{m}f_e(w_j)v_0$$

*as a Bethe solution of $S_e(z)\psi(\lambda,t_1,\dots,t_m) = q_e(z)\psi(\lambda,t_1,\dots,t_m)$.*

*b) The second kind of solutions with $w_k = z_i$ for all $i \in I$ for some fixed $I \subseteq \{1,\dots,n\}$ obtained by Proposition 2.27 yields the Bethe solution*

$$\psi(\lambda,t_1,\dots,t_m) =$$
$$\alpha(z_1,\dots,z_n,w_1,\dots,w_m,c)e^{c\lambda}\prod_{j=1}^{m'}f_e(w_j)\prod_{k\in I}(f^{(k)})^{\Lambda_k+1}v_0.$$

**Proof:**

For the proof, we need the following facts given throughout the preceding section: $f_e(z) = C\frac{\prod_{i=1}^{n}\theta(z-y_i)}{\theta(z-z_i)}$, $f_e^{(i)} = \text{Res}_{\,z=z_i}f_e(z)$ and $\lambda = \sum_{i=1}^{n}(y_i-z_i)$. By Proposition 2.15, the function $\psi(\lambda,t_1,\dots,t_n)$ could be written in terms of the new variables as $C^m\prod_{i=1}^{n}v(y_i)$, where $v(y_i)$ solved the corresponding differential equation $S_e(y_i)v(y_i) = q_e(y_i)v(y_i)$.
Let us show how to obtain with these results the first identity written in the proposition. The second one is then obtained similarly.

$$C^m\prod_{i=1}^{n}v(y_i) = C^m\prod_{i=1}^{m}\left(e^{cy_i}\prod_{j=1}^{m}\theta(y_i-w_j)\right) =$$
$$(-1)^{mn}e^{c\sum_{i=1}^{n}z_i}e^{c(\sum_{i=1}^{n}(y_i-z_i))}\prod_{j=1}^{m}\left(C\prod_{i=1}^{n}\frac{\theta(w_j-y_i)}{\theta(w_j-z_i)}\theta(w_j-z_i)\right) =$$
$$\alpha(z_1,\dots,z_n,w_1,\dots,w_m,c)e^{c\lambda}\prod_{j=1}^{m}f_e(w_j)v_0 = \psi(\lambda,t_1,\dots,t_n).$$

**Remark:**

Note that only $\prod_{k\in I}(f^{(k)})^{\Lambda_k+1}v_0$ with $I=\emptyset$ has a nontrivial projection on the Verma module, in our realisation $\mathbb{C}[t_1,\dots,t_n]/(\sum_{i=1}^{n}t_i^{\Lambda_i+1}\mathbb{C}[t_1,\dots,t_n])$.

### 2.6.2 Completeness of the Bethe eigenvectors

Let us first give the necessary definitions to understand the theorem and then write down the theorem.

**Definition 2.29** ($\mathcal{H}(\chi)$) *Let $\chi \in \Gamma^*$ be given. Then $\mathcal{H}(\chi)$ is the following space of functions*

$$\mathcal{H}(\chi) = \{\phi : \lambda \to \phi(\lambda) \in V[0] \,|\, \phi \text{ meromorphic in } \lambda,$$
$$\phi(\lambda+1) = \chi(1)\phi(\lambda),\ \phi(\lambda+\tau) = \chi(\tau)e^{\pi i \sum_{i=1}^n z_i h^{(i)}}\phi(\lambda)\}.$$

**Remark:**
Since the operators $e_e(z), f_e(z), \frac{\partial}{\partial\lambda} - \frac{1}{2}h_e(z)$ preserve the space $\mathcal{H}(\chi)$, so do $S_e(z)$ and $H_e^i$ for $i = 1,\dots,n$. Thus, it is sensible to look for solutions of $H_e^i\phi(\lambda) = \mu_i\phi(\lambda)$ for all $i = 1,\dots,n$ which are of the form $\phi(\lambda) \in \mathcal{H}(\chi)$.

**Definition 2.30** ($\Sigma(\chi)$) *Let $\chi \in \Gamma^*$ be given. Then*

$$\Sigma(\chi) = \{(\mu_c, \mu_1, \dots, \mu_n) \in \mathbb{C}^{n+1} \,|\, \text{there exists a nontrivial } \phi(\lambda) \in \mathcal{H}(\chi)$$
$$\text{with } H_e^i\phi(\lambda) = \mu_i\phi(\lambda),\ H_e^c\phi(\lambda) = \mu_c\phi(\lambda)\}.$$

**Theorem 2.31** *Let $\chi \in \Gamma^*$ be fixed. Let $\sigma \in \Gamma^*$ be given by the property $\sigma(1) = -1$ and $\sigma(\tau) = -1$.*
*Then $(\mu_c, \mu_1, \dots, \mu_n) \in \Sigma(\chi)$ if and only if $\sum_{i=1}^n \mu_i = 0$ and the separated problem*

$$\left(\frac{\partial}{\partial y} - \sum_{i=1}^n \frac{\Lambda_i}{2}\frac{\theta'}{\theta}(y - z_i)\right)^2 v(y) = q_e(y)v(y)$$

*admits a nontrivial solution $v(y) \in \Theta_m(\sigma^m\chi)$.*
*To this solution, there corresponds a Bethe eigenvector (cf. Proposition 2.28).*

**Proof of the Theorem:**
Let us first look at the if-direction. Let us suppose that to the set of eigenvalues $(\mu_c, \mu_1, \dots, \mu_n)$ there exists a nontrivial function $\phi(\lambda) \in \mathcal{H}(\chi)$ which solves $H_e^i\phi(\lambda) = \mu_i\phi(\lambda)$ for $i = 1,\dots,n$ and $H_e^c\phi(\lambda) = \mu_c\phi(\lambda)$. Thus, we can write $\phi(\lambda, t_1, \dots, t_n) = \sum_{m_1+\dots+m_n=m} \phi_{m_1\dots m_n}(\lambda)\prod_i (t_i)^{m_i}$. In particular, since $\phi(\lambda) \in \mathcal{H}(\chi)$, $\phi(\lambda) \in V[0]$. Thus, $\phi(\lambda, t_1, \dots, t_n) = C^m v(y_1, \dots, y_n) = \sum_{m_i} \phi_{m_1,\dots,m_n}(\sum_{i=1}^n (y_i - z_i)) \prod_i (C\frac{\prod_{j=1}^n \theta(z_i - y_j)}{\prod_{j=1, j\neq i}^n \theta(z_i - z_j)})^{m_i}$. It remains to be shown that $v(y_1, \dots, v_n)$ is indeed an elliptic polynomial in the $y_i$ for all $i = 1,\dots,n$.. ($\phi(\lambda)$ was only required to be meromorphic in $\lambda$.) To this end, we first have to show that $v(y_1, \dots, y_n)$ is holomorphic in the $y_i$, $i = 1,\dots,n$ and then that $v(y_1, \dots, y_n)$ as a function of each $y_i$ for $i = 1,\dots,n$ is indeed an element of $\Theta_m(\sigma^m\chi)$. For the first hypothesis it is sufficient to show that the coefficients $\phi_{m_1\dots m_n}(\sum_{i=1}^n (y_i - z_i))$ are not singular at $\lambda = \sum_{i=1}^n (y_i - z_i) = 0$, as this is the only possible pole for the $\phi_{m_1\dots m_n}(\sum_{i=1}^n (y_i - z_i))$ due to the strucuture of the differential equation $H_e^i\phi(\lambda) = \mu_i\phi(\lambda)$ they obey. This is proven by the following argument: As $v(y_1, \dots, y_n)$ solves the separated equations for $i = 1,\dots,n$, it may only have poles that occur in these equations, i.e. at the points $z_i$ for $i = 1,\dots,n$. For a generic choice of the $y_1, \dots, y_n$, $\sum_{i=1}^n y_i - \sum_{j\neq k, j=1}^n z_j \neq z_k$, thus avoiding a pole at $\sum_{i=1}^n (y_i - z_i) = 0$.

The second hypothesis can be shown by a straightforward calculation for every function $y_i \to v(y_1, \ldots, y_i, \ldots, y_n)$, where $i = 1, \ldots, n$, by the fact that $C^m v(y_1, \ldots, y_i, \ldots, y_n) = C^m \sum_i \phi_{m_1,\ldots,m_n}(\sum_{i=1}^n (y_i - z_i)) \prod_i (\frac{\prod_{j=1}^n \theta(z_i - y_j)}{\prod_{j=1, j\neq i}^n \theta(z_i - z_j)})$ and that we know the character of every $\phi_{m_1,\ldots,m_n}(\sum_{i=1}^n (y_i - z_i))$. Now, by Propositions 2.24, 2.26 and 2.27, we see that a Bethe solution corresponding to $v(y_1, \ldots, y_n)$ indeed exists.
Let us conversely suppose that we have a Bethe solution given, i.e. by Proposition 2.28 a solution of the separated cquation $v(y) \in \Theta_m(\sigma^m \chi)$. The Bcthe solution is a nontrivial solution of the $n + 1$ eigenvalue problems, so we only have to check that it is in $\mathcal{H}(\chi)$. Let us write the Bethe solution as a polynomial in the $t_i$, $i = 1, \ldots, n$. $\psi(\lambda, t_1, \ldots, t_n) = e^{c\lambda} \prod_{j=1}^{\tilde{m}} (\sum_{i=1}^n \frac{\theta(\lambda - w_j + z_i)\theta'(0)}{\theta(w_j - z_i)\theta(\lambda)} t_i) \prod_{i \in I} t_i^{\Lambda_i + 1}$. It is meromorphic in the variable $\lambda$ and an element of $V[0]$. Note that the character $\sigma^m \chi$ of $v(y_i)$ for $i = 1, \ldots, n$ can be calculated directly. Thus, by a straightforward calculation we can show that $\psi(\lambda, t_1, \ldots, t_n) \in \mathcal{H}(\chi)$.

# 3 Introduction to the difference case

In the following two chapters, we want to study the eigenvalue problem arising with the SOS eight-vertex model of statistical mechanics studied by Baxter [4], Andrews, Baxter and Forrester [1] and Date et al. [11] with antiperiodic boundary conditions.
More explicitly, in the first chapter we aim at finding common eigenvalues and eigenvectors of a commuting family of operators depending on a complex parameter called antiperiodic transfer matrices of the SOS model. This notion and other notions we need in order to understand the model – Boltzmann weights, transfer matrices, partition function – will be described quite heuristically in the first section if this chapter.
The most complex part we need to know to understand the SOS model as treated here is some of the representation theory of the elliptic quantum group $E_{\tau,\eta}(sl_2)$ as described in [25], since the SOS model comes along with a certain finite-dimensional irreducible representation of $E_{\tau,\eta}(sl_2)$. Since the exposition of the representation theory as we will need it involves a lot of previous definitions, we will deal with it in the second section of the chapter.
In the second section, we will make precise our statements on the notions of the SOS model which we introduced in the first section by connecting these with the representation theory of $E_{\tau,\eta}(sl_2)$.
In the third section, we define our main tool enabling us to study the antiperiodic SOS model, the so-called auxiliary representation of $E_{\tau,\eta}(sl_2)$ (cf. Proposition 4.33). This representation is a generalization of Sklyanin's ideas to the elliptic case [46, 44]. We need this representation to perform separation of variables, a method we have to choose since we cannot treat the antiperiodic SOS model by conventional algebraic Bethe ansatz (ABA), as described in the general introduction. This is so, as the notion of a highest weight representation, necessary for the ABA, cannot be suitably used in this context. (Contrary to the periodic SOS case, where a solution in the context of representation theory of the elliptic quantum group has been found by Felder and Varchenko [27].) To achieve separation of variables, we first need to know that the two involved representations of the elliptic quantum group, the auxiliary representation and the representation of the SOS model, are in fact isomorphic. What this means and how the isomorphism is constructed will be shown in the fifth section of this chapter.
The use of the method of separation of variables will be visible in the fourth section, where we will discuss the results on possible common eigenvalues and eigenvectors of the antiperiodic SOS model. We obtain these results by studying the eigenvalue problem of a family of transfer matrices of the auxiliary representation. Studying the latter transfer matrices is an advantage compared to studying the antiperiodic SOS transfer matrices, since solving the original eigenvalue problem of the SOS transfer matrices involves solving a nonlinear difference equation in $n$ variables, whereas solving the eigenvalue problem for the auxiliary transfer matrices consists in solving $n$ structurally identical linear difference equations in one (so-called separated) variable per equation. These linear difference equations are called the separated equations.
The fourth section closes with two theorems (4.54 and 4.55): the first one on possible common eigenvalues of the family of antiperiodic SOS transfer matrices, the second one on possible common eigenfunctions of the family of antiperiodic SOS transfer matrices.
The last section of the first chapter connects to the preceding one on the elliptic Gaudin model. In Proposition 4.59, we show that the operator $S_e(z)$ of Proposition 2.23, there

described in the separated variables $(y_1, \dots, y_n, C) \in \{\mathcal{S}^n(E_\tau) \times \mathbb{C}/0\}$ can be seen as a limit of the auxiliary transfer matrix of the SOS eight-vertex model which is also naturally formulated in the separated variables appearing in the quantum case.
In the second chapter, we try to elucidate the structure of the objects abstractly given in the first chapter. Thus, we perform a calculation for the simplest nontrivial case of the SOS model (n=3). We explicitly compute the isomorphism and give an eigenvector with corresponding eigenvalue.

# 4 The SOS eight-vertex model

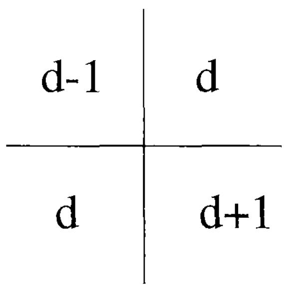
## 4.1 Basic notions of the SOS eight-vertex model

**Synopsis:**
In this section, we aim at introducing the basic notions of the eight-vertex SOS model, without using the representation theory of the elliptic quantum group $E_{\tau,\eta}(sl_2)$. This enables us to do two things: on the one hand we will see what problem we finally want to solve in a none too complicated manner, on the other hand we are reassured in the section where we will describe the SOS model in terms of representation theory to rediscover the notions we already introduced here.
In the first definition, we will define the Boltzmann weights of the eight-vertex SOS model. In the second definition we will define the row-to-row transfer matrix of the SOS model with antiperiodic boundary conditions in analogy to [4, 27]. In the third definition we will pose the common eigenvalue problem of the family of transfer matrices of the SOS model with antiperiodic boundary conditions, which is the problem we finally want to solve.

**Definition 4.1 (Boltzmann weights [11])** *The Boltzmann weights of the eight-vertex SOS model are given by*

$$\begin{aligned}
W_e(d+1,d+2,d+1,d|z) &= \theta(z+2\eta), \\
W_e(d+1,d,d+1,d|z) &= -\frac{\theta(z-2\eta d)\theta(2\eta)}{\theta(2\eta d)}, \\
W_e(d+1,d,d-1,d|z) &= \frac{\theta(z)\theta(2\eta(d-1))}{\theta(2\eta d)}, \\
W_e(d-1,d,d+1,d|z) &= \frac{\theta(z)\theta(2\eta(d+1))}{\theta(2\eta d)}, \\
W_e(d-1,d,d-1,d|z) &= \frac{\theta(2\eta d+z)\theta(2\eta)}{\theta(2\eta d)}, \\
W_e(d-1,d-2,d-1,d|z) &= \theta(z+2\eta).
\end{aligned}$$

**Remark:**
These Boltzmann weights were obtained in [11], p. 210, as a solution to the vertex-IRF transformation, from the eight-vertex model towards an SOS model, established by Baxter in [4], solving the equation of the vertex-IRF transformation which was given in the introduction. They correspond to not yet normalized transition probabilities between four adjacent faces from the values attached to a face (called the height) of the upper two faces to those of the lower two faces. The heights of adjacent faces are to differ by plus or minus one. E.g. the Boltzmann weight $W_e(d-1,d,d,d+1)$ corresponds to the combination of faces



That these Boltzmann weights obey the star-triangle-relation, the condition which is necessary to ensure commutativity of the transfer matrices, will be obtained later on as a corollary.

**Definition 4.2 (Row-to-row transfer matrix of the antiperiodic SOS model)** *Let us consider a lattice consisting of $n$ rows of faces with antiperiodic boundary conditions, i.e. the height of the $n+1$th face is the opposite of the height of the first face. The row-to-row transfer matrix is much easier to define if we describe its action onto an antiperiodic path, i.e. an entity with $n+1$ entries, where each entry corresponds to the height of the corresponding face. Such a path looks like $|a_1, \dots, a_i, \dots, a_n, a_{n+1} = -a_1 >$. Then, the transfer matrix describes with what probabilities a fixed path, i.e. a fixed assignment of heights attached to a row of faces, changes into any other possible path, i.e. into any other possible assignment of heights to the subsequent row of faces. The probabilities are given by products of Boltzmann weights, as a probability of how the heights of two faces on the one row change into the heights of two faces in the subsequent rows are described by the Boltzmann weights (cf. above). Thus, the transfer matrix reads*

$$T_{SOS,e}(z)|a_1, \dots, a_{n+1} = -a_1 >= \\ \sum_{b_1,\dots,b_n,b_{n+1}=-b_1} \prod_{i=1}^{n} W(a_{i+1}, a_i, b_i, b_{i+1}|z)|b_1, \dots, b_{n+1} = -b_1 > .$$

*It is a matrix on the space of antiperiodic paths. The parameter $z \in \mathbb{C}$ comes about by the definition of the Boltzmann weights. Thus, we rather obtain a family of transfer matrices depending on this parameter.*

**Remark:**
This treatment is made rigorous in the third section of this chapter. There, we e.g. show that the set of antiperiodic paths defines a basis of a $2^n-$ dimensional vector space. This row-to-row transfer matrix corresponds by its structure to the ones obtained in [27] and [4], p. 28. The only difference is the presence of antiperiodic boundary conditions instead of periodic ones, or – as we formulated it here – $b_1 = -b_{n+1}$ instead of $b_1 = b_{n+1}$.

**Definition 4.3 (Common eigenvalue problem)** *A solution to the common eigenvalue problem of the family of transfer matrices is given by a pair*

$$(\sum_{a_1,\dots,a_n} \alpha_{a_1,\dots,a_n}|a_1, \dots, a_n, a_{n+1} = -a_1 >, \epsilon_{SOS}(z))$$

*which solves*

$$T_{SOS,e}(z) \sum_{a_1,\dots,a_n} \alpha_{a_1,\dots,a_n}|a_1, \dots, a_{n+1} = -a_1 >= \\ \epsilon_{SOS}(z)(\sum_{a_1,\dots,a_n} \alpha_{a_1,\dots,a_n}|a_1, \dots, a_{n+1} = -a_1 >).$$

*The formula $\sum_{a_1,\dots,a_n} \alpha_{a_1,\dots,a_n}|a_1, \dots, a_{n+1} = -a_1 >$ indicates a linear combination of antiperiodic paths, hence attachments of heights to faces, each attachment with the antiperiodic boundary conditions preserved. In the sequel, we restrict the eigenvalues we are looking for to be elliptic polynomials (cf. Appendix 2). This makes sense since ... .*

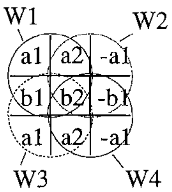
**Remark:**
Why it is sensible to study this problem was emphasized in the introduction of the thesis. To further stress its importance, we will give a heuristic definition of the partition function in terms of the above defined transfer matrix

**Definition 4.4 (Partition function)** *The partition function of the SOS model is in terms of its row-to-row transfer matrix given by*

$$\mathcal{Z}_{SOS,M}(z) = tr\left(T^M_{SOS,e}(z)\right),$$

*where the trace is taken over the space of antiperiodic paths where which row-to-row transfer matrix is an endomorphism of. The intepretation is the following: the partition function $\mathcal{Z}_{SOS,N}$ describes the sum over all possible attachments of heights to faces, i.e. over all antiperiodic paths, of (normalized) probabilities of the following events: we start with a given attachment of heights to the n faces (the antiperiodic boundary conditions understood) and after M row-to-row transitions we are to return to the same attachment of heights to faces which we started with.*

**Remark:**
This can be visualized by the following picture for the simplest case $n = 2$ and $M = 2$.




By the partition function $\mathcal{Z}_{SOS,2}$ we would obtain a sum of all products of $n \times M = 4$ Boltzmann weights $W1 \cdot W2 \cdot W3 \cdot W4$ depending on allowed – i.e. $o_2 = o_1 \pm 1$ with $o = a, b$ – attachements of heights to faces.
Being in possession of the common eigenvalues and eigenvectors of the family of antiperiodic SOS transfer matrices enables us to compute the above partition function of the model as well as other physical interesting quantities as for example the magnetization.

## 4.2 The setting corresponding to the SOS eight-vertex model

**Synopsis:**
In this section, we will present the basic representation theoretical notions concerning $E_{\tau,\eta}(sl_2)$ which we will need in the sequel. First, we define some notation (Definition 4.5) and the notion of a diagonalizable $\mathcal{H}-$module (Definition 4.5). The latter will be needed to define a representation. Then we introduce the R-matrix of the elliptic quantum group (Definition 4.6), which gives us the basic structure of this quantum group. The R-matrix is very important as for the representations of $E_{\tau,\eta}(sl_2)$ (Definition 4.8), as it defines relations every representation has to obey, the RLL-relations. Then, we give some examples of representations (Proposition 4.10). The examples are mostly finite dimensional irreducible representations, because these are the ones that can be used to construct representations corresponding to (higher dimensional analogues of) the eight-vertex SOS model. The construction of the representation corresponding to the eight-vertex SOS model also heavily relies on the fact that we can build shifted tensor products of representations of $E_{\tau,\eta}(sl_2)$ to obtain new representations of $E_{\tau,\eta}(sl_2)$ (Proposition 4.9).
We then continue with a slight generalization of the notion of a representation: the functional representation and its operator algebra (Definition 4.12 and Definition 4.14). For a functional representation the diagonalizable $\mathcal{H}$-module is replaced by a suitable space of functions. We introduce the notion of the quantum determinant (Definition 4.15) which

will be needed in the sequel as we can replace any of the four entries of the L-operator (cf. introduction) of a given representation of $E_{\tau,\eta}(sl_2)$ by the quantum determinant to equivalently describe this representation.
We proceed by introducing the notion of a highest weight representation of $E_{\tau,\eta}(sl_2)$ (Definition 4.16). We discuss this notion for the following reason: we want to show that the representation of $E_{\tau,\eta}(sl_2)$ to be attached to the eight-vertex SOS model is isomorphic to the auxiliary representation. We then state a theorem on the shifted tensor product of finite-dimensional irreducible highest weight representations (Proposition 4.18). Finally, we give a Theorem [25] stating that finite-dimensional irreducible highest weight representations of $E_{\tau,\eta}(sl_2)$ are isomorphic if their highest weights coincide (Proposition 4.19).

### 4.2.1 Introduction

**Remark:**
The functions appearing in this chapter correspond to the ones defined in the chapter on the differential elliptic Gaudin model.

**Definition 4.5 (Basic notions)**

a) *Let $\mathcal{H} = \mathbb{C}h$ be the one-dimensional Lie algebra generated by one generator $h$. Let $V_i, i = 1, \dots, n$ be modules over $\mathcal{H}$.*
*$V_i$ is called a diagonalizable $\mathcal{H}$-module if $V_i$ is the direct sum of finite dimensional eigenspaces $V_i[\mu]$ of $h$ which are labeled by the eigenvalues $\mu$ of $h$: $V_i = \oplus_\mu V_i[\mu]$. We can for example take $V = \mathbb{C}^2$ and split it into two disjoint subspaces $V[1] = \{\alpha\, e[1] \,|\, \alpha \in \mathbb{C}\}$ and $V[-1] = \{\alpha\, e[-1] \,|\, \alpha \in \mathbb{C}\}$ by identifying*

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; e[1] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; \text{and } e[-1] = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

b) *Let $V_i, i = 1, \dots, n$ be diagonalizable $\mathcal{H}$-modules. We may consider their tensor product $V_1 \otimes \dots \otimes V_n$. For $X \in End(V_i)$ we denote by $X^{(i)} \in End(V_1 \otimes \dots \otimes V_n)$ the operator*

$$X^{(i)} = 1 \otimes \dots \otimes \underbrace{X}_{ith\ place} \otimes \dots \otimes 1\,.$$

*If $X \in End(V_i \otimes V_j)$, we define $X^{(ij)} \in End(V_1 \otimes \dots \otimes V_n)$ analogously.*

c) *Let $v \in V_1 \otimes \dots \otimes V_n$. We may define $h^{(i)} \in End(V_1 \otimes \dots \otimes V_n)$ by the above notation. Let $X = X(h^{(1)}, \dots, h^{(n)})$ be a function taking values in $End(V_1 \otimes \dots \otimes V_n)$. If $h^{(i)}v = \mu_i v$ for all $i = 1, \dots, n$, then $X(h^{(1)}, \dots, h^{(n)})v = X(\mu_1, \dots, \mu_n)v$.*

d) *Let $V$ be a diagonalizable $\mathcal{H}$-module and $\mathbb{I}$ the identity matrix on it. Let $A \in End\ (V^{\otimes j})$. Then we can define $A^{(n-j+1\dots n)} \in End\ (V^{\otimes n})$ by*

$$A^{(n-j+1\dots n)} = \underbrace{\mathbb{I} \otimes \dots \otimes \mathbb{I}}_{first\ j\ copies\ of\ V} \otimes A.$$

**Definition 4.6 (R-matrix)** *Let $V = V[-1] \oplus V[1]$ be a two-dimensional complex vector space.*
*Let the elliptic R-matrix $R_e \in End(V \otimes V)$ depending on the $z, \lambda \in \mathbb{C}$ be defined by*

$$R_e(z,\lambda) = \begin{pmatrix} \theta(z+2\eta) & 0 & 0 & 0 \\ 0 & \frac{\theta(z)\theta(\lambda+2\eta)}{\theta(\lambda)} & -\frac{\theta(z-\lambda)\theta(2\eta)}{\theta(\lambda)} & 0 \\ 0 & \frac{\theta(\lambda+z)\theta(2\eta)}{\theta(\lambda)} & \frac{\theta(z)\theta(\lambda-2\eta)}{\theta(\lambda)} & 0 \\ 0 & 0 & 0 & \theta(z+2\eta) \end{pmatrix}, \tag{51}$$

*where $\theta(z) \equiv \theta(z,\tau)$ with the two parameters $\tau, \eta \in \mathbb{C}$, $Im(\tau) > 0$.*
*We identified $e[1] \otimes e[1] = (1\,0\,0\,0)^T, e[1] \otimes e[-1] = (0\,1\,0\,0)^T, e[-1] \otimes e[1] = (0\,0\,1\,0)^T$ and $e[-1] \otimes e[-1] = (0\,0\,0\,1)^T$.*

**Proposition 4.7 (QDYBE)** *The elliptic R-matrix obeys the dynamical quantum Yang–Baxter equation*

$$\begin{aligned} &R^{(12)}(z-w,\lambda-2\eta h^{(3)})R^{(13)}(z,\lambda)R^{(23)}(w,\lambda-2\eta h^{(1)}) \\ &\quad = R^{(23)}(w,\lambda)R^{(13)}(z,\lambda-2\eta h^{(2)})R^{(12)}(z-w,\lambda), \end{aligned} \tag{52}$$

*where the notation is as defined above. This relation is defined on $End(V^{\otimes 3})$.*

### 4.2.2 Representations, functional representations, operator algebras

**Remark:**
We now looking at representations of the elliptic quantum group $E_{\tau,\eta}(sl_2)$. This will be done in two different ways. The first definition will deal with diagonalizable $\mathcal{H}$-modules, the second one will be a slight generalization.

**Definition 4.8 (Representation [25])** *A representation of the elliptic quantum group $E_{\tau,\eta}(sl_2)$ is a pair $(W, L_e)$, where $W$ is a diagonalizable $\mathcal{H}$-module $W = \oplus_{\mu\in\mathbb{C}} W[\mu]$ and $L_e = L_e(z,\lambda) \in End(V \otimes W)$ is a linear map commuting with $h^{(1)} + h^{(2)}$ meromorphic in $z, \lambda \in \mathbb{C}$ called the L-operator.*
*The L-operator obeys the relation*

$$\begin{aligned} &R_e^{(12)}(z-w,\lambda-2\eta h)L_e^{(1)}(z,\lambda)L_e^{(2)}(w,\lambda-2\eta h^{(1)}) \\ &\quad = L_e^{(2)}(w,\lambda)L_e^{(1)}(z,\lambda-2\eta h^{(2)})R_e^{(12)}(z-w,\lambda). \end{aligned} \tag{53}$$

*This relation is called the dynamical RLL-relation.*

**Remark:**
The $L$-operator is usually written in the form

$$L_e(z,\lambda) = \begin{pmatrix} a_e(z,\lambda) & b_e(z,\lambda) \\ c_e(z,\lambda & d_e(z,\lambda) \end{pmatrix}, \tag{54}$$

where $a_e(z,\lambda), b_e(z,\lambda), c_e(z,\lambda), d_e(z,\lambda) \in \mathrm{End}(W)$ are meromorphic in $z, \lambda \in \mathbb{C}$ and obey the dynamical $RLL$-relation, which explicitly yields the following sixteen conditions:

$$
\begin{aligned}
a_e(z,\lambda)a_e(w,\lambda-2\eta) &= a_e(w,\lambda)a_e(z,\lambda-2\eta),\\
\theta(z-w+2\eta)a_e(z,\lambda)b_e(w,\lambda-2\eta) &= b_e(w,\lambda)a_e(z,\lambda+2\eta)\alpha(z-w,\lambda)\\
&\quad +a_e(w,\lambda)b_e(z,\lambda-2\eta)\beta(z-w,-\lambda),\\
\theta(z-w+2\eta)b_e(z,\lambda)a_e(w,\lambda+2\eta) &= b_e(w,\lambda)a_e(z,\lambda+2\eta)\beta(z-w,\lambda)\\
&\quad +a_e(w,\lambda)b_e(z,\lambda-2\eta)\alpha(z-w,-\lambda),\\
b_e(z,\lambda)b_e(w,\lambda+2\eta) &= b_e(w,\lambda)b_e(z,\lambda+2\eta),\\
\theta(z-w+2\eta)c_e(w,\lambda)a_e(z,\lambda-2\eta) &= \beta(z-w,\lambda-2\eta h)c_e(z,\lambda)a_e(w,\lambda-2\eta)\\
&\quad +\alpha(z-w,\lambda-2\eta h)a_e(z,\lambda)c_e(w,\lambda-2\eta),\\
\beta(z-w,\lambda-2\eta h)c_e(z,\lambda)b_e(w,\lambda-2\eta) &+ \alpha(z-w,\lambda-2\eta h)a_e(z,\lambda)d_e(w,\lambda-2\eta)\\
=\alpha(z-w,\lambda)d_e(w,\lambda)a_e(z,\lambda+2\eta) &+ \beta(z-w,-\lambda)c_e(w,\lambda)b_e(z,\lambda-2\eta),\\
\beta(z-w,\lambda-2\eta h)d_e(z,\lambda)a_e(w,\lambda+2\eta) &+ \alpha(z-w,\lambda-2\eta h)b_e(z,\lambda)c_e(w,\lambda+2\eta)\\
=\beta(z-w,\lambda)d_e(w,\lambda)a_e(z,\lambda+2\eta) &+ \alpha(z-w,-\lambda)c_e(w,\lambda)b_e(z,\lambda-2\eta),\\
\theta(z-w+2\eta)d_e(w,\lambda)b_e(z,\lambda+2\eta) &= \beta(z-w,\lambda-2\eta h)d_e(z,\lambda)b_e(w,\lambda+2\eta)\\
&\quad +\alpha(z-w,\lambda-2\eta h)b_e(z,\lambda)d_e(w,\lambda+2\eta),\\
\theta(z-w+2\eta)a_e(w,\lambda)c_e(z,\lambda-2\eta) &= \alpha(z-w,2\eta h-\lambda)c_e(z,\lambda)a_e(w,\lambda-2\eta)\\
&\quad +\beta(z-w,2\eta h-\lambda)a_e(z,\lambda)c_e(w,\lambda-2\eta),\\
\alpha(z-w,2\eta h-\lambda)c_e(z,\lambda)b_e(w,\lambda-2\eta) &+ \beta(z-w,2\eta h-\lambda)a_e(z,\lambda)d_e(w,\lambda-2\eta)\\
=\alpha(z-w,\lambda)b_e(w,\lambda)c_e(z,\lambda+2\eta) &+ \beta(z-w,-\lambda)a_e(w,\lambda)d_e(z,\lambda-2\eta),
\end{aligned}
$$

$$
\begin{aligned}
\alpha(z-w,2\eta h-\lambda)d_e(z,\lambda)a_e(w,\lambda+2\eta) &+ \beta(z-w,2\eta h-\lambda)b_e(z,\lambda)c_e(w,\lambda+2\eta)\\
=\beta(z-w,\lambda)b_e(w,\lambda)c_e(z,\lambda+2\eta) &+ \alpha(z-w,-\lambda)a_e(w,\lambda)d_e(z,\lambda-2\eta),\\
\theta(z-w+2\eta)b_e(w,\lambda)d_e(z,\lambda+2\eta) &= \alpha(z-w,2\eta h-\lambda)d_e(z,\lambda)b_e(w,\lambda+2\eta)\\
&\quad +\beta(z-w,2\eta h-\lambda)b_e(z,\lambda)d_e(w,\lambda+2\eta),\\
c_e(z,\lambda)c_e(w,\lambda-2\eta) &= c_e(w,\lambda)c_e(z,\lambda-2\eta),\\
\theta(z-w+2\eta)c_e(z,\lambda)d_e(w,\lambda-2\eta) &= \alpha(z-w,\lambda)d_e(w,\lambda)c_e(z,\lambda+2\eta)\\
&\quad +\beta(z-w,-\lambda)c_e(w,\lambda)d_e(z,\lambda-2\eta),\\
\theta(z-w+2\eta)d_e(z,\lambda)c_e(w,\lambda+2\eta) &= \beta(z-w,\lambda)d_e(w,\lambda)c_e(z,\lambda+2\eta)+\\
&\quad \alpha(z-w,-\lambda)c_e(w,\lambda)d_e(z,\lambda-2\eta),\\
d_e(z,\lambda)d_e(w,\lambda+2\eta) &= d_e(w,\lambda)d_e(z,\lambda+2\eta)\,,
\end{aligned}
$$

where $\alpha(z,\lambda) = \frac{\theta(z)\theta(\lambda+2\eta)}{\theta(\lambda)}$ and $\beta(z,\lambda) = -\frac{\theta(z-\lambda)\theta(2\eta)}{\theta(\lambda)}$.

**Proposition 4.9 (Shifted tensor product [25])** *Let two representations of the elliptic quantum group $E_{\tau,\eta}(sl_2)$ $(W_1, L_{1,e}(z,\lambda))$ and $(W_2, L_{2,e}(z,\lambda))$ be given.*
*A new representation is given by the following tensor product of the two representations:* $(W_1 \otimes W_2, L_{1,e}^{(01)}(z,\lambda-2\eta h^{(2)})L_{2,e}^{(02)}(z,\lambda))$.

**Remark:**
The explicit $L$-operator of the tensor product is given by

$$\begin{aligned}
a^{(12)}_{1\otimes 2,e}(z,\lambda) &= a^{(1)}_{1,e}(z,\lambda-2\eta h^{(2)})a^{(2)}_{2,e}(z,\lambda)+b^{(1)}_{1,e}(z,\lambda-2\eta h^{(2)})c^{(2)}_{2,e}(z,\lambda),\\
b^{(12)}_{1\otimes 2,e}(z,\lambda) &= a^{(1)}_{1,e}(z,\lambda-2\eta h^{(2)})b^{(2)}_{2,e}(z,\lambda)+b^{(1)}_{1,e}(z,\lambda-2\eta h^{(2)})d^{(2)}_{2,e}(z,\lambda),\\
c^{(12)}_{1\otimes 2,e}(z,\lambda) &= c^{(1)}_{1,e}(z,\lambda-2\eta h^{(2)})a^{(2)}_{2,e}(z,\lambda)+d^{(1)}_{1,e}(z,\lambda-2\eta h^{(2)})c^{(2)}_{2,e}(z,\lambda),\\
d^{(12)}_{1\otimes 2,e}(z,\lambda) &= c^{(1)}_{1,e}(z,\lambda-2\eta h^{(2)})b^{(2)}_{2,e}(z,\lambda)+d^{(1)}_{1,e}(z,\lambda-2\eta h^{(2)})d^{(2)}_{2,e}(z,\lambda).
\end{aligned}$$

**Proposition 4.10 (Examples [25])**

*a)* *$(W=V, L_e=R_e(z-z_0,\lambda))$, where $z_0\in\mathbb{C}$, is called the fundamental representation of $E_{\tau,\eta}(sl_2)$.*

*b)* *Let $V_\Lambda$ be an infinite dimensional complex vector space with basis $e_k, k\in\mathbb{N}$. We define an action of $h$ by $f(h)e_k=f(\Lambda-2k)e_k$ for $f(h)\in End(V_\Lambda)$.*
*The pair $(W=V_\Lambda, L=L_{\Lambda,e}(z-z_0))$ is a representation of $E_{\tau,\eta}(sl_2)$, called the evaluation Verma module $V_{\Lambda,e}(z_0)$ of $E_{\tau,\eta}(sl_2)$. $L_{\Lambda,e}(z-z_0,\lambda)$ is defined as follows in terms of*
*$a_{\Lambda,e}(\lambda,z), b_{\Lambda,e}(\lambda,z), c_{\Lambda,e}(\lambda,z), d_{\Lambda,e}(\lambda,z)$:*

$$\begin{aligned}
a_{\Lambda,e}(\lambda,z-z_0)e_k &= \frac{\theta(z-z_0+(\Lambda+1-2k)\eta)\theta(\lambda+2k\eta)}{\theta(\lambda)}e_k,\\
b_{\Lambda,e}(\lambda,z-z_0)e_k &= \frac{\theta(-\lambda+z-z_0+(\Lambda-2k-1)\eta)\theta(2\eta)}{\theta(\lambda)}e_{k+1},\\
c_{\Lambda,e}(\lambda,z-z_0)e_k &= \frac{\theta(\lambda+z-z_0-\eta-(\Lambda-2k)\eta)\theta(2k\eta)}{\theta(\lambda)}e_{k-1},\\
d_{\Lambda,e}(\lambda,z-z_0)e_k &= \frac{\theta(z_0-z+(-\Lambda+2k)\eta+\eta)\theta(\lambda-2(\Lambda-k)\eta)}{\theta(\lambda)}e_k.
\end{aligned}$$

*c)* *If $\Lambda=n+(m+l\tau)/2\eta, l,m,n\in\mathbb{N}$, the representation $(V_\Lambda, L_{\Lambda,e})$ has a finite dimensional irreducible quotient module of dimension $n+1$. This representation will be denoted $W_{\Lambda,e}(z_0)$.*

*d)* *If $\Lambda=1$, this finite dimensional quotient module is isomorphic to the fundamental representation mentioned above.*

**Remark:**
The notion of a representation of $E_{\tau,\eta}(sl_2)$ may be further generalized to the notion of a functional representation of $E_{\tau,\eta}(sl_2)$.
To be able to understand its definition, we first need to define a suitable space of functions.

**Definition 4.11** ($\mathcal{F}_n^\lambda$)
$\mathcal{F}_n^\lambda=\{f(x_1,\dots,x_n,\lambda):\mathbb{C}^{n+1}\to\mathbb{C}\,|\,f$ *holomorphic in* $x_i$ *for* $i=1,\dots,n$ *and* $f$ *meromorphic in* $\lambda\}$.

**Definition 4.12 (Functional Representation [25])** *Let $\mathcal{F}_1^\lambda$ be the complex vector space of all complex valued functions meromorphic in $\lambda\in\mathbb{C}$ and holomorphic in $\mu\in\mathbb{C}$ (instead of $x_1$).*

*A functional representation of* $E_{\tau,\eta}(sl_2)$ *is a pair* $(W, L_e)$ *where* $W \subseteq \mathcal{F}_1^\lambda$ *and* $L = L_e(z, \mu, \lambda) = L_e(z, \lambda)$ *is a function holomorphic in* $z, \mu \in \mathbb{C}$ *and meromorphic in* $\lambda \in \mathbb{C}$*: It acts as a difference operator on* $V \otimes W$*, commutes with* $h \otimes 1 + 1 \otimes h$ *and obeys the elliptical* $RLL$*-relations.*
*The operator* $h$*, the weight, acts by multiplication by the continuous variable* $\mu \in \mathbb{C}$ *:* $h\, v(\lambda, \mu) = \mu\, v(\lambda, \mu)$*, where* $v(\lambda, \mu) \in W$*.*

**Proposition 4.13 (Examples [25])**

a) $(W = \mathcal{F}_1^\lambda, L^{\mathcal{F}}_{\Lambda,e}(z - z_0))$ *defines a functional representation of* $E_{\tau,\eta}(sl_2)$ *depending on two parameters* $\Lambda, z_0 \in \mathbb{C}$*. It is called the universal evaluation module* $V^{\mathcal{F}}_{\Lambda,e}(z_0)$*.* $L^{\mathcal{F}}_{\Lambda,e}(z - z_0)$ *is defined as follows*

$$
\begin{aligned}
(\bar{a}_{\Lambda,e}(z,\lambda,h)v)(z,\lambda,\mu) &= \frac{\theta(z - z_0 + \mu\eta + \eta)\theta(\lambda - (\mu - \Lambda)\eta)}{\theta(\lambda)} v(\lambda - 2\eta, \mu), \\
(\bar{b}_{\Lambda,e}(z,\lambda,h)v)(z,\lambda,\mu) &= -\frac{\theta(-\lambda + z - z_0 + \mu\eta - \eta)\theta((\mu + \Lambda)\eta)}{\theta(\lambda)} \times \\
\times v(\lambda + 2\eta, \mu - 2), & \\
(\bar{c}_{\Lambda,e}(z,\lambda,h)v)(z,\lambda,\mu) &= \frac{\theta(\lambda + z - z_0 - \mu\eta - \eta)\theta((\Lambda - \mu)\eta)}{\theta(\lambda)} \times \\
\times v(\lambda - 2\eta, \mu + 2), & \\
(\bar{d}_{\Lambda,e}(z,\lambda,h)v)(z,\lambda,\mu) &= \frac{\theta(z - z_0 - \mu\eta + \eta)\theta(\lambda - (\mu + \Lambda)\eta)}{\theta(\lambda)} v(\lambda + 2\eta, \mu),
\end{aligned}
$$

*where* $v(\lambda, \mu) \in \mathcal{F}$*.*

b) *If we restrict* $\mathcal{F}_1^\lambda$ *to* $\mathcal{F}_R = \{v \in \mathcal{F} \,|\, v = v(\mu),\ \mu \in \{\Lambda - 2k | k \in \mathbb{Z}\}\} \subset \mathcal{F}_1$ *and set* $v(\lambda, \Lambda - 2k) = e_k$*,* $e_k$ *defining the basis of an infinite dimensional complex vector space, the functional representation* $(\mathcal{F}_R, L^{\mathcal{F}_R}_{\Lambda,e}(z - z_0))$ *the is the evaluation Verma module of* $E_{\tau,\eta}(sl_2)$ $V_{\Lambda,e}(z_0)$*.*
*The L-operator* $L^{\mathcal{F}_R}_{\Lambda,e}(z - z_0)$ *looks the same as the operator defined in a), but its action is restricted onto* $\mathcal{F}_R$*.*

**Proof:**

a) The proof mainly consists in checking the $RLL$-relations (cf. [25]).

b) This part is done by comparison.

**Remark:**
Since the entries of the functional L-operator act as difference operators on the elements $v(\mu, \lambda) \in W \subseteq \mathcal{F}_1^\lambda$, we can write them down this way. The set of entries of the functional L-operator written down as difference operators plus the operator $h$ are called the operator algebra of the functional representation.
Since any representation of $E_{\tau,\eta}(sl_2)$ can also be conceived as a suitably restricted functional representation, we see that this way we also obtain the operator algebra of a given representation of $E_{\tau,\eta}(sl_2)$.

**Definition 4.14 (Operator Algebra [25])**

*a) Let us suppose a functional representation* $(W \subseteq \mathcal{F}_1^\lambda, L_e^{\mathcal{F}}(z,\lambda))$ *as given by*
$\bar{a}_e(z,\lambda)v(\mu,\lambda) = a_e(z,\lambda,h)(T_\lambda^{-2\eta}v(\mu,\lambda))$, $\bar{b}_e(z,\lambda)v(\mu,\lambda) = b_e(z,\lambda,h)(T_\lambda^{+2\eta}v(\mu,\lambda))$,
$\bar{c}_e(z,\lambda)v(\mu,\lambda) = c_e(z,\lambda,h)(T_\lambda^{-2\eta}v(\mu,\lambda))$, $\bar{d}_e(z,\lambda)v(\mu,\lambda) = d_e(z,\lambda,h)(T_\lambda^{+2\eta}v(\mu,\lambda))$,
*where every operator is an element of End* $(W)$, *and* $hv(\mu,\lambda) = \mu v(\mu,\lambda)$, *where the* $o_e(z,\lambda,h)$, $o = a,b,c,d$ *are difference operators in the weight* $h$ *whose coefficients are functions meromorphic in the complex variable* $\lambda$ *and holomorphic in all* $h$ *and* $z \in \mathbb{C}$.
*Then its operator algebra is the algebra generated by the operators*
$h, \bar{a}_e(\lambda,z), \bar{b}_e(\lambda,z), \bar{c}_e(\lambda,z), \bar{d}_e(\lambda,z) \in End(W)$.

*b) If we have a functional representation involving several complex continuous weights* $\mu_1,\dots,\mu_n \in \mathbb{C}$ *given in terms of the functional L-operator* $L_e(z,h^{(1)},\dots,h^{(n)},\lambda)$, *each of whose entries* $\bar{o}_e(z,h^{(1)},\dots,h^{(n)},\lambda)$, $o = a,b,c,d$, *acts as a difference operator on a (sub-)space (of)* $\mathcal{F}_n^\lambda$ *of all functions meromorphic in the complex variable* $\lambda$ *and holomorphic in the complex variables* $\mu_1,\dots,\mu_n$ *and* $z$, *its operator algebra is generated by the operators* $\bar{o}_e(z,h^{(1)},\dots,h^{(n)},\lambda) = o_e(z,h^{(1)},\dots,h^{(n)},\lambda)T_\lambda^{+2\eta\mu}$, *where* $\mu \in \{-1,1\}$, *for* $o = a,b,c,d$ *– where* $o_e(z,h^{(1)},\dots,h^{(n)},\lambda)$ *are to be difference operators in the* $\mu_i$ *whose coefficients are functions meromorphic in* $\lambda$ *and holomorphic in all other variables – and the operators* $h^{(i)}$, $i = 1,\dots,n$, *with* $h^{(i)}v(\lambda,\mu_1,\dots,\mu_n) = \mu_i v(\lambda,\mu_1,\dots,\mu_n)$.

**Proposition 4.15 (Quantum determinant [25])**

*a) The following element of the operator algebra is a central element:*

$$\overline{Det}_e(z,\lambda) = Det_e(z,\lambda) =$$
$$\frac{\theta(\lambda)}{\theta(\lambda-2\eta h)}\left(\bar{a}_e(z-2\eta)\bar{d}_e(z) - \bar{c}_e(z-2\eta)\bar{b}_e(z)\right). \tag{55}$$

*It is denoted the quantum determinant.* $\overline{Det}_e(z,\lambda) \in End\ (W)$.

*b) Let* $(W_1, L_{1,e}(z,\lambda))$ *and* $(W_2, L_{2,e}(z,\lambda))$ *two finite dimensional irreducible representations of* $E_{\tau,\eta}(sl_2)$ *with quantum determinant* $Det_{1,e}(z,\lambda) = Det^f_{1,e}(z,\lambda)\mathbb{I}_{W_1}$ *and* $Det_{2,e}(z,\lambda) = Det^f_{2,e}(z,\lambda)\mathbb{I}_{W_1}$ *respectively, where* $\mathbb{I}_{W_i}$ *denotes the identity matrix on* $W_i$ *and* $Det^f_{i,e}(z,\lambda)$ *is a function not depending on the weights of the correponding representation for* $i = 1,2$.
*Then the quantum determinant of* $(W_1 \otimes W_2, L^{(01)}_{1,e}(z,\lambda-2\eta h_2)L^{(02)}_{2,e}(z,\lambda))$ *is given by* $Det_e(z,\lambda) = Det^f_{1,e}(z,\lambda)\ Det^f_{2,e}(z,\lambda)\mathbb{I}_{W_1\otimes W_2}$, *where* $\mathbb{I}_{W_1\otimes W_2}$ *denotes the identity matrix on* $W_1 \otimes W_2$.

**Proof:**
This is shown by commuting the quantum determinant with all generators of the operator algebra of the corresponding (functional) representation $E_{\tau,\eta}(sl_2)$.

### 4.2.3 Highest weight representations

**Remark:**
To deal with the Gaudin eigenvalue problem in the differential case, we needed the notion of a highest weight representation of $sl_2$. A similar notion can be defined for the elliptic quantum group $E_{\tau,\eta}(sl_2)$.

**Definition 4.16 (Highest weight)** *A representation $(W, L_e(z,\lambda))$ of the elliptic quantum group $E_{\tau,\eta}(sl_2)$ is a highest weight representation if it has the following properties: $W$ contains a nontrivial element $v_{h.w.} \in W$ such that*

$$\begin{aligned} c_e(z,\lambda)v_{h.w.} &= 0 \text{ for all } z,\lambda, \quad f(h)v_{h.w.} = f(\omega)v_{h.w.}, \\ a_e(z,\lambda)v_{h.w.} &= \Delta^+_{e,h.w.}(z,\lambda)v_{h.w.}, \quad d_e(z,\lambda)v_{h.w.} = \Delta^-_{e,h.w.}(z,\lambda)v_{h.w.} \end{aligned} \tag{56}$$

*for some $\omega \in \mathbb{C}$ and $\Delta^+_{e,h.w.}(z,\lambda), \Delta^-_{e,h.w.}(z,\lambda) \in$ End $(W)$.*
*The triple $(\omega, \Delta^-_{e,h.w.}(z,\lambda), \Delta^+_{e,h.w.}(z,\lambda))$ is called the highest weight of the highest weight representation $(W, L_e(z,\lambda))$.*

**Remark:**
The previous notion can be generalized to functional highest weight representations of $E_{\tau,\eta}(sl_2)$. The corresponding highest weight triple structurally stays the same, whereas $v_{h.w.} = v_{h.w.}(\mu,\lambda) \in W \subseteq \mathcal{F}_1^\lambda$, $v_{h.w.} \not\equiv 0$.

**Proposition 4.17 (Examples)**

a) *The representation $V_{\Lambda,e}(z_0)$ is a highest weight representation with highest weight $(\Lambda, \Delta^+_{e,h.w.}(z,\lambda) = \theta(z - z_0 + \Lambda\eta + \eta), \Delta^-_{e,h.w.}(z,\lambda) = \theta(z - z_0 - \Lambda\eta + \eta)\frac{\theta(\lambda-2\Lambda\eta)}{\theta(\lambda)})$.*

b) *The representation $V^{\mathcal{F}}_{\Lambda,e}(z_0)$ is a highest weight representation with highest weight $(\Lambda, \Delta^+_{e,h.w.}(z,\lambda) = \theta(z - z_0 + \Lambda\eta + \eta), \Delta^-_{e,h.w.}(z,\lambda) = \theta(z - z_0 - \Lambda\eta + \eta)\frac{\theta(\lambda-2\Lambda\eta)}{\theta(\lambda)})$.*

c) *For $\Lambda \in \mathbb{N}$ the representation $W_{\Lambda,e}(z_0)$ is a finite dimensional irreducible highest weight representation with highest weight $(\Lambda, \Delta^+_{e,h.w.}(z,\lambda) = \theta(z - z_0 + \Lambda\eta + \eta), \Delta^-_{e,h.w.}(z,\lambda) = \theta(z - z_0 - \Lambda\eta + \eta)\frac{\theta(\lambda-2\Lambda\eta)}{\theta(\lambda)})$.*

**Proof:**
The proposition is proven the following way

a) We choose $v_{h.w.} = e_0 \in V_\Lambda$. Then $c_{\Lambda,e}(z)v_{h.w.} = 0$. The highest weight triple is obtained by checking $f(h), a_{\Lambda,e}(z), d_{\Lambda,e}(z)$ on $v_{h.w.}$.

b) We choose $v_{h.w.} = v(\Lambda,\lambda) = \tilde{v}(\lambda)\,\delta_{\Lambda,\mu} \in \mathcal{F}_1^\lambda$. Then $c_{\Lambda,e}(z)v_{h.w.} = 0$. The highest weight triple is obtained by checking $f(h), a_{\Lambda,e}(z), d_{\Lambda,e}(z)$ on $v_{h.w.}$.

c) The calculation is that of the first item remembering the restricted range of definition of $W_{\Lambda,e}(z_0)$. The irreducibility is shown in [25].

**Remark:**
By the next proposition, we see that a tensor product of finite-dimensional highest weight representations is again a highest-weight representation.

**Proposition 4.18 ([25])** *Let* $(z_1, \dots, z_n) \in \mathbb{C}^n - diag$. *Let* $\Lambda_i \in \mathbb{N}$ *for all* $i = 1, \dots, n$. *Then the tensor product of n irreducible, finite dimensional,highest weight representations* $\otimes_{i=1}^n W_{\Lambda_i,e}(z_i)$ *is again a finite dimensional irreducible highest weight representation with highest weight*

$$(\sum_{i=1}^n \Lambda_i, \Delta^+_{e,h.w.}(z,\lambda) = \prod_{i=1}^n \Delta^+_{e,h.w.,i}(z - z_i, \lambda - 2\eta \sum_{j=i+1}^n h^{(j)}),$$
$$\Delta^-_{e,h.w.}(z,\lambda) = \prod_{i=1}^n \Delta^-_{e,h.w.,i}(z - z_i, \lambda - 2\eta \sum_{j=i+1}^n h^{(j)})) \quad (57)$$

*and highest weight vector* $\otimes_{i=1}^n v_{h.w.,i} \in \otimes_{i=1}^n V_{\Lambda_i}$, *where the highest weight functions* $\Delta^+_{r,h.w.,i}(z,\lambda), \Delta^-_{r,h.w.,i}(z,\lambda)$ *are described in Proposition 4.17.*

**Proof ([25]):**
The statement is proven analogously to the statements of Proposition 4.17, the highest weight vector being $\prod_{i=1}^n \delta_{\mu_i,\Lambda_i}$.
**Remark:**
The following theorem is a very important result for our purposes, as will be seen in Proposition 4.44.

**Proposition 4.19 (Isomorphism Theorem [25])** *Two finite dimensional irreducible highest weight representations of* $E_{\tau,\eta}(sl_2)$ *are isomorphic if their highest weights coincide.*

**Proof:**
The proof is given in [25].

## 4.3 The eigenvalue problem corresponding to the SOS eight-vertex model

**Synopsis:**
In this chapter, the emphasis is on introducing two notions: the representation correponding to the eight-vertex SOS model (Definition 4.21) and the family of commuting transfer matrices of the eight-vertex SOS model with antiperiodic boundary conditions (Definition 4.26).
First, we define how to obtain the Boltzmann weights corresponding to the ones of the eight-vertex SOS model [10] by means of the elliptic R-matrix (Definition 4.21 a)). Then we describe the representation that comes along with the SOS model (Definition 4.21 b)), consisting of a tensor product of $n$ shifted fundamental representations of $E_{\tau,\eta}(sl_2)$. After this, we want to define the family of commuting transfer matrices of the SOS model with antiperiodic boundary conditions. To ensure commutativity we have to choose $\lambda_0 = \eta \sum_{i=1}^n h_i$. Note that we can properly define this notion only if $n$ is odd due to possible poles of the transfer matrix if $\lambda_0 = 0$ which can only occur if $n$ is even. We furthermore want the transfer matrix to act on a space of antiperiodic paths $P_n$. So, we first have to define the notion of an antiperiodic path (Definition 4.25 a)), show that the antiperiodic paths thus defined form a basis of a space of antiperiodic paths isomorphic to the space which an SOS transfer matrix naturally acts on and describe the isomorphism (Definition 4.25 b) and c)). We then show that a transfer matrix of the SOS model with antiperiodic boundary conditions is indeed well-defined on the space $P_n$

(Proposition 4.26).
In Definition 4.30, we explicitly pose the common eigenvalue problem of the family of commuting transfer matrices of the SOS model with antiperiodic boundary conditions. The last proposition of the section shows that the family of SOS transfer matrices is indeed commutative.
Note that in what concerns the notions heuristically defined in the first sectio of this section, they coincide with what we define by representation theory in this chapter.

### 4.3.1 The SOS model in terms of the representation theory of $E_{\tau,\eta}(sl_2)$

**Remark:**
First, we want to redefine the basic notions describing the SOS model, i.e. its Boltzmann weights and transfer matrix.
In the first subsection, we show how the Boltzmann weights of the model emerge out of the elliptic R-matrix and how to thus attach a representation of $E_{\tau,\eta}(sl_2)$ to the SOS model. (In a second subsection, we show that the attached representation is a highest weight representation and compute its highest weight.)
Note that we are treating here the simplest case of the SOS model, i.e. the case built out of fundamental representations of $E_{\tau,\eta}(sl_2)$ only. It is called of order $n$ if it involves a tensor product of fundamental representations of $E_{\tau,\eta}(sl_2)$, i.e. $(V^{\otimes n}, R^{(01)}(z, \lambda - 2\eta\sum_{i=2}^{n} h^{(i)})) \otimes \ldots \otimes R^{(0n)}(z,\lambda))$. The fundamental representations involve $n$ weights $h_i$, where $h_i \in \{-1, 1\}$.
As we will see by Proposition 4.31 we also have to discretize the value of the parameter $\lambda$ to be a function of the weights, $\lambda_0 = \eta\sum_{i=1}^{n} h_i$.
If we extended the models to include tensor products of higher (finite) dimensional representations of $E_{\tau,\eta}(sl_2)$ we would obtain the SOS models of [11], as stated in [27].

**Definition 4.20 (Weights)** *We can attach $n$ weights $h_i \in \{-1,1\}$ for $i = 1,\ldots,n$ to a given element of the tensor product basis of $V^{\otimes n}$, $e[\sigma_1]\otimes\ldots\otimes e[\sigma_n]$, where $\sigma_i \in \{-1,1\}$ for $i = 1,\ldots,n$ by setting*

$$h_i\, e[\sigma_1] \otimes \ldots \otimes e[\sigma_i] \otimes \ldots \otimes e[\sigma_n] = \sigma_i\, e[\sigma_1] \otimes \ldots \otimes e[\sigma_i] \otimes \ldots \otimes e[\sigma_n]$$

*for $i = 1,\ldots,n$.*

Now, we can define the basic notions of the SOS model.

**Definition 4.21 (Boltzmann Weights, L-operator)**

*a) The Boltzmann weights $W_e(c,b,a,d|z)$ of the SOS model are defined by the following formula*

$$\begin{aligned} &R_e(z, \lambda = -2\eta d)\, e[\,c-d\,] \otimes e[\,b-c\,] = \\ &\textstyle\sum_a W_e(c,b,a,d\,|\,z)\, e[\,b-a\,] \otimes e[\,a-d\,], \end{aligned} \qquad (58)$$

*where the terms $c-d, b-c, b-a, a-d \in \{-1,1\}$ and $z \in \mathbb{C}$. The expressions $e[a-b]\otimes e[c-d]$ are to be considered as the standard tensor product basis of $V \otimes V$.*

*Written down explicitly the Boltzmann weights read*

$$\begin{aligned}
W_e(d+1,d+2,d+1,d|z) &= \theta(z+2\eta),\\
W_e(d+1,d,d+1,d|z) &= -\frac{\theta(z-2\eta d)\theta(2\eta)}{\theta(2\eta d)},\\
W_e(d+1,d,d-1,d|z) &= \frac{\theta(z)\theta(2\eta(d-1))}{\theta(2\eta d)},\\
W_e(d-1,d,d+1,d|z) &= \frac{\theta(z)\theta(2\eta(d+1))}{\theta(2\eta d)},\\
W_e(d-1,d,d-1,d|z) &= \frac{\theta(2\eta d+z)\theta(2\eta)}{\theta(2\eta d)},\\
W_e(d-1,d-2,d-1,d|z) &= \theta(z+2\eta).
\end{aligned}$$

*The Boltzmann weights thus defined coincide with the ones obtained by Date et al. [10] for the eight-vertex SOS-model as we defined it in Definition 4.1.*

*b) The L-operator of the SOS model is given by*

$$L_{SOS,e}(z,z_1,\dots,z_n,\lambda) = R_e^{(01)}(z-z_1,\lambda-2\eta\sum_{i=2}^{n} h_i)$$
$$R_e^{(02)}(z-z_2,\lambda-2\eta\sum_{i=3}^{n} h_i)\dots R_e^{(0n)}(z-z_n,\lambda), \tag{59}$$

*where* $(z_1,\dots,z_n)\in\mathbb{C}^n - diag.$ $L_{SOS,e}(z,z_1,\dots,z_n,\lambda)\in$ *End* $(V^{\otimes(n+1)})$.

**Remark:**
The Boltzmann weights defined above translate the dynamical Yang–Baxter–relation of Proposition 4.7 into the star–triangle–relation mentioned in the (general) introduction, as was shown in [27]. For what follows the definition of the operator $\bar{L}_{SOS,e}(z,z_1,\dots,z_n,\lambda)$ will be useful. To understand this definition, we must define the following space of functions:

**Definition 4.22** $(M(\mathbb{C},V^{\otimes n}))$

$$M(\mathbb{C},V^{\otimes n}) = \{f:\mathbb{C}\to V^{\otimes n},\ \lambda\to f(\lambda)\,|\,f \text{ meromorphic in } \lambda\}. \tag{60}$$

**Definition 4.23** $(\bar{L}_{SOS,e}(z,z_1,\dots,z_n,\lambda))$

$$\begin{aligned}
\bar{L}_{SOS,e}(z,z_1,\dots,z_n,\lambda) &= \begin{pmatrix} \bar{a}_{SOS,e}(z,z_1,\dots,z_n,\lambda) & \bar{b}_{SOS,e}(z,z_1,\dots,z_n,\lambda)\\ \bar{c}_{SOS,e}(z,z_1,\dots,z_n,\lambda) & \bar{d}_{SOS,e}(z,z_1,\dots,z_n,\lambda)\end{pmatrix}\\
&= L_{SOS,e}(z,z_1,\dots,z_n,\lambda)\begin{pmatrix} T_\lambda^{-2\eta} & 0\\ 0 & T_\lambda^{+2\eta}\end{pmatrix}.
\end{aligned}$$

*The so-defined operator is a matrix on* $V$ *with entries in End* $(M(\mathbb{C},V^{\otimes n}))$.

### 4.3.2 The representation attached to the SOS model as a highest-weight representation

**Remark:**
Here, we show that the representation which we attached to the SOS model in Definition 4.21 b) is a highest weight representation in the sense of Definition 4.16

**Proposition 4.24** *Let $(z_1, \dots, z_n) \in \mathbb{C}^n - diag$. Then the representation of $E_{\tau,\eta}(sl_2)$ attached to the SOS model, namely $(V^{\otimes n}, L_{SOS,e}(z, z_1, \dots, z_n, \lambda) = R_e^{(01)}(z - z_1, \lambda - 2\eta \sum_{i=2}^n h_i) R_e^{(02)}(z - z_2, \lambda - 2\eta \sum_{i=3}^n h_i) \dots R_e^{(0n)}(z - z_n, \lambda))$, is a finite-dimensional irreducible highest weight representation with highest weight*

$$\left( n, \Delta_{SOS}^{+}(z, \lambda) = \prod_{i=1}^{n} \theta(z - z_i + 2\eta), \Delta_{SOS}^{-}(z, \lambda) = \prod_{i=1}^{n} \theta(z - z_i) \frac{\theta(\lambda - 2\eta n)}{\theta(\lambda)} \right). \tag{61}$$

**Proof:**
This proposition is a corollary of Proposition 4.18 with $\Lambda_i \in \mathbb{N}$ set to $\Lambda_i = 1$ for $i = 1, \dots, n$.

### 4.3.3 The family of transfer matrices of the SOS model with antiperiodic boundary conditions

We proceed by showing how the antiperiodic boundary conditions appear. Then, we turn to the definition of an antiperiodic path and show that the space of antiperiodic paths is isomorphic to $V^{\otimes n}$. Then, we define the family of antiperiodic SOS transfer matrices by representation theory and show that it is an endomorphism of the path space. Finally, we pose the common eigenvalue problem.
By the next proposition the definition of the Boltzmann weights will be confirmed if we compare the transfer matrix of the SOS model with antiperiodic boundary conditions as given here with the one given in Definition 4.2.
In order to make sense of the definition of the SOS transfer matrix, we first need to define two special bases of the complex vector space $V^{\otimes n}$.

**Lemma 4.25** $(P_n, I_{CA})$

a) *Let us consider $n+1$ numbers $a_1, \dots, a_{n+1} \in \frac{\mathbb{Z}}{2}$ subject to the conditions $|a_i - a_{i+1}| = 1$, $i = 1, \dots n$. We define the vector*

$$|a_1, \dots, a_{n+1}> = e[a_1 - a_2] \otimes \dots \otimes e[a_n - a_{n+1}] \delta_{\lambda, 2\eta a_{n+1}}.$$

*For every fixed $a_1 \in \frac{\mathbb{Z}}{2}$, this defines a basis of $V^{\otimes n}$.*

b) *Let us consider $n+1$ numbers $a_1, \dots, a_{n+1} \in \frac{\mathbb{Z}}{2}$ subject to the conditions $|a_i - a_{i+1}| = 1$, $i = 1, \dots$ and $a_{n+1} = -a_1$. If we consider the vectors*

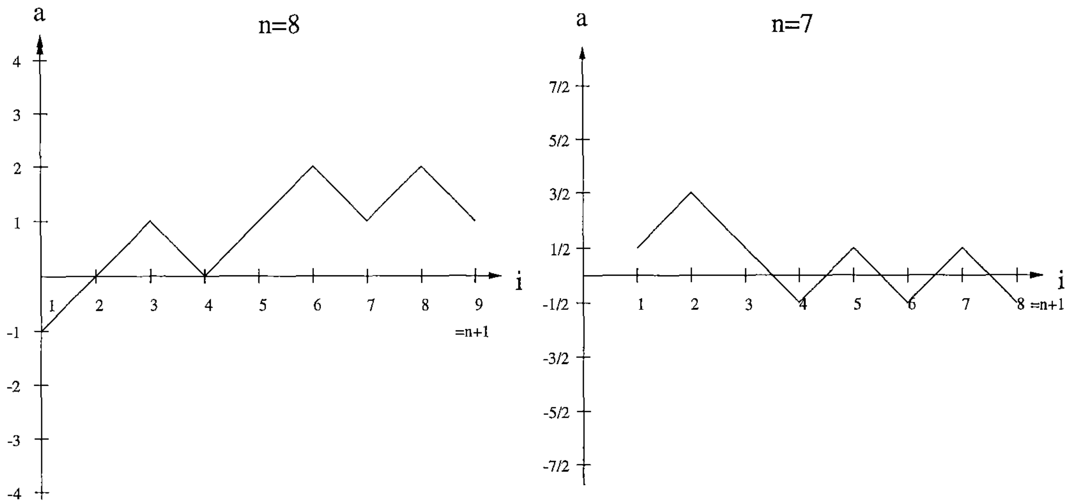
$$|a_1, \dots, a_{n+1}> = e[a_1 - a_2] \otimes \dots \otimes e[a_n + a_1] \delta_{\lambda, -2\eta a_1}$$

*for all possible $a_1, \dots, a_n, a_{n+1}$, we obtain a basis of $V^{\otimes n}$, called the basis of antiperiodic paths $P_n$.*

*c) To each element of the standard tensor product basis* $\{e[\sigma_1] \otimes \ldots \otimes e[\sigma_n] | \sigma_i \in \{-1,1\}$ *for all* $i = 1, \ldots, n\}$ *of* $V^{\otimes n}$*, we can attach an antiperiodic path* $|a_1, \ldots, a_n >$ *by means of the isomorphism* $I_{CA} : V^{\otimes n} \to P_n$, $I_{CA} e[\sigma_1] \otimes \ldots \otimes e[\sigma_n] = |a_1, \ldots, a_{n+1} = -a_1 >$*, where* $a_i = \frac{1}{2}(-\sum_{j=1}^{i-1} \sigma_j + \sum_{j=i}^{n} \sigma_j)$ *for all* $i = 1, \ldots, n+1$.

**Remark:**

It is helpful to consider the vector $|a_1, \ldots, a_{n+1} >$ as a path in $\frac{\mathbb{Z}}{2}$ as is shown below for (even) $n = 8$ and (odd) $n = 7$.




Here, the axis labelled with an $a$ indicates possible values of the $a_i, i = 1, \ldots, n+1$. In case of n being even, $a_1$ and $a_{n+1}$ differ by an even integer or zero. If $n$ is odd, the $a_1$ and $a_{n+1}$ differ by an odd integer.

**Proof:**

First let us remark that $V^{\otimes n}$ is of dimension $2^n$ since $V$ is two-dimensional. We may attach to it a basis $e[\sigma^1] \otimes \ldots \otimes e[\sigma^n]$, where $\sigma^i \in \{-1, 1\}$, $i = 1, \ldots, n$.

a) Let us now start with a fixed $a_1 \in \frac{\mathbb{Z}}{2}$. By the condition $|a_1 - a_2| = 1$, we get two possible values of $a_2 : a_2 = a_1 \pm 1$. From there, we get by $|a_2 - a_3| = 1$ four possible values of $a_3$. Iterating this procedure another $n-1$ times, we see that we have $2^n$ possible different combinations of $(a_1, \ldots, a_{n+1})$ subject to the conditions $|a_i - a_{i+1}| = 1$, $i = 1, \ldots$. Due to these conditions, we can attach to each combination a vector $e[\sigma_1] \otimes \ldots \otimes e[\sigma_n]$, where $\sigma_i \in \{-1, 1\}$, $i = 1, \ldots, n$. This construnction works for every $a_1 \in \frac{\mathbb{Z}}{2}$.

b) Let us again start with some fixed $a_1 \in \frac{\mathbb{Z}}{2}$ and construct the $2^n$ vectors $|a_1, \ldots, a_{n+1} >$ as shown in the first part of the lemma. To implement the additional condition $a_1 = -a_{n+1}$, we have to readjust the value of $a_1$ to $\tilde{a}_1 = a_1 - \frac{a_1 + a_{n+1}}{2}$ and $\tilde{a}_i = a_i - \frac{a_1 + a_{n+1}}{2}$, $i = 2, \ldots, n+1$. This then implies that $\tilde{a}_{n+1} = -\tilde{a}_1$. We can do this for all of the $2^n$ vectors of fixed $a_1$, hence we get a basis of $V^{\otimes n}$. Especially, if $a_1 = a_{n+1}$, we set $a_1 = 0 = \pm a_{n+1}$. Note that to the vector $|\tilde{a}_1, \ldots, \tilde{a}_{n+1} = -\tilde{a}_1 >$ we still relate the same vector $e[\sigma^1] \otimes \ldots e[\sigma^n]$ as to the vector $|a_1, \ldots, a_{n+1} >$, since $a_i - a_{i+1} = \tilde{a}_i - \tilde{a}_{i+1}$, $i = 1, \ldots, n$.

c) The construction of the isomorphism is a corollary of the construction of the basis of antiperiodic paths $P_n$. Let us check that $a_1 = -a_{n+1}$: by definition of the $a_i$, $a_1 = \frac{1}{2}\sum_{j=1}^n \sigma_j$ and $a_{n+1} = -\frac{1}{2}\sum_{j=1}^n \sigma_j$.

**Definition 4.26 (Antiperiodic SOS transfer matrix)** *The transfer matrix of the SOS model with antiperiodic boundary conditions is given by*

$$\begin{aligned}\bar{T}_{SOS,e}(z,z_1,\dots,z_n,\lambda_0) &= \textstyle\sum_{\mu\in\{-1,1\}} tr_{(0)}^{V[\mu]} K^{(0)} \bar{L}_{SOS,e}(z,z_1,\dots,z_n,\lambda_0)\\ &= \bar{b}_{SOS,e}(z,z_1,\dots,z_n,\lambda_0) + \bar{c}_{SOS,e}(z,z_1,\dots,z_n,\lambda_0),\end{aligned} \tag{62}$$

*where $\lambda \in \mathbb{C}$ is fixed to $\lambda_0 = \eta\sum_{i=1}^n h^{(i)}$ and the matrix $K$ is given by*

$$K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

**Proposition 4.27** *The previous definition of $\bar{T}_{SOS,e}(z,z_1,\dots,z_n)$ defines the row–to–row transfer matrix of the eight–vertex SOS model (cf. the figure in the introduction 1.1.2) as we defined it in Definition 4.2, where $|a_1,\dots,a_n,a_{n+1} = -a_1 >$ corresponds to the height configuration of a row with antiperiodic boundary conditions,*

$$\bar{T}_{SOS,e}(z,z_1,\dots,z_n,\lambda_0)|a_1,\dots,-a_1 >=$$

$$\sum_{b_1,\dots,b_{n+1}=-b_1}\left(\prod_{i=1}^n W_e(a_{i+1},a_i,b_i,b_{i+1}|z)\right)|b_1,\dots,b_{n+1}=-b_1 > . \tag{63}$$

*Thus, $\bar{T}_{SOS,e}(z,z_1,\dots,z_n,\lambda_0) \in$ End $(P_n)$. This coincides with Definition 4.2.*

**Proof:**
Let us first note that $\lambda_0 = \eta\sum_{i=1}^n h_i = \sum_{i=1}^n(a_i - a_{i+1})\eta = (a_1 - a_{n+1})\eta = -2\eta a_{n+1}$ by the definition of the weights $h_i$. This agrees with the fixing of $\lambda$ for an antiperiodic path: $\delta_{\lambda,-2\eta a_{n+1}}$.
Furthermore, $e^*[b-a]$ is defined as the dual basis element to $e[b-a]$: $e^*[b-a]e[b-a] = 1$. With the above conventions, the action of the antiperiodic SOS transfer matrix on a path is given by

$$\begin{aligned}&\bar{T}_{SOS,e}(z,z_1,\dots,z_n)|a_1,\dots,a_{n+1} = -a_1 >=\\ &\sum_\mu tr_0^{V[\mu]} R_e^{(01)}(z-z_1,\lambda-2\eta\sum_{i=2}^n h^{(i)})\dots\\ &R^{(0n)}(z-z_n,\lambda)T_\lambda^{-2\eta\mu}\delta_{-2\eta a_{n+1},\lambda}e[a_1-a_2]\otimes\dots\otimes e[a_n-a_{n+1}] =\\ &\sum_\mu e^{(0)*}[\mu]K^{(0)}R_e^{(01)}(z-z_1,\lambda-2\eta\sum_{i=2}^n h^{(i)})\dots R^{(0n)}(z-z_n,\lambda)\\ &e^{(0)}[\mu]\otimes e[a_1-a_2]\otimes\dots\otimes e[a_n-a_{n+1}]\delta_{-2\eta a_{n+1}+2\eta\mu,\lambda} =\\ &\sum_{b_{n+1}} e^{(0)*}[a_{n+1}-b_{n+1}]K^{(0)}R_e^{(01)}(z-z_1,\lambda-2\eta\sum_{i=2}^n h^{(i)})\dots R^{(0n)}(z-z_n,\lambda)\\ &e^{(0)}[a_{n+1}-b_{n+1}]\otimes e[a_1-a_2]\otimes\dots\otimes e[a_n-a_{n+1}]\delta_{-2\eta a_{n+1}+2\eta(a_{n+1}-b_{n+1}),\lambda} =\end{aligned}$$

$$\sum_{b_{n+1},b_n} e^{(0)*}[a_{n+1}-b_{n+1}]K^{(0)}R_e^{(01)}(z-z_1,\lambda-2\eta\sum_{i=2}^{n}h^{(i)})$$

$$W_e(a_{n+1},a_n,b_n,b_{n+1}|z-z_n)e^{(0)}[a_n-b_n]\otimes e[a_1-a_2]\otimes\ldots\otimes e[b_n-b_{n+1}]=$$

$$\ldots=\sum_{b_1,\ldots,b_{n+1}}(\prod_{i=1}^{n}W_e(a_{i+1},a_i,b_i,b_{i+1}|z-z_i))e^{(0)*}[a_{n+1}-b_{n+1}]K^{(0)}\delta_{\lambda,-2\eta b_{n+1}}$$

$$e^{(0)}[a_1-b_1]\otimes e[b_1-b_2]\otimes\ldots e[b_n-b_{n+1}]=$$

$$\sum_{b_1,\ldots,b_{n+1}}(\prod_{i=1}^{n}W_e(a_{i+1},a_i,b_i,b_{i+1}|z-z_i))\delta_{\lambda,-2\eta b_{n+1}}$$

$$e^{(0)*}[a_{n+1}-b_{n+1}]e^{(0)}[b_1-a_1]\otimes e[b_1-b_2]\otimes\ldots e[b_n-b_{n+1}]=$$

$$\sum_{b_1,\ldots,b_{n+1}}(\prod_{i=1}^{n}W_e(a_{i+1},a_i,b_i,b_{i+1}|z-z_i))|b_1,\ldots,b_{n+1}>,$$

if $b_{n+1}=-b_n$ is obeyed.

**Corollary 4.28** *Since $\bar{T}_{SOS,e}(z,z_1,\ldots,z_n,\lambda_0)\in$ End $(P_n)$, by means of the isomorphism $I_{CA}:V^{\otimes n}\to P_n$, we can define*

$$\bar{T}^{\mathbb{C}}_{SOS,e}(z,z_1,\ldots,z_n,\lambda_0)=I_{CA}^{-1}\bar{T}_{SOS,e}(z,z_1,\ldots,z_n,\lambda_0)I_{CA}\in \textit{ End }(V^{\otimes n}).$$

everywhere

**Remark:**

Let us now proceed to the common eigenvalue problem we want to solve. It is of course completely similar to the one obtained in Definition 4.3.

**Definition 4.29** *Let $\epsilon_{SOS}(z)$ be an elliptic polynomial, an element of $\Theta_n(\chi)$ with some character $\chi\in\Gamma^*$, as defined in the Appendix, and $\sum_{a_1,\ldots,a_n}\alpha_{a_1,\ldots,a_n}$ $|a_1,\ldots,a_n,a_{n+1}=-a_1>\in P_n$, where every $\alpha_{a_1\ldots a_n}\in\mathbb{C}$. We are looking for a pair*

$$(\epsilon_{SOS}(z),\sum_{a_1,\ldots,a_n}\alpha_{a_1,\ldots,a_n}|a_1,\ldots,a_n,a_{n+1}=-a_1>)$$

*obeying*

$$\bar{T}_{SOS,e}(z,z_1,\ldots,z_n,\lambda_0)\sum_{a_1,\ldots,a_n}\alpha_{a_1,\ldots,a_n}|a_1,\ldots,a_{n+1}=-a_1> \tag{64}$$

$$=\epsilon_{SOS}(z)\sum_{a_1,\ldots,a_n}\alpha_{a_1,\ldots,a_n}|a_1,\ldots,a_{n+1}=-a_1>,$$

*where $\sum_{a_1,\ldots,a_n}\alpha_{a_1,\ldots,a_n}|a_1,\ldots,a_{n+1}=-a_1>\in P_n$.*

**Remark:**

The periodic case of the SOS model may be treated by algebraic Bethe ansatz as shown by Felder and Varchenko in [27].

To ensure that the solutions thus obtained are indeed common solutions of a commuting family of transfer matrices we need the following lemmas.

**Lemma 4.30** *Let $(W, L_e(z,\lambda))$ be a representation or functional representation of $E_{\tau,\eta}(sl_2)$, with*

$$\bar{L}_e(z,\lambda) = \begin{pmatrix} a_e(z,\lambda) & b_e(z,\lambda) \\ c_e(z,\lambda) & d_e(z,\lambda) \end{pmatrix} \begin{pmatrix} T_\lambda^{-2\eta} & 0 \\ 0 & T_\lambda^{+2\eta} \end{pmatrix}.$$

*Let $\lambda \in \mathbb{C}$ be fixed to $\lambda_0 = \eta h$. Then, the family of transfer matrices defined by*

$$\bar{T}_{L_e}(z,\lambda_0) = \bar{b}_e(z,\lambda_0) + \bar{c}_e(z,\lambda_0) \equiv tr_{(0)} K^{(0)} L_e(z,\lambda_0) \text{ for } z \in \mathbb{C} \tag{65}$$

*is commutative. $K$ is given by $K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.*

**Proof:**
This is shown by using the elliptic RLL relations.
We have to check that

$$[\bar{T}_{L_e}(z,\lambda_0), \bar{T}_{L_e}(w,\lambda_0)] = 0 \text{ for all } z, w \in \mathbb{C}.$$

**Proposition 4.31 (Commutativity of the SOS transfer matrices)** *The transfer matrices of the antiperiodic SOS model commute, i.e.*

$$[\bar{T}_{SOS,e}(z, z_1, \dots, z_n, \lambda_0), \bar{T}_{SOS,e}(w, z_1, \dots, z_n, \lambda_0)] = 0 \quad \text{for all } z, w \in \mathbb{C}, \tag{66}$$

*if $\lambda_0 = \sum_{i=1}^n \eta h_i$.*

## 4.4 Generalizing Sklyanin's results: The auxiliary representation

**Synopsis:**
In this section, we introduce the so-called auxiliary representation of $E_{\tau,\eta}(sl_2)$. It is our main tool in order to achieve the solution of the eigenvalue problem of the SOS transfer matrix with antiperiodic boundary conditions. The origin of this construction will be described in the remark below. First, we give the definition of the auxiliary representation and show that it is indeed a functional representation of the elliptic quantum group. Then, we define the corresponding family of transfer matrices, denoted the auxiliary transfer matrices with antiperiodic boundary conditions or just the antiperiodic auxiliary transfer matrices. At last, we construct an isomorphism $I_{FC}$ that allows to write the auxiliary transfer matrices on "non-functional spaces" and show that the family of transfer matrices is commutative. Thus, it makes sense to treat the common eigenvalue problem of the auxiliary transfer matrices, also denoted the auxiliary eigenvalue problem. In the next section, we will show that the auxiliary representation is isomorphic to the representation attached to the SOS model (cf. Theorem 4.21 b)). This was already suggested by Proposition 4.19. Also, the family of SOS transfer matrices with antiperiodic boundary conditions will be connected by an isomorphism to the family of antiperiodic auxiliary transfer matrices (cf. Corollary 4.51). This will enable us to perform the separation of variables. By the construction of the two isomorphisms, the following section will in a sense complete the one which we just began.
**Remark:**
In [46, 44] Sklyanin achieved the solution of the XXX model as described in [20, 21, 37]

with various, including periodic and antiperiodic, boundary conditions. To achieve this, the main tool he uses is the auxiliary representation of the Yangian $\mathcal{Y}(sl_2)$ (cf. Appendix 1). The auxiliary representation he uses is shown to be isomorphic to the representation of $\mathcal{Y}(sl_2)$ attached to the XXX model and also the corresponding transfer matrices can be connected by the isomorphism. Thus, the common eigenvalue problem of the XXX transfer matrices, i.e. the solution of the XXX model, is connected to solving the common eigenvalue problem of the family of auxiliary transfer matrices. At this point he can use the main advantage of the auxiliary representation: its transfer matrix evaluated at $n$ (ausgezeichnet) points yields a system of $n$ difference equations, the separated equations, which are one-dimensional problems. They yield Bethe ansatz type equations in the course of their solution. By a suitable interpolation, we can out of their solution find the common eigenvalue of the auxiliary transfer matrices and by the knowledge of the isomorphism also of the original eigenvalue problem of the XXX transfer matrices.
Here, we generalize Sklyanin's ideas to the case of $E_{\tau,\eta}(sl_2)$. The succession of the steps will be the following: introduction of the auxiliary representation and the commuting family of auxiliary transfer matrices in this section, construction of the isomorphism relating the original SOS and the auxiliary transfer matrix in section 4.5, describing the original and the auxiliary common eigenvalue problem as well as the system of separated equations emerging from the auxiliary eigenvalue problem in section 4.6.
To be able to define the auxiliary representation, which we describe here as the operator algebra of a functional representation of $E_{\tau,\eta}(sl_2)$, we first have to define the spaces of functions on which this representation will act.

### 4.4.1 Introducing the auxiliary representation

**Definition 4.32** ($\mathcal{F}_n^{\lambda_0^\alpha}, \mathcal{F}_D^\lambda, \mathcal{F}_D^{\lambda_0^\alpha}, \mathcal{F}_n, \mathcal{F}_D$) *Let $(z_1, \dots, z_n) \in (E_\tau)^n -$ diag and $\Lambda_i \in \mathbb{N}$ for $i = 1, \dots, n$.*
*Let $S_i = \{-z_i - \Lambda_i\eta, -z_i - \Lambda_i\eta + 2\eta, \dots, -z_i + \Lambda_i\eta\}$ for $i = 1, \dots, n$, where $S_i \cap S_i = \emptyset$ for $i \neq j$ and all $i, j = 1, \dots, n$.*
*Let $D = \{(x_1, \dots, x_n) | x_i \in S_i$ for all $i = 1, \dots, n\}$. With these definitions understood, we can define the following spaces of functions.*

*a)* $\mathcal{F}_n^{\lambda_0^\alpha} = \{f(x_1, \dots, x_n, \lambda) : \mathbb{C}^{n+1} \to \mathbb{C} \,|\, f \in \mathcal{F}_n^\lambda, \lambda$ *is restricted to* $\lambda_0 = \sum_{i=1}^n x_i + \alpha\}$,

*b)* $\mathcal{F}_n = \{f(x_1, \dots, x_n) : \mathbb{C}^n \to \mathbb{C} \,|\, f$ *holomorphic in* $x_i$ *for* $i = 1, \dots, n\}$,

*c)* $\mathcal{F}_D^\lambda = \mathcal{F} / \{f \in \mathcal{F} \,|\, f(x_1, \dots, x_n) = 0$ *for all* $(x_1, \dots, x_n) \in D\}$,

*d)* $\mathcal{F}_D^{\lambda_0^\alpha} = \mathcal{F}_n^{\lambda_0^\alpha} / \{f \in \mathcal{F}_n^{\lambda_0^\alpha} \,|\, f(x_1, \dots, x_n) = 0$ *for all* $(x_1, \dots, x_n) \in D\}$,

*e)* $\mathcal{F}_D = \mathcal{F}_n / \{f \in \mathcal{F}_n \,|\, f(x_1, \dots, x_n) = 0$ *for all* $(x_1, \dots, x_n) \in D\}$.

**Remark:**
The following operator algebra, denoted the operator algebra of the auxiliary representation, will be our main tool in solving the SOS 8-vertex model by separation of variables.

**Proposition 4.33** *Let $(z_1, \dots, z_n) \in \mathbb{C}^n -$ diag. Let $\Lambda_i \in \mathbb{N}$ for all $i = 1, \dots, n$. Let $S_i$ for $i = 1, \dots, n$ be defined as above and $S_i \cap S_j = \emptyset$ for all $j \neq i$ for $i, j = 1, \dots, n$. Let*

$(x_1, \dots, x_n) \in D$.
*Let*

$$\Delta^-_{n,e}(z) = \prod_{i=1}^{n} \theta(z + z_i + \Lambda_i \eta) \text{ and } \Delta^+_{n,e}(z) = \prod_{i=1}^{n} \theta(z + z_i - \Lambda_i \eta)\,. \tag{67}$$

*Let the difference operators* $T^{\pm}_{x_i} \in End(\mathcal{F}^{\lambda}_{D}), i = 1, \dots, n$, *and* $T^{a}_{\lambda} \in End(\mathcal{F}^{\lambda}_{D})$ *be given by*

$$\Delta^{\pm}_{n,e}(z)(T^{\pm 2\eta}_{x_i} f)(\lambda, x_1, \dots, x_n) = \Delta^{\pm}_{n,e}(z) f(\lambda, x_1, \dots, x_i \pm 2\eta, \dots, x_n), \tag{68}$$

$$(T^{a}_{\lambda} f)(\lambda, x_1, \dots, x_n) = f(\lambda + a, x_1, \dots, x_i, \dots, x_n)\,, \tag{69}$$

*where* $a \in \mathbb{C}$.
*Let* $s = \sum_{i=1}^{n} (-x_i - z_i)$.
*With the above conventions, the operators*

$$\begin{aligned}
\bar{a}_{aux,e}(z, z_1, \dots, z_n, \lambda, \Lambda_1, \dots, \Lambda_n) &= a_{aux,e}(z, z_1, \dots, z_n, \lambda, \Lambda_1, \dots, \Lambda_n) T^{-2\eta}_{\lambda} \\
&= \prod_{i=1}^{n} \theta(z + x_i) \frac{\theta(\lambda + \sum_{i=1}^{n}(-x_i - z_i + \Lambda_i \eta))}{\theta(\lambda)} T^{-2\eta}_{\lambda}, \qquad (70)\\
\bar{b}_{aux,e}(z, z_1, \dots, z_n, \lambda, \Lambda_1, \dots, \Lambda_n) &= b_{aux,e}(z, z_1, \dots, z_n, \lambda, \Lambda_1, \dots, \Lambda_n) T^{+2\eta}_{\lambda} \\
&= \sum_{i=1}^{n} \frac{\theta(\lambda - (z + x_i))}{\theta(\lambda)} \prod_{j \neq i, j=1}^{n} \frac{\theta(z + x_j)}{\theta(x_j - x_i)} \times \\
&\times \Delta^{+}_{n,e}(x_i) T^{+2\eta}_{x_i} T^{+2\eta}_{\lambda}, \qquad (71)\\
\bar{c}_{aux,e}(z, z_1, \dots, z_n, \lambda, \Lambda_1, \dots, \Lambda_n) &= c_{aux,e}(z, z_1, \dots, z_n, \lambda, \Lambda_1, \dots, \Lambda_n) T^{-2\eta}_{\lambda} \\
&= -\sum_{i=1}^{n} \frac{\theta(+\lambda + z + x_i + 2s)}{\theta(\lambda)} \prod_{j \neq i, j=1}^{n} \frac{\theta(z + x_j)}{\theta(x_i - x_j)} \times \\
&\times \Delta^{-}_{n,e}(x_i) T^{-2\eta}_{x_i} T^{-2\eta}_{\lambda}, \qquad (72)\\
\overline{Det}_{aux,e}(z, z_1, \dots, z_n, \lambda, \Lambda_1, \dots, \Lambda_n) &= Det_{aux,e}(z, z_1, \dots, z_n, \Lambda_1, \dots, \Lambda_n) \\
&= \prod_{i=1}^{n} \theta(z - z_i + \Lambda_i \eta) \theta(z - z_i - \Lambda_i \eta - 2\eta)\,. \qquad (73)
\end{aligned}$$

*define an operator algebra obeying the elliptic RLL-relations.*
*Note that the operator*
$\bar{d}_{aux,e}(z, z_1, \dots, z_n, \lambda, \Lambda_1, \dots, \Lambda_n) = d_{aux,e}(z, z_1, \dots, z_n, \lambda, \Lambda_1, \dots, \Lambda_n) T^{+2\eta}_{\lambda}$
*is defined implicitly by Proposition 4.15 and that all appearing operators are elements of End* $(\mathcal{F}^{\lambda}_{D})$.

**Proof:**
Let us check the second of the RLL-relations as an example. It reads:

$$\begin{aligned}
\theta(z - w + 2\eta) a_e(z, \lambda) b_e(z, \lambda - 2\eta) &= \alpha(z - w, \lambda) b_e(w, \lambda) a_e(z, \lambda + 2\eta) \\
&\quad + \beta(z - w, -\lambda) a_e(w, \lambda) b_e(z, \lambda - 2\eta).
\end{aligned}$$

The remaining relations can be checked by similar means, where the relation $d_e(z, \lambda) d_e(w, \lambda + 2\eta) = d_e(w, \lambda) d_e(z, \lambda + 2\eta)$ can be deduced by the preceding relations. Concerning the relation we want to prove, we first can argue – since $\bar{b}_{n,e}(z, \lambda)$ consists of terms $(\bar{b}_{n,e}(z, \lambda))_i = \frac{\theta(z + x_i - \lambda)}{\theta(\lambda)} \prod_{j \neq i, j=1}^{n} \frac{\theta(z + x_j)}{\theta(x_i - x_j)} \Delta^{-}_{n,e}(x_i) T^{+2\eta}_{x_i} T^{+2\eta}_{\lambda}$ acting as a difference operator in the variable $x_i$ only – that it suffices to check the following $n$ sums:

$$\theta(z-w+2\eta)\frac{\theta(z+x_j)\theta(w+x_j-\lambda+2\eta)}{\theta(\lambda)\theta(\lambda-2\eta)} = \frac{\theta(w-\lambda+x_j)\theta(z+x_j+2\eta)}{\theta(\lambda)} \times$$
$$\times\frac{\theta(z-w)}{\theta(\lambda)} + \frac{\theta(2\eta)\theta(z-w+\lambda)}{\theta(\lambda)}\frac{\theta(w+x_j)\theta(z+x_j-\lambda+2\eta)}{\theta(\lambda)\theta(\lambda-2\eta)}$$

for all $j=1,\dots,n$. Note that the missing factors of the operators $a_{n,e}(z,\lambda)$ and $b_{n,e}(z,\lambda)$ are left invariant by the action of the difference operators.

We can formally write the sum as $f_1(z,\lambda) = f_2(z,\lambda) + f_3(z,\lambda)$. Let us first check the transformation properties of each of the summands $f_i(z,\lambda)$, $i=1,2,3$, under $\lambda \to \lambda+1$ and $\lambda \to \lambda+\tau$. If the first transformation is performed, we obtain $f_i(z,\lambda+1) = -f_i(z,\lambda)$ for $i=1,2,3$. The second transformation yields $f_i(z,\lambda+\tau) = e^{-\pi i\tau+2\pi i(\lambda+w+x_j)}f_i(z,\lambda)$ for $i=1,2,3$.

The residues of the sum can be taken at values $\lambda=0$ and $\lambda=2\eta$. The first calclation reduces to

$$\theta(z-w)\theta(w+x_j)\theta(z+x_j+2\eta) - \frac{\theta(2\eta)\theta(z-w)\theta(w+x_j)\theta(z+x_j+2\eta)}{\theta(2\eta)} = 0,$$

whereas the second one yields

$$\frac{\theta(z-w+2\eta)\theta(z+x_j)\theta(w+x_j)}{\theta(2\eta)} = \frac{\theta(z-w+2\eta)\theta(w+x_j)\theta(z+x_j)}{\theta(2\eta)^2}.$$

Both equations are obviously true.

The only zero at $\lambda = w+x_j+2\eta$ leads to

$$0 = -\theta(z-w)\theta(2\eta)\theta(z+x_j+2\eta) + \frac{\theta(2\eta)\theta(z+x_j+2\eta)\theta(w+x_j)\theta(z-w)}{\theta(w+x_j)},$$

which is also a true statement. By the coincidence of transformation behaviour, residues and zeroes of the left and right hand side of the sum, the equality of both sides is proven. This proves the correctness of the indicated RLL relation.

Note that to show that every operator $\bar{o}_{aux,e}(z,z_1,\dots,z_n,\lambda,\Lambda_1,\dots,\Lambda_n)$ for $o=a,b,c$, Det is an element of $\mathcal{F}_D^\lambda$, it suffices to show that by each operator a function belonging to $\mathcal{F}_0^\lambda = \{f \in \mathcal{F}_n^\lambda \,|\, f(x_1,\dots,x_n)=0 \text{ for all } (x_1,\dots,x_n)\in D\}$ is mapped onto another function $F_{\bar{o}_{aux,e}} \in \mathcal{F}_0^\lambda$.

For the operators $\bar{a}_{aux,e}(z,z_1,\dots,z_n,\lambda,\Lambda_1,\dots,\Lambda_n)$ and the quantum determinant this is easily shown since the action of those operators does not change the value of $(x_1,\dots,x_n)\in D$ once it is fixed.

Let us show that also the operators $\bar{b}_{aux,e}(z,z_1,\dots,z_n,\lambda,\Lambda_1,\dots,\Lambda_n)$ and $\bar{c}_{aux,e}(z,z_1,\dots,z_n,\lambda,\Lambda_1,\dots,\Lambda_n)$ define functions $F_{\bar{b}_{aux,e}}(x_1,\dots,x_n,\lambda)$, $F_{\bar{c}_{aux,e}}(x_1,\dots,x_n,\lambda)$ which are elements of $\mathcal{F}_0^\lambda$. To this end, let $f(x_1,\dots,x_n,\lambda) \in \mathcal{F}_0^\lambda$ be a function vanishing at every $(x_1,\dots,x_n)\in D$, i.e. $x_i \in \{-z_i-\Lambda_i\eta, -z_i-\Lambda_i\eta+2\eta,\dots,-z_i+\Lambda_i\eta\}$ for every $i=1,\dots,n$. Then consider the function

$$F_{\bar{b}_{aux,e}}(x_1,\dots,x_n,\lambda) = \bar{b}_{aux,e}(z,z_1,\dots,z_n,\lambda,\Lambda_1,\dots,\Lambda_n)f(x_1,\dots,x_n,\lambda)$$
$$= -\left(\sum_{i=1}^n \frac{\theta(\lambda-(z+x_i))}{\theta(\lambda)} \prod_{j\neq i,j=1}^n \frac{\theta(z+x_j)}{\theta(x_j-x_i)}\times\right.$$
$$\left.\times\Delta_{n,e}^-(x_i)T_{x_i}^{+2\eta}T_\lambda^{+2\eta}f(x_1,\dots,x_i+2\eta,x_n,\lambda+2\eta)\right).$$

The only possible cases to get a function $F_{\bar{b}_{aux,e}}(x_1,\dots,x_n,\lambda)$ that does not vanish at a point in $D$ is when at least one of the $x_i$ has the value $x_i=-z_i+\Lambda_i\eta$, as in this case $F_{\bar{b}_{aux,e}}(x_1,\dots,x_n,\lambda)$ involves $f(x_1,\dots,-z_i+\Lambda_i\eta+2\eta,\dots,x_n)$, hence it has to be evaluated outside $D$, where we do not know about its value. But in this case, $x_i=-z_i+\Lambda_i\eta$, the coefficient $\Delta^-_{n,e}(x_i)=\theta(x_i+z_i-\Lambda_i\eta)\prod_{j=1,j\neq i}^n\theta(x_i+z_j-\Lambda_j\eta)$ vanishes, thus making sure that $F_{\bar{b}_{aux,e}}(x_1,\dots,x_n,\lambda)$ vanishes. Hence, $F_{\bar{b}_{aux,e}}(x_1,\dots,x_n,\lambda)\in\mathcal{F}_0^\lambda$, since it vanishes at every $(x_1,\dots,x_n)\in D$.
The same argument can be applied to the function $F_{\bar{c}_{aux,e}}(x_1,\dots,x_n,\lambda)$, where the critical cases occur when at least one of the $x_i$ takes the value $x_i=-z_i-\Lambda_i\eta$ which is exactly the value at which the coefficient $\Delta^+_{n,e}(x_i)$ vanishes.

**Corollary 4.34** *For $n=1$, the operators $\bar{a}_{aux,e}(z,z_1,\lambda,\Lambda),\bar{b}_{aux,e}(z,z_1,\lambda,\Lambda)$, $\bar{c}_{aux,e}(z,z_1,\lambda,\Lambda),\bar{d}_{aux,e}(z,z_1,\lambda,\Lambda)\in End(\mathcal{F}_D^\lambda)$ as entries of an L-operator are equal to the universal evaluation module $V_{\Lambda_1}^{\mathcal{F}_R}(z_1)$.*

**Proof:**
If for $n=1$ we replace $z+\eta$ by $z$, the operators of Proposition 4.12 reduce to

$$\begin{aligned}
\bar{a}_{aux,e}(z,z_1,\lambda) &= \frac{\theta(z-z_1+x_1)\theta(\lambda-x_1+\Lambda_1\eta)}{\theta(\lambda)}T_\lambda^{-2\eta},\\
\bar{b}_{aux,e}(z,z_1,\lambda) &= \frac{\theta(\lambda-z+z_1-h_1)\theta(h_1+\Lambda_1\eta)}{\theta(\lambda)}T_{x_1}^{+2\eta}T_\lambda^{+2\eta},\\
\bar{c}_{aux,e}(z,z_1,\lambda) &= \frac{\theta(\lambda+z-z_1-h_1)\theta(-h_1+\Lambda_1\eta)}{\theta(\lambda)}T_{x_1}^{-2\eta}T_\lambda^{-2\eta},\\
\bar{d}_{aux,e}(z,z_1,\lambda) &= \frac{\theta(z-z_1-h_1)\theta(\lambda-h_1-\Lambda_1\eta)}{\theta(\lambda)}T_\lambda^{+2\eta},\\
f(\frac{h_1}{\eta})v(\mu) &= f(\frac{1}{\eta}(x_1+z_1))v(\mu)\equiv f(\mu)v(\mu),\ \text{where } v(\mu)\in\mathcal{F}_R.
\end{aligned}$$

If we keep in mind the definition of $h_1$ given by the last equation and compare with the defintion of the universal evalution module, we get the desired result up to a normalizing factor.

### 4.4.2 The auxiliary transfer matrix

**Remark:**
Let us now define the auxiliary transfer matrix and the isomorphism enabling us to compare it to the transfer matrix of the antiperiodic SOS model.

**Definition 4.35 (Auxiliary Transfer Matrix)** *Let*

$$(\mathcal{F}_D^\lambda,\bar{L}_{aux,e}(z,z_1,\dots,z_n,\lambda,\Lambda_1,\dots,\Lambda_n))$$

*be the representation of Proposition 4.12.*
*Let $\lambda\in\mathbb{C}$ be fixed to $\lambda_0=\sum_{i=1}^n(x_i+z_i)$.*
*Then the auxiliary transfer matrix is given by*

$$\begin{aligned}
&\bar{T}_{aux,e}(z,z_1,\dots,z_n,\lambda_0,\Lambda_1,\dots,\Lambda_n)=\\
&tr_{(0)}K^{(0)}\bar{L}_{aux,e}(z,z_1,\dots,z_n,\lambda_0,\Lambda_1,\dots,\Lambda_n)\\
&=(\bar{b}_{aux,e}+\bar{c}_{aux,e})(z,z_1,\dots,z_n,\lambda_0,\Lambda_1,\dots,\Lambda_n).
\end{aligned}$$

**Remark:**
The explicit form of the transfer matrix as an operator on $\mathcal{F}_D^{\lambda_0}$ is given by

$$\bar{T}_{aux,e}(z, z_1, \ldots, z_n, \lambda_0) =$$
$$\sum_{i=1}^{n} \frac{\theta(\lambda_0 - (z + x_i))}{\theta(\lambda_0)} \left( \prod_{i=1}^{n} \theta(z - z_i + \Lambda_i \eta) T_{x_i}^{-2\eta} T_{\lambda}^{-2\eta} + \prod_{i=1}^{n} \theta(z - z_i - \Lambda_i \eta) T_{x_i}^{+2\eta} T_{\lambda}^{+2\eta} \right).$$

The auxiliary representation we will need to treat the eigenvalue problem of the antiperiodic SOS-model is the one with $\Lambda_1 = \ldots = \Lambda_n = 1$. Hence $D = \{(x_1, \ldots, x_n) \,|\, x_i \in \{-z_i - \eta - z_i + \eta\}$ for all $i = 1, \ldots, n\}$.
Let us denote the auxiliary representation in this case
$(\mathcal{F}_D^{\lambda}, \bar{L}_{aux,e}(z, z_1, \ldots, z_n, \lambda, 1, \ldots, 1)) = (\mathcal{F}_D^{\lambda}, \bar{L}_{aux,e}(z, z_1, \ldots, z_n, \lambda))$.
In this case we can establish the following proposition.

**Proposition 4.36** ($I_{FC}$) *Let $\Lambda_1 = \ldots = \Lambda_n = 1$.*

*a) For every $\alpha \in \mathbb{C}$ a basis of $\mathcal{F}_D^{\lambda_0^{\alpha}}$ is given by the $2^n$ equivalence classes of functions $[f_{\sigma_1 \ldots \sigma_n}] = [\prod_{i=1}^{n} \delta_{x_i, -z_i + \sigma_i \eta}]$ where $\sigma_i \in \{-1, 1\}$ for all $i = 1, \ldots, n$. Here, by $[\prod_{i=1}^{n} \delta_{x_i, -z_i + \sigma_i \eta}]$ we mean an equivalence class which has a value 1 at $(-z_1 + \sigma_1 \eta, \ldots, -z_n + \sigma_n \eta)$ and vanishes everywhere else on $D$. Note that since everywhere outside $D$ it has to obey no restrictions, a representatn can be chosen as a meromorphic function of its variables.*

*b) The function $I_{FC} : \mathcal{F}_D^{\lambda_0^{\alpha}} \to V^{\otimes n}, I_{FC}([f_{\sigma_1 \ldots \sigma_n}]) = e[\sigma_1] \otimes \ldots \otimes e[\sigma_n]$ defines an isomorphism between $\mathcal{F}_D^{\lambda_0^{\alpha}}$ and $V^{\otimes n}$ for every $\alpha \in \mathbb{C}$.*

*c) The function $I_{FC} : \mathcal{F}_D^{\lambda} \to M(\mathbb{C}, V^{\otimes n}), I_{FC}([f_{\sigma_1 \ldots \sigma_n}]) = e[\sigma_1] \otimes \ldots \otimes e[\sigma_n]$ defines an isomorphism between $\mathcal{F}_D^{\lambda}$ and $M(\mathbb{C}, V^{\otimes n})$.*

**Proof:**

a) A specific element $[u(x_1, \ldots, x_n)] \in \mathcal{F}_D^{\lambda_0^{\alpha}}$ is characterized by its values on the $2^n$ different points in $D$, since $\lambda_0^{\alpha}$ can be calculated out of these and every equivalence class of functions only depends on the values on $D$. It thus can be written as $[u(x_1, \ldots, x_n, \lambda_0^{\alpha})] = \sum_{\sigma_i \in \{-1,1\}, i=1}^{n} u(-z_1 + \sigma_1 \eta, \ldots, -z_n + \sigma_n \eta, \lambda_0^{\alpha})[f_{\sigma_1 \ldots \sigma_n}]$, where $[f_{\sigma_1 \ldots \sigma_n}]$ is an equivalence class of functions characterized by the property that is one at exactly one point of $D$, namely at $(-z_1 + \sigma_1 \eta, \ldots - z_n + \sigma_n \eta)$ and zero at the other points of $D$. It can be written $[f_{\sigma_1 \ldots \sigma_n}] = [\prod_{i=1}^{n} \delta_{x_i, -z_i + \sigma_i \eta}]$ where $\sigma_i \in \{-1, 1\}$ for all $i = 1, \ldots, n$.

b) This is so since every element of $V^{\otimes n}$ can be written as
$\sum_{i=1, \sigma_i \in \{-1,1\}}^{n} u_{\sigma_1, \ldots, \sigma_n} e[\sigma_1] \otimes \ldots \otimes e[\sigma_n]$
and every element in $\mathcal{F}_D^{\lambda_0^{\alpha}}$ as
$\sum_{i=1, \sigma_i \in \{-1,1\}}^{n} \tilde{u}_{\sigma_1, \ldots, \sigma_n}(\lambda_0^{\alpha})[f_{\sigma_1 \ldots \sigma_n}]$.

c) This is so since every element of $M(\mathbb{C}, V^{\otimes n})$ can be written as
$\sum_{i=1, \sigma_i \in \{-1,1\}}^{n} u_{\sigma_1, \ldots, \sigma_n}(\lambda) e[\sigma_1] \otimes \ldots \otimes e[\sigma_n]$
and every element in $\mathcal{F}_D^{\lambda}$ as
$\sum_{i=1, \sigma_i \in \{-1,1\}}^{n} \tilde{u}_{\sigma_1, \ldots, \sigma_n}(\lambda)[f_{\sigma_1 \ldots \sigma_n}]$.

**Remark:**

a) By means of the above isomorphism $I_{FC}$, we can define

$$\bar{T}^{\mathbb{C}}_{aux,e}(z, z_1, \dots, z_n, \lambda_0) = I_{FC}\bar{T}_{aux,e}(z, z_1, \dots, z_n, \lambda_0)I^{-1}_{FC}.$$

It is an element of End $(V^{\otimes n})$.

b) Let $\mathbb{I}_2$ be the identity matrix on $V$.
By means of the above isomorphism $I_{FC}$, we can define

$$\bar{L}^{\mathbb{C}}_{aux,e}(z, z_1, \dots, z_n, \lambda) = (\mathbb{I}_2 \otimes I_{FC})\bar{L}_{aux,e}(z, z_1, \dots, z_n, \lambda)(\mathbb{I}_2 \otimes I^{-1}_{FC}).$$

It is a matrix on $V$ with entries in End $(M(\mathbb{C}, V^{\otimes n}))$.

**Corollary 4.37** $(M(\mathbb{C}, V), \bar{R}_e(z - z_1, \lambda)) = (M(\mathbb{C}, V), \bar{L}^{\mathbb{C}}_{aux,e}(z, z_1, \lambda))$.

**Proof:**
We can use Corollary 4.34 and 4.36 and the fact that both operators act on $M(\mathbb{C}, V)$.

**Proposition 4.38 (Commutativity of the auxiliary transfer matrices)** *The auxiliary transfer matrices commmute, i.e.*

$$[\bar{T}_{aux,e}(z, z_1, \dots, z_n, \lambda_0), \bar{T}_{aux,e}(z, z_1, \dots, z_n, \lambda_0)] = 0 \qquad (74)$$
$$\textit{for all } z, w \in \mathbb{C},$$

*if* $\lambda_0 = \sum_{i=1}^n (x_i + z_i)$.

**Proof:**
This is proven by using Lemma 4.30.

**Definition 4.39 (Auxiliary eigenvalue problem)** *A solution to the auxiliary eigenvalue problem*

$$\bar{T}_{aux,e}(z, z_1, \dots, z_n, \lambda_0)u(x_1, \dots, x_n) = \epsilon(z)u(x_1, \dots, x_n),$$

*is by definition a pair* $(\epsilon(z), u(x_1, \dots, x_n))$ *where* $u(x_1, \dots, x_n)$ *is a function on the lattice* $D$*, and* $\epsilon(z)$ *is an elliptic polynomial, as defined in Appendix 2.*
*Solving the auxiliary eigenvalue problem is considerably simpler than solving the actual eigenvalue problem of the SOS model, since the auxiliary problem can be split into solving* $n$ *linear difference equations, each of them involving the variable* $x_i$ *- which appeared in the definition of the auxiliary representation - for* $i = 1, \dots, n$ *only.*

### 4.4.3 Establishing the isomorphism between the SOS and the auxiliary representation abstractly

**Remark:**
Keeping in mind Proposition 4.19 which we stated in the section on highest weight representations we are now able to state the isomorphism of the representation attached to the SOS model and the auxiliary representation abstractly. Note that an explicit construction will be given in the next section.

**Lemma 4.40** *Let $(\mathcal{F}_D, L_{aux,e}(z, z_1, \dots, z_n, \Lambda_1, \dots, \Lambda_n))$ be the auxiliary representation defined in Proposition 4.12 with $\Lambda_i \in \mathbb{N}$ for all $i = 1, \dots, n$.*
*Then $(\mathcal{F}_D, L_{aux,e}(z, z_1, \dots, z_n, \Lambda_1, \dots, \Lambda_n))$ is isomorphic to $\otimes_{i=1}^n W_{\Lambda_i,e}(z_i)$.*

**Proof:**
We have to compare the highest weights of both representations, if for the auxiliary representation $z$ is shifted to $z + \eta$.
If we choose the highest weight vector of the auxiliary representation to be the following, $[\prod_{i=1}^n \delta_{x_i, +\Lambda_i\eta - z_i}] \in \mathcal{F}_D$, where by $[\prod_{i=1}^n \delta_{x_i, +\Lambda_i\eta - z_i}]$ we denote the equivalence class of functions in $\mathcal{F}_D$ which is one at $(-z_1 + \Lambda_1\eta, \dots, -z_n + \Lambda_n\eta)$ and zero everywhere else on $D$, we obtain the same triple as the one of $\otimes_{i=1}^n W_{\Lambda_i,e}(z_i)$
**Remark:**
The following proposition is important as it abstractly establishes the isomorphism between the two representations we will need for the separation of variables. However, we are not going to need it in the sequel, since the isomorphism between the two representations will be constructed explicitly.

**Proposition 4.41** *If we set $\Lambda_i = 1$ for $i = 1, \dots, n$ in the auxiliary representation, the representation attached to the SOS model in Proposition 4. , namely $(V^{\otimes n}, \otimes_{i=1}^n R_e^{(0i)}(z - z_i, \lambda - 2\eta \sum_{j=i+1}^n h^{(j)}))$, and the corresponding auxiliary representation, which was called $(\mathcal{F}_D, L_{aux,e}(z, z_1, \dots, z_n, \lambda))$ before, are isomorphic.*

**Proof:**
The proposition is a corollary of Lemma 4.40.

## 4.5 The isomorphism establishing separation of variables for the SOS model

**Synopsis:**
In this section, we develop the isomorphism between the auxiliary representation with $\Lambda_i = 1$ for $i = 1, \dots, n$ and the representation of the SOS model. This isomorphism was already motivated by our discussion of highest weight representations of $E_{\tau,\eta}(sl_2)$ in the first section of this chapter.
Since the isomorphism relies on an inductive procedure, we first show how to relate an auxiliary representation of $n$ weights to a (shifted) tensor product of a fundamental representation and an auxiliary representation of $n-1$ weights (Proposition 4.43). We then show how to relate the representation of the SOS model and the auxiliary representation with $\Lambda_i = 1$ for $i = 1, \dots, n$ by using the isomorphism shown in Proposition 4.43 (Theorem 4.44). Then we show that we can use the isomorphism shown in Theorem 4.44 to relate the family of transfer matrices of the SOS model with antiperiodic boundary conditions and the family of auxiliary transfer matrices. This is a nontrivial proof since, even if the isomorphism of Theorem 4.44 is valid for every $\lambda \in \mathbb{C}$, it is not evident that the restriction to $\lambda_0 = \eta \sum_{i=1}^n h_i$ is left invariant by it. This is shown by the lemmas preceding Corollary 4.51.
**Remark:**
In the case of the elliptic Gaudin magnet we were able to construct an isomorphism between the old weights of the Verma module of the Lie algebra $sl_2$ and the additional parameter $\lambda$, $\{(t_1, \dots, t_n, \lambda) | \sum_{i=1}^n t_i \neq 0\}$, and some new weights $\{y_1, \dots, y_n, C \in \mathcal{S}^n(E_\tau) \times \mathbb{C}^*\}$ as shown in Proposition 2.7.
This isomorphism diagonalized the operator $f_e(z)$, i.e. the new weights corresponded to its zeroes.
In analogy to the differential case, we will now construct an isomorphism between the representation associated to the SOS model $(M(\mathbb{C}, V^{\otimes n}), \bar{L}_{SOS,e}(z, z_1, \dots, z_n, \lambda))$ and the

auxiliary representation $(M(\mathbb{C}, V^{\otimes n}), \bar{L}^{\mathbb{C}}_{aux,e}(z,, z_1, \ldots, z_n, \lambda))$, which was already suggested by Proposition 4.41.
The isomorphism is constructed in a way as to diagonalize $\bar{a}_{SOS,e}(z, z_1, \ldots, z_n, \lambda)$.
**Remark (Auxiliary Representation $\Lambda_1 = \ldots = \Lambda_n = 1$):**
By the definitions of Proposition 4.33, let the operator algebra of the auxiliary functional representation be given by the following operators

$$\begin{aligned}
\bar{a}_{aux,e}(z, z_1, \ldots, z_n, \lambda) &= \prod_{i=1}^{n} \theta(z - z_i + \eta + x_i) \frac{\theta(\lambda + \sum_{j=1}^{n}(-x_j + \eta))}{\theta(\lambda)} T_\lambda^{-2\eta}, \\
\bar{b}_{aux,e}(z, z_1, \ldots, z_n, \lambda) &= -\sum_{i=1}^{n} \frac{\theta(\lambda - (z - z_i + x_i + \eta))}{\theta(\lambda)} \prod_{j \neq i, j=1}^{n} \frac{\theta(z - z_j + x_j + \eta)}{\theta(x_i - z_i + z_j - x_j)} \\
&\quad \prod_{j=1}^{n} \theta(x_i - z_i + z_j - \eta) \, T_\lambda^{+2\eta} T_{x_i}^{+2\eta}, \\
\bar{c}_{aux,e}(z, z_1, \ldots, z_n, \lambda) &= \sum_{i=1}^{n} \frac{\theta(\lambda + (z - z_i + x_i + \eta) - 2\sum_{j=1}^{n} x_j)}{\theta(\lambda)} \\
&\quad \prod_{\substack{j \neq i \\ j=1}}^{n} \frac{\theta(z - z_j + x_j + \eta)}{\theta(x_i - z_i + z_j - x_j)} \prod_{j=1}^{n} \theta(x_i - z_i + z_j + \eta) \, T_\lambda^{-2\eta} T_{x_i}^{-2\eta}, \\
\bar{\mathrm{Det}}_{aux,e}(z, z_1, \ldots, z_n, \lambda) &= \prod_{i=1}^{n} \theta(z - z_i - 2\eta) \, \theta(z - z_i + 2\eta).
\end{aligned}$$

The operator $\bar{d}_{aux,e}(z, z_1, \ldots, z_n, \lambda)$ is defined implicitly by means of the quantum determinant. The values of the weights are restricted to $D = \{(-z_1 + x_1, \ldots, -z_n + x_n) | x_i \in \{-\eta, \eta\},\ i = 1, \ldots, n\}$.
**Remark:**

a) Note that this expression coincides with the one given in Proposition 4.33 with $-z_i + x_i \to x_i, z + \eta \to z$, for all $i = 1, \ldots, n$.

b) Note that one can write $\bar{a}_{aux,e}(z, z_1, \ldots, z_n, \lambda)$, $\bar{b}_{aux,e}(z, z_1, \ldots, z_n, \lambda)$, $\bar{c}_{aux,e}(z, z_1, \ldots, z_n, \lambda)$, $\bar{d}_{aux,e}(z, z_1, \ldots, z_n, \lambda)$ as an L-operator

$$\bar{L}_{aux,e}(z, \lambda) = \begin{pmatrix} \bar{a}_{aux,e}(z, z_1, \ldots, z_n, \lambda) & \bar{b}_{aux,e}(z, z_1, \ldots, z_n, \lambda) \\ \bar{c}_{aux,e}(z, z_1, \ldots, z_n, \lambda) & \bar{d}_{aux,e}(z, z_1, \ldots, z_n, \lambda) \end{pmatrix}.$$

This operator acts as a matrix on $V$ with entries in $M(\mathbb{C}, V^{\otimes n})$.

c) By Proposition 4.41, the representation $(\mathcal{F}_D, L_{aux,e}(z, z_1, \ldots, z_n, \lambda))$ is isomorphic to the tensor product representation $(V^{\otimes n}, \otimes_{i=1}^{n} R^{(0i)}(z - z_i, \lambda - 2\eta \sum_{j=i+1}^{n} h_i))$.

The next two theorems will show this isomorphism explicitly.

**Definition 4.42** $(\mathcal{A}_{n,e}(z_1, \ldots, z_n, \lambda), A_{n,e}(z_1, \ldots, z_n, \lambda))$

a) *Let $\pi_e = \pi_e(z_1 - 2\eta) = \prod_{i=2}^{n} \theta(z_1 - z_i - 2\eta)$. Let $h = \sum_{i=2}^{n} x_i$ and $f(\lambda, h) = \frac{\theta(\lambda - 2\eta h + 2\eta)}{\theta(\lambda + 2\eta)}$. Let $\mathbb{I}_{n-1}$ be the identity matrix on $V^{\otimes(n-1)}$. Then $\mathcal{A}_{n,e}(z_1, \ldots, z_n, \lambda) \in$*

*End* $(V^{\otimes n}) \subset$ *End* $(M(\mathbb{C}, V^{\otimes n}))$ *is given by*

$$\mathcal{A}_{n,e}(z_1, \dots, z_n, \lambda) = \tag{75}$$
$$\begin{pmatrix} \mathbb{I}_{n-1} & 0 \\ a^{-1}(z_1 - 2\eta, \lambda)c(z_1 - 2\eta, \lambda + 2\eta) & \pi_e f(\lambda, h) a^{-1}(z_1 - 2\eta, \lambda) \end{pmatrix},$$

*where we put* $o_{aux,e}(z, z_2, \dots, z_n, \lambda) = o(z, \lambda)$ *for* $o = a, b, c, d$.

*b)* $A_{n,e}(z_1, \dots, z_n, \lambda) \in$ *End* $(V^{\otimes n}) \subset$ *End* $(M(\mathbb{C}, V^{\otimes n}))$ *is given by*

$$A_{n,e}(z_1, \dots, z_n, \lambda) = \mathcal{A}_{2,e}^{(n-1n)}(z_{n-1}, z_n, \lambda) \dots \times$$
$$\mathcal{A}_{i,e}^{(n-i+1\dots n)}(z_{n-i+1}, \dots, z_n, \lambda) \dots \mathcal{A}_{n,e}^{(1\dots n)}(z_1, \dots, z_n, \lambda), \tag{76}$$

*where we used the notation defined at the beginning of the chapter.*

**Proposition 4.43** *Let* $h = \sum_{i=2}^{n} x_i$. *Then*

$$(\mathbb{I}_2 \otimes \mathcal{A}_{n,e}^{-1}) R_e^{(01)}(z - z_1, \lambda - 2h) \bar{L}_{aux,e}^{\mathbb{C}(02\dots n)}(z, z_2, \dots, z_n, \lambda)(\mathbb{I}_2 \otimes \mathcal{A}_{n,e}) =$$
$$\bar{L}_{aux,e}^{\mathbb{C}(01\dots n)}(z, z_1, \dots, z_n, \lambda),$$

*where we put* $\mathcal{A}_{n,e}(z_1, \dots, z_n, \lambda) = \mathcal{A}_{n,e}$.

**Remark:**
If we write down the above identity for each entry of $\bar{L}_{aux,e}^{\mathbb{C}(01\dots n)}(z, \dots, \lambda)$ separately, we get

$$\mathcal{A}_{n,e}^{-1}(a_e^{(1)}(z - z_1, \lambda - 2h)\bar{a}_{aux,e}^{\mathbb{C}(2\dots n)}(z, z_2, \dots, z_n, \lambda)$$
$$+ b_e^{(1)}(z - z_1, \lambda - 2h)\bar{c}_{aux,e}^{\mathbb{C}(2\dots n)}(z, z_2, \dots, z_n, \lambda))\mathcal{A}_{n,e} = \bar{a}_{aux,e}^{\mathbb{C}(1\dots n)}(z, z_1, \dots, z_n, \lambda),$$
$$\mathcal{A}_{n,e}^{-1}(a_e^{(1)}(z - z_1, \lambda - 2h)\bar{b}_{aux,e}^{\mathbb{C}(2\dots n)}(z, z_2, \dots, z_n, \lambda)$$
$$+ b_e^{(1)}(z - z_1, \lambda - 2h)\bar{d}_{aux,e}^{\mathbb{C}(2\dots n)}(z, z_2, \dots, z_n, \lambda))\mathcal{A}_{n,e} = \bar{b}_{aux,e}^{\mathbb{C}(1\dots n)}(z, z_1, \dots, z_n, \lambda),$$
$$\mathcal{A}_{n,e}^{-1}(c_e^{(1)}(z - z_1, \lambda - 2h)\bar{a}_{aux,e}^{\mathbb{C}(2\dots n)}(z, z_2, \dots, z_n, \lambda)$$
$$+ d_e^{(1)}(z - z_1, \lambda - 2h)\bar{c}_{aux,e}^{\mathbb{C}(2\dots n)}(z, z_2, \dots, z_n, \lambda))\mathcal{A}_{n,e} = \bar{c}_{aux,e}^{\mathbb{C}(1\dots n)}(z, z_1, \dots, z_n, \lambda),$$
$$\mathcal{A}_{n,e}^{-1}(c_e^{(1)}(z - z_1, \lambda - 2h)\bar{b}_{aux,e}^{\mathbb{C}(2\dots n)}(z, z_2, \dots, z_n, \lambda)$$
$$+ d_e^{(1)}(z - z_1, \lambda - 2h)\bar{d}_{aux,e}^{\mathbb{C}(2\dots n)}(z, z_2, \dots, z_n, \lambda))\mathcal{A}_{n,e} = \bar{d}_{aux,e}^{\mathbb{C}(1\dots n)}(z, z_1, \dots, z_n, \lambda),$$

where we put $\mathcal{A}_{n,e}(z_1, \dots, z_n, \lambda) = \mathcal{A}_{n,e}$.

**Proof:**
Throughout the proof, we will write $\mathcal{A}_e$ instead of $\mathcal{A}_{n,e}(z_1, \dots, z_n, \lambda)$, since $n$ stays fixed. We have to check that the L-operator of the tensor product $R_e(z - z_1, \lambda - 2\eta h) \otimes \bar{L}_{aux,e}^{\mathbb{C}}(z, z_2, \dots, z_n, \lambda) \equiv \bar{L}_{\otimes}(z, z_1, \dots, z_n, \lambda)$ coincides with the L-operator $(\mathbb{I}_2 \otimes \mathcal{A}_{n,e}) L_{aux,e}^{\mathbb{C}}(z, z_1, \dots, z_n, \lambda)(\mathbb{I}_2 \otimes \mathcal{A}_{n,e})^{-1}$.
For the sake of simplicity we put $o_{aux,e}(z, z_2, \dots, z_n, \lambda) = o(z, \lambda)$ for $o = a, b, c, d$ and $f(h, \lambda) = \frac{\theta(\lambda - 2\eta h + 2\eta)}{\theta(\lambda + 2\eta)}$. Also, if an operator $o(z, \lambda)$, e.g. $a_{aux,e}^{\mathbb{C}}(z, z_1, \dots, z_n, \lambda)$, can be split into a part depending on $\lambda$ and one independent of this parameter, let us denote the $\lambda$-independent part by $o_T(z)$.

Let us first check $\bar{a}^{\mathbb{C}}_{\otimes}(z, z_1, \dots, z_n, \lambda)$. First note that with the operator $a^{-1}(z,\lambda)$, defined by $a^{-1}(z,\lambda)T^{+2\eta}_{\lambda}(\bar{a}(z,\lambda)) \equiv \mathbb{I}_{n-1}$, the inverse of $\mathcal{A}_{n,e}$ reads

$$\mathcal{A}^{-1}_{n,e} = \begin{pmatrix} \mathbb{I}_{n-1} & 0 \\ -\pi_e^{-1} f(\lambda,h)^{-1} c(z_1 - 2\eta, \lambda + 2\eta) & \pi_e^{-1} f(\lambda,h)^{-1} a(z_1 - 2\eta, \lambda+2\eta) \end{pmatrix}.$$

We then obtain

$$\mathcal{A}^{-1}_{n,e}\bar{a}^{\mathbb{C}}_{\otimes}(z, z_1, \dots, z_n, \lambda)\mathcal{A}_{n,e} =$$
$$\begin{pmatrix} 1_{n-1} & 0 \\ -\pi_e^{-1} f(\lambda,h)^{-1} c(z_1 - 2\eta, \lambda + 2\eta) & \pi_e^{-1} f(\lambda,h)^{-1} a(z_1 - 2\eta, \lambda+2\eta) \end{pmatrix} \times$$
$$\begin{pmatrix} \theta(z - z_1 + 2\eta)a(z,\lambda) & 0 \\ -\frac{\theta(2\eta)\theta(z-z_1-\lambda+2\eta h)}{\theta(\lambda-2\eta h)} c(z,\lambda) & \frac{\theta(z-z_1)\theta(\lambda-2\eta h+2\eta)}{\theta(\lambda-2\eta h)} a(z,\lambda) \end{pmatrix} T^{-2\eta}_{\lambda} \times$$
$$\begin{pmatrix} 1_{n-1} & 0 \\ a^{-1}(z_1 - 2\eta, \lambda)c(z_1 - 2\eta, \lambda+2\eta) & \pi_e f(\lambda,h) a^{-1}(z_1 - 2\eta, \lambda) \end{pmatrix} =$$
$$\begin{pmatrix} 1_{n-1} & 0 \\ -\pi_e^{-1} f(\lambda,h)^{-1} c(z_1 - 2\eta, \lambda + 2\eta) & \pi_e^{-1} f(\lambda,h)^{-1} a(z_1 - 2\eta, \lambda+2\eta) \end{pmatrix} \times$$
$$\begin{pmatrix} \theta(z - z_1 + 2\eta)a(z,\lambda) & 0 \\ \tilde{a}_{21}(z) & \pi_e f(\lambda - 2\eta, h)\frac{\theta(z-z_1)\theta(\lambda-2\eta h+2\eta)}{\theta(\lambda-2\eta h)} a(z,\lambda)a^{-1}(z_1 - 2\eta, \lambda - 2\eta) \end{pmatrix} T^{-2\eta}_{\lambda} =$$
$$\begin{pmatrix} \theta(z - z_1 + 2\eta)a(z,\lambda) & 0 \\ a_{21}(z) & \frac{f(\lambda-2\eta,h)}{f(\lambda,h)}\frac{\theta(z-z_1)\theta(\lambda-2\eta h+2\eta)}{\theta(\lambda-2\eta h)} a(z_1 - 2\eta, \lambda+2\eta)a(z,\lambda)a^{-1}(z_1 - 2\eta, \lambda - 2\eta) \end{pmatrix} T^{-2\eta}_{\lambda}$$

with

$$\begin{aligned} \tilde{a}_{21}(z) &= -\frac{\theta(2\eta)\theta(z-z_1-\lambda+2\eta h)}{\theta(\lambda-2\eta h)} c(z,\lambda) + \\ & \frac{\theta(z-z_1)\theta(\lambda-2\eta h+2\eta)}{\theta(\lambda-2\eta h)} a(z,\lambda)a^{-1}(z_1 - 2\eta, \lambda - 2\eta)c(z_1 - 2\eta, \lambda) \end{aligned}$$

and

$$\begin{aligned} a_{21}(z) = (\pi_e f(\lambda,h))^{-1} \Big(&-\theta(z - z_1 + 2\eta)c(z_1 - 2\eta, \lambda + 2\eta)a(z,\lambda) \\ &-\frac{\theta(2\eta)\theta(z-z_1-\lambda+2\eta h)}{\theta(\lambda-2\eta h)} a(z_1 - 2\eta, \lambda+2\eta)c(z,\lambda) + \\ \frac{\theta(z-z_1)\theta(\lambda-2\eta h+2\eta)}{\theta(\lambda-2\eta h)} & a(z_1 - 2\eta, \lambda + 2\eta)a(z,\lambda)a^{-1}(z_1 - 2\eta, \lambda - 2\eta)c(z_1 - 2\eta, \lambda)\Big) \end{aligned}$$

which by the first RLL relation and the ninth relation, where $z' = z_1 - 2\eta, w' = z$ and $\lambda' = \lambda + 2\eta$, can be simplified to

$$a_{21}(z) = 0.$$

We still have to compare the following entry of the matrix

$$\begin{aligned} a_{22}(z) &= \frac{f(\lambda-2\eta,h)}{f(\lambda,h)}\frac{\theta(z-z_1)\theta(\lambda-2\eta h+2\eta)}{\theta(\lambda-2\eta h)} a(z_1 - 2\eta, \lambda+2\eta)a(z,\lambda)a^{-1}(z_1 - 2\eta, \lambda - 2\eta) = \\ & \frac{\theta(\lambda+2\eta)\theta(\lambda-2\eta h)}{\theta(\lambda-2\eta h+2\eta)\theta(\lambda)}\frac{\theta(z-z_1)\theta(\lambda-2\eta h+2\eta)}{\theta(\lambda-2\eta h)} a_T(z)\frac{\theta(\lambda - \eta\sum_{j=2}^n x_j + \sum_{j=2}^n \Lambda_j\eta + 2\eta)}{\theta(\lambda+2\eta)} = \\ & \frac{\theta(z-z_1)\theta(\lambda - \sum_{j=1}^n x_j + \sum_{j=1}^n \Lambda_j \eta)}{\theta(\lambda)} a_T(z), \end{aligned}$$

since $x_1 = -\eta$ and $\Lambda_1 = 1$.
If we rewrite this as a matrix, we get

$$\mathcal{A}_e^{-1} a_{\otimes}^{\mathbb{C}}(z, z_1, \dots, z_n\lambda)\mathcal{A}_e =$$
$$\begin{pmatrix} \theta(z-z_1)a_T(z)\frac{\theta(\lambda-\sum_{j=1}^n x_j+\sum_{j=1}^n \Lambda_j\eta)}{\theta(\lambda)} & 0 \\ 0 & \theta(z-z_1)a_T(z)\frac{\theta(\lambda-\sum_{j=1}^n x_j+\sum_{j=1}^n \Lambda_j\eta)}{\theta(\lambda)} \end{pmatrix} T_\lambda^{-2\eta}.$$

If we compare this to the operator $a_{aux,e}^{\mathbb{C}}(z, z_1, \dots, z_n, \lambda)$, we notice that both operators coincide.
Let us next calculate the conjugation of the operator $\bar{b}_{\otimes}^{\mathbb{C}}(z, z_1, \dots, z_n, \lambda) = a_1(z, \lambda - 2h) \otimes \bar{b}_{aux,e}^{\mathbb{C}}(z, z_2, \dots, z_n, \lambda) + b_1(z, \lambda - 2h) \otimes \bar{d}_{aux,e}^{\mathbb{C}}(z, z_2, \dots, z_n, \lambda)$:

$$\mathcal{A}_{n,e}^{-1}\bar{b}_{\otimes}^{\mathbb{C}}(z, z_1, \dots, z_n, \lambda)\mathcal{A}_{n,e} =$$
$$\begin{pmatrix} 1 & 0 \\ -(\pi_e f(\lambda,h)^{-1})c(z_1-2\eta,\lambda+2\eta) & (\pi_e f(\lambda,h)^{-1}a(z_1-2\eta,\lambda+2\eta) \end{pmatrix} \times$$
$$\begin{pmatrix} \theta(z-z_1+2\eta)b(z,\lambda) & 0 \\ -\frac{\theta(2\eta)\theta(z-z_1-\lambda+2\eta h)}{\theta(\lambda-2\eta h)}d(z,\lambda) & \frac{\theta(z-z_1)\theta(\lambda-2\eta h+2\eta)}{\theta(\lambda-2\eta h)}b(z,\lambda) \end{pmatrix} T_\lambda^{+2\eta} \times$$
$$\begin{pmatrix} 1 & 0 \\ a^{-1}(z_1-2\eta,\lambda)c(z_1-2\eta,\lambda+2\eta) & \pi_e f(\lambda,h)a^{-1}(z_1-2\eta,\lambda) \end{pmatrix} =$$
$$\begin{pmatrix} 1 & 0 \\ -(\pi_e f(\lambda,h)^{-1})c(z_1-2\eta,\lambda+2\eta) & (\pi_e f(\lambda,h)^{-1}a(z_1-2\eta,\lambda+2\eta) \end{pmatrix} \times$$
$$\begin{pmatrix} \theta(z-z_1+2\eta)b(z,\lambda) & 0 \\ \tilde{b}_{21}(z) & \pi_e f(\lambda+2\eta,h+2\eta)\frac{\theta(z-z_1)\theta(\lambda-2\eta h+2\eta)}{\theta(\lambda-2\eta h)}b(z,\lambda)a^{-1}(z_1-2\eta,\lambda+2\eta) \end{pmatrix} T_\lambda^{+2\eta} =$$
$$\begin{pmatrix} \theta(z-z_1+2\eta)b(z) & 0 \\ b_{21}(z) & \frac{f(\lambda+2\eta,h+2\eta)}{f(\lambda,h)}\frac{\theta(z-z_1)\theta(\lambda-2\eta h+2\eta)}{\theta(\lambda-2\eta h)}a(z_1-2\eta,\lambda+2\eta)b(z,\lambda)a^{-1}(z_1-2\eta,\lambda+2\eta) \end{pmatrix} T_\lambda^{+2\eta}$$

with

$$\begin{aligned} \tilde{b}_{21}(z) &= -\frac{\theta(2\eta)\theta(z-z_1-\lambda+2\eta h)}{\theta(\lambda-2\eta h)}d(z,\lambda) \\ &\quad +\frac{\theta(z-z_1)\theta(\lambda-2\eta h+2\eta)}{\theta(\lambda-2\eta h)}b(z,\lambda)a^{-1}(z_1-2\eta,\lambda+2\eta)c(z_1-2\eta,\lambda+4\eta), \end{aligned}$$

and

$$\begin{aligned} b_{21}(z) = \pi_e^{-1}(f(\lambda,h))^{-1}\Big(&-\theta(z-z_1+2\eta)c(z_1-2\eta,\lambda+2\eta)b(z,\lambda) \\ &-\frac{\theta(2\eta)\theta(z-z_1-\lambda+2\eta h)}{\theta(\lambda-2\eta h)}a(z_1-2\eta,\lambda+2\eta)d(z,\lambda) \\ &+\frac{\theta(z-z_1)\theta(\lambda-2\eta h+2\eta)}{\theta(\lambda-2\eta h)}a(z_1-2\eta,\lambda+2\eta)b(z,\lambda)a^{-1}(z_1-2\eta,\lambda+2\eta)c(z_1-2\eta,\lambda+4\eta)\Big). \end{aligned}$$

This can be simplified as follows

$$\begin{aligned} b_{21}(z) = \pi_e^{-1}(f(\lambda,h))^{-1}\frac{\theta(\lambda-2\eta h+2\eta)}{\theta(\lambda-2\eta h)}\Big(&-\frac{\theta(z-z_1+2\eta)\theta(\lambda-2\eta h)}{\theta(\lambda-2\eta h+2\eta)}c(z_1-2\eta,\lambda+2\eta)b(z,\lambda) \\ &-\frac{\theta(2\eta)\theta(z-z_1+2\eta-(\lambda-2\eta h+2\eta))}{\theta(\lambda-2\eta h+2\eta)}a(z_1-2\eta,\lambda+2\eta)d(z,\lambda) \\ +\Big(\frac{\theta(z-z_1+2\eta)\theta(\lambda+4\eta)}{\theta(\lambda+2\eta)}b(z,\lambda+2\eta)a(z_1-2\eta,\lambda+4\eta) &- \frac{\theta(z_1-z+\lambda)\theta(2\eta)}{\theta(\lambda+2\eta)}a(z,\lambda+2\eta)b(z_1-2\eta,\lambda)\Big) \\ &a^{-1}(z_1-2\eta,\lambda+2\eta)c(z_1-2\eta,\lambda+4\eta)\Big), \end{aligned}$$

where we used the second RLL relation with $z' = z_1 - 2\eta, w' = z, \lambda' = \lambda + 2\eta$,

$$= \pi_e^{-1}(f(\lambda,h))^{-1}\frac{\theta(\lambda - 2\eta h + 2\eta)}{\theta(\lambda - 2\eta h)}\frac{\theta(z - z_1 - \lambda)}{\theta(\lambda + 2\eta)}\theta(2\eta)\left(-a(z,\lambda+2\eta)d(z_1 - 2\eta,\lambda)\right.$$
$$\left. + a(z,\lambda+2\eta)b(z_1-2\eta,\lambda)a^{-1}(z_1-2\eta,\lambda+2\eta)c(z_1-2\eta,\lambda+4\eta)\right),$$

where we used the tenth RLL relation with $z' = z_1 - 2\eta, w' = z, \lambda' = \lambda + 2\eta$,

$$= \pi_e^{-1}(f(\lambda,h))^{-1}\frac{\theta(\lambda - 2\eta h + 2\eta)}{\theta(\lambda - 2\eta h)}\frac{\theta(z - z_1 - \lambda)}{\theta(\lambda + 2\eta)}\theta(2\eta)a(z,\lambda+2\eta)$$
$$(-d(z_1-2\eta,\lambda)a(z_1,\lambda+2\eta) + b(z_1-2\eta,\lambda)c(z_1,\lambda+2\eta))\, a^{-1}(z_1,\lambda),$$

where we used the fifth RLL relation with $z' = z_1 - 2\eta, w' = z_1 - 2\eta, \lambda = \lambda + 4\eta$,

$$= -\pi_e^{-1}\frac{\theta(\lambda - 2\eta h + 2\eta)}{\theta(\lambda - 2\eta h)}(f(\lambda,h))^{-1}\frac{\theta(z - z_1 - \lambda)\theta(2\eta)}{\theta(\lambda + 2\eta)}\frac{\theta(\lambda - 2\eta h)}{\theta(\lambda)}\times$$
$$\times Det^{\mathbb{C}}_{aux,e}(z,z_2,\ldots,z_n)a(z,\lambda+2\eta)a^{-1}(z_1,\lambda),$$

where we used Proposition 6.5,

$$= -\frac{\theta(\lambda - z + z_1)\theta(-2\eta)}{\theta(\lambda)}a_T(z)a_T^{-1}(z_1)\prod_{j=2}^{n}\theta(z_1 - z_j + 2\eta),$$

where we used the definition of $f(\lambda,h), Det^{\mathbb{C}}_{aux,e}(z,z_2,\ldots,z_n)$ and $\pi_e$. If we compare this to the term of $b_{aux,e}(z,z_1,\ldots,z_n,\lambda) = I_{FC}b^{\mathbb{C}}_{aux,e}(z,z_1,\ldots,z_n,\lambda)$
$I_{FC}^{-1}$ which is proportional to $T_{x_1}^{+2\eta}T_{\lambda}^{+2\eta}$ and take into account that $x_1 = -\eta$, we perceive that $b_{21}(z) = (b^{\mathbb{C}}_{aux,e})_1(z,z_1,\ldots,z_n,\lambda)$, where $(b^{\mathbb{C}}_{aux,e})_i(z,z_1,\ldots,z_n,\lambda)$ is the $i$th summand of the operator $b^{\mathbb{C}}_{aux,e}(z,z_1,\ldots,z_n,\lambda)$.
Let us now check $b_{11}(z)$ and the corresponding term of $b^{\mathbb{C}}_{aux,e}(z,z_1,\ldots,z_n,\lambda)$.

$$b_{11}(z) = \theta(z - z_1 + 2\eta)b(z,\lambda),$$

whereas the corresponding expression of $b^{\mathbb{C}}_{aux,e}(z,z_1,\ldots,z_n,\lambda)$ is given by

$$I_{FC}^{-1}(b^{\mathbb{C}}_{aux,e})_{11}(z,z_1,\ldots,z_n)I_{FC} = \sum_{i=1}^{n}\frac{\theta(\lambda - z + z_i)}{\theta(\lambda)}\prod_{j\neq i,j=2}^{n}\left(\frac{\theta(z - z_j + x_j + \eta)}{\theta(x_i - z_i + z_j - x_j)}\right)\times$$
$$\prod_{j=2}^{n}\theta(x_i - z_i + z_j - \eta)\left(\frac{\theta(z - z_1 + x_1 + \eta)\theta(x_i - z_i + z_1 - \eta)}{\theta(x_i - z_i + z_1 - x_1)}\right)|_{x_1=\eta}T_{\lambda}^{+2\eta}T_{x_i}^{+2\eta} =$$
$$\sum_{i=1}^{n}\frac{\theta(\lambda - z + z_i)}{\theta(\lambda)}\prod_{j\neq i,j=2}^{n}\left(\frac{\theta(z - z_j + x_j + \eta)}{\theta(x_i - z_i + z_j - x_j)}\right)\times$$
$$\times\prod_{j=2}^{n}\theta(x_i - z_i + z_j - \eta)\frac{\theta(z - z_1 + 2\eta)\theta(x_i - z_i + z_1 - \eta)}{\theta(x_i - z_i + z_1 - \eta)}T_{\lambda}^{+2\eta}T_{x_i}^{+2\eta} = \theta(z - z_1 + 2\eta)I_{FC}^{-1}b(z,\lambda)I_{FC}$$
$$= I_{FC}^{-1}\theta(z - z_1 + 2\eta)b(z,\lambda)I_{FC}.$$

Let us finally check $b_{22}(z)$ which is to coincide with

$$I_{FC}^{-1}(b^{\mathbb{C}}_{aux,e})_{22}(z,z_1,\dots,z_n)I_{FC}=\sum_{i=1}^{n}\frac{\theta(\lambda-z+z_i)}{\theta(\lambda)}\prod_{j\neq i,j=2}^{n}\left(\frac{\theta(z-z_j+x_j+\eta)}{\theta(x_i-z_i+z_j-x_j)}\right)\times$$
$$\prod_{j=2}^{n}\theta(x_i-z_i+z_j-\eta)\left(\frac{\theta(z-z_1+x_1+\eta)\theta(x_i-z_i+z_1-\eta)}{\theta(x_i-z_i+z_1-x_1)}\right)|_{x_1=-\eta}T_{\lambda}^{+2\eta}T_{x_i}^{+2\eta}=$$
$$\sum_{i=1}^{n}\frac{\theta(\lambda-z+z_i)}{\theta(\lambda)}\prod_{j\neq i,j=2}^{n}\left(\frac{\theta(z-z_j+x_j+\eta)}{\theta(x_i-z_i+z_j-x_j)}\right)\times$$
$$\prod_{j=2}^{n}\theta(x_i-z_i+z_j-\eta)\left(\frac{\theta(z-z_1)\theta(x_i-z_i+z_1-\eta)}{\theta(x_i-z_i+z_1+\eta)}\right)T_{\lambda}^{+2\eta}T_{x_i}^{+2\eta}.$$

It reads

$$b_{22}(z)=\frac{f(\lambda+2\eta,h+2\eta)}{f(\lambda,h)}\frac{\theta(\lambda-2\eta h+2\eta)}{\theta(\lambda-2\eta h)}a(z_1-2\eta,\lambda+2\eta)b(z,\lambda)\times$$
$$\times a^{-1}(z_1-2\eta,\lambda+2\eta)=\frac{\theta(\lambda+2\eta)}{\theta(\lambda+4\eta)}\theta(z-z_1)\frac{\theta(\lambda+4\eta)}{\theta(\lambda+2\eta-\sum_{j=1}^{n}x_j-2\eta+\sum_{j=2}^{n}\Lambda_j\eta)}\times$$
$$\times\frac{\theta(\lambda-\sum_{j=1}^{n}x_j+\sum_{j=2}^{n}\Lambda_j\eta)}{\theta(\lambda+2\eta)}a_T(z_1-2\eta)b(z,\lambda)a_T^{-1}(z_1-2\eta)=$$
$$I_{FC}(-\theta(z-z_1)\sum_{i=2}^{n}\frac{\theta(\lambda-z+z_i)}{\theta(\lambda)}\left(\prod_{j\neq i,j=2}^{n}\frac{\theta(z-z_j+x_j+\eta)}{\theta(x_i-z_i+z_j-x_j)}\right)\times$$
$$\times\prod_{j=2}^{n}\theta(x_i-z_i+z_j-\eta)\frac{\theta(z_1-2\eta-z_i+x_i+\eta)}{\theta(z_1-2\eta-z_i+x_i+2\eta+\eta)}T_{\lambda}^{+2\eta}T_{x_i}^{+2\eta})I_{FC}^{-1},$$

where we used the definition of $b^{\mathbb{C}}_{aux,e}(z,z_2,\dots,z_n,\lambda)\equiv b(z,\lambda)$. By our calculation $b_{22}(z)=(b^{\mathbb{C}}_{aux,e})_{22}(z,z_1,\dots,z_n,\lambda)$.
Since $(b^{\mathbb{C}}_{aux,e})_{12}(z,z_1,\dots,z_n,\lambda)=0=b_{12}(z)$, we obtain $\bar{b}^{\mathbb{C}}_{aux,e}(z,z_1,\dots,z_n,\lambda)$ $=\mathcal{A}_e^{-1}\bar{b}^{\mathbb{C}}_{\otimes}(z,z_1,\dots,z_n,\lambda)\mathcal{A}_e$.
Finally, we have to check that $\bar{c}^{\mathbb{C}}_{aux,e}(z,z_1,\dots,z_n,\lambda)=\mathcal{A}_e^{-1}\bar{c}^{\mathbb{C}}_{\otimes}(z,z_1,\dots,z_n,\lambda)\mathcal{A}_e$, where

$$c^{\mathbb{C}}_{\otimes}(z,z_1,\dots,z_n,\lambda)=c_1(z,\lambda-2\eta h)\otimes a^{\mathbb{C}}_{aux,e}(z,z_2,\dots,z_n,\lambda)+$$
$$d_1(z,\lambda-2\eta h)\otimes c^{\mathbb{C}}_{aux,e}(z,z_2,\dots,z_n,\lambda)\ :$$
$$\mathcal{A}_e^{-1}c^{\mathbb{C}}_{\otimes}(z,z_1,\dots,z_n,\lambda)\mathcal{A}_e=$$
$$\begin{pmatrix}1 & 0\\ -\pi_e^{-1}(f(\lambda,h))^{-1}c(z_1-2\eta,\lambda+2\eta) & \pi_e^{-1}(f(\lambda,h))^{-1}a(z_1-2\eta,\lambda+2\eta)\end{pmatrix}\times$$
$$\begin{pmatrix}\frac{\theta(z-z_1)\theta(\lambda-2\eta h-2\eta)}{\theta(\lambda-2\eta h)}c(z,\lambda) & \frac{\theta(2\eta)\theta(\lambda-2\eta h+z-z_1)}{\theta(\lambda-2\eta h)}a(z,\lambda)\\ 0 & \theta(z-z_1+2\eta)c(z,\lambda)\end{pmatrix}T_{\lambda}^{-2\eta}\times$$
$$\begin{pmatrix}1 & 0\\ a^{-1}(z_1-2\eta,\lambda)c(z_1-2\eta,\lambda+2\eta) & \pi_e f(\lambda,h)a^{-1}(z_1-2\eta,\lambda)\end{pmatrix}=$$

$$\begin{pmatrix} 1 & 0 \\ -\pi_e^{-1}(f(\lambda,h))^{-1}c(z_1-2\eta,\lambda+2\eta) & \pi_e^{-1}(f(\lambda,h))^{-1}a(z_1-2\eta,\lambda+2\eta) \end{pmatrix} \times$$

$$\begin{pmatrix} \tilde{c}_{11}(z) & \pi_e f(\lambda-2\eta,h)\frac{\theta(2\eta)\theta(\lambda-2\eta h+z-z_1)}{\theta(\lambda-2\eta h)}a(z,\lambda)a^{-1}(z_1-2\eta,\lambda-2\eta) \\ \tilde{c}_{21}(z) & \pi_e f(\lambda-2\eta,h-2\eta)\theta(z-z_1+2\eta)c(z,\lambda)a^{-1}(z_1-2\eta,\lambda-2\eta) \end{pmatrix} T_\lambda^{-2\eta} =$$

$$\begin{pmatrix} c_{11}(z) & c_{12}(z) \\ c_{21}(z) & c_{22}(z) \end{pmatrix} T_\lambda^{-2\eta}$$

with

$$\tilde{c}_{11}(z) = \frac{\theta(z-z_1)\theta(\lambda-2\eta h-2\eta)}{\theta(\lambda-2\eta h)}c(z,\lambda) +$$

$$\frac{\theta(2\eta)\theta(\lambda-2\eta h+z-z_1)}{\theta(\lambda-2\eta h)}a(z,\lambda)a^{-1}(z_1-2\eta,\lambda-2\eta)c(z_1-2\eta,\lambda),$$

$$\tilde{c}_{21}(z) = \theta(z-z_1+2\eta)c(z,\lambda)a^{-1}(z_1-2\eta,\lambda-2\eta)c(z_1-2\eta,\lambda),$$

and

$$c_{11}(z) = \frac{\theta(z-z_1)\theta(\lambda-2\eta h-2\eta)}{\theta(\lambda-2\eta h)}c(z,\lambda) +$$

$$\frac{\theta(2\eta)\theta(\lambda-2\eta h+z-z_1)}{\theta(\lambda-2\eta h)}a(z,\lambda)a^{-1}(z_1-2\eta,\lambda-2\eta)c(z_1-2\eta,\lambda),$$

$$c_{12}(z) = \pi_e f(\lambda-2\eta,h)\frac{\theta(2\eta)\theta(\lambda-2\eta h+z-z_1)}{\theta(\lambda-2\eta h)}a(z,\lambda)a^{-1}(z_1-2\eta,\lambda-2\eta),$$

$$c_{21}(z) = (\pi_e f(\lambda,h))^{-1}\left(-\frac{\theta(z-z_1)\theta(\lambda-2\eta h+2\eta)}{\theta(\lambda-2\eta h+4\eta)}c(z_1-2\eta,\lambda+2\eta)c(z,\lambda)\right.$$

$$-\frac{\theta(2\eta)\theta(\lambda-2\eta h+z-z_1+4\eta)}{\theta(\lambda-2\eta h+4\eta)}c(z_1-2\eta,\lambda+2\eta)a(z,\lambda)a^{-1}(z_1-2\eta,\lambda-2\eta)c(z_1-2\eta,\lambda)$$

$$\left.+\theta(z-z_1+2\eta)a(z_1-2\eta,\lambda+2\eta)c(z,\lambda)a^{-1}(z_1-2\eta,\lambda-2\eta)c(z_1-2\eta,\lambda)\right),$$

$$c_{22}(z) = \frac{f(\lambda-2\eta,h-2\eta)}{f(\lambda,h)}\left(\theta(z-z_1+2\eta)a(z_1-2\eta,\lambda+2\eta)c(z,\lambda)a^{-1}(z_1-2\eta,\lambda-2\eta)\right.$$

$$\left.-\frac{\theta(2\eta)\theta(\lambda-2\eta h+z-z_1+4\eta)}{\theta(\lambda-2\eta h+4\eta)}c(z_1-2\eta,\lambda+2\eta)a(z,\lambda)a^{-1}(z_1-2\eta,\lambda-2\eta)\right).$$

Now let us check that all of the four operators correspond to its counterpart, the operator $(c^{\mathbb{C}}_{aux,e})_{ij}(z,z_1,\dots,z_n,\lambda)$, for $i,j=1,2$.
The simplest calculation is the simplification of $c_{21}(z)$, yielding

$$c_{21}(z) = (\pi_e f(\lambda,h))^{-1}\frac{\theta(\lambda-2\eta h+2\eta)}{\theta(\lambda-2\eta h+4\eta)}\left(-\theta(z-z_1)c(z,\lambda+2\eta)c(z_1-2\eta,\lambda)\right.$$

$$-\frac{\theta(2\eta)\theta(\lambda-2\eta h+z-z_1+4\eta)}{\theta(\lambda-2\eta h+2\eta)}c(z_1-2\eta,\lambda+2\eta)a(z,\lambda)a^{-1}(z_1-2\eta,\lambda-2\eta)c(z_1-2\eta,\lambda)$$

$$\left.+\frac{\theta(z-z_1+2\eta)\theta(\lambda-2\eta h+4\eta)}{\theta(\lambda-2\eta h+2\eta)}a(z_1-2\eta,\lambda+2\eta)c(z,\lambda)a^{-1}(z_1-2\eta,\lambda-2\eta)c(z_1-2\eta,\lambda)\right) =$$

$$(\pi_e f(\lambda,h))^{-1}\frac{\theta(\lambda-2\eta h+2\eta)}{\theta(\lambda-2\eta h+4\eta)}\left(-\theta(z-z_1)c(z,\lambda+2\eta)a(z_1-2\eta,\lambda)-\right.$$

$$\frac{\theta(2\eta)\theta(\lambda-2\eta h+2\eta+z-z_1+2\eta)}{\theta(\lambda-2\eta h+2\eta)}c(z_1-2\eta,\lambda+2\eta)a(z,\lambda)$$

$$\left.+\frac{\theta(z-z_1+2\eta)\theta(\lambda-2\eta h+4\eta)}{\theta(\lambda-2\eta h+2\eta)}a(z_1-2\eta,\lambda+2\eta)c(z,\lambda)\right)a^{-1}(z_1-2\eta,\lambda-2\eta)c(z_1-2\eta,\lambda)=0,$$

where for the first line we use the thirteenth relation and for the last equality we need the fifth relation with $w' = z, z' = z_1 - 2\eta, \lambda' = \lambda + 2\eta$. This coincides with $(c^{\mathbb{C}}_{aux,e})_{21}(z, z_1, \ldots, z_n, \lambda)$.
The operator $I_{FC}(c^{\mathbb{C}}_{aux,e})_{12}(z, z_1, \ldots, z_n, \lambda)I_{FC}^{-1}$ is given by the expression proportional to $T_{x_1}^{-2\eta}$ of $c^{\mathbb{C}}_{aux,e}(z, z_1, \ldots, z_n, \lambda)$ :

$$I_{FC}(c^{\mathbb{C}}_{aux,e})_{12}(z)I_{FC}^{-1} = \frac{\theta(\lambda + z - z_1 - 2\eta h)}{\theta(\lambda)} \prod_{j=2}^{n} \frac{\theta(z - z_j + x_j + \eta)}{\theta(x_1 - z_1 - z_j + x_j)} \theta(2\eta) \prod_{j=2}^{n} \theta(x_1 - z_1 + z_j + \eta) T_{\lambda}^{-2\eta} T_{x_1}^{-2\eta}$$

with $x_1 = \eta$.
Let us compare this to

$$\prod_{j=2}^{n} \theta(z_1 - z_j - 2\eta) \frac{\theta(\lambda - 2\eta h)}{\theta(\lambda)} \frac{\theta(2\eta)\theta(\lambda - 2\eta h + z - z_1)}{\theta(\lambda - 2\eta h)} a_T(z) a_T^{-1}(z_1 - 2\eta) =$$

$$\frac{\theta(\lambda - 2\eta h + z - z_1)}{\theta(\lambda)} \prod_{j=2}^{n} \left(\frac{\theta(z - z_j + x_j + \eta)}{\theta(z_1 - z_j + x_j - \eta)} \theta(z_1 - z_j - 2\eta)\right) \theta(2\eta) =$$

$$\theta(2\eta) \frac{\theta(\lambda - 2\eta h + z - z_1)}{\theta(\lambda)} \prod_{j=2}^{n} \left(\frac{\theta(z - z_j + x_j + \eta)}{\theta(-z_1 + z_j - x_j + x_1)} \theta(x_1 - z_1 + z_j + \eta)\right).$$

This coincides with the coefficient of $I_{FC}(c^{\mathbb{C}}_{aux,e})_{12}(z)I_{FC}^{-1}$.
The operator $(c^{\mathbb{C}}_{aux,e})_{11}(z, z_1, \ldots, z_n, \lambda)$ reads

$$I_{FC}(c^{\mathbb{C}}_{aux,e})_{11}(z)I_{FC}^{-1} = \sum_{i=2}^{n} \frac{\theta(\lambda + z - z_i + x_i + \eta - 2\sum_{j=2}^{n} x_j - 2x_1)}{\theta(\lambda)} \left( \prod_{j \neq i, j=2}^{n} \frac{\theta(z - z_j + x_j + \eta)}{\theta(x_i - z_i + z_j - x_j)} \right.$$
$$\left. \prod_{j=2}^{n} \theta(x_i - z_i + z_j + \eta) \right) \left( \frac{\theta(z - z_1 + x_1 + \eta)}{\theta(x_i - z_i + z_1 - x_1)} \theta(x_i - z_i + z_1 + \eta) \right)_{x_1 = \eta} T_{x_i}^{-2\eta} T_{\lambda}^{-2\eta} =$$
$$\sum_{i=2}^{n} \frac{\theta(\lambda + z - z_i + x_i + \eta - 2\sum_{j=2}^{n} x_j - 2x_1)}{\theta(\lambda)} \left( \prod_{j \neq i, j=2}^{n} \frac{\theta(z - z_j + x_j + \eta)}{\theta(x_i - z_i + z_j - x_j)} \right.$$
$$\left. \prod_{j=2}^{n} \theta(x_i - z_i + z_j + \eta) \right) \left( \frac{\theta(z - z_1 + 2\eta)}{\theta(x_i - z_i + z_1 - \eta)} \theta(x_i - z_i + z_1 + \eta) \right) T_{x_i}^{-2\eta} T_{\lambda}^{-2\eta}.$$

The simplification of $c_{11}(z)$ yields

$$c_{11}(z) = \left( \frac{\theta(z - z_1)\theta(\lambda - 2\eta h - 2\eta)}{\theta(\lambda - 2\eta h)} c(z, \lambda) a(z_1, \lambda - 2\eta) \right.$$
$$\left. + \frac{\theta(2\eta)\theta(\lambda - 2\eta h + z - z_1)}{\theta(\lambda - 2\eta h)} a(z, \lambda) c(z_1, \lambda - 2\eta) \right) a^{-1}(z_1, \lambda - 4\eta)$$

— where we use the fifth relation with $z' = z_1 - 2\eta, w' = z_1, \lambda = \lambda - 2\eta$ —

$$= \theta(z - z_1 + 2\eta) a(z_1, \lambda) c(z, \lambda - 2\eta) a^{-1}(z_1, \lambda - 4\eta)$$

— where we use the ninth relation with $w' = z_1, z' = z, \lambda' = \lambda$ —

$$= \frac{\theta(z - z_1 + 2\eta)\theta(\lambda - 2\eta)\theta(\lambda - \sum_{j=2}^{n} x_j + \sum_{j=2}^{n} \eta)}{\theta(\lambda - 2\eta)\theta(\lambda)\theta(\lambda - 2\eta - \sum_{j=2}^{n} x_j + 2\eta + \sum_{j=2}^{n} \eta)} \sum_{i=2}^{n} \theta(\lambda - 2\eta + z - z_i + x_i + \eta - 2\sum_{j=2}^{n} x_j)$$
$$\left( \prod_{j \neq i, j=2}^{n} \frac{\theta(z - z_j + x_j + \eta)}{\theta(x_i - z_i + z_j - x_j)} \theta(x_i - z_i + z_j + \eta) \right) \frac{\theta(z_1 - z_i + x_i + \eta)}{\theta(z_1 - z_i + x_i - \eta)}.$$

This is equivalent to $(c^{\mathbb{C}}_{aux,e})_{11}(z)$.
The operator $(c^{\mathbb{C}}_{aux,e})_{22}(z)$ reads

$$I_{FC}(c^{\mathbb{C}}_{aux,e})_{22}(z)I_{FC}^{-1} = \sum_{i=2}^{n} \frac{\theta(\lambda + z - z_i + x_i + \eta - 2\sum_{j=2}^{n} x_j - 2x_1)}{\theta(\lambda)} \left( \prod_{j\neq i,j=2}^{n} \frac{\theta(z - z_j + x_j + \eta)}{\theta(x_i - z_i + z_j - x_j)} \right.$$
$$\left. \prod_{j=2}^{n} \theta(x_i - z_i + z_j + \eta) \right) \left( \frac{\theta(z - z_1 + x_1 + \eta)}{\theta(x_i - z_i + z_1 - x_1)} \theta(x_i - z_i + z_1 + \eta) \right)_{x_1 = -\eta} T_{x_i}^{-2\eta} T_{\lambda}^{-2\eta} =$$
$$\sum_{i=2}^{n} \frac{\theta(\lambda + z - z_i + x_i + \eta - 2\sum_{j=2}^{n} x_j + 2\eta)}{\theta(\lambda)} \left( \prod_{j\neq i,j=2}^{n} \frac{\theta(z - z_j + x_j + \eta)}{\theta(x_i - z_i + z_j - x_j)} \right.$$
$$\left. \prod_{j=2}^{n} \theta(x_i - z_i + z_j + \eta) \right) \left( \frac{\theta(z - z_1)\theta(x_i - z_i + z_1 + \eta)}{\theta(x_i - z_i + z_1 + \eta)} \right) T_{x_i}^{-2\eta} T_{\lambda}^{-2\eta} =$$
$$\theta(z - z_1) \frac{\theta(\lambda + 2\eta)}{\theta(\lambda)} c_{aux,e}(z, z_2, \dots, z_n, \lambda + 2\eta).$$

If we use the definition of $f(\lambda, h)$ the operator $c_{22}(z)$ simplifies to

$$\frac{\theta(\lambda + 2\eta)}{\theta(\lambda)} \left( \frac{\theta(z - z_1 + 2\eta)\theta(\lambda - 2\eta h + 4\eta)}{\theta(\lambda - 2\eta h + 2\eta)} a(z_1 - 2\eta, \lambda + 2\eta) c(z, \lambda) \right.$$
$$\left. - \frac{\theta(2\eta)\theta(\lambda - 2\eta h + z - z_1 + 4\eta)}{\theta(\lambda - 2\eta h + 2\eta)} c(z_1 - 2\eta, \lambda + 2\eta) a(z, \lambda) \right) a^{-1}(z_1 - 2\eta, \lambda - 2\eta)$$

— where the fifth relation with $z' = z_1 - 2\eta, w' = z, \lambda' = \lambda + 2\eta$ is used —

$$= \frac{\theta(\lambda + 2\eta)}{\theta(\lambda)} \theta(z - z_1) c(z, \lambda + 2\eta) a(z_1 - 2\eta, \lambda) a^{-1}(z_1 - 2\eta, \lambda - 2\eta)$$
$$= \frac{\theta(\lambda + 2\eta)}{\theta(\lambda)} \theta(z - z_1) c(z, \lambda + 2\eta) = (c^{\mathbb{C}}_{aux,e})_{22}(z, z_1, \dots, z_n, \lambda),$$

what coincides with the coefficient of what was calculated before. Since all possible operators $(c^{\mathbb{C}}_{aux,e})_{ij}(z, z_1, \dots, z_n, \lambda), i, j = 1, 2$, coincide, we conclude that the operator $\bar{c}^{\mathbb{C}}_{aux,e}(z, z_1, \dots, z_n, \lambda) = \mathcal{A}_{n,e}^{-1} \bar{c}^{\mathbb{C}}_{\otimes}(z, z_1, \dots, z_n, \lambda) \mathcal{A}_{n,e}$.
Since the determinant is multiplicative by Lemma 4.15 b) and the determinant determines the operator $\bar{d}_{aux,e}(z, z_1, \dots, z_n, \lambda)$, this concludes the proof.

**Theorem 4.44** *Let $\bar{L}_{SOS,e}(z, z_1, \dots, z_n, \lambda)$ be the operator defined at the beginning of the section. Then*

$$(\mathbb{I}_2 \otimes A_{n,e}(z_1, \dots, z_n, \lambda)^{-1} \bar{L}_{SOS,e}(z, z_1, \dots, z_n, \lambda)(\mathbb{I}_2 \otimes A_{n,e}(z_1, \dots, z_n, \lambda))$$
$$= \bar{L}^{\mathbb{C}}_{aux,e}(z, z_1, \dots, z_n, \lambda). \tag{77}$$

**Proof of the Theorem:**
The proof is by induction using Proposition 4.43 and the fact that $\bar{L}^{\mathbb{C}}_{aux,e}(z, z_1, \lambda) = \bar{R}_e(z - z_1, \lambda)$. We are going to prove the hypothesis for each entry of the L-operator $\bar{L}_{SOS,e}(z, z_1, \dots, z_n, \lambda)$ separately.
The first hypothesis of the induction we have to prove, the case $n = 2$, is given by the

four identities – where we put $\mathcal{A}_{2,e}(z_1,z_2,\lambda)=\mathcal{A}_{2,e}$ –

$$\begin{aligned}
\bar{a}^{\mathbb{C}}_{aux,e}(z,z_1,z_2,\lambda) &= (\mathcal{A}_{2,e}^{-1})^{(12)}\bar{a}_{SOS,e}(z,z_1,z_2,\lambda)(\mathcal{A}_{2,e})^{(12)}\\
&= (\mathcal{A}_{2,e}^{-1})^{(12)}\left(a_e(z-z_1,\lambda-2x_2)\otimes a_e(z-z_2,\lambda)\right.\\
&+ \left. b_e(z-z_1,\lambda-2x_2)\otimes c_e(z-z_2,\lambda)\right)T_\lambda^{-2\eta}(\mathcal{A}_{2,e})^{(12)}\\
&= (\mathcal{A}_{2,e}^{-1})^{(12)}\Big(a_e(z-z_1,\lambda-2x_2)\otimes \bar{a}^{\mathbb{C}}_{aux,e}(z,z_2,\lambda)\\
&+ b_e(z-z_1,\lambda-2x_2)\otimes \bar{c}^{\mathbb{C}}_{aux,e}(z,z_2,\lambda)\Big)(\mathcal{A}_{2,e})^{(12)},\\
\bar{b}^{\mathbb{C}}_{aux,e}(z,z_1,z_2,\lambda) &= (\mathcal{A}_{2,e}^{-1})^{(12)}\bar{b}_{SOS,e}(z,\lambda)(\mathcal{A}_{2,e})^{(12)}\\
&= (\mathcal{A}_{2,e}^{-1})^{(12)}\left(a_e(z-z_1,\lambda-2x_2)\otimes b_e(z-z_2,\lambda)\right.\\
&+ \left. b_e(z-z_1,\lambda-2x_2)\otimes d_e(z-z_2,\lambda)\right)T_\lambda^{+2\eta}(\mathcal{A}_{2,e})^{(12)}\\
&= (\mathcal{A}_{2,e}^{-1})^{(12)}\Big(a_e(z-z_1,\lambda-2x_2)\otimes \bar{b}^{\mathbb{C}}_{aux,e}(z,z_2,\lambda)\\
&+ b_e(z-z_1,\lambda-2x_2)\otimes \bar{d}^{\mathbb{C}}_{aux,e}(z,z_2,\lambda)\Big)(\mathcal{A}_{2,e})^{(12)},\\
\bar{c}^{\mathbb{C}}_{aux,e}(z,z_1,z_2,\lambda) &= (\mathcal{A}_{2,e}^{-1})^{(12)}\bar{c}_{SOS,e}(z,z_1,z_2,\lambda)(\mathcal{A}_{2,e})^{(12)}\\
&= (\mathcal{A}_{2,e}^{-1})^{(12)}\left(c_e(z-z_1,\lambda-2x_2)\otimes a_e(z-z_2,\lambda)\right.\\
&+ \left. d_e(z-z_1,\lambda-2x_2)\otimes c_e(z-z_2,\lambda)\right)T_\lambda^{-2\eta}(\mathcal{A}_{2,e})^{(12)}\\
&= (\mathcal{A}_{2,e}^{-1})^{(12)}\Big(c_e(z-z_1,\lambda-2x_2)\otimes \bar{a}^{\mathbb{C}}_{aux,e}(z,z_2,\lambda)\\
&+ d_e(z-z_1,\lambda-2x_2)\otimes \bar{c}^{\mathbb{C}}_{aux,e}(z,z_2,\lambda)\Big)(\mathcal{A}_{2,e})^{(12)},\\
\bar{d}^{\mathbb{C}}_{aux,e}(z,z_1,z_2,\lambda) &= (\mathcal{A}_{2,e}^{-1})^{(12)}\bar{d}_{SOS,e}(z,z_1,z_2,\lambda)(\mathcal{A}_{2,e})^{(12)}\\
&= (\mathcal{A}_{2,e}^{-1})^{(12)}\left(c_e(z-z_1,\lambda-2x_2)\otimes b_e(z-z_2,\lambda)\right.\\
&+ \left. d_e(z-z_1,\lambda-2x_2)\otimes d_e(z-z_2,\lambda)\right)T_\lambda^{-2\eta}(\mathcal{A}_{2,e})^{(12)}\\
&= (\mathcal{A}_{2,e}^{-1})^{(12)}\Big(c_e(z-z_1,\lambda-2x_2)\otimes \bar{b}^{\mathbb{C}}_{aux,e}(z,z_2,\lambda)\\
&+ d_e(z-z_1,\lambda-2x_2)\otimes \bar{d}^{\mathbb{C}}_{aux,e}(z,z_2,\lambda)\Big)(\mathcal{A}_{2,e})^{(12)},
\end{aligned}$$

where we used the definition of the entries of $L_{SOS,e}(z,z_1,z_2,\lambda)$ by means of the tensor product, the identity of $R^{\mathbb{C}}_{aux,e}(z,z_1,\lambda)$ with $R_e(z-z_1,\lambda)$, the fact that by definition $\mathcal{A}_{2,e}(z_1,z_2,\lambda)=A_{2,e}(z_1,z_2,\lambda)$ and Proposition 4.43.
Let us now assume that the corresponding identities for $n$ are valid, i.e. the following four statements hold true:

$$\begin{aligned}
&(A_{n,e}^{(2\dots(n+1))})^{-1}(z_2,\dots,z_{n+1},\lambda)\bar{o}_{SOS,e}(z,z_2,\dots,z_{n+1},\lambda)\times\\
&\quad\times A_{n,e}^{(2\dots(n+1))}(z_2,\dots,z_{n+1},\lambda)=\bar{o}^{\mathbb{C}}_{aux,e}(z,z_2,\dots,z_{n+1},\lambda),
\end{aligned}$$

for $o=a,b,c,d$.
We now want to prove the following four identities:

$$\begin{aligned}
&(A_{n+1,e}^{(1\dots n+1)})^{-1}(z_1,\dots,z_{n+1},\lambda)\bar{o}_{SOS,e}(z,z_1,z_2,\dots,z_{n+1},\lambda)\times\\
&\quad\times A_{n+1,e}^{(1\dots n+1)}(z_1,\dots,z_{n+1},\lambda)=\bar{o}^{\mathbb{C}}_{aux,e}(z,z_1,\dots,z_{n+1},\lambda),
\end{aligned}$$

for $o = a, b, c, d$.
Let us prove one identity, e.g. the one involving the operators $\bar{b}_{SOS,e}(z, z_1, \dots, z_{n+1}, \lambda)$ and $\bar{b}_e^{aux}(z, z_1, \dots, z_{n+1}, \lambda)$. The other three identities are proven in exactly the same fashion.

$$\bar{b}^{\mathbb{C}}_{aux,e}(z, z_1, \dots, z_{n+1}, \lambda) = (\mathcal{A}^{-1}_{(n+1),e})^{(1\dots(n+1))}$$
$$\left( a_e(z - z_1, \lambda - 2\sum_{j=2}^{n+1} x_j) \otimes \bar{b}^{\mathbb{C}}_{aux,e}(z, z_2, \dots, z_{n+1}, \lambda) + \right.$$
$$\left. b_e(z - z_1,)\lambda - 2\sum_{j=2}^{n+1} x_j) \otimes \bar{d}^{\mathbb{C}}_{aux,e}(z, z_2, \dots, z_{n+1}, \lambda) \right) (\mathcal{A}_{(n+1),e})^{(1\dots(n+1))},$$

what is obtained by Proposition 4.43 and by $\mathcal{A}_{n+1,e}(z_1, \dots, z_{n+1}, \lambda) = \mathcal{A}_{n+1,e}$ and equals by our previous assumptions

$$= (\mathcal{A}^{-1}_{n+1,e})^{(1\dots n+1)} (A^{-1}_{n,e})^{(2\dots n+1)}(z_2, \dots, z_{n+1}, \lambda)$$
$$\left( a_e(z - z_1, \lambda - 2\sum_{j=2}^{n+1} x_j) \otimes \bar{b}^{\mathbb{C}}_{aux,e}(z, z_2, \dots, z_{n+1}, \lambda) + \right.$$
$$\left. b_e(z - z_1, \lambda - 2\sum_{j=2}^{n+1} x_j) \otimes \bar{d}^{\mathbb{C}}_{aux,e}(z, z_2, \dots, z_{n+1}, \lambda) \right)$$
$$A^{(2\dots n+1)}_{n,e}(z_2, \dots, z_{n+1}, \lambda)(\mathcal{A}_{n+1,e})^{(1\dots n+1)} =$$

$$(A^{-1}_{(n+1),e})^{(1\dots(n+1))}(z_1, \dots, z_{n+1}, \lambda)$$
$$\left( a_e(z - z_1, \lambda - 2\sum_{j=2}^{n+1} x_j) \otimes b_{SOS,e}(z, z_2, \dots, z_{n+1}, \lambda) + \right.$$
$$\left. b_e(z - z_1, \lambda - 2\sum_{j=2}^{n+1} x_j) \otimes d_{SOS,e}(z, z_2, \dots, z_{n+1}, \lambda) \right) T^{+2\eta}_{\lambda}$$
$$A^{(1\dots(n+1))}_{(n+1),e}(z_1, \dots, z_{n+1}, \lambda),$$

where we used the definition of the entries of $\bar{L}_{SOS,e}(z, z_2, \dots, z_{n+1}, \lambda)$,

$$= (A^{-1}_{n+1,e})^{(1\dots n+1)}(z_1, \dots, z_{n+1}, \lambda)\bar{b}_{SOS,e}(z, z_1, \dots, z_{n+1}, \lambda)A^{(1\dots n+1)}_{n+1,e}(z_1, \dots, z_{n+1}, \lambda),$$

where we used the definition of the tensor product of $(\bar{L}^{SOS}_{n+1,e})$

$$\bar{L}^{(01\dots n+1)}_{SOS,e}(z, z_1, \dots, z_{n+1}, \lambda) =$$
$$R^{(01)}_e(z - z_1, \lambda - 2\sum_{j=2}^{n+1} x_j)\bar{L}^{(02\dots n+1)}_{SOS,e}(z, z_2, \dots, z_{n+1}, \lambda).$$

Thus, the identity involving $\bar{b}_{SOS,e}(z, z_1, \dots, z_{n+1}, \lambda)$ and $\bar{b}^{\mathbb{C}}_{aux,e}(z, z_1, \dots, z_{n+1}, \lambda)$ is shown. Since the other relations can be shown similarly, this completes the proof.

**Remark:**
The following corollary is very important since it involves the transfer matrix of the SOS model with antiperiodic boundary conditions and connects it to the auxiliary antiperiodic transfer matrix.
We will need it in the next section.

**Corollary 4.45** *For all $\lambda \in \mathbb{C}$ for which the following identity is defined,*

$$(A_{n,e}^{-1})^{(1\dots n)} \left(\bar{b}_{SOS,e}(z, z_1, \dots, z_n, \lambda) + \bar{c}_{SOS,e}(z, z_1, \dots, z_n, \lambda)\right) \times$$
$$\times A_{n,e}^{(1\dots n)} = \left(\bar{b}^{\mathbb{C}}_{aux,e}(z, z_1, \dots, z_n, \lambda) + \bar{c}^{\mathbb{C}}_{aux,e}(z, z_1, \dots, z_n, \lambda)\right)$$

*where* $A_{n,e}^{(1\dots n)}(z_1, \dots, z_n, \lambda) = A_{n,e}^{(1\dots n)}$.

**Proof:**
The Corollary is easily proven while looking at the proof of Theorem 4.44, where we explicitly proved that

$$(A_{n,e}^{-1})^{(1\dots n)}(z_1, \dots, z_n, \lambda)\bar{b}_{SOS,e}(z, z_1, \dots, z_n, \lambda)A_{n,e}^{(1\dots n)}(z_1, \dots, z_n, \lambda) = \bar{b}^{\mathbb{C}}_{aux,e}(z, z_1, \dots, z_n, \lambda)$$

and

$$(A_{n,e}^{-1})^{(1\dots n)}(z_1, \dots, z_n, \lambda)\bar{c}_{SOS,e}(z, z_1, \dots, z_n, \lambda)A_{n,e}^{(1\dots n)}(z_1, \dots, z_n, \lambda) = \bar{c}^{\mathbb{C}}_{aux,e}(z, z_1, \dots, z_n, \lambda).$$

**Remark:**
To show that

**Proposition 4.46**

$$(A_{n,e}^{-1})^{(1\dots n)}\bar{T}_{SOS,e}(z, z_1, \dots, z_n, \lambda_0)A_{n,e}^{(1\dots n)} = \bar{T}^{\mathbb{C}}_{aux,e}(z, z_1, \dots, z_n, \lambda_0) \tag{78}$$

is obeyed for $\lambda_0 = \sum_{i=1}^n x_i$, i.e. for the restriction of $\lambda$ to a function of the set $(x_1, \dots, x_n)$ we need four more lemmas. They show that every operator used in the isomorphism, i.e. $\bar{b}_{aux,e}(z, z_1, \dots, z_n, \lambda), \bar{c}_{aux,e}(z, z_1, \dots, z_n, \lambda)$, $A_n(z_1, \dots, z_n, \lambda)$ and $\mathcal{A}_n(z_1, \dots, z_n, \lambda)$, preserves functions of $\lambda$ while restricted to $\lambda_0$. These functions are either elements of $\mathcal{F}_D^{\lambda_0}$ or $M(\mathbb{C}, V^{\otimes n})|_{\lambda=\lambda_0} \simeq V^{\otimes n}$ and are fixed to a specific value $(\lambda_0^\alpha)^{\mathbb{C}}$ of $\lambda_0^\alpha$.

**Lemma 4.47** *Let $\alpha \in \mathbb{C}$. Let $u(x_1, \dots, x_n, \lambda_0^\alpha) \in \mathcal{F}_D^{\lambda_0^\alpha}$. Then*

$$(\bar{b}_{aux,e}(z, z_1, \dots, z_n, \lambda_0^\alpha)u(x_1, \dots, x_n, \lambda_0^\alpha)) \in \mathcal{F}_D^{\lambda_0^\alpha},$$
$$(\bar{c}_{aux,e}(z, z_1, \dots, z_n, \lambda_0^\alpha)u(x_1, \dots, x_n, \lambda_0^\alpha)) \in \mathcal{F}_D^{\lambda_0^\alpha}.$$

**Proof:**
By definition, $\bar{b}_{aux,e}(z, z_1, \dots, z_n, \lambda_0^\alpha)$ acts on $u(x_1, \dots, x_n, \lambda_0^\alpha)$ as $u(x_1', \dots, x_n', \lambda_0^\alpha + 2\eta)$, where $\sum_{i=1}^n x_i' = \sum_{i=1}^n x_i + 2\eta$. Hence, $\lambda_0^\alpha = \sum_{i=1}^n x_i + 2\eta + \alpha = \sum_{i=1}^n x_i' + \alpha = (\lambda_0^\alpha)'$.
Hence, $(\bar{b}_{aux,e}(z, z_1, \dots, z_n, \lambda_0^\alpha)u(x_1, \dots, x_n, \lambda_0^\alpha)) \in \mathcal{F}_D^{\lambda_0^\alpha}$.
The proof concerning $\bar{c}_{aux,e}(z, z_1, \dots, z_n, \lambda_0^\alpha)$ is analogous, switching $+2\eta$ to $-2\eta$.

**Lemma 4.48** *Let $\alpha \in \mathbb{C}$. Let $\sigma_i \in \{-1,1\}$ for all $i = 1,\dots,n$. Let $\tilde{\lambda}_0^\alpha = \sum_{i=1}^n h_i\eta + \alpha$, i.e. $\tilde{\lambda}_0^\alpha e[\sigma_1] \otimes \ldots \otimes e[\sigma_n] = (\eta\sum_{i=1}^n \sigma_i + \alpha)e[\sigma_1] \otimes \ldots \otimes e[\sigma_n]$.*
*Let $u(\tilde{\lambda}_0^\alpha)e[\sigma_1] \otimes \ldots \otimes e[\sigma_n] \in M(\mathbb{C}, V^{\otimes n})$ with $\tilde{\lambda}_0^\alpha$ acting as $\eta\sum_{i=1}^n \sigma_i + \alpha = (\lambda_0^\alpha)^{\mathbb{C}} \in \mathbb{C}$ Then,*

$$\mathcal{A}_{n,e}(z_1,\dots,z_n,\tilde{\lambda}_0^\alpha)u((\lambda_0^\alpha)^{\mathbb{C}})e[\sigma_1] \otimes \ldots \otimes e[\sigma_n] = \bar{u}((\lambda_0^\alpha)^{\mathbb{C}})e[\sigma_1'] \otimes \ldots \otimes e[\sigma_n'].$$

**Proof:**
Let $\mathcal{A}_{n,e}(z_1,\dots,z_n,\tilde{\lambda}_0^\alpha)$ be given by

$$\mathcal{A}_{n,e}(z_1,\dots,z_n,\tilde{\lambda}_0^\alpha) = \begin{pmatrix} \mathbb{I}_{n-1} & 0 \\ a^{-1}(z_1-2\eta,\tilde{\lambda}_0^\alpha)c(z_1-2\eta,\tilde{\lambda}_0^\alpha+2\eta) & \pi_e f(\tilde{\lambda}_0^\alpha,h)a^{-1}(z_1-2\eta,\tilde{\lambda}_0^\alpha) \end{pmatrix}$$

with the notation of the definition of $\mathcal{A}_{n,e}(z_1,\dots,z_n,\tilde{\lambda}_0^\alpha)$ understood.
We can either choose a vector $u(\tilde{\lambda}_0^\alpha)e[\eta] \otimes e[\sigma_2] \otimes \ldots \otimes e[\sigma_n]$ or $u(\tilde{\lambda}_0^\alpha)e[-\eta] \otimes e[\sigma_2] \otimes \ldots \otimes e[\sigma_n]$. In the first case $\tilde{\lambda}_0^\alpha$ acts as $(\lambda_0^\alpha)^{\mathbb{C}} = \eta + \eta\sum_{i=2}^n \sigma_i$, in the second case as $(\lambda_0^\alpha)^{\mathbb{C}} = -\eta + \eta\sum_{i=2}^n \sigma_i$. Let us see how the operator $\mathcal{A}_{n,e}(z_1,\dots,z_n,\tilde{\lambda}_0^\alpha)$ acts on each of these vectors.
Let us first look at the case $\sigma_1 = -1$ corresponding to $(\lambda_0^\alpha)^{\mathbb{C}} = -\eta + \eta\sum_{i=2}^n \sigma_i$.

$$\begin{pmatrix} \mathbb{I}_{n-1} & 0 \\ a^{-1}(z_1-2\eta,\tilde{\lambda}_0^\alpha)c(z_1-2\eta,\tilde{\lambda}_0^\alpha+2\eta) & \pi_e f(\tilde{\lambda}_0^\alpha,h)a^{-1}(z_1-2\eta,\tilde{\lambda}_0^\alpha) \end{pmatrix} \times \begin{pmatrix} 0 \\ u(\tilde{\lambda}_0^\alpha)e[\sigma_2] \otimes \ldots \otimes e[\sigma_n] \end{pmatrix} = \begin{pmatrix} 0 \\ \pi_e f(\tilde{\lambda}_0^\alpha,h)a^{-1}(z_1-2\eta,\tilde{\lambda}_0^\alpha)u(\tilde{\lambda}_0^\alpha)e[\sigma_2] \otimes \ldots \otimes e[\sigma_n] \end{pmatrix}.$$

As we want it to be, the non-zero entry of the vector is a function restricted to $(\lambda_0^\alpha)^{\mathbb{C}} = -\eta + \eta\sum_{i=2}^n \sigma_i + \alpha$, since every operator $\tilde{\lambda}_0^\alpha$ is evaluated on the same vector $e[-\eta] \otimes e[\sigma_2] \otimes \ldots \otimes e[\sigma_n]$.
Let us now turn to the case with $\sigma_1 = 1$ or $(\lambda_0^\alpha)^{\mathbb{C}} = \eta + \sum_{i=2}^n \sigma_i\eta$.. It reads

$$\begin{pmatrix} \mathbb{I}_{n-1} & 0 \\ a^{-1}(z_1-2\eta,\tilde{\lambda}_0^\alpha)c(z_1-2\eta,\tilde{\lambda}_0^\alpha+2\eta) & \pi_e f(\tilde{\lambda}_0^\alpha,h)a^{-1}(z_1-2\eta,\tilde{\lambda}_0^\alpha) \end{pmatrix} \times \begin{pmatrix} u(\tilde{\lambda}_0^\alpha)e[\sigma_2] \otimes \ldots \otimes e[\sigma_n] \\ 0 \end{pmatrix} = \begin{pmatrix} u(\tilde{\lambda}_0^\alpha)e[\sigma_2] \otimes \ldots \otimes e[\sigma_n] \\ (a^{-1}(z_1-2\eta,\tilde{\lambda}_0^\alpha)c(z_1-2\eta,\tilde{\lambda}_0^\alpha+2\eta)u(\tilde{\lambda}_0^\alpha)e[\sigma_2] \otimes \ldots \otimes e[\sigma_n]) \end{pmatrix}.$$

The upper entry of the vector shows the right behaviour, since evaluating $\tilde{\lambda}_0^\alpha$ on $e[\eta] \otimes e[\sigma_2] \otimes \ldots \otimes e[\sigma_n]$ yields $(\lambda_0^\alpha)^{\mathbb{C}} = \eta + \sum_{i=2}^n \sigma_i\eta + \alpha$. It thus remains to check the lower entry.
By the definition of $c_{aux,e}(z,z_1,\dots,z_n,\lambda)$ and $I_{FC}$, $c_{aux,e}(z,z_1,\dots,z_n,\lambda)$ maps a vector $e[\sigma_2]\otimes\ldots\otimes e[\sigma_n]$ to vectors of the form $e[\sigma_2']\otimes\ldots\otimes e[\sigma_n']$ with $\sum_{i=2}^n \sigma_i' = \sum_{i=2}^n \sigma_i + 2$. Thus, $\tilde{\lambda}_0^\alpha$ in $u(\tilde{\lambda}_0^\alpha)$ should obtain a value equal to $-\eta + \sum_{i=2}^n \sigma_i' + \alpha = -\eta + \sum_{i=2}^n \sigma_i + 2\eta + \alpha = \eta + \sum_{i=2}^n \sigma_i + \alpha = (\lambda_0^\alpha)^{\mathbb{C}}$.

The same applies to the coefficient of $c_{aux,e}(z_1-2\eta, z_2, \ldots, z_n, \tilde{\lambda}_0^\alpha+2\eta)$, since here we have to evaluate $(\tilde{\lambda}_0^\alpha+2\eta)e[-\eta]\otimes e[\sigma_2]\otimes\ldots\otimes e[\sigma_n]$ resulting in $(-\eta+\sum_{i=2}^n \sigma_i\eta+\alpha)+2\eta = (\lambda_0^\alpha)^{\mathbb{C}}$. Finally, the value of $\tilde{\lambda}_0^\alpha$ in the operator $a^{-1}(z_1-2\eta, \tilde{\lambda}_0^\alpha)$ is automatically correct, since it is evaluated on the vector shifted by $c(z_1-2\eta, \lambda_0^\alpha)$ which we called $e[-\eta]\otimes e[\sigma_2']\otimes\ldots e[\sigma_n']$. Thus, we see that in both cases, hence generally, a function $u(\lambda)$ which was fixed to a specific value $(\lambda_0^\alpha)^{\mathbb{C}}$ on the hyperplane $\tilde{\lambda}_0^\alpha = \sum_{i=1}^n h_i\eta+\alpha$ is mapped by $\mathcal{A}_{n,e}(z_1, \ldots, z_n, \tilde{\lambda}_0^\alpha)$ to a function $\bar{u}(\lambda)$ with $\lambda = (\lambda_0^\alpha)^{\mathbb{C}}$.

**Lemma 4.49** *Let $\alpha \in \mathbb{C}$. Let $\tilde{\lambda}_0^\alpha$ be defined as in Lemma 4.48.*
*Let $u((\lambda_0^\alpha)^{\mathbb{C}})e[\sigma_1]\otimes\ldots\otimes e[\sigma_n] \in M(\mathbb{C}, (\mathbb{C}^2)^{\otimes n})$, where $(\lambda_0^\alpha)^{\mathbb{C}} = \eta\sum_{i=1}^n \sigma_i + \alpha$ and $\sigma_i \in \{-1, 1\}$ for all $i = 1, \ldots, n$. Then,*

$$A_{n,e}(z_1, \ldots, z_n, \tilde{\lambda}_0^\alpha)u((\lambda_0^\alpha)^{\mathbb{C}})e[\sigma_1]\otimes\ldots\otimes e[\sigma_n] = \bar{u}((\lambda_0^\alpha)^{\mathbb{C}})e[\sigma_1']\otimes\ldots\otimes e[\sigma_n']).$$

**Proof:**
The proof is by induction. The proof of the case $n = 2$ is a corollary of the preceding lemma.
Let us now assume that the statement for fixed $n-1 \in \mathbb{N}$ holds true, i.e. that

$$(A_{n-1,e})^{(2\ldots n)}(z_2, \ldots, z_n, \lambda_0^{\alpha'})u((\lambda_0^{\alpha'})^{\mathbb{C}})\, e[\sigma_2]\otimes\ldots\otimes e[\sigma_n] = \bar{u}((\lambda_0^{\alpha'})^{\mathbb{C}})\, e[\sigma_2']\otimes\ldots\otimes e[\sigma_n']$$

with $\tilde{\lambda}_0^{\alpha'} = \eta\sum_{i=2}^n h_i + \alpha$ and $(\lambda_0^{\alpha'})^{\mathbb{C}} = \eta\sum_{i=2}^n \sigma_i + \alpha$. Let us then prove the statement for $n$.
By definition, $A_{n,e}(z_1, \ldots, z_n, \tilde{\lambda}_0^\alpha) = A_{n-1,e}^{(2\ldots n)}(z_2, \ldots, z_n, \tilde{\lambda}_0^\alpha)\mathcal{A}_{n,e}^{(1\ldots n)}(z_1, \ldots, z_n, \tilde{\lambda}_0^\alpha)$. Thus, with $\tilde{\lambda}_0^\alpha = \eta\sum_{i=1}^n h_i + \alpha$

$$\begin{aligned} &A_{n,e}(z_1, \ldots, z_n, \tilde{\lambda}_0^\alpha)u((\lambda_0^\alpha)^{\mathbb{C}})e[\sigma_1]\otimes\ldots\otimes e[\sigma_n] = \\ &A_{n-1,e}^{(2\ldots n)}(z_2, \ldots, z_n, \tilde{\lambda}_0^\alpha)(\mathcal{A}_{n,e}^{(1\ldots n)}(z_1, \ldots, z_n, \tilde{\lambda}_0^\alpha)u((\lambda_0^\alpha)^{\mathbb{C}})e[\sigma_1]\otimes\ldots\otimes e[\sigma_n]) = \\ &A_{n-1,e}^{(2\ldots n)}(z_2, \ldots, z_n, \tilde{\lambda}_0^\alpha)\bar{u}((\lambda_0^\alpha)^{\mathbb{C}} = \tilde{\lambda}_0^\alpha)e[\sigma_1']\otimes\ldots\otimes e[\sigma_n'] \end{aligned}$$

with $(\lambda_0^\alpha)^{\mathbb{C}} = \sum_{i=1}^n \eta\sigma_i + \alpha = \eta\sum_{i=1}^n \sigma_i' + \alpha$ by the preceding lemma. Now, we can either obtain a vector $\bar{u}((\lambda_0^\alpha)^{\mathbb{C}})e[\eta]\otimes e[\sigma_2']\otimes\ldots\otimes e[\sigma_n']$, which will be treated in the first case, or $\bar{u}((\lambda_0^\alpha)^{\mathbb{C}})e[-\eta]\otimes e[\sigma_2']\otimes\ldots\otimes e[\sigma_n']$, which will be treated in the second case. Since $A_{n-1,e}^{(2\ldots n)}(z_2, \ldots, z_n, \tilde{\lambda}_0^\alpha)$ does not affect the value of $\sigma_1$, in the first case $\tilde{\lambda}_0^\alpha = \eta+\sum_{i=2}^n h_i + \alpha = \sum_{i=2}^n h_i + (\alpha+\eta) = \tilde{\lambda}_0^{\alpha+\eta'}$ and $(\lambda_0^\alpha)^{\mathbb{C}} = \eta+\sum_{i=2}^n \sigma_i' + \alpha = (\lambda_0^{\alpha+\eta'})^{\mathbb{C}}$. In the second case, $\tilde{\lambda}_0^\alpha = -\eta+\sum_{i=2}^n h_i + \alpha = \sum_{i=2}^n h_i + (\alpha-\eta) = \tilde{\lambda}_0^{\alpha-\eta'}$ and $(\lambda_0^\alpha)^{\mathbb{C}} = -\eta+\sum_{i=2}^n \sigma_i' + \alpha = (\lambda_0^{\alpha-\eta'})^{\mathbb{C}}$.
Hence, in the first case, we can apply our assumption with $\alpha' = \alpha+\eta$, to get

$$A_{n-1,e}^{(2\ldots n)}(z_2, \ldots, z_n, \tilde{\lambda}_0^{\alpha+\eta'})\bar{u}((\lambda_0^{\alpha+\eta'})^{\mathbb{C}})e[\eta]\otimes(e[\sigma_2']\otimes\ldots\otimes e[\sigma_n']) = \hat{u}((\lambda_0^{\alpha+\eta'})^{\mathbb{C}})e[\eta]\otimes(e[\sigma_2'']\otimes\ldots\otimes e[\sigma_n''])$$

with $(\lambda_0^{\alpha+\eta'})^{\mathbb{C}} = (\lambda_0^\alpha)^{\mathbb{C}}$.
In the second case, we get with $\alpha' = \alpha-\eta$

$$A_{n-1,e}^{(2\ldots n)}(z_2, \ldots, z_n, \tilde{\lambda}_0^{\alpha-\eta'})\bar{u}((\lambda_0^{\alpha-\eta'})^{\mathbb{C}})e[-\eta]\otimes(e[\sigma_2']\otimes\ldots\otimes e[\sigma_n']) = \hat{u}((\lambda_0^{\alpha-\eta'})^{\mathbb{C}})e[-\eta]\otimes(e[\sigma_2'']\otimes\ldots\otimes e[\sigma_n''])$$

with $(\lambda_0^{\alpha-\eta'})^{\mathbb{C}} = (\lambda_0^{\alpha})^{\mathbb{C}}$.
Thus, in both cases $\lambda$ stays restricted after the action of $A_{n,e}(z_1, \ldots, z_n, \tilde{\lambda}_0^{\alpha})$ to $(\lambda_0^{\alpha})^{\mathbb{C}} = \eta \sum_{i=1}^{n} \sigma_i + \alpha$. This completes the proof.

**Lemma 4.50** *Let $\alpha \in \mathbb{C}$ and $\tilde{\lambda}_0^{\alpha} = \sum_{i=1}^{n} \eta h_i + \alpha$.*
*Let $u((\lambda_0^{\alpha})^{\mathbb{C}})e[\sigma_1] \otimes \ldots \otimes e[\sigma_n] \in M(\mathbb{C}, V^{\otimes n})$, where $\sigma_i \in \{-1, 1\}$ for all $i = 1, \ldots, n$ and $(\lambda_0^{\alpha})^{\mathbb{C}} = \eta \sum_{i=1}^{n} \sigma_i + \alpha$. Then,*

$$A_{n,e}^{-1}(z_1, \ldots, z_n, \tilde{\lambda}_0^{\alpha}) u((\lambda_0^{\alpha})^{\mathbb{C}})e[\sigma_1] \otimes \ldots \otimes e[\sigma_n] = \bar{u}((\lambda_0^{\alpha})^{\mathbb{C}})e[\sigma_1'] \otimes \ldots \otimes e[\sigma_n'].$$

**Proof:**
This is an implication of the fact that $A_{n,e}^{-1}(z_1, \ldots, z_n, \tilde{\lambda}_0^{\alpha}) A_{n,e}(z_1, \ldots, z_n, \tilde{\lambda}_0^{\alpha})$ $u((\lambda_0^{\alpha})^{\mathbb{C}})e[\sigma_1] \otimes \ldots \otimes e[\sigma_n] = u((\lambda_0^{\alpha})^{\mathbb{C}})e[\sigma_1] \otimes \ldots \otimes e[\sigma_n]$ and the preceding Lemma 4.48.

**Corollary 4.51** *By Lemma 4.47, 4.48 and 4.49 for $\alpha = 0$, the Proposition 4.46 is proven. Thus,*

$$\begin{aligned} A_{n,e}^{-1}(z_1, \ldots, z_n, \lambda_0) \bar{T}_{SOS,e}^{\mathbb{C}}(z, z_1, \ldots, z_n, \lambda_0) A_{n,e}(z_1, \ldots, z_n, \lambda_0) \\ = \bar{T}_{aux,e}^{\mathbb{C}}(z, z_1, \ldots, z_n, \lambda_0) \end{aligned}$$

*for $\lambda_0 = \sum_{i=1}^{n} x_i = \eta \sum_{i=1}^{n} h_i$.*

## 4.6 Solving the eigenvalue problem of the antiperiodic SOS model

**Synopsis:**
In this chapter, we deal with solutions of the common eigenvalue problem of the family of commuting transfer matrices of the eight-vertex SOS model with antiperiodic boundary conditions (cf. Definition 4.26). A solution to this problem was given in Definition 4.29 as a pair $(\epsilon_{SOS}(z), \sum_{a_1 \ldots a_n} \alpha_{a_1 \ldots a_n} | a_1, \ldots, a_{n+1} = -a_1 >)$, where the eigenvalue was to be an elliptic polynomial (cf. Appendix 2) and the eigenvector an antiperiodic path in $P_n$. Both entities are to solve

$$\begin{aligned} \bar{T}_{SOS,e}(z, z_1, \ldots, z_n, \lambda_0) \textstyle\sum_{a_1 \ldots a_n} \alpha_{a_1 \ldots a_n} | a_1, \ldots, a_{n+1} = -a_1 >= \\ \epsilon_{SOS}(z) \textstyle\sum_{a_1 \ldots a_n} \alpha_{a_1 \ldots a_n} | a_1, \ldots, a_{n+1} = -a_1 >, \end{aligned}$$

where the family of transfer matrices of the SOS model with antiperiodic boundary conditions, well defined for $n \in 2\mathbb{N}+1$ fundamental representations, is given in Definition 4.26

$$\bar{T}_{SOS,e}(z, z_1, \ldots, z_n, \lambda_0) = \sum_{\mu} tr_{(0)}^{V[\mu]} K^{(0)} \bar{L}_{SOS,e}(z, z_1, \ldots, z_n, \lambda_0)$$

with $\lambda_0 = \eta \sum_{i=1}^{n} (x_i + z_i)$ and

$$K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In Corollary 4.51 we showed how to relate the family of SOS and the family of auxiliary transfer matrices by the isomorphism $A_{n,e}(z_1, \ldots, z_n, \lambda_0)$. A common solution to the auxiliary eigenvalue problem was given in Definition 4.39 as a pair $(\epsilon(z), u(x_1, \dots, x_n))$, where $\epsilon(z)$ is to be an elliptic polynomial and $u(x_1, \dots, x_n)$ a function on $D$, solving

$$\bar{T}_{aux,e}(z, z_1, \dots, z_n, \lambda_0)u(x_1, \dots, x_n) = \epsilon(z)u(x_1, \dots, x_n).$$

In this section, we first show how to obtain out of the family of commuting auxiliary transfer matrices the (equivalent) system of separated equations (Definition 4.52) out of whose solutions we can build a solution of the eigenvalue problem of the auxiliary transfer matrix.

Then, we show what conditions an elliptic polynomial (cf. Appendix 2) has to obey to be a common eigenvalue of the family of auxiliary transfer matrices (Proposition 4.53). In Corollary 4.54, we use Corollary 4.51 to state the conditions which an elliptic polynomial has to satisfy in order to be a common eigenvalue of the family of SOS transfer matrices. In Theorem 4.55, we finally show how to obtain a common eigenvector of the family of SOS transfer matrices out of a common eigenvector of the family of auxiliary transfer matrices, also by using Corollary 4.51.

**Remark:**

Note that we have to restrict ourselves to the case of $n \in \mathbb{N}$ being an **odd** integer to avoid poles of the auxiliary transfer matrix which could occurr at $\lambda_0 = \sum_{i=1}^{n}(x_i + z_i)$ if $n$ was an even integer.

By means of the isomorphism constructed in the fifth section of this chapter, we were able to relate it to the auxiliary transfer matrix given by

$$\bar{T}_{aux,e}(z, z_1, \dots, z_n, \lambda_0) = \sum_{i=1}^{n} \frac{\theta(z + x_i - \lambda_0)}{\theta(\lambda_0)} \prod_{j=1, j\neq i}^{n} \frac{\theta(z + x_j)}{\theta(x_i - x_j)} \times$$

$$\times \left( \prod_{j=1}^{n} \theta(x_i + z_j + \eta) T_{x_i}^{-2\eta} + \prod_{j=1}^{n} \theta(x_i + z_j - \eta) T_{x_i}^{+2\eta} \right).$$

Note that the operators $T_{\lambda}^{+2\eta}$ and $T_{\lambda}^{-2\eta}$ are omitted in the transfer matrix as written above. This can be justified by looking at Lemma 4.47, where it was shown that the action of $\bar{b}_{aux,e}(z, z_1, \dots, z_n, \lambda_0)$ and $\bar{b}_{aux,e}(z, z_1, \dots, z_n, \lambda_0)$ on a function $\Phi(x_1, \dots, x_n, (\lambda_0)^{\mathbb{C}}) \in \mathcal{F}_D^{\lambda_0}$ with a fixed $(\lambda_0)^{\mathbb{C}} \in \mathbb{C}$ led to a function $\bar{\Phi}(x_1, \dots, x_n, (\lambda^0)^{\mathbb{C}})$ with the same value of $\lambda = (\lambda^0)^{\mathbb{C}}$.

Thus, we may evaluate this family of operators on a possible common eigenfunction $\Phi(x_1, \dots, x_n) \in \mathcal{F}_D$ in order to get a possible common eigenvalue $\epsilon(z)$.

If we evaluate a transfer matrix $T_{aux,e}(z, z_1, \dots, z_n, \lambda_0)$ at the $n$ points $z = -x_i$ for $i = 1, \dots, n$, we get the separated equations.

**Definition 4.52 (Separated equations)** *The separated equations are given by*

$$\Big(\textstyle\prod_{j=1}^{n} \theta(x_i + z_j + \eta)\Phi(x_1, \dots, x_i - 2\eta, \dots, x_n) + \prod_{j=1}^{n} \theta(x_i + z_j - \eta) \times$$
$$\times \Phi(x_1, \dots, x_i + 2\eta, \dots, x_n)\Big) = \epsilon(-x_i)\Phi(x_1, \dots, x_i, \dots, x_n) \qquad (79)$$

*for all* $i = 1, \dots, n$.

All separated equations that appear show the same structure of a linear difference equation in one variable $x_i$ for all $i = 1, \dots, n$. Note that this is a considerable simplification

compared to the nonlinear difference equation in $n$ variables defined by the original eigenvalue problem of the antiperiodic SOS transfer matrix.
Since only one variable $x_i$ is affected at a time, we can write $\Phi(x_1,\dots,x_i,\dots,x_n) = \prod_{i=1}^n \phi(x_i)$ yielding for the $i$th separated equation

$$(\theta(z+z_j+\eta)\phi(z-2\eta)+\theta(z+z_j-\eta)\phi(z+2\eta)) = \epsilon(-z)\phi(z), \tag{80}$$

where we substituted $x_i = z$. This can be done for all occuring cases $i=1,\dots,n$.
**Remark:**
The equation appearing in this form is also known as a Baxter equation (cf. [47], [31] and the introduction).
Let us now state the theorems on common eigenvalues and eigenvectors of the family of auxiliary antiperiodic transfer matrices and by means of the isomorphism also of the transfer matrices of the SOS with antiperiodic boundary conditions.

**Proposition 4.53 (Eigenvalues of the auxiliary transfer matrix)** *Suppose that $\Lambda_i = 1$ for $i=1,\dots,n$ with $n$ being an odd integer, $\eta \notin \Gamma$, $z_i \neq z_j + 2\eta l$ for $l \in \{0,1\}$. Let $\bar{T}_{aux,e}(z,z_1,\dots,z_n,\lambda_0)$ be the auxiliary transfer matrix as defined above, the $x_i, i = 1,\dots,n$, being restricted to $\{-z_i-\eta,-z_i+\eta\}$ and $\lambda_0 = \sum_{i=1}^n (x_i+z_i)$.*
*Then a function $\epsilon(z)$ is a common eigenvalue of the family of transfer matrices $T_{aux,e}(z)$, $z \in \mathbb{C}$, if and only if*

*i) $\epsilon(z) \in \Theta_n(\chi)$ with $\chi(1) = (-1)^n$, $\chi(\tau) = (-1)^n e^{2\pi i \sum_{i=1}^n z_i}$ and*

*ii) $\epsilon(z)$ obeys the quadratic relations*

$$\epsilon(z_i+\eta)\epsilon(z_i-\eta) = \prod_{k=1}^n \theta(z_k - z_i - 2\eta)\theta(z_k - z_i + 2\eta) \tag{81}$$

*for $i=1,\dots,n$.*

**Proof:** [30]
Let us first prove the if part. For this, let us suppose that $\Phi(x_1,\dots,x_n)$ defined on $D$ is a common eigenfunction of the $\bar{T}_{aux,e}(z,z_1,\dots,z_n)$. In particular, it is not identically zero on $D$. From the transformation properties of $T_{aux,z}$ we see that a possible common eigenvalue $\epsilon(z)$ has to be an element of $\Theta_n(\chi)$ with $\chi(1) = (-1)^n$ and $\chi(\tau) = (-1)^n e^{2\pi i \sum_{i=1}^n z_i}$. Setting $z = -x_i$ in the equation $T_{aux,e}(z)u(x_1,\dots,x_n) = \epsilon(z)u(x_1,\dots,x_n)$ yields the separated equations as described above. Due to their structure they yield while setting $x_i$ to either one of its two possible values

$$\prod_{k=1}^n \theta(z_k - z_i - 2\eta)\Phi(x_1,\dots,-z_i+\eta,\dots,x_n) = \epsilon(z_i+\eta)\Phi(x_1,\dots,-z_i-\eta,\dots,x_n),$$
$$\prod_{k=1}^n \theta(z_k - z_i + 2\eta)\Phi(x_1,\dots,-z_i-\eta,\dots,x_n) = \epsilon(z_i-\eta)\Phi(x_1,\dots,-z_i+\eta,\dots,x_n),$$

for $i=1,\dots,n$. Since $\Phi(x_1,\dots,x_n)$ does not vanish identically zero on $D$ the left hand side of one of the above equations is non-zero, leading to both sides of both equations being non-zero. Thus, we obtain the second property of $\epsilon(z)$.

For the only-if-part, let us start with an elliptic polynomial $\epsilon(z)$ obeing the two conditions indicated above. $\epsilon(z)$ obeying the second condition means that the system of equations

$$\prod_{k=1}^{n}\theta(x+z_k+\eta)Q_i(x-2\eta)+\prod_{k=1}^{n}\theta(x+z_k-\eta)Q_i(x+2\eta)=\epsilon(-x)Q_i(x),$$
$$\text{for } x=-z_i+\eta,-z_i-\eta$$

admits a non-trivial solution for every $i=1,\dots,n$. Hence, $\Phi(x_1,\dots,x_n)=\prod_{i=1}^{n}Q_i(x_i)$ obeys the system of separated equations. Thus - and by $\epsilon(z)\in\Theta_n(\chi)$ - on $D$ $(T_{aux,e}(z)-\epsilon(z))\Phi(x_1,\dots,x_n)$ defines a function in $\Theta_n(\chi)$ with respect to $z$ that vanishes at $n$ points $-x_i$. Thus, by Proposition E.2 of the second Appendix, it vanishes identically. This is due to the fact that the possible non-vanishing condition, cf. Proposition E.2, cannot hold since $\sum_{i=1}^{n} z_i\neq-\sum_{i=1}^{n}x_i$ due to $x_i=-z_i\pm\eta$ and $n$ being odd.

**Corollary 4.54 (Eigenvalues of the antiperiodic SOS model)**
*$\epsilon_{SOS}(z)$ is a common eigenvalue of the family of transfer matrices of the SOS model with antiperiodic boundary conditions if and only if it obeys the two conditions stated in Proposition 4.53.*

**Proof:**
For the proof we apply Proposition 4.46 and the isomorphism $I_{FC}$, in order to get $I_{CA}A_{n,e}(z_1,\dots,z_n,\lambda_0)\bar{T}^{\mathbb{C}}_{aux,e}(z,z_1,\dots,z_n,\lambda_0)A^{-1}_{n,e}(z_1,\dots,z_n,\lambda_0)I^{-1}_{CA}=\bar{T}_{SOS,e}(z,z_1,\dots,z_n,\lambda_0)$.

**Theorem 4.55 (Eigenfunctions of the antiperiodic SOS model)**
*Let $e^*[\sigma_1]\otimes\dots e^*[\sigma_n]$ the dual basis to the standard tensor product basis of $V^{\otimes n}$, i.e. $(e^*[\sigma_1]\otimes\dots e^*[\sigma_n])(e[\sigma_1']\otimes\dots\otimes e[\sigma_n'])=\prod_{i=1}^{n}\delta_{\sigma_i,\sigma_i'}$ for all $\sigma_i,\sigma_i'\in\{-1,1\}$ for $i=1,\dots,n$. Let $\Phi(x_1,\dots,x_n)=\prod_{i=1}^{n}\phi(x_i)=\sum_{i=1,\sigma_i\in\{-1,1\}}^{n}\prod_{i=1}^{n}\phi(-z_i+\sigma_i\eta)f_{\sigma_1\dots\sigma_n}\in\mathcal{F}_n^D$ a common eigenfunction of the family $\bar{T}_{aux,e}(z,z_1,\dots,z_n,\lambda_0)$, where $f_{\sigma_1\dots\sigma_n}$ was defined in the preceding section. I.e. every $\phi(x_i)$ solves the associated separated equation as defined before for $i=1,\dots,n$.*
*Then a common eigenfunction of the family of commuting transfer matrices of the SOS model with antiperiodic boundary conditions $\bar{T}_{SOS,e}(z,z_1,\dots,z_n,\lambda_0)$ is given by*

$$P_\Phi=\sum_{i=1,\sigma_i\in\{-1,1\}}^{n}\left(\prod_{i=1}^{n}\phi(-z_i+\sigma_i\eta)\right)\sum_{j=1,\sigma_j'\in\{-1,1\}}^{n}(A_{n,e})^{\sigma_1'\dots\sigma_n'}_{\sigma_1\dots\sigma_n}\times$$
$$\times|\sum_{i=1}^{n}\frac{\sigma_i'}{2},\dots,-\sum_{j=1}^{i-1}\frac{\sigma_j'}{2}+\sum_{j=i}^{n}\frac{\sigma_j'}{2},\dots,-\sum_{i=1}^{n}\frac{\sigma_i'}{2}> \tag{82}$$

*with*

$$(A_{n,e})^{\sigma_1'\dots\sigma_n'}_{\sigma_1\dots\sigma_n}=e^*[\sigma_1']\otimes\dots\otimes e^*[\sigma_n'](A_{n,e}(z_1,\dots,z_n,\lambda_0))e[\sigma_1]\otimes\dots\otimes e[\sigma_n].$$

**Proof of the Theorem:**
Let $A_{n,e}(z_1,\dots,z_n,\lambda_0)=A_{n,e}$ throughout the proof. Let $\Phi(x_1,\dots,x_n)\in\mathcal{F}_n^D$ be a common eigenfunction of $\bar{T}_{aux,e}(z,z_1,\dots,z_n,\lambda_0)\in$ End $(\mathcal{F}_n^D)$, i.e.

$$\bar{T}_{aux,e}(z,z_1,\dots,z_n,\lambda_0)\Phi(x_1,\dots,x_n)=\epsilon_{SOS}(z)\Phi(x_1,\dots,x_n).$$

Then we successively obtain $\bar{T}_{SOS,e}(z, z_1, \dots, z_n, \lambda_0) \in$ End $(P_n)$ as

$$\bar{T}_{aux,e}(z, z_1, \dots, z_n, \lambda_0) = I_{FC}^{-1} A_{n,e}^{-1} I_{CA}^{-1} \bar{T}_{SOS,e}(z, z_1, \dots, z_n, \lambda_0) I_{CA} A_{n,e} I_{FC},$$

where we first applied the isomorphism $I_{FC} : \mathcal{F}_n^D \to V^{\otimes n}$, then Proposition 4.38, then the isomorphism $I_{CA} : V^{\otimes n} \to P_n$.
If we apply this identity to the auxiliary eigenvalue problem, we get

$$\bar{T}_{SOS,e}(z, z_1, \dots, z_n, \lambda_0) I_{CA} A_{n,e} I_{FC} \sum_{i=1,\sigma_i \in \{-1,1\}}^{n} \left( \prod_{i=1}^{n} \phi(-z_i + \sigma_i \eta) \right) f_{\sigma_1 \dots \sigma_n}$$
$$= \epsilon_{SOS}(z) I_{CA} A_{n,e} I_{FC} \sum_{i=1,\sigma_i \in \{-1,1\}}^{n} \left( \prod_{i=1}^{n} \phi(-z_i + \sigma_i \eta) \right) f_{\sigma_1 \dots \sigma_n}.$$

With the definition of

$$(A_{n,e})_{\sigma_1 \dots \sigma_n}^{\sigma'_1 \dots \sigma'_n} = e^*[\sigma'_1] \otimes \dots \otimes e^*[\sigma'_n](A_{n,e}(z_1, \dots, z_n, \lambda_0)) e[\sigma_1] \otimes \dots \otimes e[\sigma_n]$$

and

$$I_{CA}(e[\sigma_1] \otimes \dots e[\sigma_n]) = | \sum_{i=1}^{n} \frac{\sigma_i}{2}, \dots, -\sum_{j=1}^{i-1} \frac{\sigma_j}{2} + \sum_{j=i}^{n} \frac{\sigma_j}{2}, \dots, -\sum_{i=1}^{n} \frac{\sigma_i}{2} >,$$

we find

$$I_{CA} A_{n,e} I_{FC} \sum_{i=1,\sigma_i \in \{-1,1\}}^{n} \left( \prod_{i=1}^{n} \phi(-z_i + \sigma_i \eta) \right) f_{\sigma_1 \dots \sigma_n} =$$
$$\sum_{i=1,\sigma_i \in \{-1,1\}}^{n} \left( \prod_{i=1}^{n} \phi(-z_i + \sigma_i \eta) \right) \sum_{j=1,\sigma'_j \in \{-1,1\}}^{n} (A_{n,e})_{\sigma_1 \dots \sigma_n}^{\sigma'_1 \dots \sigma'_n} \times$$
$$\times | \sum_{i=1}^{n} \frac{\sigma'_i}{2}, \dots, -\sum_{j=1}^{i-1} \frac{\sigma'_j}{2} + \sum_{j=i}^{n} \frac{\sigma'_j}{2}, \dots, -\sum_{i=1}^{n} \frac{\sigma'_i}{2} > .$$

## 4.7 Limiting cases of the SOS eight-vertex model

**Synopsis:**
In this section, we want to show how to obtain the operator $S_e(Z)$ in separated variables (cf. Definition 2.23) as a limiting case of the family of commuting auxiliary antiperiodic transfer matrices (cf. Definition 4.35).
**Remark:**
For this section, we need a slight generalisation of the auxiliary representation defined in Proposition 4.33.

**Corollary 4.56** *Let the operators $\bar{a}_{e,aux}(z, z_1, \dots, z_n, \lambda), \bar{b}_{e,aux}(z, z_1, \dots, z_n, \lambda)$, $\bar{c}_{e,aux}(z, z_1, \dots, z_n, \lambda), \bar{d}_{e,aux}(z, z_1, \dots, z_n, \lambda)$ be the ones defined by Proposition 4.12 acting on the space $\mathcal{F}_n^{\lambda}$. Then they define the operator algebra of a functional representation of $E_{\tau,\eta}(sl_2)$.*
*This is a corollary of Proposition 4.33 in the sense that we may imitate the proof where we nowhere needed the fact that the weights $x_i$ for $i = 1, \dots, x_n$ then took values in a discrete set.*

**Definition 4.57** *In this case the auxiliary transfer matrix is given by*

$$\bar{T}_{aux,e}(z) = \sum_{i=1}^{n} \frac{\theta(\lambda - x_i - z)}{\theta(\lambda)} \prod_{j\neq i, j=1}^{n} \frac{\theta(z + x_j)}{\theta(x_j - x_i)} \times$$

$$\times \left( \prod_{j=1}^{n} \theta(x_i + z_j + \Lambda_j \eta) T_{x_i}^{-2\eta} T_{\lambda}^{-2\eta} + \prod_{j=1}^{n} \theta(x_i + z_j - \Lambda_j \eta) T_{x_i}^{+2\eta} T_{\lambda}^{+2\eta} \right). \tag{83}$$

**Proposition 4.58** *For $\lambda$ restricted to $\lambda_0 = \sum_{i=1}^{n}(x_i + z_i)$ the transfer matrices defined above commute.*
*The proof is given in [30].*
*The transfer matrices can now be considered acting on a space $\mathcal{F}_n^{\lambda_0} \simeq \mathcal{F}_n$.*
*This reduces the transfer matrices to*

$$T_{aux,e}(z) = \sum_{i=1}^{n} \frac{\theta(\lambda - x_i - z)}{\theta(\lambda)} \prod_{j\neq i, j=1}^{n} \frac{\theta(z + x_j)}{\theta(x_j - x_i)} \times$$

$$\times \left( \prod_{j=1}^{n} \theta(x_i + z_j + \Lambda_j \eta) T_{x_i}^{-2\eta} + \prod_{j=1}^{n} \theta(x_i + z_j - \Lambda_j \eta) T_{x_i}^{+2\eta} \right).$$

**Remark:**
We want to analyse the elliptic Gaudin limit.

**Proposition 4.59** *Let $\eta \to 0$ for $T_{aux,e}(z)$. We then obtain an expansion $T_{aux,e}(z) = T_0(z) + 4\eta^2 T_1(z) + h.o.t.$, where*

$$T_1(z,\lambda) = -\sum_{i=1}^{n} \frac{\theta(\lambda - z - x_i)}{\theta(\lambda)} \prod_{j\neq i, j=1}^{n} \frac{\theta(z + x_j)}{\theta(x_i - x_j)} \prod_{j=1}^{n} \theta(x_i + z_j)$$

$$\left( (\partial_{x_i} - \sum_{j=1}^{n} \frac{\Lambda_j}{2} \frac{\theta'}{\theta}(x_i + z_j))^2 - \sum_{j=1}^{n} c^{(j)} \wp(x_i + z_j) \right) =$$

$$\left( S_e(z) - \sum_{i=1}^{n} c^{(i)} \wp(z - z_i) \right) \prod_{i=1}^{n} \theta(z - z_i), \tag{84}$$

*where in the last expression we set $y_j = -x_j, j = 1, \ldots, n$, in the expression $S_e(z)$ of Proposition 2.23.*

**Proof:**
The proof is straightforward, taking into account the expression $S_e(z)$ calculated in Proposition 2.23.
To calculate any term, we have to look at the expressions

$$\left( \prod_{i=1}^{n} \theta(x_k + z_i + \Lambda_i \eta) T_{x_k}^{-2\eta} + \prod_{i=1}^{n} \theta(x_k + z_i - \Lambda_i \eta) T_{x_k}^{+2\eta} \right)$$

for $k = 1, \ldots, n$ evaluated at $\eta = 0$ only, since the other appearing terms involve no dependence on $\eta$.

For the term of second order, it suffices to look at $2\frac{1}{2}\eta^2\frac{\partial^2}{\partial\eta^2}(\prod_{i=1}^n \theta(x_k+z_i+\Lambda_i\eta)T_{x_k}^{-2\eta})$, since the term $\prod_{i=1}^n \theta(x_k+z_i+-\Lambda_i\eta)T_{x_k}^{+2\eta}$ is symmetric under $\eta\to-\eta$ to the term $\prod_{i=1}^n \theta(x_k+z_i+\Lambda_i\eta)T_{x_k}^{-2\eta})$.
We get

$$2\frac{1}{2}\eta^2\frac{\partial^2}{\partial\eta^2}(\prod_{i=1}^n \theta(x_k+z_i+\Lambda_i\eta)T_{x_k}^{-2\eta})|_{\eta=0} =$$

$$4\eta^2\left(\sum_{i,j=1}^n \frac{\Lambda_i\Lambda_j}{4}\frac{\theta'}{\theta}(x_k+z_i)\frac{\theta'}{\theta}(x_k+z_j)-\sum_{i=1}^n\frac{\Lambda_i^2}{4}\wp(x_k+z_i)\right.$$
$$\left.-\sum_{i=1}^n\Lambda_i\frac{\theta'}{\theta}(x_k+z_i)\frac{\partial}{\partial x_k}+\frac{\partial^2}{\partial x_k^2}\right)\prod_{i=1}^n\theta(x_k+z_i)=$$
$$4\eta^2\left((\frac{\partial}{\partial x_k}-\sum_{i=1}^n\frac{\Lambda_i}{2}\frac{\theta'}{\theta}(x_k+z_i))^2-\sum_{i=1}^n\frac{\Lambda_i(\Lambda_i+2)}{4}\wp(x_k+z_i)\right)\prod_{i=1}^n\theta(x_k+z_i)$$

for every $k=1,\dots,n$. This yields the first sum indicated in the proposition and by Proposition 2.23 with $x_i=-y_i$ for $i=1,\dots,n$ it also yields the second one.

# 5 The Antiperiodic SOS Model: $n=3$

In this chapter, we want to look closer at the steps of solving the eigenvalue problem for the SOS model with antiperiodic boundary conditions with 3 spin-$\frac{1}{2}$ particles. Hence, we will work with the auxiliary representation of Proposition 4.33, given by $(M(\mathbb{C},V^{\otimes 3}),\bar{L}^{\mathbb{C}}_{aux,e}(z,z_1,z_2,z_3,\lambda))$ with $\Lambda_1=\Lambda_2=\Lambda_3=1$, the tensor product of three fundamental representations as described in the definition of the L-operator of the SOS model $(M(\mathbb{C},V^{\otimes 3}),L^{(0123)}_{SOS,e}(z,z_1,z_2,z_3,\lambda)=R^{(01)}_e(z-z_1,\lambda-2\eta(h_2+h_3))R^{(02)}_e(z-z_2,\lambda-2\eta h_3)\bar{R}^{(03)}_e(z-z_3,\lambda))$ and the corresponding isomorphism of Proposition 4.44 connecting the auxiliary representation with the L-operator of the SOS model.
We proceed in several steps: first, we construct the auxiliary representation for $n=2$ $(M(\mathbb{C},V^{\otimes 2}),\bar{L}^{\mathbb{C}}_{aux,e}(z,z_1,z_2,\lambda))$ and show that the isomorphism of Proposition 4.44 is correct. Note that the example $n=2$ is of no use in solving the antiperiodic eigenvalue problem of the SOS model, since this problem can only be properly treated for an odd number of underlying fundamental representations.
We then verify that the isomorphism of Proposition 4.44 correctly reproduces the auxiliary representation for $n=3$. We also compute the basis of $V^{\otimes 3}$ in which the operator $\bar{a}^{\mathbb{C}}_{aux,e}(z,z_1,z_2,z_3,\lambda)$ is diagonal.
Note that the representation $(M(\mathbb{C},V^{\otimes 3}),\bar{L}^{\mathbb{C}}_{aux,e}(z,z_1,z_2,z_3,\lambda))$ is the simplest non-trivial example of the antiperiodic SOS eigenvalue problem treated by functional Bethe ansatz.
Finally, we compute one eigenvector of the antiperiodic SOS model for $n=3$ explicitly. We also show that the eigenvalue obtained by this eigenvector obeys the - necessary and sufficient - condition on eigenvalues given in Theorem 4.54 $\epsilon(z_i)\epsilon(z_i-2\eta)=\prod_{j=1}^3\theta(z_i-z_j-2\eta)\theta(z_i-z_j+2\eta)$.

## 5.1 A preliminary step: Computing the auxiliary representation for $n=2$

**Synopsis:**
We first give $(M(\mathbb{C},V),\bar{R}_e(z-z_1,\lambda))$, i.e. the basic operator to construct the auxiliary representation and the representation connected to the SOS model from.
Then, we formulate the isomorphism of Theorem 4.44 in the case $n=2$ (Lemma 5.1). We proceed by writing down the L-operators which we want to compare by the isomorphism: $(M(\mathbb{C},V^{\otimes 2}),\bar{L}_e^{\otimes 2(012)}=R_e^{(01)}(z-z_1,\lambda-2\eta h_2)\bar{R}_e^{(02)}(z-z_2,\lambda))$ (Lemma 5.2) and $(M(\mathbb{C},V^{\otimes 2}),\bar{L}^{\mathbb{C}}_{aux,e}(z,z_1,z_2,z_3,\lambda))$ (Lemma 5.3).
In Proposition 5.4 we then show that they are indeed related by the isomorphism of Theorem 4.44 in the case $n=2$. In Proposition 5.5, we show that by this isomorphism a basis of $V^{\otimes 2}$ (for $\lambda\neq 0$) is given and the operator $\bar{a}_e^{\otimes 2}(z,z_1,z_2,\lambda)$ is diagonal in this basis. Diagonalizing this operator was one of the main objectives of the isomorphism.
Let us write down the fundamental representation of $E_{\tau,\eta}(sl_2)$ $(M(\mathbb{C},V),\bar{R}_e(z-z_1,\lambda))$ in matrix form. Remember that it acts on the space $V$ which is a two-dimensional complex vector space with basis $e[-1],e[1]$. We need this representation to formulate the isomorphism described by Proposition 4.44 in the case $n=2$. It is given by

$$\begin{aligned}
\bar{a}_e(\lambda,z-z_1) &= \begin{pmatrix}\theta(z-z_1+2\eta) & 0\\ 0 & \theta(z-z_1)\frac{\theta(\lambda+2\eta)}{\theta(\lambda)}\end{pmatrix} T_\lambda^{-2\eta},\\
\bar{b}_e(\lambda,z-z_1) &= \begin{pmatrix}0 & 0\\ \frac{\theta(\lambda-z+z_1)\theta(2\eta)}{\theta(\lambda)} & 0\end{pmatrix} T_\lambda^{+2\eta},\\
\bar{c}_e(\lambda,z-z_1) &= \begin{pmatrix}0 & \frac{\theta(z-z_1+\lambda)\theta(2\eta)}{\theta(\lambda)}\\ 0 & 0\end{pmatrix} T_\lambda^{-2\eta},\\
\bar{d}_e(\lambda,z-z_1) &= \begin{pmatrix}\theta(z-z_1)\frac{\theta(\lambda-2\eta)}{\theta(\lambda)} & 0\\ 0 & \theta(z-z_1+2\eta)\end{pmatrix} T_\lambda^{+2\eta},\\
\bar{D}et_e(z-z_1) &= \theta(z-z_1+2\eta)\theta(z-z_1+2\eta)\mathbb{I}_1,
\end{aligned}$$

where the determinant can be calculated by using the formula given in Proposition 4.15 a). This representation coincides with $(M(\mathbb{C},V),\bar{L}^{\mathbb{C}}_{aux,e}(z,z_1,\lambda))$ by Proposition 4.41.
Let us now write down the isomorphism of Proposition 4.42 in the case $n=2$, i.e. the matrix $\mathcal{A}_{2,e}(z_1,z_2,\lambda)=A_{2,e}(z_1,z_2\lambda)$.

**Lemma 5.1** *In the case $n=2$ the matrix $\mathcal{A}_{2,e}(z_1,z_2,\lambda)\in$ End $(V^{\otimes 2})\subset$ End $(M(\mathbb{C},V^{\otimes 2}))$ is given by*

$$\mathcal{A}_{2,e}(z_1,z_2,\lambda)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&\frac{\theta(\lambda+z_1-z_2)\theta(2\eta)}{\theta(\lambda+2\eta)\theta(z_1-z_2)}&\frac{\theta(z_1-z_2-2\eta)\theta(\lambda)}{\theta(\lambda+2\eta)\theta(z_1-z_2)}&0\\0&0&0&1\end{pmatrix}. \tag{85}$$

*Its inverse is given by*

$$(\mathcal{A}_{2,e})^{-1}(z_1,z_2,\lambda)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&-\frac{\theta(\lambda+z_1-z_2)\theta(2\eta)}{\theta(z_1-z_2-2\eta)\theta(\lambda)}&\frac{\theta(z_1-z_2)\theta(\lambda+2\eta)}{\theta(z_1-z_2-2\eta)\theta(\lambda)}&0\\0&0&0&1\end{pmatrix}. \tag{86}$$

**Proof:**
This is proven by filling in the appropriate operators into the definition 4.42. The correct form of the inverse $(\mathcal{A}_{2,e})^{-1}(z_1, z_2, \lambda)$ is checked by multiplying with its inverse involving no residual calculations at all.
Let us now write down the representation $(M(\mathbb{C}, V^{\otimes 2}), R_e(z - z_1, \lambda - 2\eta h_2) \otimes \bar{R}_e(z - z_2, \lambda) = \bar{L}_e^{\otimes 2}(z, z_1, z_2, \lambda))$ which consists of the operators $\bar{a}_e^{\otimes 2}(z, z_1, z_2, \lambda)$, $\bar{b}_{1,e}^{\otimes 2}(z, z_1, z_2, \lambda)$, $\bar{c}_e^{\otimes 2}(z, z_1, z_2, \lambda), \bar{d}_e^{\otimes 2}(z, z_1, z_2, \lambda)$ in order to compare it to the auxiliary representation $(M(\mathbb{C}, V^{\otimes 2}), L_{aux,e}^{\mathbb{C}}(z, z_1, z_2, \lambda))$.

**Lemma 5.2** *The entries of $\bar{L}_e^{\otimes 2}(z, z_1, z_2, \lambda)$ are given by*

$$\bar{a}_e^{\otimes 2}(z, z_1, z_2, \lambda) =$$

$$a_e(z - z_1, \lambda - 2\eta h_2) \otimes \bar{a}_e(z - z_2, \lambda) + b_{1,e}(z - z_1, \lambda - 2\eta h_2) \otimes \bar{c}_e(z - z_2, \lambda) =$$

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{pmatrix} T_\lambda^{-2\eta},$$

*with*

$$\begin{aligned}
a_{11} &= \theta(z - z_1 + 2\eta)\theta(z - z_2 + 2\eta), \\
a_{22} &= \frac{\theta(z - z_1 + 2\eta)\theta(z - z_2)\theta(\lambda + 2\eta)}{\theta(\lambda)}, \\
a_{32} &= \frac{(\theta(2\eta))^2\theta(\lambda + z - z_2)\theta(\lambda - z + z_1 - 2\eta)}{\theta(\lambda - 2\eta)\theta(\lambda)}, \\
a_{33} &= \frac{\theta(z - z_2 + 2\eta)\theta(z - z_1)\theta(\lambda)}{\theta(\lambda - 2\eta)}, \\
a_{44} &= \frac{\theta(z - z_1)\theta(z - z_2)\theta(\lambda + 4\eta)}{\theta(\lambda)},
\end{aligned}$$

$$\bar{b}_e^{\otimes 2}(z, z_1, z_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ b_{21} & 0 & 0 & 0 \\ b_{31} & 0 & 0 & 0 \\ 0 & b_{42} & b_{43} & 0 \end{pmatrix} T_\lambda^{+2\eta},$$

*with*

$$\begin{aligned}
b_{21} &= \frac{\theta(z - z_1 + 2\eta)\theta(\lambda - z + z_2)\theta(2\eta)}{\theta(\lambda)}, \\
b_{31} &= \frac{\theta(z - z_2)\theta(\lambda - z + z_1 - 2\eta)\theta(2\eta)}{\theta(\lambda)}, \\
b_{42} &= \frac{\theta(z - z_2 + 2\eta)\theta(\lambda - z + z_1 + 2\eta)\theta(2\eta)}{\theta(\lambda + 2\eta)}, \\
b_{43} &= \frac{\theta(2\eta)\theta(\lambda - z + z_2)\theta(z - z_1)\theta(\lambda + 4\eta)}{\theta(\lambda)\theta(\lambda + 2\eta)},
\end{aligned}$$

$$\bar{c}_e^{\otimes 2}(z, z_1, z_2) = \begin{pmatrix} 0 & c_{12} & c_{13} & 0 \\ 0 & 0 & 0 & c_{24} \\ 0 & 0 & 0 & c_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix} T_\lambda^{-2\eta},$$

*with*

$$\begin{aligned}
c_{12} &= \frac{\theta(z-z_1)\theta(\lambda-4\eta)\theta(2\eta)\theta(\lambda+z-z_2)}{\theta(\lambda-2\eta)\theta(\lambda)}, \\
c_{13} &= \frac{\theta(\lambda-2\eta+z-z_1)\theta(2\eta)\theta(z-z_2+2\eta)}{\theta(\lambda-2\eta)}, \\
c_{24} &= \frac{\theta(z-z_2)\theta(\lambda+2\eta+z-z_1)\theta(2\eta)}{\theta(\lambda)}, \\
c_{34} &= \frac{\theta(z-z_1+2\eta)\theta(2\eta)\theta(\lambda+z-z_2)}{\theta(\lambda)},
\end{aligned}$$

$$\bar{D}et_e^{\otimes 2}(z,z_1,z_2) = \theta(z-z_1-2\eta)\theta(z-z_2-2\eta)\theta(z-z_1+2\eta)\theta(z-z_2+2\eta)\mathbb{I}_1.$$

**Proof:**
This lemma is proved by straightforward calculation. The determinant is either checked by the multiplicative property of the quantum determinant (cf. [25]) or also by straightforward computation by means of the formula given in Proposition 4.15 involving also the operator

$$\begin{gathered}
d_e^{\otimes 2}(z,z_1,z_2,\lambda) = \\
c_e(z-z_1,\lambda-2\eta h_2)\otimes b_e(z-z_2,\lambda) + d_e(z-z_1,\lambda-2\eta h_2)\otimes d_e(z-z_2,\lambda).
\end{gathered}$$

The needed calculation consists in comparison of the transformation properties, zeroes and residues of the left and right hand side of the formula given by Proposition 4.15 a).

**Lemma 5.3** *The operators of the auxiliary representation in the case $n = 2$ are given by*

$$\bar{a}^{\mathbb{C}}_{aux,e}(z,z_1,z_2,\lambda) = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{pmatrix} T_\lambda^{-2\eta},$$

*with*

$$\begin{aligned}
a_{11} &= \theta(z-z_1+2\eta)\theta(z-z_2+2\eta), \\
a_{22} &= \theta(z-z_1+2\eta)\frac{\theta(z-z_2)\theta(\lambda+2\eta)}{\theta(\lambda)}, \\
a_{33} &= \theta(z-z_2+2\eta)\frac{\theta(z-z_1)\theta(\lambda+2\eta)}{\theta(\lambda)}, \\
a_{44} &= \frac{\theta(z-z_1)\theta(z-z_2)\theta(\lambda+4\eta)}{\theta(\lambda)},
\end{aligned}$$

$$\bar{b}^{\mathbb{C}}_{aux,e}(z,z_1,z_2,\lambda) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ b_{21} & 0 & 0 & 0 \\ b_{31} & 0 & 0 & 0 \\ 0 & b_{42} & b_{43} & 0 \end{pmatrix} T_\lambda^{+2\eta},$$

*with*

$$
\begin{aligned}
b_{21} &= \frac{\theta(\lambda - z + z_1)\theta(z - z_2 + 2\eta)\theta(2\eta)}{\theta(\lambda)}, \\
b_{31} &= \frac{\theta(\lambda - z + z_1)\theta(z - z_2 + 2\eta)\theta(2\eta)}{\theta(\lambda)}, \\
b_{42} &= \frac{\theta(\lambda - z + z_1)\theta(z - z_2)\theta(2\eta)\theta(z_1 - z_2 + 2\eta)}{\theta(z_1 - z_2)\theta(\lambda)}, \\
b_{43} &= \frac{\theta(\lambda - z + z_2)\theta(2\eta)\theta(z_1 - z_2 - 2\eta)}{\theta(z_1 - z_2)\theta(\lambda)},
\end{aligned}
$$

$$
\bar{c}^{\mathbb{C}}_{aux,e}(z, z_1, z_2, \lambda) = \begin{pmatrix} 0 & c_{12} & c_{13} & 0 \\ 0 & 0 & 0 & c_{24} \\ 0 & 0 & 0 & c_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix} T_\lambda^{-2\eta},
$$

*with*

$$
\begin{aligned}
c_{12} &= \frac{\theta(2\eta)\theta(\lambda + z - z_2 - 2\eta)\theta(z - z_1 + 2\eta)\theta(z_1 - z_2 + 2\eta)}{\theta(\lambda)\theta(z_1 - z_2)}, \\
c_{13} &= \frac{\theta(2\eta)\theta(\lambda + z - z_1 - 2\eta)\theta(z - z_2 + 2\eta)\theta(z_1 - z_2 - 2\eta)}{\theta(\lambda)\theta(z_1 - z_2)}, \\
c_{24} &= \frac{\theta(\lambda + z - z_1 + 2\eta)\theta(z - z_2)\theta(2\eta)}{\theta(\lambda)}, \\
c_{34} &= \frac{\theta(\lambda + z - z_2 + 2\eta)\theta(z - z_1)\theta(2\eta)}{\theta(\lambda)},
\end{aligned}
$$

$$
\begin{aligned}
\bar{D}et^{\mathbb{C}}_{aux,e}(z, z_1, z_2) &= \theta(z - z_1 - 2\eta)\theta(z - z_2 - 2\eta) \times \\
&\times \ \theta(z - z_1 + 2\eta)\theta(z - z_2 + 2\eta)\mathbb{I}_2.
\end{aligned}
$$

**Proof:**
This is shown by appropriately writing down the definitions of Proposition 4.33, taking into account that for $n = 2$ the auxiliary representation acts on $M(\mathbb{C}, V^{\otimes 2})$, where $V^{\otimes 2}$ is a complex vector space of four dimensions with a canonical basis given by $e[\sigma_1] \otimes e[\sigma_2]$, $\sigma_i \in \{-1, 1\}$ for $i = 1, 2$, which can be identified with $V^4$.
**Remark:**
Now we can compare each of the operators of Lemma 5.2 after conjugation by the operator $\mathcal{A}_{2,e}(z_1, z_2, \lambda)$ of Lemma 5.1 to its counterpart of Lemma 5.3.

**Proposition 5.4**

$$
\begin{aligned}
(\mathcal{A}_{2,e})^{-1}\bar{a}_e^{\otimes 2}(z, z_1, z_2, \lambda)\mathcal{A}_{2,e} &= \bar{a}^{\mathbb{C}}_{aux,e}(z, z_1, z_2, \lambda), && (87) \\
(\mathcal{A}_{2,e})^{-1}\bar{b}_e^{\otimes 2}(z, z_1, z_2, \lambda)\mathcal{A}_{2,e} &= \bar{b}^{\mathbb{C}}_{aux,e}(z, z_1, z_2, \lambda), && (88) \\
(\mathcal{A}_{2,e})^{-1}\bar{c}_e^{\otimes 2}(z, z_1, z_2, \lambda)\mathcal{A}_{2,e} &= \bar{c}^{\mathbb{C}}_{aux,e}(z, z_1, z_2, \lambda), && (89) \\
(\mathcal{A}_{2,e})^{-1}\bar{D}et_e^{\otimes 2}(z, z_1, z_2)\mathcal{A}_{2,e} &= \bar{D}et^{\mathbb{C}}_{aux,e}(z, z_1, z_2), && (90)
\end{aligned}
$$

*where* $\mathcal{A}_{2,e} = \mathcal{A}_{2,e}(z_1, z_2, \lambda)$.

**Proof:**
Let us first check the simplest equality, the fourth one. Since the determinant is a function not depending on the weights $x_1, x_2$ it can be written

$$\bar{Det}_e^{\otimes 2}(z, z_1, z_2) = Det_e^{\otimes 2}(z, z_1, z_2)\, \mathbb{I}_4.$$

The calculation hence reduces to the fact that $(\mathcal{A}_e^2)^{-1}(z_1, z_2, \lambda)\mathcal{A}_e^2(z_1, z_2, \lambda) = \mathbb{I}_4$ on the one hand and the fact that $Det_e^{\otimes 2}(z, z_1, z_2) = Det_{aux,e}^{\mathbb{C}}(z, z_1, z_2)$ by Lemmas 5.2 and 5.3. The most important thing is to show that the isomorphism $\mathcal{A}_e^2(z_1, z_2, \lambda)$ indeed diagonalizes the operator $\bar{a}_1^{\otimes 2}(z, z_1, z_2, \lambda)$. Let us show this. The way the first equality is shown coincides with how the other equalities are obtained.

$$(\mathcal{A}_e^2)^{-1}(z_1, z_2, \lambda)\bar{a}_e^{\otimes 2}(z, z_1, z_2, \lambda)\mathcal{A}_e^2(z_1, z_2, \lambda) =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{\theta(\lambda+z_1-z_2)\theta(2\eta)}{\theta(z_1-z_2-2\eta)\theta(\lambda)} & \frac{\theta(z_1-z_2)\theta(\lambda+2\eta)}{\theta(z_1-z_2-2\eta)\theta(\lambda)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11}^{\otimes} & 0 & 0 & 0 \\ 0 & a_{22}^{\otimes} & 0 & 0 \\ 0 & a_{32}^{\otimes} & a_{33}^{\otimes} & 0 \\ 0 & 0 & 0 & a_{44}^{\otimes} \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{\theta(\lambda+z_1-z_2-2\eta)\theta(2\eta)}{\theta(\lambda)\theta(z_1-z_2)} & \frac{\theta(z_1-z_2-2\eta)\theta(\lambda-2\eta)}{\theta(\lambda)\theta(z_1-z_2)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} T_\lambda^{-2\eta} =$$

$$\begin{pmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & a_{32} & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{pmatrix} T_\lambda^{-2\eta},$$

with

$$\begin{aligned}
a_{11}^{\otimes} &= \theta(z - z_1 + 2\eta)\theta(z - z_2 + 2\eta), \\
a_{22}^{\otimes} &= \frac{\theta(z - z_1 + 2\eta)\theta(z - z_2)\theta(\lambda + 2\eta)}{\theta(\lambda)}, \\
a_{32}^{\otimes} &= \frac{(\theta(2\eta))^2\theta(\lambda + z - z_2)\theta(\lambda - z + z_1 - 2\eta)}{\theta(\lambda - 2\eta)\theta(\lambda)}, \\
a_{33}^{\otimes} &= \frac{\theta(z - z_2 + 2\eta)\theta(z - z_1)\theta(\lambda)}{\theta(\lambda - 2\eta)}, \\
a_{44}^{\otimes} &= \frac{\theta(z - z_1)\theta(z - z_2)\theta(\lambda + 4\eta)}{\theta(\lambda)}, \\
a_{11} &= \theta(z - z_1 + 2\eta)\theta(z - z_2 + 2\eta), \\
a_{22} &= \frac{\theta(z - z_1 + 2\eta)\theta(z - z_2)\theta(\lambda + 2\eta)}{\theta(\lambda)}, \\
a_{33} &= \frac{\theta(z - z_1)\theta(z - z_2 + 2\eta)\theta(\lambda + 2\eta)}{\theta(\lambda)}, \\
a_{44} &= \frac{\theta(z - z_1)\theta(z - z_2)\theta(\lambda + 4\eta)}{\theta(\lambda)}
\end{aligned}$$

$$a_{32} = -\frac{\theta(\lambda+z_1-z_2)\theta(2\eta)\theta(z-z_1+2\eta)\theta(z-z_2)\theta(\lambda+2\eta)}{\theta(\lambda)^2\theta(z_1-z_2-2\eta)}$$
$$+\frac{\theta(\lambda+2\eta)\theta(z_1-z_2)\theta(2\eta)^2\theta(\lambda+z-z_2)\theta(\lambda-z+z_1-2\eta)}{\theta(z_1-z_2-2\eta)\theta(\lambda)^2\theta(\lambda-2\eta)}$$
$$+\frac{\theta(\lambda+2\eta)\theta(\lambda+z_1-z_2-2\eta)\theta(z-z_1)\theta(z-z_2+2\eta)\theta(2\eta)}{\theta(z_1-z_2-2\eta)\theta(\lambda)\theta(\lambda-2\eta)}$$

We want to show that $a_{32}=0$. First, we show that all summands transform the same way under $\lambda\to\lambda+1$, $\lambda\to\lambda+\tau$. Indeed $\lambda\to\lambda+1$ leaves every summand unchanged and $\lambda\to\lambda+\tau$ multiplies every summand by a factor $e^{-2\pi i(z_1-z_2+2\eta)}$ by means of the transformation properties of the odd Jacobi theta function.
The residues at $\lambda=2\eta$ and $\lambda=0$ vanish identically. Let us check this for $\lambda=2\eta$. (The second case is slightly more complicated due to derivatives caused by the second power of $\theta(\lambda)$.)

$$\mathrm{Res}_{\lambda=2\eta}(a_{32}) = -\frac{\theta(4\eta)\theta(z_1-z_2)\theta(z-z_2+2\eta)\theta(z-z_1)}{\theta(z_1-z_2-2\eta)}+$$
$$\frac{\theta(4\eta)\theta(z_1-z_2)\theta(z-z_1)\theta(z-z_2+2\eta)}{\theta(z_1-z_2-2\eta)}=0.$$

Furthermore, $a_{32}|_{\lambda=-2\eta}=0$. Thus, $a_{32}\equiv 0$.
By comparing the operator thus calculated to $a^{\mathbb{C}}_{aux,e}(z,z_1,z_2,\lambda)$ we see that both coincide.
Let us now show the second identity.

$$(\mathcal{A}_{2,e})^{-1}(z_1,z_2,\lambda)\bar{b}_e^{\otimes 2}(z,z_1,z_2,\lambda)\mathcal{A}_{2,e}(z_1,z_2,\lambda)=$$
$$\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & -\frac{\theta(\lambda+z_1-z_2)\theta(2\eta)}{\theta(z_1-z_2-2\eta)\theta(\lambda)} & \frac{\theta(z_1-z_2)\theta(\lambda+2\eta)}{\theta(z_1-z_2-2\eta)\theta(\lambda)} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0\\ b^{\otimes}_{21} & 0 & 0 & 0\\ b^{\otimes}_{31} & 0 & 0 & 0\\ 0 & b^{\otimes}_{42} & b^{\otimes}_{43} & 0\end{pmatrix}\times$$
$$\times\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & \frac{\theta(\lambda+z_1-z_2+2\eta)\theta(2\eta)}{\theta(\lambda+4\eta)\theta(z_1-z_2)} & \frac{\theta(z_1-z_2-2\eta)\theta(\lambda+2\eta)}{\theta(\lambda+4\eta)\theta(z_1-z_2)} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}T_\lambda^{+2\eta}=$$
$$\begin{pmatrix} 0 & 0 & 0 & 0\\ \frac{\theta(\lambda-z+z_2)\theta(z-z_1+2\eta)\theta(2\eta)}{\theta(\lambda)} & 0 & 0 & 0\\ b_{31} & 0 & 0 & 0\\ 0 & b_{42} & \frac{\theta(2\eta)\theta(\lambda-z+z_2)\theta(z-z_2)\theta(z_1-z_2-2\eta)}{\theta(\lambda)\theta(z_1-z_2)} & \end{pmatrix}T_\lambda^{+2\eta},$$

with

$$b^{\otimes}_{21} = \frac{\theta(z-z_1+2\eta)\theta(\lambda-z+z_2)\theta(2\eta)}{\theta(\lambda)},$$
$$b^{\otimes}_{31} = \frac{\theta(z-z_2)\theta(\lambda-z+z_1-2\eta)\theta(2\eta)}{\theta(\lambda)},$$
$$b^{\otimes}_{42} = \frac{\theta(z-z_2+2\eta)\theta(\lambda-z+z_1+2\eta)\theta(2\eta)}{\theta(\lambda+2\eta)},$$
$$b^{\otimes}_{43} = \frac{\theta(2\eta)\theta(\lambda-z+z_2)\theta(z-z_1)\theta(\lambda+4\eta)}{\theta(\lambda)\theta(\lambda+2\eta)},$$

and

$$b_{31} = -\frac{\theta(\lambda + z_1 - z_2)\theta(2\eta)^2\theta(\lambda - z + z_2)\theta(z - z_1 + 2\eta)}{\theta(z_1 - z_2 - 2\eta)\theta(\lambda)^2} + \frac{\theta(\lambda + 2\eta)\theta(z_1 - z_2)\theta(\lambda - z + z_1 - 2\eta)}{\theta(z_1 - z_2 - 2\eta)\theta(\lambda)^2},$$

$$b_{42} = \frac{\theta(2\eta)\theta(\lambda - z + z_1 + 2\eta)\theta(z - z_2 + 2\eta)}{\theta(\lambda + 2\eta)} + \frac{\theta(2\eta)^2\theta(\lambda - z + z_2)\theta(z - z_1)\theta(\lambda + z_1 - z_2 + 2\eta)}{\theta(\lambda)\theta(\lambda + 2\eta)\theta(z_1 - z_2)}.$$

If we compare the single term coefficients of $(\mathcal{A}_{2,e})^{-1}(z_1, z_2, \lambda)\bar{b}_e^{\otimes 2}(z, z_1, z_2, \lambda)\, \mathcal{A}_{2,e}(z_1, z_2, \lambda)$ with the corresponding entries of $\bar{b}^{\mathbb{C}}_{aux,e}(z, z_1, z_2, \lambda)$ in Lemma 8.3, we perceive that they are identical.
It remains to be checked that

$$b_{31} = \frac{\theta(\lambda - z + z_1)\theta(2\eta)\theta(z - z_2 + 2\eta)}{\theta(\lambda)}$$

and

$$b_{42} = \frac{\theta(\lambda + z - z_1)\theta(z - z_2)\theta(z_1 - z_2 + 2\eta)}{\theta(\lambda)\theta(z_1 - z_2)}.$$

Let us verify the second of the above identities. The first is shown analogously.
Each summand of its left hand side and the term on the right hand side transform identically under $\lambda \to \lambda + 1$ and are to be multiplied by $e^{-2\pi i(-z+z_1)}$ if $\lambda \to \lambda + \tau$. The zeroes of the right hand side are at $\lambda = z - z_1$ and $z = z_2$ and are easily shown to be zeroes of the left hand side. There are possible residues occurring at $\lambda = -2\eta$ and $\lambda = 0$. They read:

$$\mathrm{Res}_{\lambda=-2\eta}(b_{42}) = -\theta(2\eta)\theta(z - z_1)\theta(z - z_2 + 2\eta) + \frac{\theta(z_1 - z_2)\theta(2\eta)\theta(z - z_2 + 2\eta)\theta(z - z_1)}{\theta(z_1 - z_2)} = 0$$

and

$$\mathrm{Res}_{\lambda=0}(b_{42}) = -\frac{\theta(2\eta)\theta(z - z_2)\theta(z - z_1)\theta(z_1 - z_2 + 2\eta)}{\theta(z_1 - z_2)} = \mathrm{Res}_{\lambda=0}\left(\frac{\theta(\lambda + z - z_1)\theta(z - z_2)\theta(z_1 - z_2 + 2\eta)}{\theta(\lambda)\theta(z_1 - z_2)}\right).$$

Hence, the left and the right hand side of the equation coincide. Thus, the second identity of Proposition 5.4 holds true.
The third identity is proved by similar means.

**Proposition 5.5** *For $\lambda \neq 0$, a basis of $V^{\otimes 2}$ is given by*

$$B_2 = \{\mathcal{A}_{2,e}(z_1, z_2, \lambda)\, e[\sigma_1] \otimes e[\sigma_2] \,|\, \sigma_i \in \{-1, 1\} \text{ for } i = 1, 2\}.$$

*In this basis the operator $\bar{a}_e^{\otimes 2}(z, z_1, z_2, \lambda)$ is diagonal.*

**Proof:**
For every $\lambda \neq 0$ $\mathcal{A}_{2,e}(z_1, z_2, \lambda)$ is a regular matrix on $V^{\otimes 2}$.
We know that the operator $(\mathcal{A}_{2,e})^{-1}(z_1, z_2, \lambda)\bar{a}_e^{\otimes 2}(z, z_1, z_2, \lambda)\mathcal{A}_{2,e}(z_1, z_2, \lambda)$ is diagonal in the basis $e[\sigma_1] \otimes e[\sigma_2]$, i.e.

$$(\mathcal{A}_{2,e})^{-1}(z_1, z_2, \lambda)\bar{a}_e^{\otimes 2}(z, z_1, z_2, \lambda)\mathcal{A}_{2,e}(z_1, z_2, \lambda)(e[\sigma_1] \otimes e[\sigma_2]) \\ = \alpha_{\sigma_1,\sigma_2}(\lambda)(e[\sigma_1] \otimes e[\sigma_2]),$$

where $\alpha_{\sigma_1,\sigma_2}(\lambda)$ indicates the eigenvalue of the operator depending on the vector $e[\sigma_1] \otimes e[\sigma_2]$ the operator acts on. This identity leads to

$$\bar{a}_e^{\otimes 2}(z, z_1, z_2, \lambda)(\mathcal{A}_{2,e}(z_1, z_2, \lambda)(e[\sigma_1] \otimes e[\sigma_2])) \\ = \alpha_{\sigma_1,\sigma_2}(\lambda)\mathcal{A}_{2,e}(z_1, z_2, \lambda)(e[\sigma_1] \otimes e[\sigma_2]),$$

showing that $\bar{a}_e^{\otimes 2}(z, z_1, z_2, \lambda)$ is diagonal in the basis $B_2$.

**Corollary 5.6** *The new basis is explicitly given by the following four vectors*

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{\theta(\lambda+z_1-z_2)\theta(2\eta)}{\theta(\lambda+2\eta)\theta(z_1-z_2)} \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ \frac{\theta(z_1-z_2-2\eta)\theta(\lambda)}{\theta(\lambda+2\eta)\theta(z_1-z_2)} \\ 0 \end{pmatrix},$$

$$v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Note that we will need these expressions for the case $n = 3$.

## 5.2 Computing the auxiliary representation for $n = 3$

**Synopsis:**
Here, we repeat the steps of the case $n = 2$ for the case $n = 3$.
First, we define the isomorphism $\mathcal{A}_{3,e}(z_1, z_2, z_3, \lambda)$ as used in Proposition 4.43 (Lemma 5.7). Then, we give the representations we want to compare: in Lemma 5.8, we define $(M(\mathbb{C}, V^{\otimes 3}), \bar{L}_e^{\otimes 3(0123)}(z, z_1, z_2, z_3, \lambda) = R_e^{(01)}(z - z_1, \lambda - 2\eta(h_2 + h_3))L_e^{\otimes 2(023)}(z, z_2, z_3, \lambda))$ and in Lemma 5.9 $(M(\mathbb{C}, V^{\otimes 3}), \bar{L}_{aux,e}^{\mathbb{C}(0123)}(z, z_1, z_2, z_3, \lambda))$. In Proposition 5.10, we show that by means of the iosmorphism given in Lemma 5.7 both representations are indeed isomorphic. This is a special case, $n = 3$, of Proposition 4.343. Finally, we show that in the basis of $V^{\otimes 3}$ given by the isomorphism $A_{3,e}(z_1, z_2, z_3, \lambda)$ needed in Theorem 4.44 – where we start with the standard tensor product basis of $V^{\otimes 3}$ – is indeed a basis in which $\bar{a}_{SOS,e}(z, z_1, z_2, z_3, \lambda)$ is diagonal (Lemma 5.11).
**Remark:**
Having calculated the auxiliary representation in the case $n = 2$, we may reiterate the same steps in the case $n = 3$ using what we obtained before.
Let us first state the isomorphism of Proposition 4.43 in the case $n = 3$.

**Lemma 5.7** *The isomorphism* $\mathcal{A}_{3,e}(z_1, z_2, z_3, \lambda) \in End\ (V^{\otimes 3}) \subset End\ (M(\mathbb{C}, V^{\otimes 3}))$ *is given by*

$$\mathcal{A}_{3,e}(z_1, z_2, z_3, \lambda) =$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & a_{52} & a_{53} & 0 & a_{55} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{64} & 0 & a_{66} & 0 & 0 \\ 0 & 0 & 0 & a_{74} & 0 & 0 & a_{77} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

*with*

$$\begin{aligned}
a_{52} &= \frac{\theta(2\eta)\theta(\lambda + z_1 - z_3 - 2\eta)\theta(z_2 - z_3 + 2\eta)}{\theta(\lambda + 2\eta)\theta(z_2 - z_3)\theta(z_1 - z_3)}, \\
a_{53} &= \frac{\theta(2\eta)\theta(\lambda + z_1 - z_2 - 2\eta)\theta(z_2 - z_3 - 2\eta)}{\theta(\lambda + 2\eta)\theta(z_1 - z_2)\theta(z_2 - z_3)}, \\
a_{64} &= \frac{\theta(2\eta)\theta(\lambda + z_1 - z_2 + 2\eta)}{\theta(\lambda + 4\eta)\theta(z_1 - z_2)}, \\
a_{74} &= \frac{\theta(2\eta)\theta(\lambda + z_1 - z_3 + 2\eta)}{\theta(\lambda + 4\eta)\theta(z_1 - z_3)}, \\
a_{55} &= \frac{\theta(\lambda - 2\eta)\theta(z_1 - z_2 - 2\eta)\theta(z_1 - z_3 - 2\eta)}{\theta(\lambda + 2\eta)\theta(z_1 - z_2)\theta(z_1 - z_3)}, \\
a_{66} &= \frac{\theta(\lambda + 2\eta)\theta(z_1 - z_2 - 2\eta)}{\theta(\lambda + 4\eta)\theta(z_1 - z_2)}, \\
a_{77} &= \frac{\theta(\lambda + 2\eta)\theta(z_1 - z_3 - 2\eta)}{\theta(\lambda + 4\eta)\theta(z_1 - z_3)}.
\end{aligned}$$

*Its inverse is given by*

$$(\mathcal{A}_{3,e})^{-1}(z_1, z_2, z_3, \lambda) =$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & a_{52}^{-1} & a_{53}^{-1} & 0 & a_{55}^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{64}^{-1} & 0 & a_{66}^{-1} & 0 & 0 \\ 0 & 0 & 0 & a_{74}^{-1} & 0 & 0 & a_{77}^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

*with*

$$
\begin{aligned}
a^{-1}_{52} &= -\frac{\theta(2\eta)\theta(\lambda+z_1-z_3-2\eta)\theta(z_1-z_2)\theta(z_2-z_3+2\eta)}{\theta(\lambda-2\eta)\theta(z_1-z_2-2\eta)\theta(z_1-z_3-2\eta)\theta(z_2-z_3)},\\
a^{-1}_{53} &= -\frac{\theta(2\eta)\theta(\lambda+z_1-z_2-2\eta)\theta(z_1-z_3)\theta(z_2-z_3-2\eta)}{\theta(\lambda-2\eta)\theta(z_2-z_3)\theta(z_1-z_2-2\eta)\theta(z_1-z_3-2\eta)},\\
a^{-1}_{64} &= -\frac{\theta(2\eta)\theta(\lambda+z_1-z_2+2\eta)}{\theta(\lambda+2\eta)\theta(z_1-z_2-2\eta)},\\
a^{-1}_{74} &= -\frac{\theta(2\eta)\theta(\lambda+z_1-z_3+2\eta)}{\theta(\lambda+2\eta)\theta(z_1-z_3-2\eta)},\\
a^{-1}_{55} &= \frac{\theta(z_1-z_2)\theta(z_1-z_3)\theta(\lambda+2\eta)}{\theta(\lambda-2\eta)\theta(z_1-z_2-2\eta)\theta(z_1-z_3-2\eta)},\\
a^{-1}_{66} &= \frac{\theta(z_1-z_2)\theta(\lambda+4\eta)}{\theta(z_1-z_2-2\eta)\theta(\lambda+2\eta)},\\
a^{-1}_{77} &= \frac{\theta(z_1-z_3)\theta(\lambda+4\eta)}{\theta(z_1-z_3-2\eta)\theta(\lambda+2\eta)}.
\end{aligned}
$$

**Proof:**
The proof consists in filling the appropriate terms into Definition 4.42. Multiplication of $\mathcal{A}_{3,e}(z_1,z_2,z_3,\lambda)$ with $(\mathcal{A}_{3,e})^{-1}(z_1,z_2,z_3,\lambda)$ shows that the inverse was properly chosen. Let us now continue by describing the tensor product representation $(M(\mathbb{C}, V^{\otimes 3}), \bar{L}^{\otimes 3}_e(z,z_1,z_2,z_3,\lambda) = R_e(z-z_1,\lambda-2\eta(h_2+h_3)) \otimes \bar{L}_{aux,e}(z,z_1,z_2,\lambda))$ and the auxiliary representation $(M(\mathbb{C},V^{\otimes 3}), \bar{L}^{\mathbb{C}}_{aux,e}(z,z_1,z_2,z_3,\lambda))$as in the previous case.

**Lemma 5.8** *The entries of* $\bar{L}^{\otimes 3}_e(z,z_1,z_2,z_3,\lambda)$ *– namely* $\bar{a}^{\otimes 3}_e(z,z_1,z_2,z_3,\lambda)$*,*
$\bar{b}^{\otimes 3}_e(z,z_1,z_2,z_3,\lambda)$*,*
$\bar{c}^{\otimes 3}_e(z,z_1,z_2,z_3,\lambda), \bar{D}et^{\otimes 3}_e(z,z_1,z_2,z_3)$ *are given by the following four expressions*

$$
\begin{gathered}
\bar{a}^{\otimes 3}_e(z,z_1,z_2,z_3,\lambda) = \\
a_e(z-z_1,\lambda-2(x_2+x_3)) \otimes \bar{a}^{\mathbb{C}}_{aux,e}(z,z_2,z_3,\lambda) + \\
b_e(z-z_1,\lambda-2(x_2+x_3)) \otimes \bar{c}^{\mathbb{C}}_{aux,e}(z,z_2,z_3,\lambda) = \\
\begin{pmatrix}
a_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & a_{22} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & a_{33} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & a_{44} & 0 & 0 & 0 & 0\\
0 & a_{52} & a_{53} & 0 & a_{55} & 0 & 0 & 0\\
0 & 0 & 0 & a_{64} & 0 & a_{66} & 0 & 0\\
0 & 0 & 0 & a_{74} & 0 & 0 & a_{77} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{88}
\end{pmatrix} T^{-2\eta}_{\lambda}.
\end{gathered}
$$

*with*

$$a_{11} = \theta(z - z_1 + 2\eta)\theta(z - z_2 + 2\eta)\theta(z - z_3 + 2\eta),$$
$$a_{22} = \theta(z - z_1 + 2\eta)\theta(z - z_2 + 2\eta)\frac{\theta(z - z_3)\theta(\lambda + 2\eta)}{\theta(\lambda)},$$
$$a_{33} = \theta(z - z_1 + 2\eta)\theta(z - z_2)\frac{\theta(z - z_3 + 2\eta)\theta(\lambda + 2\eta)}{\theta(\lambda)},$$
$$a_{44} = \frac{\theta(z - z_1 + 2\eta)\theta(z - z_2)\theta(z - z_3)\theta(\lambda + 4\eta)}{\theta(\lambda)},$$
$$a_{55} = \frac{\theta(z - z_1)\theta(z - z_2 + 2\eta)\theta(z - z_3 + 2\eta)\theta(\lambda - 2\eta)}{\theta(\lambda - 4\eta)},$$
$$a_{66} = \frac{\theta(z - z_1)\theta(z - z_2 + 2\eta)\theta(z - z_3)\theta(\lambda + 2\eta)^2}{\theta(\lambda)^2},$$
$$a_{77} = \frac{\theta(z - z_1)\theta(z - z_2)\theta(z - z_3 + 2\eta)\theta(\lambda + 2\eta)^2}{\theta(\lambda)^2},$$
$$a_{88} = \frac{\theta(z - z_1)\theta(z - z_2)\theta(z - z_3)\theta(\lambda + 6\eta)}{\theta(\lambda)},$$
$$a_{52} = \frac{\theta(\lambda + z - z_3 - 2\eta)\theta(z - z_2 + 2\eta)\theta(2\eta)^2\theta(\lambda - 4\eta - z + z_1)\theta(z_2 - z_3 + 2\eta)}{\theta(\lambda)\theta(\lambda - 4\eta)\theta(z_2 - z_3)},$$
$$a_{53} = \frac{\theta(\lambda + z - z_2 - 2\eta)\theta(z - z_3 + 2\eta)\theta(2\eta)^2\theta(\lambda - 4\eta - z + z_1)\theta(z_2 - z_3 - 2\eta)}{\theta(\lambda)\theta(\lambda - 4\eta)\theta(z_2 - z_3)},$$
$$a_{64} = \frac{\theta(\lambda + z - z_2 + 2\eta)\theta(z - z_3)\theta(2\eta)^2\theta(\lambda - z + z_1)}{\theta(\lambda)^2},$$
$$a_{74} = \frac{\theta(\lambda + z - z_3 + 2\eta)\theta(z - z_2)\theta(2\eta)^2\theta(\lambda - z + z_1)}{\theta(\lambda)^2},$$

$$\textit{and} \quad \bar{b}_e^{\otimes 3}(z, z_1, z_2, z_3, \lambda) =$$
$$a_e(z - z_1, \lambda - 2(x_2 + x_3)) \otimes \bar{b}^{\mathbb{C}}_{aux,e}(z, z_2, z_3, \lambda) +$$
$$b_e(z - z_1, \lambda - 2(x_2 + x_3)) \otimes \bar{d}^{\mathbb{C}}_e(z, z_2, z_3, \lambda),$$

*where the operator is not written down as a matrix due to the complicated structure of* $\bar{d}^{\mathbb{C}}_e(z, z_2, z_3, \lambda)$*. The last two operators read:*

$$\bar{c}_e^{\otimes 3}(z, z_1, z_2, z_3, \lambda) =$$
$$\begin{pmatrix} 0 & c_{12} & c_{13} & 0 & c_{15} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{24} & 0 & c_{26} & 0 & 0 \\ 0 & 0 & 0 & c_{34} & 0 & 0 & c_{37} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{48} \\ 0 & 0 & 0 & 0 & 0 & c_{56} & c_{57} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{68} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{78} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} T_\lambda^{-2\eta}.$$

*with*

$$c_{12} = \frac{\theta(z-z_1)\theta(\lambda-6\eta)\theta(2\eta)\theta(\lambda+z-z_3-2\eta)\theta(z-z_2+2\eta)\theta(z_2-z_3+2\eta)}{\theta(\lambda-4\eta)\theta(\lambda)\theta(z_2-z_3)},$$

$$c_{13} = \frac{\theta(z-z_1)\theta(\lambda-6\eta)\theta(2\eta)\theta(2\eta)\theta(\lambda+z-z_2-2\eta)\theta(z-z_3+2\eta)\theta(z_2-z_3-2\eta)}{\theta(\lambda-4\eta)\theta(\lambda)\theta(z_2-z_3)},$$

$$c_{24} = \frac{\theta(z-z_1)\theta(\lambda-2\eta)\theta(\lambda+z-z_2+2\eta)\theta(z-z_3)\theta(2\eta)}{\theta(\lambda)^2},$$

$$c_{34} = \frac{\theta(z-z_1)\theta(\lambda-2\eta)\theta(\lambda+z-z_3+2\eta)\theta(z-z_2)\theta(2\eta)}{\theta(\lambda)^2},$$

$$c_{15} = \frac{\theta(2\eta)\theta(\lambda-4\eta+z-z_1)\theta(z-z_2+2\eta)\theta(z-z_3+2\eta)}{\theta(\lambda-4\eta)},$$

$$c_{26} = \frac{\theta(2\eta)\theta(\lambda+z-z_1)\theta(z-z_2+2\eta)\theta(z-z_3)\theta(\lambda+2\eta)}{\theta(\lambda)^2},$$

$$c_{37} = \frac{\theta(2\eta)\theta(\lambda+z-z_1)\theta(z-z_2)\theta(z-z_3+2\eta)\theta(\lambda+2\eta)}{\theta(\lambda)^2},$$

$$c_{48} = \frac{\theta(2\eta)\theta(\lambda+z-z_1+4\eta)\theta(z-z_2)\theta(z-z_3)}{\theta(\lambda)},$$

$$c_{56} = \frac{\theta(2\eta)\theta(z-z_1+2\eta)\theta(\lambda+z-z_3-2\eta)\theta(z-z_2+2\eta)\theta(z_2-z_3+2\eta)}{\theta(\lambda)\theta(z_2-z_3)},$$

$$c_{57} = \frac{\theta(2\eta)\theta(z-z_1+2\eta)\theta(\lambda+z-z_2-2\eta)\theta(z-z_3+2\eta)\theta(z_2-z_3-2\eta)}{\theta(\lambda)\theta(z_2-z_3)},$$

$$c_{68} = \frac{\theta(z-z_1+2\eta)\theta(\lambda+z-z_2+2\eta)\theta(z-z_3)\theta(2\eta)}{\theta(\lambda)},$$

$$c_{78} = \frac{\theta(z-z_1+2\eta)\theta(\lambda+z-z_3+2\eta)\theta(z-z_3)\theta(2\eta)}{\theta(\lambda)}$$

*and*

$$\begin{aligned}\bar{D}et_e^{\otimes 3}(z,z_1,z_2,z_3) &= \theta(z-z_1+2\eta)\theta(z-z_2+2\eta)\theta(z-z_3+2\eta)\\ &\quad \theta(z-z_1-2\eta)\theta(z-z_2-2\eta)\theta(z-z_3-2\eta)\mathbb{I}_3.\end{aligned}$$

**Proof:**
This is a straightforward calculation, taking into account the multiplicative property of the quantum determinant.

**Lemma 5.9** *The auxiliary representation* $(M(\mathbb{C},V^{\otimes 3}),\bar{L}^{\mathbb{C}}_{aux,e}(z,z_1,z_2,z_3,\lambda))$ *is given by*

$$\bar{a}^{\mathbb{C}}_{aux,e}(z,z_1,z_2,z_3,\lambda) = \begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{33} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{55} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{77} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{88} \end{pmatrix} T_\lambda^{-2\eta}$$

*with*

$$\begin{aligned}
a_{11} &= \theta(z-z_1+2\eta)\theta(z-z_2+2\eta)\theta(z-z_3+2\eta),\\
a_{22} &= \frac{\theta(z-z_1+2\eta)\theta(z-z_2+2\eta)\theta(z-z_3)\theta(\lambda+2\eta)}{\theta(\lambda)},\\
a_{33} &= \frac{\theta(z-z_1+2\eta)\theta(z-z_3+2\eta)\theta(z-z_2)\theta(\lambda+2\eta)}{\theta(\lambda)},\\
a_{44} &= \frac{\theta(z-z_1+2\eta)\theta(z-z_2)\theta(z-z_3)\theta(\lambda+4\eta)}{\theta(\lambda)},\\
a_{55} &= \frac{\theta(z-z_1)\theta(z-z_2+2\eta)\theta(z-z_3+2\eta)\theta(\lambda+2\eta)}{\theta(\lambda)},\\
a_{66} &= \frac{\theta(z-z_1)\theta(z-z_2+2\eta)\theta(z-z_3)\theta(\lambda+4\eta)}{\theta(\lambda)},\\
a_{77} &= \frac{\theta(z-z_1)\theta(z-z_2)\theta(z-z_3+2\eta)\theta(\lambda+4\eta)}{\theta(\lambda)},\\
a_{88} &= \frac{\theta(z-z_1)\theta(z-z_2)\theta(z-z_3)\theta(\lambda+6\eta)}{\theta(\lambda)},
\end{aligned}$$

$$\bar{b}^{\mathbb{C}}_{aux,e}(z,z_1,z_2,z_3,\lambda) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
b_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
b_{31} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & b_{42} & b_{43} & 0 & 0 & 0 & 0 & 0\\
b_{51} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & b_{62} & 0 & 0 & b_{65} & 0 & 0 & 0\\
0 & 0 & b_{73} & 0 & b_{75} & 0 & 0 & 0\\
0 & 0 & 0 & b_{84} & 0 & b_{86} & b_{87} & 0
\end{pmatrix} T_\lambda^{+2\eta}$$

*with*

$$\begin{aligned}
b_{21} &= \frac{\theta(\lambda-z+z_3)\theta(2\eta)\theta(z-z_1+2\eta)\theta(z-z_2+2\eta)}{\theta(\lambda)},\\
b_{31} &= \frac{\theta(\lambda-z+z_2)\theta(2\eta)\theta(z-z_1+2\eta)\theta(z-z_3+2\eta)}{\theta(\lambda)},\\
b_{42} &= \frac{\theta(z_2-z_3+2\eta)\theta(\lambda-z+z_2)\theta(2\eta)\theta(z-z_1+2\eta)\theta(z-z_3)}{\theta(z_2-z_3)\theta(\lambda)},\\
b_{43} &= \frac{\theta(z_2-z_3-2\eta)\theta(\lambda-z+z_3)\theta(2\eta)\theta(z-z_1+2\eta)\theta(z-z_2)}{\theta(z_2-z_3)\theta(\lambda)},\\
b_{51} &= \frac{\theta(\lambda-z+z_1)\theta(2\eta)\theta(z-z_2+2\eta)\theta(z-z_3+2\eta)}{\theta(\lambda)},\\
b_{62} &= \frac{\theta(\lambda-z+z_1)\theta(2\eta)\theta(z-z_2+2\eta)\theta(z-z_3)\theta(z_1-z_3+2\eta)}{\theta(z_1-z_3)\theta(\lambda)},\\
b_{65} &= \frac{\theta(\lambda-z+z_3)\theta(2\eta)\theta(z-z_1)\theta(z-z_2+2\eta)\theta(z_1-z_3-2\eta)}{\theta(z_1-z_3)\theta(\lambda)},\\
b_{73} &= \frac{\theta(\lambda-z+z_1)\theta(2\eta)\theta(z-z_2)\theta(z-z_3+2\eta)\theta(z_1-z_2+2\eta)}{\theta(z_1-z_2)\theta(\lambda)},\\
b_{75} &= \frac{\theta(\lambda-z+z_2)\theta(z-z_1)\theta(z-z_2+2\eta)\theta(z_1-z_2-2\eta)\theta(2\eta)}{\theta(z_1-z_2)\theta(\lambda)},
\end{aligned}$$

$$
\begin{aligned}
b_{84} &= \frac{\theta(z_1-z_2+2\eta)\theta(z_1-z_3+2\eta)\theta(2\eta)\theta(\lambda-z+z_1)}{\theta(z_1-z_2)\theta(z_2-z_3)} \times \\
&\times \frac{\theta(z-z_2)\theta(z-z_3)}{\theta(\lambda)}, \\
b_{86} &= \frac{\theta(\lambda-z+z_2)\theta(2\eta)\theta(z-z_1)\theta(z-z_3)}{\theta(z_1-z_3)\theta(z_2-z_3)} \times \\
&\times \frac{\theta(z_1-z_2-2\eta)\theta(z_2-z_3+2\eta)}{\theta(\lambda)}, \\
b_{87} &= \frac{\theta(\lambda-z+z_3)\theta(2\eta)\theta(z-z_1)\theta(z-z_2)}{\theta(z_1-z_3)\theta(z_2-z_3)} \times \\
&\times \frac{\theta(z_1-z_3-2\eta)\theta(z_2-z_3-2\eta)}{\theta(\lambda)}.
\end{aligned}
$$

$$
\bar{c}^{\mathbb{C}}_{aux,e}(z,z_1,z_2,z_3,\lambda) = \begin{pmatrix}
0 & c_{12} & c_{13} & 0 & c_{15} & 0 & 0 & 0 \\
0 & 0 & 0 & c_{24} & 0 & c_{26} & 0 & 0 \\
0 & 0 & 0 & c_{34} & 0 & 0 & c_{37} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{48} \\
0 & 0 & 0 & 0 & 0 & c_{56} & c_{57} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{68} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{78} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} T_\lambda^{-2\eta}
$$

*with*

$$
\begin{aligned}
c_{12} &= \frac{\theta(2\eta)\theta(\lambda+z-z_3-4\eta)\theta(z-z_1+2\eta)\theta(z-z_2+2\eta)}{\theta(z_1-z_2)\theta(z_2-z_3)} \times \\
&\times \frac{\theta(z_1-z_3+2\eta)\theta(z_2-z_3+2\eta)}{\theta(\lambda)}, \\
c_{13} &= \frac{\theta(2\eta)\theta(\lambda+z-z_2-4\eta)\theta(z-z_1+2\eta)\theta(z-z_3+2\eta)}{\theta(z_1-z_2)\theta(z_2-z_3)} \times \\
&\times \frac{\theta(z_1-z_2+2\eta)\theta(z_2-z_3-2\eta)}{\theta(\lambda)}, \\
c_{15} &= \frac{\theta(2\eta)\theta(\lambda+z-z_1-4\eta)\theta(z-z_2+2\eta)\theta(z-z_3+2\eta)}{\theta(z_1-z_2)\theta(z_1-z_3)} \times \\
&\times \frac{\theta(z_1-z_2-2\eta)\theta(z_1-z_3-2\eta)}{\theta(\lambda)}, \\
c_{24} &= \frac{\theta(2\eta)\theta(\lambda+z-z_2)\theta(z-z_3)\theta(z-z_1+2\eta)\theta(z_1-z_2+2\eta)}{\theta(\lambda)\theta(z_1-z_2)}, \\
c_{26} &= \frac{\theta(2\eta)\theta(\lambda+z-z_1)\theta(z-z_2+2\eta)\theta(z-z_3)\theta(z_1-z_2-2\eta)}{\theta(z_1-z_2)\theta(\lambda)}, \\
c_{34} &= \frac{\theta(2\eta)\theta(\lambda+z-z_3)\theta(z-z_1+2\eta)\theta(z-z_2)\theta(z_1-z_3+2\eta)}{\theta(\lambda)\theta(z_1-z_3)}, \\
c_{37} &= \frac{\theta(2\eta)\theta(\lambda+z-z_1)\theta(z-z_2)\theta(z-z_3+2\eta)\theta(z_1-z_3-2\eta)}{\theta(z_1-z_3)\theta(\lambda)},
\end{aligned}
$$

$$
\begin{aligned}
c_{48} &= \frac{\theta(2\eta)\theta(\lambda+z-z_1+4\eta)\theta(z-z_2)\theta(z-z_3)}{\theta(\lambda)}, \\
c_{56} &= \frac{\theta(2\eta)\theta(\lambda+z-z_3)\theta(z-z_1)\theta(z-z_2+2\eta)\theta(z_2-z_3+2\eta)}{\theta(z_2-z_3)\theta(\lambda)}, \\
c_{57} &= \frac{\theta(2\eta)\theta(\lambda+z-z_2)\theta(z-z_1)\theta(z-z_3+2\eta)\theta(z_2-z_3-2\eta)}{\theta(z_2-z_3)\theta(\lambda)}, \\
c_{68} &= \frac{\theta(2\eta)\theta(\lambda+z-z_2+4\eta)\theta(z-z_1)\theta(z-z_3)}{\theta(\lambda)}, \\
c_{78} &= \frac{\theta(2\eta)\theta(\lambda+z-z_3+4\eta)\theta(z-z_1)\theta(z-z_2)}{\theta(\lambda)},
\end{aligned}
$$

$$
\begin{aligned}
\bar{D}et^{\mathbb{C}}_{aux,e}(z,z_1,z_2,z_3) &= \theta(z-z_1-2\eta)\theta(z-z_2-2\eta)\theta(z-z_3-2\eta)\times \\
&\times\ \theta(z-z_1+2\eta)\theta(z-z_2+2\eta)\theta(z-z_3+2\eta)\,\mathbb{I}_3.
\end{aligned}
$$

**Proof:**
This is a rewriting of the definition of the auxiliary representation of Proposition 4.33 for the case $n=3$.

**Proposition 5.10**

$$
\begin{aligned}
(\mathcal{A}_{3,e})^{-1}\bar{a}_e^{\otimes 3}(z,z_1,z_2,z_3,\lambda)\mathcal{A}_{3,e} &= \bar{a}^{\mathbb{C}}_{aux,e}(z,z_1,z_2,z_3,\lambda), &(91)\\
(\mathcal{A}_{3,e})^{-1}\bar{b}_e^{\otimes 3}(z,z_1,z_2,z_3,\lambda)\mathcal{A}_{3,e} &= \bar{b}^{\mathbb{C}}_{aux,e}(z,z_1,z_2,z_3,\lambda), &(92)\\
(\mathcal{A}_{3,e})^{-1}\bar{c}_e^{\otimes 3}(z,z_1,z_2,z_3,\lambda)\mathcal{A}_{3,e} &= \bar{c}^{\mathbb{C}}_{aux,e}(z,z_1,z_2,z_3,\lambda), &(93)\\
(\mathcal{A}_{3,e})^{-1}\bar{D}et_e^{\otimes 3}(z,z_1,z_2,z_3)\mathcal{A}_{3,e} &= \bar{D}et^{\mathbb{C}}_{aux,e}(z,z_1,z_2,z_3), &(94)
\end{aligned}
$$

*with* $\mathcal{A}_{3,e}=\mathcal{A}_{3,e}(z_1,z_2,z_3,\lambda)$.

**Proof:**
Throughout the proof, let us write $\mathcal{A}_{3,e}$ instead of $\mathcal{A}_{3,e}(z_1,z_2,z_3,\lambda)$.
Let us start with the easiest identity. $Det_e^{\otimes 3}(z,z_1,z_2,z_3)$ is a function independent of $x_1,x_2,x_3$. Hence, we may write

$$
\begin{aligned}
(\mathcal{A}_{3,e})^{-1}\bar{D}et_{1,e}^{\otimes 3}(z,z_1,z_2,z_3)\mathcal{A}_{3,e} &= Det_{1,e}^{\otimes 3}(z,z_1,z_2,z_3)(\mathcal{A}_{3,e})^{-1}\mathbb{I}_3\mathcal{A}_{3,e}\\
&= \bar{D}et^{\mathbb{C}}_{aux,e}(z,z_1,z_2,z_3),
\end{aligned}
$$

since the formulas of the determinant of the tensored representation and the auxiliary representation coincide.
Let us now first check the identity involving the operator $\bar{a}^{\mathbb{C}}_{aux,e}(z,z_1,z_2,z_3,\lambda)$, then the one involving $\bar{c}^{\mathbb{C}}_{aux,e}(z,z_1,z_2,z_3,\lambda)$. The remaining identity can the be checked by identical methods.

$$
(\mathcal{A}_e^3)^{-1}a_{1,e}^{\otimes 3}(z,z_1,z_2,z_3,\lambda)\mathcal{A}_e^3 =
$$

$$
\begin{pmatrix}
a_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & a_{22} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & a_{33} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & a_{44} & 0 & 0 & 0 & 0\\
0 & a_{52} & a_{53} & 0 & a_{55} & 0 & 0 & 0\\
0 & 0 & 0 & a_{64} & 0 & a_{66} & 0 & 0\\
0 & 0 & 0 & a_{74} & 0 & 0 & a_{77} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{88}
\end{pmatrix} T_\lambda^{-2\eta}.
$$

If the steps of the multiplication are performed, we perceive that the diagonal entries of the above matrix coincide with the entries of $a^3_{aux,e}(z, z_1, z_2, z_3, \lambda)$ as given in Lemma 5.9.
The off-diagonal entries are as follows and are to be shown to equal zero:

$$a_{52} = -\frac{\theta(2\eta)\theta(\lambda + z_1 - z_3 - 2\eta)\theta(z_1 - z_2)\theta(z_2 - z_3 + 2\eta)\theta(z - z_1 + 2\eta)\theta(z - z_2 + 2\eta)\theta(z - z_3)\theta(\lambda + 2\eta)}{\theta(\lambda)\theta(\lambda - 2\eta)\theta(z_1 - z_2 - 2\eta)\theta(z_1 - z_3 - 2\eta)\theta(z_2 - z_3)}$$
$$+\frac{\theta(2\eta)^2\theta(z_1 - z_2)\theta(z_1 - z_3)\theta(\lambda + 2\eta)\theta(\lambda + z - z_3 - 2\eta)\theta(z - z_2 + 2\eta)\theta(\lambda - 4\eta - z + z_1)\theta(z_2 - z_3 + 2\eta)}{\theta(\lambda - 2\eta)\theta(z_1 - z_2 - 2\eta)\theta(z_1 - z_3 - 2\eta)\theta(\lambda)\theta(\lambda - 4\eta)\theta(z_2 - z_3)}$$
$$+\frac{\theta(2\eta)\theta(z_1 - z_2)\theta(\lambda + 2\eta)\theta(z - z_1)\theta(z - z_2 + 2\eta)\theta(z - z_3 + 2\eta)\theta(\lambda + z_1 - z_3 - 4\eta)\theta(z_2 - z_3 + 2\eta)}{\theta(z_1 - z_2 - 2\eta)\theta(z_1 - z_3 - 2\eta)\theta(\lambda - 4\eta)\theta(z_2 - z_3)\theta(\lambda)},$$
$$a_{53} = -\frac{\theta(2\eta)\theta(\lambda + z_1 - z_2 - 2\eta)\theta(z_1 - z_3)\theta(z_2 - z_3 - 2\eta)\theta(z - z_1 + 2\eta)\theta(z - z_2)\theta(z - z_3 + 2\eta)\theta(\lambda + 2\eta)}{\theta(\lambda - 2\eta)\theta(z_2 - z_3)\theta(z_1 - z_2 - 2\eta)\theta(z_1 - z_3 - 2\eta)\theta(\lambda)}$$
$$+\frac{\theta(z_1 - z_2)\theta(z_1 - z_3)\theta(\lambda + 2\eta)\theta(2\eta)^2\theta(\lambda + z - z_2 - 2\eta)\theta(z - z_3 + 2\eta)\theta(z_2 - z_3 - 2\eta)\theta(\lambda - 4\eta - z + z_1)}{\theta(\lambda - 2\eta)\theta(z_1 - z_3 - 2\eta)\theta(z_1 - z_2 - 2\eta)\theta(\lambda)\theta(\lambda - 4\eta)\theta(z_2 - z_3)}$$
$$+\frac{\theta(z_1 - z_3)\theta(\lambda + 2\eta)\theta(z - z_1)\theta(z - z_2 + 2\eta)\theta(z - z_3 + 2\eta)\theta(2\eta)\theta(\lambda + z_1 - z_2 - 4\eta)\theta(z_2 - z_3 - 2\eta)}{\theta(z_1 - z_3 - 2\eta)\theta(z_1 - z_2 - 2\eta)\theta(\lambda)\theta(\lambda - 4\eta)\theta(z_2 - z_3)},$$
$$a_{64} = -\frac{\theta(2\eta)\theta(\lambda + z_1 - z_2 + 2\eta)\theta(z - z_1 + 2\eta)\theta(z - z_2)\theta(z - z_3)\theta(\lambda + 4\eta)}{\theta(\lambda + 2\eta)\theta(\lambda)\theta(z_1 - z_2 - 2\eta)}$$
$$+\frac{\theta(z_1 - z_2)\theta(\lambda + 4\eta)\theta(\lambda + z - z_2 + 2\eta)\theta(z - z_3)\theta(2\eta)^2\theta(\lambda - z + z_1)}{\theta(z_1 - z_2 - 2\eta)\theta(\lambda + 2\eta)\theta(\lambda)^2}$$
$$+\frac{\theta(z_1 - z_2)\theta(\lambda + 4\eta)\theta(z - z_1)\theta(z - z_2 + 2\eta)\theta(z - z_3)\theta(2\eta)\theta(\lambda + z_1 - z_2)}{\theta(z_1 - z_2 - 2\eta)\theta(\lambda)^2\theta(z_1 - z_2)},$$

$$a_{74} = -\frac{\theta(\lambda + z_1 - z_3 + 2\eta)\theta(2\eta)\theta(z - z_1 + 2\eta)\theta(z - z_2)\theta(z - z_3)\theta(\lambda + 4\eta)}{\theta(\lambda + 2\eta)\theta(z_1 - z_3 - 2\eta)\theta(\lambda)}$$
$$+\frac{\theta(z_1 - z_3)\theta(\lambda + 4\eta)\theta(\lambda + z - z_3 + 2\eta)\theta(z - z_2)\theta(2\eta)^2\theta(\lambda - z + z_1)}{\theta(z_1 - z_3 - 2\eta)\theta(\lambda + 2\eta)\theta(\lambda)^2}$$
$$+\frac{\theta(\lambda + 4\eta)\theta(z - z_1)\theta(z - z_2)\theta(z - z_3 + 2\eta)\theta(2\eta)\theta(\lambda + z_1 - z_3)}{\theta(z_1 - z_3 - 2\eta)\theta(\lambda)^2}.$$

Let us perform the necessary calculation of one of those entries, e.g. $a_{52}$. If $\lambda \to \lambda + \tau$, then each summand is multiplied by a factor $e^{-2\pi i(z_1 - z_3 + 2\eta)}$, whereas $\lambda \to \lambda + 1$ each summand stays unchanged, both properties due to the transformation properties of the odd Jacobi theta function.
Let us now check the residues which are at $\lambda = 4\eta, \lambda = 2\eta, \lambda = 0$.

$$\mathrm{Res}_{\lambda=4\eta}(a_{52}) = -\frac{\theta(z_1 - z_2)\theta(z_1 - z_3)\theta(6\eta)\theta(2\eta)\theta(2\eta + z - z_3)\theta(z - z_2 + 2\eta)\theta(z - z_1)\theta(z_2 - z_3 + 2\eta)}{\theta(z_1 - z_2 - 2\eta)\theta(z_1 - z_3 - 2\eta)\theta(4\eta)\theta(z_2 - z_3)}$$
$$+\frac{\theta(z_1 - z_2)\theta(6\eta)\theta(z - z_1)\theta(z - z_2 + 2\eta)\theta(z - z_3 + 2\eta)\theta(2\eta)\theta(z_1 - z_3)\theta(z_2 - z_3 + 2\eta)}{\theta(z_1 - z_2 - 2\eta)\theta(z_1 - z_3 - 2\eta)\theta(z_2 - z_3)\theta(4\eta)} = 0,$$
$$\mathrm{Res}_{\lambda=2\eta}(a_{52}) = -\frac{\theta(z_1 - z_3)\theta(z_1 - z_2)\theta(z_2 - z_3 + 2\eta)\theta(z - z_1 + 2\eta)\theta(z - z_2 + 2\eta)\theta(z - z_3)\theta(4\eta)}{\theta(z_1 - z_2 - 2\eta)\theta(z_1 - z_3 - 2\eta)\theta(z_2 - z_3)}$$
$$+\frac{\theta(z_1 - z_2)\theta(z_1 - z_3)\theta(4\eta)\theta(z - z_3)\theta(z - z_2 + 2\eta)\theta(z - z_1 + 2\eta)\theta(z_2 - z_3 + 2\eta)}{\theta(z_1 - z_2 - 2\eta)\theta(z_1 - z_3 - 2\eta)\theta(z_2 - z_3)} = 0,$$
$$\mathrm{Res}_{\lambda}(a_{52}) = \frac{\theta(2\eta)\theta(z_1 - z_3 - 2\eta)\theta(z_1 - z_2)\theta(z_2 - z_3 + 2\eta)\theta(z - z_1 + 2\eta)\theta(z - z_2 + 2\eta)\theta(z - z_3)}{\theta(z_1 - z_2 - 2\eta)\theta(z_1 - z_3 - 2\eta)\theta(z_2 - z_3)}$$
$$-\frac{\theta(z_1 - z_3)\theta(z_1 - z_2)\theta(2\eta)^2\theta(z - z_3 - 2\eta)\theta(z - z_1 + 4\eta)\theta(z_2 - z_3 + 2\eta)\theta(z - z_2 + 2\eta)}{\theta(4\eta)\theta(z_1 - z_2 - 2\eta)\theta(z_1 - z_3 - 2\eta)\theta(z_2 - z_3)}$$
$$-\frac{\theta(z_1 - z_2)\theta(2\eta)^2\theta(z - z_1)\theta(z - z_2 + 2\eta)\theta(z - z_3 + 2\eta)\theta(z_1 - z_3 - 4\eta)\theta(z_2 - z_3 + 2\eta)}{\theta(z_1 - z_2 - 2\eta)\theta(z_1 - z_3 - 2\eta)\theta(4\eta)\theta(z_2 - z_3)}.$$

The last case can be shown to equal zero if we take into account the common transformation behaviour of each summand under transformations of $z \to z + \tau$, yielding multiplication by $e^{-2\pi i(3z - z_1 - z_2 - z_3 + 2\eta)}$, and $z \to z + 1$ yielding multiplication by $(-1)$.

The two last summands can then be shown to equal the negative first one by looking at the first one's zeroes at $z = z_2 - 2\eta, z = z_1 - 2\eta, z = z_3$ which are also zeroes of the sum of the last two summands.
The zero of $a_{52}$ are at $\lambda = -2\eta, \lambda = z_3 - z_1 + 2\eta$. Thus, we see that the negative first summand of $a_{52}$ equals the sum of the last two summands or put differently $a_{52} = 0$.
The entries of the matrix called $a_{53}, a_{64}, a_{74}$ can be checked by the same means.
Thus, we conceive that the first identity of Proposition 5.10 holds true.
Let us now look at $(\mathcal{A}_{3,e})^{-1}\bar{c}_e^{\otimes 3}(z, z_1, z_2, z_3, \lambda)\mathcal{A}_{3,e}$. If we perform the matrix mulitplication we obtain

$$(\mathcal{A}_{3,e})^{-1}\bar{c}_e^{\otimes 3}(z, z_1, z_2, z_3, \lambda)\mathcal{A}_{3,e} = \begin{pmatrix} 0 & c_{12} & c_{13} & 0 & c_{15} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{24} & 0 & c_{26} & 0 & 0 \\ 0 & 0 & 0 & c_{34} & 0 & 0 & c_{37} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{48} \\ 0 & 0 & 0 & c_{54} & 0 & c_{56} & c_{57} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{68} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{78} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with

$$c_{12} = \frac{\theta(z-z_1)\theta(\lambda-6\eta)\theta(2\eta)\theta(\lambda+z-z_3-2\eta)\theta(z-z_2+2\eta)\theta(z_2-z_3+2\eta)}{\theta(\lambda-4\eta)\theta(\lambda)\theta(z_2-z_3)}$$
$$+\frac{\theta(2\eta)^2\theta(\lambda-4\eta+z-z_1)\theta(z-z_2+2\eta)\theta(z-z_3+2\eta)\theta(\lambda+z_1-z_3-4\eta)\theta(z_2-z_3+2\eta)}{\theta(\lambda-4\eta)\theta(\lambda)\theta(z_2-z_3)\theta(z_1-z_3)},$$
$$c_{13} = \frac{\theta(z-z_1)\theta(\lambda-6\eta)\theta(2\eta)\theta(\lambda+z-z_2-2\eta)\theta(z-z_3+2\eta)\theta(z_2-z_3-2\eta)}{\theta(\lambda-4\eta)\theta(\lambda)\theta(z_2-z_3)}$$
$$+\frac{\theta(2\eta)^2\theta(\lambda-4\eta+z-z_1)\theta(z-z_2+2\eta)\theta(z-z_3+2\eta)\theta(\lambda+z_1-z_2-4\eta)\theta(z_2-z_3-2\eta)}{\theta(\lambda-4\eta)\theta(\lambda)\theta(z_1-z_2)\theta(z_2-z_3)},$$
$$c_{15} = \frac{\theta(2\eta)\theta(\lambda-4\eta+z-z_1)\theta(z-z_2+2\eta)\theta(z-z_3+2\eta)\theta(z_1-z_2-2\eta)\theta(z_1-z_3-2\eta)}{\theta(\lambda)\theta(z_1-z_2)\theta(z_1-z_3)},$$
$$c_{24} = \frac{\theta(z-z_1)\theta(\lambda-2\eta)\theta(\lambda+z-z_2+2\eta)\theta(z-z_3)\theta(2\eta)}{\theta(\lambda)^2}$$
$$+\frac{\theta(2\eta)^2\theta(\lambda+z-z_1)\theta(z-z_2+2\eta)\theta(z-z_3)\theta(\lambda+z_1-z_2)}{\theta(\lambda)^2\theta(z_1-z_2)},$$
$$c_{26} = \frac{\theta(2\eta)\theta(\lambda+z-z_1)\theta(z-z_2+2\eta)\theta(z-z_3)\theta(z_1-z_2-2\eta)}{\theta(\lambda)\theta(z_1-z_2)},$$
$$c_{34} = \frac{\theta(z-z_1)\theta(\lambda-2\eta)\theta(\lambda+z-z_3+2\eta)\theta(z-z_2)\theta(2\eta)}{\theta(\lambda)^2}$$
$$+\frac{\theta(2\eta)^2\theta(\lambda+z-z_1)\theta(z-z_3+2\eta)\theta(z-z_2)\theta(\lambda+z_1-z_3)}{\theta(\lambda)^2\theta(z_1-z_3)},$$
$$c_{37} = \frac{\theta(2\eta)\theta(\lambda+z-z_1)\theta(z-z_2)\theta(z-z_3+2\eta)\theta(z_1-z_3-2\eta)}{\theta(\lambda)\theta(z_1-z_3)},$$
$$c_{48} = \frac{\theta(2\eta)\theta(\lambda+z-z_1+4\eta)\theta(z-z_2)\theta(z-z_2)\theta(z-z_3)}{\theta(\lambda)},$$
$$c_{54} = -\frac{\theta(2\eta)^2\theta(\lambda+z_1-z_3-2\eta)\theta(z_2-z_3+2\eta)\theta(z-z_1)\theta(\lambda+z-z_2+2\eta)\theta(z-z_3)}{\theta(z_1-z_2-2\eta)\theta(z_1-z_3-2\eta)\theta(z_2-z_3)\theta(\lambda)^2}$$
$$-\frac{\theta(2\eta)\theta(\lambda+z_1-z_3-2\eta)\theta(z_2-z_3+2\eta)\theta(2\eta)^2\theta(\lambda+z-z_1)\theta(z-z_2+2\eta)\theta(z-z_3)\theta(\lambda+z_1-z_2)}{\theta(\lambda-2\eta)\theta(z_1-z_2-2\eta)\theta(z_1-z_3-2\eta)\theta(z_2-z_3)\theta(\lambda)^2}$$

$$-\frac{\theta(2\eta)^2\theta(\lambda+z_1-z_2-2\eta)\theta(z_1-z_3)\theta(z_2-z_3-2\eta)\theta(z-z_1)\theta(\lambda+z-z_3+2\eta)\theta(z-z_2)}{\theta(z_2-z_3)\theta(z_1-z_2-2\eta)\theta(z_1-z_3-2\eta)\theta(\lambda)^2}$$
$$-\frac{\theta(2\eta)^3\theta(\lambda+z_1-z_2-2\eta)\theta(z_2-z_3-2\eta)\theta(\lambda+z-z_1)\theta(z-z_2)\theta(z-z_3+2\eta)\theta(\lambda+z_1-z_3)}{\theta(\lambda-2\eta)\theta(z_1-z_2-2\eta)\theta(z_1-z_3-2\eta)\theta(\lambda)^2\theta(z_2-z_3)}$$
$$+\frac{\theta(z_1-z_3)\theta(z-z_1+2\eta)\theta(2\eta)^2\theta(\lambda+z-z_3-2\eta)\theta(z_2-z_3+2\eta)\theta(\lambda+z_1-z_2)\theta(z-z_1+2\eta)}{\theta(z_1-z_2-2\eta)\theta(z_1-z_3-2\eta)\theta(\lambda)\theta(z_2-z_3)}$$
$$+\frac{\theta(z_1-z_2)\theta(z-z_1+2\eta)\theta(\lambda+z-z_2-2\eta)\theta(z-z_3+2\eta)\theta(z_2-z_3-2\eta)\theta(2\eta)^2\theta(\lambda+z_1-z_3)}{\theta(\lambda-2\eta)\theta(z_1-z_2-2\eta)\theta(z_1-z_3-2\eta)\theta(\lambda)\theta(z_2-z_3)},$$

$$c_{56}=-\frac{\theta(2\eta)^2\theta(\lambda+z_1-z_3-2\eta)\theta(z_2-z_3+2\eta)\theta(\lambda+z-z_1)\theta(z-z_2+2\eta)\theta(z-z_3)}{\theta(\lambda-2\eta)\theta(z_1-z_3-2\eta)\theta(z_2-z_3)\theta(\lambda)}$$
$$+\frac{\theta(z_1-z_3)\theta(z-z_1+2\eta)\theta(2\eta)\theta(\lambda+z-z_3-2\eta)\theta(z-z_2+2\eta)\theta(z_2-z_3+2\eta)}{\theta(z_1-z_3-2\eta)\theta(\lambda-2\eta)\theta(z_2-z_3)},$$

$$c_{57}=-\frac{\theta(2\eta)^2\theta(\lambda+z_1-z_2-2\eta)\theta(z_2-z_3-2\eta)\theta(\lambda+z-z_1)\theta(z-z_2)\theta(z-z_3+2\eta)}{\theta(\lambda-2\eta)\theta(z_2-z_3)\theta(z_1-z_2--2\eta)\theta(\lambda)}$$
$$+\frac{\theta(z_1-z_2)\theta(z-z_1+2\eta)\theta(2\eta)\theta(\lambda+z-z_2-2\eta)\theta(z-z_3+2\eta)\theta(z_2-z_3-2\eta)}{\theta(z_1-z_2-2\eta)\theta(\lambda-2\eta)\theta(z_2-z_3)},$$

$$c_{68}=-\frac{\theta(2\eta)^2\theta(\lambda+z_1-z_2+2\eta)\theta(\lambda+z-z_1+4\eta)\theta(z-z_2)\theta(z-z_3)}{\theta(\lambda+2\eta)\theta(z_1-z_2-2\eta)\theta(\lambda)}$$
$$+\frac{\theta(z_1-z_2)\theta(\lambda+4\eta)\theta(z-z_1+2\eta)\theta(\lambda+z-z_2+2\eta)\theta(z-z_3)\theta(2\eta)}{\theta(z_1-z_2-2\eta)\theta(\lambda+2\eta)\theta(\lambda)},$$

$$c_{78}=-\frac{\theta(2\eta)^2\theta(\lambda+z_1-z_3+2\eta)\theta(\lambda+z-z_1+4\eta)\theta(z-z_2)\theta(z-z_3)}{\theta(\lambda+2\eta)\theta(z_1-z_3-2\eta)\theta(\lambda)}$$
$$+\frac{\theta(z_1-z_3)\theta(\lambda+4\eta)\theta(z-z_1+2\eta)\theta(\lambda+z-z_3+2\eta)\theta(z-z_2)\theta(2\eta)}{\theta(z_1-z_3-2\eta)\theta(\lambda+2\eta)\theta(\lambda)}.$$

If we compare $c_{15}, c_{26}, c_{37}, c_{48}$ to their counterparts in $c^3_{aux,e}(z, z_1, z_2, z_3, \lambda)$ we see that they coincide. The claim is that also the remaining entries of the conjugated matrix are the same as the corresponding entries of $c^{\mathbb{C}}_{aux,e}(z, z_1, z_2, z_3, \lambda)$, in particular $c_{54} = 0$.
Let us verify this claim in one case. The other cases are treated similarly. We choose $c_{12}$. The claim reads

$$\begin{aligned} c_{12} &= \frac{\theta(z-z_1+2\eta)\theta(z-z_2+2\eta)\theta(\lambda+z-z_3-4\eta)\theta(z_1-z_3+2\eta)}{\theta(\lambda)\theta(z_1-z_3)\theta(z_2-z_3)}\times \\ &\times\ \theta(z_2-z_3+2\eta)\theta(2\eta) = (\bar{c}^{\mathbb{C}}_{aux,e}(z,z_1,z_2,z_3,\lambda))_{12}. \end{aligned}$$

The transformation behaviour of $c_{12}$ is $e^{-2\pi i(z-z_3-4\eta)}$ while $\lambda\to\lambda+\tau$ and $c_{12}$ if $\lambda\to\lambda+1$. This coincides with the behaviour of $(\bar{c}^{\mathbb{C}}_{aux,e}(z,z_1,z_2,z_3,\lambda))_{12}$.
The residue of $c_{12}$ at $\lambda = 4\eta$ vanishes, whereas its residue at $\lambda = 0$ yields

$$\begin{aligned} \mathrm{Res}_{\lambda=0}(c_{12}) &= \frac{\theta(z-z_1)\theta(6\eta)\theta(2\eta)\theta(z-z_3-2\eta)\theta(z-z_2+2\eta)\theta(z_2-z_3+2\eta)}{\theta(4\eta)\theta(z_2-z_3)} \\ &-\frac{\theta(z-z_1-4\eta)\theta(2\eta)^2\theta(z-z_2+2\eta)\theta(z-z_3+2\eta)\theta(z_1-z_3-4\eta)\theta(z_2-z_3+2\eta)}{\theta(4\eta)\theta(z_1-z_3)\theta(z_2-z_3)} \\ &= \frac{\theta(2\eta)\theta(z-z_3-4\eta)\theta(z-z_1+2\eta)\theta(z-z_2+2\eta)\theta(z_1-z_3+2\eta)\theta(z_2-z_3+2\eta)}{\theta(z_1-z_3)\theta(z_2-z_3)} \\ &= \mathrm{Res}_{\lambda=0}(\bar{c}^{\mathbb{C}}_{aux,e}(z,z_1,z_2,z_3,\lambda))_{12}. \end{aligned}$$

That the above equation holds true is conceived by investigating the transformation behaviour with respect to $z\to z+\tau$ which yields a multiplication of every summand by $e^{-2\pi i(3z-z_1-z_2-z_3)}$ and $z\to z+1$ leading to a multiplication by $(-1)$.
The zeroes are also identical occurring at $z = z_2 - 2\eta, z = z_1 - 2\eta, z = z_3 + 4\eta$.

The zeroes of $c_{12}$ are at $\lambda = 4\eta + z_3 - z, z = z_1 - 2\eta, z = z_2 - 2\eta$. Hence, they coincide with the zeroes of $(c^{\mathbb{C}}_{aux,e}(z, z_1, z_2, z_3, \lambda))_{12}$. This shows that

$$c_{12} = (\bar{c}^{\mathbb{C}}_{aux,e}(z, z_1, z_2, z_3, \lambda))_{12}.$$

Note that due to symmetry while interchanging $z_2$ and $z_3$, the same calculation can be used to show that $c_{13} = (\bar{c}^{\mathbb{C}}_{aux,e}(z, z_1, z_2, z_3, \lambda))_{13}$. This proves the second identity of Proposition 5.10. The third identity can be shown by identical means. This completes the proof of Proposition 5.10.
**Remark:**
The last lemma concerns the structure of the eigenvectors of $\bar{a}_{SOS,e}(z, z_1, z_2, z_3, \lambda)$.

**Lemma 5.11** *For $\lambda \neq 0$, a basis of $V^{\otimes 3}$ is given by the set of vectors*

$$B_3 = \{\mathcal{A}_{3,e}(z_1, z_2, z_3, \lambda)(e[\sigma_1] \otimes e[\sigma_2] \otimes e[\sigma_3]) \mid \sigma_i \in \{-1, 1\}, i = 1, 2, 3\}.$$

*with $\mathcal{A}_{3,e}(z_1, z_2, z_3, \lambda) = (\mathbb{I}_2 \otimes \mathcal{A}_{2,e})(z_2, z_3, \lambda)\mathcal{A}_{3,e}(z_1, z_2, z_2, \lambda)$. In this basis the operator*

$$\begin{gathered}\bar{a}_{SOS,e}(z, z_1, z_2, z_3, \lambda) = \\ a_e(z - z_1, \lambda - 2\eta(x_2 + x_3)) \otimes \big(a_e(z - z_2, \lambda - 2\eta x_3) \otimes \bar{a}_e(z - z_3, \lambda) \\ +b_e(z - z_2, \lambda - 2\eta x_3) \otimes \bar{c}_{1,e}(z - z_3, \lambda)\big) + b_e(z - z_1, \lambda - 2\eta(x_2 + x_3)) \otimes \\ \big(c_e(z - z_2, \lambda - 2\eta x_3) \otimes \bar{a}_e(z - z_3, \lambda) + d_e(z - z_2, \lambda - 2\eta x_3) \otimes \bar{c}_e(z - z_3, \lambda)\big)\end{gathered}$$

*is diagonal, as is suggested by Proposition 4.44.*

**Proof:**
We know that in the basis $e[\sigma_1] \otimes e[\sigma_2] \otimes e[\sigma_3], \sigma_i \in \{-1, 1\}, i = 1, 2, 3$, the operator $\bar{a}^{\mathbb{C}}_{aux,e}(z, z_1, z_2, z_3, \lambda)$ is diagonal. By Proposition 5.9, it follows from

$$\begin{gathered}(\mathcal{A}_{3,e})^{-1}\bar{a}_e^{\otimes 3}(z, z_1, z_2, z_3, \lambda)\mathcal{A}_{3,e}e[\sigma_1] \otimes e[\sigma_2] \otimes e[\sigma_3] \\ = \alpha_{\sigma_1\sigma_2\sigma_3}(\lambda)e[\sigma_1] \otimes e[\sigma_2] \otimes e[\sigma_3],\end{gathered}$$

where $\alpha_{\sigma_1\sigma_2\sigma_3}(\lambda)$ denotes the eigenvalue corresponding to the basis vector in case, that

$$\begin{gathered}\bar{a}_e^{\otimes 3}(z, z_1, z_2, z_3, \lambda)\left(\mathcal{A}_{3,e}(e[\sigma_1] \otimes e[\sigma_2] \otimes e[\sigma_3])\right) \\ = \alpha_{\sigma_1\sigma_2\sigma_3}(\lambda)\mathcal{A}_{3,e}\left(e[\sigma_1] \otimes e[\sigma_2] \otimes e[\sigma_3]\right).\end{gathered}$$

Let us now look at the definition of the operator $\bar{a}_e^{\otimes 3}(z, z_1, z_2, z_3, \lambda)$ in Lemma 5.8 yielding

$$\begin{gathered}\big(a_e(z, z_1, \lambda - 2\eta(h_2 + h_3)) \otimes \bar{a}^{\mathbb{C}}_{aux,e}(z, z_2, z_3, \lambda) \\ +b_e(z, z_1, \lambda - 2\eta(h_2 + h_3)) \otimes \bar{c}^{\mathbb{C}}_{aux,e}(z, z_2, z_3, \lambda)\big)\mathcal{A}_{3,e}(e[\sigma_1] \otimes e[\sigma_2] \otimes e[\sigma_3])) \\ = \alpha_{\sigma_1\sigma_2\sigma_3}(\lambda)\mathcal{A}_{3,e}\left(e[\sigma_1] \otimes e[\sigma_2] \otimes e[\sigma_3]\right)\end{gathered}$$

and by Proposition 5.4

$$\begin{gathered}\big(a_e(z, z_1, \lambda - 2\eta(h_2 + h_3)) \otimes ((\mathcal{A}_{2,e})^{-1}\bar{a}_e^{\otimes 2}(z, z_2, z_3, \lambda)\mathcal{A}_{2,e}) \\ +b_e(z, z_1, \lambda - 2\eta(h_2 + h_3)) \otimes ((\mathcal{A}_{2,e})^{-1}\bar{c}_e^{\otimes 2}(z, z_2, z_3, \lambda)\mathcal{A}_{2,e})\big) \times \\ \times\mathcal{A}_{3,e}(e[\sigma_1] \otimes e[\sigma_2] \otimes e[\sigma_3]) = \alpha_{\sigma_1\sigma_2\sigma_3}(\lambda)\mathcal{A}_{3,e}\left(e[\sigma_1] \otimes e[\sigma_2] \otimes e[\sigma_3]\right),\end{gathered}$$

what can be rewritten as

$$\begin{aligned}
&(\mathbb{I}_1 \otimes (\mathcal{A}_{2,e})^{-1})\big(a_e(z, z_1, \lambda - 2\eta(h_2 + h_3)) \otimes \bar{a}_e^{\otimes 2}(z, z_2, z_3, \lambda) \\
&\quad + b_e(z, z_1, \lambda - 2\eta(h_2 + h_3)) \otimes \bar{c}_e^{\otimes 2}(z, z_2, z_3, \lambda)\big)(\mathbb{I}_1 \otimes \mathcal{A}_{2,e}) \\
&\quad \mathcal{A}_{3,e}(e[\sigma_1] \otimes e[\sigma_2] \otimes e[\sigma_3]) \\
&= (\mathbb{I}_1 \otimes \mathcal{A}_{2,e}))^{-1}\big(a_e(z, z_1, \lambda - 2\eta(h_2 + h_3)) \otimes \bar{a}_e^{\otimes 2}(z, z_2, z_3, \lambda) \\
&\quad + b_e(z, z_1, \lambda - 2\eta(h_2 + h_3)) \otimes \bar{c}_e^{\otimes 2}(z, z_2, z_3, \lambda)\big)(\mathbb{I}_1 \otimes \mathcal{A}_{2,e}) \\
&\quad \mathcal{A}_{3,e}(e[\sigma_1] \otimes e[\sigma_2] \otimes e[\sigma_3]) = \\
&\quad \alpha_{\sigma_1\sigma_2\sigma_3}(\lambda)\mathcal{A}_{3,e}\left(e[\sigma_1] \otimes e[\sigma_2] \otimes e[\sigma_3]\right).
\end{aligned}$$

Throughout the computation we denoted $\mathcal{A}_{3,e} = \mathcal{A}_{3,e}(z_1, z_2, z_3, \lambda)$ and $\mathcal{A}_{2,e} = \mathcal{A}_{2,e}(z_2, z_3, \lambda)$. If we look up the definition of $\bar{a}_e^{\otimes 2}(z, z_2, z_3, \lambda)$ and $\bar{c}_e^{\otimes 2}(z, z_2, z_3, \lambda)$ in Lemma 5.2, this yields the result claimed in Lemma 5.11.

**Corollary 5.12** *The basis $B_3$ of $V^{\otimes 3}$ is explicitly given by the following eight vectors*

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{\theta(\lambda+z_2-z_3)\theta(2\eta)}{\theta(\lambda+2\eta)\theta(z_2-z_3)} \\ 0 \\ \frac{\theta(2\eta)\theta(\lambda+z_1-z_3-2\eta)\theta(z_2-z_3+2\eta)}{\theta(\lambda+2\eta)\theta(z_2-z_3)\theta(z_1-z_3)} \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$v_3 = \begin{pmatrix} 0 \\ 0 \\ \frac{\theta(z_2-z_3-2\eta)\theta(\lambda)}{\theta(\lambda+2\eta)\theta(z_2-z_3)} \\ 0 \\ \frac{\theta(2\eta)\theta(\lambda+z_1-z_2-2\eta)\theta(z_2-z_3-2\eta)}{\theta(\lambda+2\eta)\theta(z_2-z_3)\theta(z_1-z_2)} \\ 0 \\ 0 \\ 0 \end{pmatrix}, v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ \frac{\theta(2\eta)\theta(\lambda+z_1-z_2+2\eta)}{\theta(\lambda+4\eta)\theta(z_1-z_2)} \\ -\frac{\theta(2\eta)\theta(\lambda+z_1-z_3)\theta(z_1-z_2+2\eta)}{\theta(\lambda+4\eta)\theta(z_1-z_2)\theta(z_1-z_3)} \\ 0 \end{pmatrix},$$

$$v_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{\theta(\lambda-2\eta)\theta(z_1-z_2-2\eta)\theta(z_1-z_3-2\eta)}{\theta(\lambda+2\eta)\theta(z_1-z_2)\theta(z_1-z_3)} \\ 0 \\ 0 \\ 0 \end{pmatrix}, v_6 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{\theta(\lambda+2\eta)\theta(z_1-z_2-2\eta)}{\theta(z_1-z_2)\theta(\lambda+4\eta)} \\ \frac{\theta(2\eta)\theta(\lambda+z_2-z_3)\theta(z_1-z_2-2\eta)}{\theta(z_1-z_2)\theta(z_2-z_3)\theta(\lambda+4\eta)} \\ 0 \end{pmatrix},$$

$$v_7 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{\theta(\lambda)\theta(z_2-z_3-2\eta)\theta(z_1-z_3-2\eta)}{\theta(\lambda+4\eta)\theta(z_2-z_3)\theta(z_1-z_3)} \\ 0 \end{pmatrix}, v_8 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

*where for the fourth vector we used the fact that the sum of the residues of the function*

$$\frac{\theta(\lambda - z_1 + 2\eta)\theta(\lambda - z_2 + 2\eta)\theta(\lambda - z_3 - 4\eta)}{\theta(\lambda - z_1)\theta(\lambda - z_2)\theta(\lambda - z_2)}$$

*- being invariant under* $\lambda \to \lambda + \tau, \lambda \to \lambda + 1$ *- vanishes.*

Note that we will use this corollary while treating the antiperiodic SOS model.

## 5.3 The antiperiodic SOS model in the case $n = 3$

**Synopsis:**
Now let us look at the antiperiodic SOS model in the case $n = 3$. We first write down the auxiliary antiperiodic transfer matrix in the case $n = 3$ (cf. below). Note that we fixed $\lambda = \lambda_0$ to ensure commutativity.
Then, we describe an eigenvector of this transfer matrix in Proposition 5.13. (This serves as a sign that finding eigenvectors of the auxiliary transfer matrix seems feasible.)
In Lemma 5.14, we find the corresponding eigenvalue and show that it indeed obeys the properties of Proposition 4.54 which are sufficient and necessary for it to be a common eigenvalue of the SOS antiperiodic transfer matrices as well. The eigenvector of the SOS transfer matrix corresponding to the one of Proposition 5.13 would then be given by Theorem 4.55.
**Remark:**
Here, we first need to verify that, since we had to restrict $\lambda$ to $\lambda = x_1 + x_2 + x_3$ with $x_i \in \{-\eta, \eta\}, i = 1, 2, 3$ , $\lambda \neq 0$.
We first want to look at the eigenvectors and eigenvalues of the antiperiodic SOS transfer matrix

$$\bar{T}^{\mathbb{C}}_{aux,e}(z, z_1, z_2, z_3, \lambda_0) = (\bar{b}^{\mathbb{C}}_{aux,e} + \bar{c}^{\mathbb{C}}_{aux,e})(z, z_1, z_2, z_3, \lambda_0),$$

where $\lambda_0 = x_1 + x_2 + x_3$, and then use the obtained results to look at the antiperiodic SOS transfer matrix

$$\bar{T}_{SOS,e}(z, z_1, z_2, z_3, \lambda_0) = \bar{b}_{SOS,e}(z, z_1, z_2, z_3, \lambda_0) + \bar{c}_{SOS,e}(z, z_1, z_2, z_3, \lambda_0),$$

where $\lambda_0 = \eta(h_1 + h_2 + h_3)$.

**Proposition 5.13** *An eigenvector to the auxiliary antiperiodic transfer matrix is given by*

$$v_{0,aux} = \sum_{\sigma_1=-1}^{1} \sum_{\sigma_2=-1}^{1} \sum_{\sigma_3=-1}^{1} e[\sigma_1] \otimes e[\sigma_2] \otimes e[\sigma_3] = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}. \tag{95}$$

*The corresponding eigenvalue reads*

$$\begin{gathered} \epsilon_0(z, z_1, z_2, z_3) = \\ \frac{\theta(2\eta)\theta(z-z_1+3\eta)\theta(z-z_2)\theta(z-z_3)\theta(z_1-z_2+2\eta)\theta(z_1-z_3+2\eta)}{\theta(3\eta)\theta(z_1-z_2)\theta(z_1-z_3)} + \\ \frac{\theta(z-z_2+3\eta)\theta(2\eta)\theta(z-z_1)\theta(z-z_3)\theta(z_1-z_2-2\eta)\theta(z_2-z_3+2\eta)}{\theta(3\eta)\theta(z_1-z_2)\theta(z_2-z_3)} + \\ \frac{\theta(z-z_3+3\eta)\theta(2\eta)\theta(z-z_1)\theta(z-z_2)\theta(z_1-z_3-2\eta)\theta(z_2-z_3-2\eta)}{\theta(3\eta)\theta(z_1-z_3)\theta(z_2-z_3)}. \end{gathered} \tag{96}$$

**Proof:**
Let us first write down the auxiliary transfer matrix

$$\begin{gathered} \bar{T}^{\mathbb{C}}_{aux,e}(z, z_1, z_2, z_3, \lambda_0) = \\ \bar{b}^{\mathbb{C}}_{aux,e}(z, z_1, z_2, z_3, \lambda_0) + \bar{c}^{\mathbb{C}}_{aux,e}(z, z_1, z_2, z_3, \lambda_0) = \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ t_{31} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t_{42} & t_{43} & 0 & 0 & 0 & 0 & 0 \\ t_{51} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t_{62} & 0 & 0 & t_{65} & 0 & 0 & 0 \\ 0 & 0 & t_{73} & 0 & t_{75} & 0 & 0 & 0 \\ 0 & 0 & 0 & t_{84} & 0 & t_{86} & t_{86} & 0 \end{pmatrix} T_\lambda^{+2\eta} \\ + \begin{pmatrix} 0 & t_{12} & t_{13} & 0 & t_{15} & 0 & 0 & 0 \\ 0 & 0 & 0 & t_{24} & 0 & t_{26} & 0 & 0 \\ 0 & 0 & 0 & t_{34} & 0 & 0 & t_{37} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{48} \\ 0 & 0 & 0 & 0 & 0 & t_{56} & t_{57} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{68} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{78} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} T_\lambda^{-2\eta} \end{gathered}$$

with

$$t_{12} = \frac{\theta(2\eta)\theta(-\eta + z - z_3)\theta(z - z_1 + 2\eta)\theta(z - z_2 + 2\eta)\theta(z_1 - z_3 + 2\eta)\theta(z_2 - z_3 + 2\eta)}{\theta(3\eta)\theta(z_1 - z_3)\theta(z_2 - z_3)},$$
$$t_{13} = \frac{\theta(2\eta)\theta(z - z_2 - \eta)\theta(z - z_1 + 2\eta)\theta(z - z_3 + 2\eta)\theta(z_1 - z_2 + 2\eta)\theta(z_2 - z_3 - 2\eta)}{\theta(3\eta)\theta(z_1 - z_2)\theta(z_2 - z_3)},$$
$$t_{15} = \frac{\theta(2\eta)\theta(z - z_1 - \eta)\theta(z - z_2 + 2\eta)\theta(z - z_3 + 2\eta)\theta(z_1 - z_2 - 2\eta)\theta(z_1 - z_3 - 2\eta)}{\theta(3\eta)\theta(z_1 - z_2)\theta(z_1 - z_3)},$$
$$t_{21} = -\frac{\theta(2\eta)\theta(z - z_3 - \eta)\theta(z - z_1 + 2\eta)\theta(z - z_2 + 2\eta)}{\theta(\eta)},$$
$$t_{24} = \frac{\theta(2\eta)\theta(z - z_2 + \eta)\theta(z - z_3)\theta(z - z_1 + 2\eta)\theta(z_1 - z_2 + 2\eta)}{\theta(\eta)\theta(z_1 - z_2)},$$
$$t_{26} = \frac{\theta(z - z_1 + \eta)\theta(z - z_2 + 2\eta)\theta(z - z_3)\theta(z_1 - z_2 - 2\eta)\theta(2\eta)}{\theta(\eta)\theta(z_1 - z_2)},$$
$$t_{31} = -\frac{\theta(z - z_2 - \eta)\theta(z - z_1 + 2\eta)\theta(z - z_3 + 2\eta)\theta(2\eta)}{\theta(\eta)},$$
$$t_{34} = \frac{\theta(z - z_3 + \eta)\theta(2\eta)\theta(z - z_2)\theta(z - z_1 + 2\eta)\theta(z_1 - z_3 + 2\eta)}{\theta(\eta)\theta(z_1 - z_3)},$$
$$t_{37} = \frac{\theta(z - z_1 + \eta)\theta(z - z_2)\theta(z - z_3 + 2\eta)\theta(z_1 - z_3 + 2\eta)\theta(2\eta)}{\theta(\eta)\theta(z_1 - z_3)},$$
$$t_{42} = \frac{\theta(2\eta)\theta(z - z_2 + \eta)\theta(z - z_1 + 2\eta)\theta(z - z_3)\theta(z_2 - z_3 + 2\eta)}{\theta(\eta)\theta(z_2 - z_3)},$$
$$t_{43} = \frac{\theta(z - z_3 + \eta)\theta(z - z_1 + 2\eta)\theta(z - z_2)\theta(2\eta)\theta(z_2 - z_3 - 2\eta)}{\theta(\eta)\theta(z_2 - z_3)},$$
$$t_{48} = -\frac{\theta(2\eta)\theta(z - z_3)\theta(z - z_2)\theta(z - z_1 + 3\eta)}{\theta(\eta)},$$
$$t_{51} = -\frac{\theta(z - z_1 - \eta)\theta(z - z_2 + 2\eta)\theta(z - z_3 + 2\eta)\theta(2\eta)}{\theta(\eta)},$$
$$t_{56} = \frac{\theta(z - z_3 + \eta)\theta(z - z_1)\theta(z - z_2 + 2\eta)\theta(2\eta)\theta(z_2 - z_3 + 2\eta)}{\theta(\eta)\theta(z_2 - z_3)},$$
$$t_{57} = \frac{\theta(2\eta)\theta(z - z_2 + \eta)\theta(z - z_1)\theta(z - z_3 + 2\eta)\theta(z_2 - z_3 - 2\eta)}{\theta(\eta)\theta(z_2 - z_3)},$$
$$t_{62} = \frac{\theta(z - z_1 + \eta)\theta(2\eta)\theta(z - z_2 + 2\eta)\theta(z - z_3)\theta(z_1 - z_3 + 2\eta)}{\theta(\eta)\theta(z_1 - z_3)},$$
$$t_{65} = \frac{\theta(z - z_3 + \eta)\theta(2\eta)\theta(z - z_1)\theta(z - z_2 + 2\eta)\theta(z_1 - z_3 - 2\eta)}{\theta(\eta)\theta(z_1 - z_3)},$$
$$t_{68} = -\frac{\theta(z - z_2 + 3\eta)\theta(2\eta)\theta(z - z_1)\theta(z - z_3)}{\theta(\eta)},$$
$$t_{73} = \frac{\theta(z - z_1 + \eta)\theta(2\eta)\theta(z - z_2)\theta(z - z_3 + 2\eta)\theta(z_1 - z_2)}{\theta(\eta)\theta(z_1 - z_2)},$$
$$t_{75} = \frac{\theta(z - z_2 + \eta)\theta(2\eta)\theta(z - z_1)\theta(z - z_3 + 2\eta)\theta(z_1 - z_2 - 2\eta)}{\theta(\eta)\theta(z_1 - z_2)},$$
$$t_{78} = -\frac{\theta(2\eta)\theta(z - z_3 + 3\eta)\theta(z - z_1)\theta(z - z_2)}{\theta(\eta)},$$

$$t_{84} = \frac{\theta(2\eta)\theta(z - z_1 + 3\eta)\theta(z - z_2)\theta(z - z_3)\theta(z_1 - z_2 + 2\eta)\theta(z_1 - z_3 + 2\eta)}{\theta(3\eta)\theta(z_1 - z_2)\theta(z_1 - z_3)},$$
$$t_{86} = \frac{\theta(z - z_2 + 3\eta)\theta(2\eta)\theta(z - z_1)\theta(z - z_3)\theta(z_1 - z_2 - 2\eta)\theta(z_2 - z_3 + 2\eta)}{\theta(3\eta)\theta(z_1 - z_2)\theta(z_2 - z_3)},$$
$$t_{87} = \frac{\theta(z - z_3 + 3\eta)\theta(2\eta)\theta(z - z_1)\theta(z - z_2)\theta(z_1 - z_3 - 2\eta)\theta(z_2 - z_3 - 2\eta)}{\theta(3\eta)\theta(z_1 - z_3)\theta(z_2 - z_3)}.$$

If we evaluate this matrix on $v_{0,aux}$, we get eight different expressions for the eigenvalue $\epsilon_0(z, z_1, z_2, z_3)$. We can check that they are all in $\Theta_3(\chi)$, $\chi(1) = -1, \chi(\tau) = -e^{+2\pi i(z_1+z_2+z_3)-6\pi i\eta}$. Hence, we have to prove that the eight values mutually coincide at three points: The first and second value coincide when evaluated at $z = z_1 - 2\eta, z = z_2 - 2\eta, z = z_3 + \eta$, hence everywhere. The third and fourth value coincide at $z = z_1 - 2\eta, z = z_2, z = z_3 - \eta$, hence everywhere. The fifth and sixth term coicide at $z = z_1, z = z_2 - 2\eta, z = z_3 - \eta$, hence everywhere. The eighth and seventh term coincide at $z = z_1, z = z_2, z = z_3 - 3\eta$, hence everywhere. The sixth and eigth term coincide at $z = z_1, z = z_3, z = z_2 - 3\eta$, hence the last four expressions are identical.

The second and fourth term coincide at $z = z_1 - 2\eta, z = z_2 - \eta, z = z_3$, hence the first four terms are the same everywhere. The second and sixth expression coincide at $z = z_1 - \eta, z = z_2 - 2\eta, z = z_3$. Hence each of the last four expressions equals each of the first four ones. Thus all expressions coincide and we can write down any of the eight expressions as the eigenvalue $\epsilon_0(z, z_1, z_2, z_3)$, e.g. the last one - which is also the one used in the proposition:

$$\epsilon_0(z, z_1, z_2, z_3) =$$
$$\frac{\theta(2\eta)\theta(z-z_1+3\eta)\theta(z-z_2)\theta(z-z_3)\theta(z_1-z_2+2\eta)\theta(z_1-z_3+2\eta)}{\theta(3\eta)\theta(z_1-z_2)\theta(z_1-z_3)}+$$
$$\frac{\theta(z-z_2+3\eta)\theta(2\eta)\theta(z-z_1)\theta(z-z_3)\theta(z_1-z_2-2\eta)\theta(z_2-z_3+2\eta)}{\theta(3\eta)\theta(z_1-z_2)\theta(z_2-z_3)}+$$
$$\frac{\theta(z-z_3+3\eta)\theta(2\eta)\theta(z-z_1)\theta(z-z_2)\theta(z_1-z_3-2\eta)\theta(z_2-z_3-2\eta)}{\theta(3\eta)\theta(z_1-z_3)\theta(z_2-z_3)}.$$

**Lemma 5.14** *The eigenvalue $\epsilon_0(z, z_1, z_2, z_3)$ obeys the two properties of Proposition 4.54:*

*a)* $\epsilon_0(z_i, z_1, z_2, z_3)\epsilon_0(z_i - 2\eta, z_1, z_2, z_3) = \prod_{j=1}^{3} \theta(z_i - z_j - 2\eta)\theta(z_i - z_j + 2\eta),$

*for all* $i = 1, 2, 3$.

*b)* $\epsilon_0(z, z_1, z_2, z_3) \in \Theta_3(\chi)$ *with* $\chi(1) = -1$ *and* $\chi(\tau) = -e^{2\pi i(z_1+z_2+z_3)}$.

**Proof:**

a) Let us write down also the first of the eight expressions obtained by the action of the transfer matrix on $v_{0,aux}$.
It reads

$$\epsilon'_0(z, z_1, z_2, z_3) =$$
$$\frac{\theta(2\eta)\theta(z-z_3-\eta)\theta(z-z_1+2\eta)\theta(z-z_2+2\eta)\theta(z_1-z_3+2\eta)\theta(z_2-z_3+2\eta)}{\theta(3\eta)\theta(z_1-z_3)\theta(z_2-z_3)}+$$
$$\frac{\theta(2\eta)\theta(z-z_2-\eta)\theta(z-z_1+2\eta)\theta(z-z_3+2\eta)\theta(z_1-z_3+2\eta)\theta(z_2-z_3-2\eta)}{\theta(3\eta)\theta(z_1-z_2)\theta(z_2-z_3)}+$$
$$\frac{\theta(2\eta)\theta(z-z_1-\eta)\theta(z-z_3+2\eta)\theta(z-z_2+2\eta)\theta(z_1-z_3-2\eta)\theta(z_1-z_2-2\eta)}{\theta(3\eta)\theta(z_1-z_3)\theta(z_1-z_2)}.$$

In Proposition 5.13, we showed that all expressions of $\epsilon_0(z, z_1, z_2, z_3)$ are indeed the same function. If we evaluate $\epsilon_0(z, z_1, z_2, z_3)$ at $z = z_i, i = 1, 2, 3$, we get

$$\epsilon_0(z_i, z_1, z_2, z_3) = \theta(z_i - z_1 + 2\eta)\theta(z_i - z_2 + 2\eta)\theta(z_i - z_3 + 2\eta).$$

If we evaluate $\epsilon'_0(z, z_1, z_2, z_3)$ at $z = z_i - 2\eta, i = 1, 2, 3$, we get

$$\epsilon'_0(z_i - 2\eta, z_1, z_2, z_3) = \theta(z_i - z_1 - 2\eta)\theta(z_i - z_2 - 2\eta)\theta(z_i - z_3 - 2\eta).$$

The product of both expressions at the same value of $i$ yields the lemma.

b) In Proposition 5.13 we showed that under $z \to z + \tau$ $\epsilon_0(z, z_1, z_2, z_3)$ transformed as $\epsilon_0(z + \tau, z_1, z_2, z_3) = e^{-6\pi i(z+\tau)+3\pi i\tau+2\pi i(z_1+z_2+z_3)}$. Taken into account that we shifted the $z$ of Proposition 4.33 to $z \to z + \eta$ in the section of the isomorphism, we see that $\epsilon_0(z, z_1, z_2, z_3)$ is indeed an element of the correct space of elliptic polynomials.

# 6 Appendix 1: An alternative approach to the XXX magnetic chain as described by Sklyanin [47]

## 6.1 The setting corresponding to the XXX chain

**Synopsis:**
Here, we mainly repeat the steps we did in the section explaining the representation theory of $E_{\tau,\eta}(sl_2)$. But we do not go as much into details, i.e. in particular we do not investigate the eigenvalue problem of a possible transfer matrix or auxiliary transfer matrix but rather restrict ourselves to stating an isomorphism between the $n$fold tensor product of fundamental representations of the Yangian $\mathcal{Y}(sl_2)$ and an auxiliary representation also found in [46, 44].
We briefly sketch the steps: First, we define some basic notation (Definition 6.1) and the notion of an $\mathcal{H}$-module which we need to define a representation of the Yangian (Definition 6.4). Then, we give the R-matrix of the Yangian which tells about its (rational) structure (Definition 6.2).
By the R-matrix, which gives the RLL-relations, and a given $\mathcal{H}$ -module we define a representation of the Yangian (Definition 6.4) and then give examples of finite-dimensional irreducible representations (Proposition 6.6) (which we could possibly use to develop generalizations of the isomorphism to be constructed). We also show (Proposition 6.5) that by means of a tensor product we can construct a new representation of the Yangian out of two given ones. This means of obtaining new representations if of course needed to construct the $n$fold tensor product of fundamental representations which we want to compare to the auxiliary representation in the next section.
In the actual section, we proceed by generalizing the notion of a representation of a Yangian to the notion of a functional representation (Definition 6.8). We first define the spaces of functions which we will need concerning the functional representations which we will need (Definition 6.7). The functional representation which we will need is given in Proposition 6.11, namely the auxiliary representation.
We end the chapter with a short digression on twisted representations of the Yangian, a notion which Sklyanin [46, 44] used to implement different boundary conditions of the XXX-model which he described.

### 6.1.1 Introduction

We first want to define the basic objects to deal with in the formulation of the eigenvalue problem corresponding to the XXX-chain as described by [47], p.67: the Yangian of $sl_2(\mathbb{C})$, denoted $\mathcal{Y}(sl_2)$ and some examples of representations of $\mathcal{Y}(sl_2)$ that will be needed afterwards. (In [46] there are also some references given concerning the origin of the treatment of the mentioned model [20, 21, 37].)

**Definition 6.1 (Basic notions)**

a) *Let $\mathcal{H} = \mathbb{C}h$ be the one-dimensional Lie-algebra generated by a generator $h$. Let $V_i, i = 1, \dots, n$ be modules over $\mathcal{H}$. $V_i$ is called a diagonalizable $\mathcal{H}$-module, if $V_i$ is the direct sum of finite dimensional eigenspaces of $h$ called $V_i[\mu]$, labeled by the eigenvalue $\mu \in \mathbb{C}$ of $h$: $V_i = \oplus_\mu V_i[\mu]$.*

*We may for example choose* $V = \mathbb{C}^2 = V[-1] \oplus V[1]$, *with* $V[\mu] = \{\alpha\, e[\mu] \,|\, \alpha \in \mathbb{C}\}$, *taking* $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $e[1] = (1\,0)^T$, $e[-1] = (0\,1)^T$.

b) *Let* $V_i, i = 1, \dots, n$ *be diagonalizable* $\mathcal{H}$*-modules. We may consider their tensor product* $V_1 \otimes \dots \otimes V_n$. *For* $X \in End\,(V_i)$ *we denote by* $X^{(i)} \in End(V_1 \otimes \dots \otimes V_n)$ *the operator*

$$X^{(i)} = 1 \otimes \dots \otimes \underbrace{X}_{ith\ place} \otimes \dots \otimes 1\,. \tag{97}$$

*If* $X \in End(V_i \otimes V_j)$, *we define* $X^{(ij)} \in End(V_1 \otimes \dots \otimes V_n)$ *analogously.*

c) *Let* $A \in End\,(V^{\otimes j})$. *Then we can construct* $A^{(n-j+1 \dots n)} \in End\,(V^{\otimes n})$ *by*

$$A^{(n-j+1\dots n)} = \underbrace{\mathbb{I} \otimes \dots \otimes \mathbb{I}}_{first\ j\ copies\ of V} \otimes A.$$

d) *Let* $v \in V_1 \otimes \dots \otimes V_n$. *We may define* $h^{(i)} \in End(V_1 \otimes \dots \otimes V_n)$ *by means of the above notation. Let* $X = X(h^{(1)}, \dots, h^{(n)})$ *be a function taking values in* $End(V_1 \otimes \dots \otimes V_n)$. *If* $h^{(i)}v = \mu_i v, i = 1, \dots, n$, *then* $X(h^{(1)}, \dots, h^{(n)})v = X(\mu_1, \dots, \mu_n)v$.

**Definition 6.2 (R-matrix)** *Let* $V$ *be a two-dimensional complex vector space with base* $e[-1], e[1]$. *Let the rational R-matrix* $R \in End(V \otimes V)$ *be given by*

$$R_r(z) = \begin{pmatrix} z + 2\eta & 0 & 0 & 0 \\ 0 & z & 2\eta & 0 \\ 0 & 2\eta & z & 0 \\ 0 & 0 & 0 & z + 2\eta \end{pmatrix}, \tag{98}$$

*with the parameter* $\eta \in \mathbb{C}$, *where we identified* $e[1] \otimes e[1] = (1\,0\,0\,0)^T, e[1] \otimes e[-1] = (0\,1\,0\,0)^T, e[-1] \otimes e[1] = (0\,0\,1\,0)^T, e[-1] \otimes e[-1] = (0\,0\,0\,1)^T$.
*This is the same R-matrix as given in [47] with* $\eta$ *here replaced by* $\frac{\eta}{2}$.

**Proposition 6.3 (QYBE [47])** *The rational R-matrix* $R_r(z)$ *obeys the quantum Yang–Baxter–relation*

$$R^{(12)}(z-w)R^{(13)}(z)R^{(23)}(w) = R^{(23)}(w)R^{(13)}(z)R^{(12)}(z-w)\,, \tag{99}$$

*where the notation is as previously noted. This relation is defined on* $End(V^{\otimes 3})$.

### 6.1.2 Representations, functional representations, operator algebras

**Definition 6.4 (Representation)** *A representation of the Yangian* $\mathcal{Y}(sl_2)$ *is a pair* $(W, L_r)$, *where* $W$ *is a diagonalizable* $\mathcal{H}$*-module* $W = \oplus_{\mu \in \mathbb{C}} W[\mu]$ *and* $L_r = L_r(z) \in End(V \otimes W)$ *is a linear map commuting with* $h^{(1)} + h^{(2)}$ *meromorphic in* $z \in \mathbb{C}$ *called the L-operator.*
*The L-operator obeys the relation*

$$R_r^{(12)}(z-w)L_r^{(13)}(z)L_r^{(23)}(w) = L_r^{(23)}(w)L_r^{(13)}(z)R_r^{(12)}(z-w)\,. \tag{100}$$

*This condition is called the RLL-relation.*

**Remark:**
The $L$-operator is usually written in the form

$$L_r(z) = \begin{pmatrix} a_r(z) & b_r(z) \\ c_r(z) & d_r(z) \end{pmatrix} \in \text{End}\,(V \otimes W), \tag{101}$$

where $a_r(z), b_r(z), c_r(z), d_r(z) \in \text{End}(W)$ are meromorphic in $z \in \mathbb{C}$ and obey the conditions defined by the $RLL$-relation.
The $RLL$-relation written in terms of the above operators yields the following sixteen expressions:

$$\begin{aligned}
a_r(z)a_r(w) &= a_r(w)a_r(z), \\
(z-w+2\eta)a_r(z)b_r(w) &= 2\eta a_r(w)b_r(z) + (z-w)b_r(w)a_r(z), \\
(z-w+2\eta)b_r(z)a_r(w) &= (z-w)a_r(w)b_r(z) + 2\eta b_r(w)a_r(z), \\
b_r(z)b_r(w) &= b_r(w)b_r(z), \\
(z-w+2\eta)c_r(w)a_r(z) &= (z-w)a_r(z)c_r(w) + 2\eta c_r(z)a_r(w) \\
(z-w)a_r(z)d_r(w) &+ 2\eta c_r(z)b_r(w) = 2\eta c_r(w)b_r(z) + (z-w)d_r(w)a_r(z), \\
(z-w)b_r(z)c_r(w) &+ 2\eta d_r(z)a_r(w) = (z-w)c_r(w)b_r(z) + 2\eta d_r(w)a_r(z), \\
(z-w+2\eta)d_r(w)b_r(z) &= (z-w)b_r(z)d_r(w) + 2\eta d_r(z)b_r(w), \\
(z-w+2\eta)a_r(w)c_r(z) &= 2\eta a_r(z)c_r(w) + (z-w)c_r(z)a_r(w), \\
(z-w)c_r(z)b_r(w) &+ 2\eta a_r(z)d_r(w) = 2\eta a_r(w)d_r(z) + (z-w)b_r(w)c_r(z), \\
(z-w)d_r(z)a_r(w) &+ 2\eta b_r(z)c_r(w) = (z-w)a_r(w)d_r(z) + 2\eta b_r(w)c_r(z), \\
(z-w+2\eta)b_r(w)d_r(z) &= 2\eta b_r(z)d_r(w) + (z-w)d_r(z)b_r(w), \\
c_r(z)c_r(w) &= c_r(w)c_r(z), \\
(z-w+2\eta)c_r(z)d_r(w) &= 2\eta c_r(w)d_r(z) + (z-w)d_r(w)c_r(z), \\
(z-w+2\eta)d_r(z)c_r(w) &= (z-w)c_r(w)d_r(z) + 2\eta d_r(w)c_r(z), \\
d_r(z)d_r(w) &= d_r(w)d_r(z)\,.
\end{aligned}$$

**Proposition 6.5 ([47])** *If we have two representations of the Yangian $\mathcal{Y}(sl_2)$ denoted $(W_1, L_{r,1}(z-z_1))$ and $(W_2, L_{r,2}(z-z_2))$, a new representation is given by the tensor product of the two representations. It reads $(W_1 \otimes W_2, L_{r,1}(z-z_1)L_{r,2}(z-z_2))$ by means of the comultiplication property of the Yangian $\mathcal{Y}(sl_2)$, as it is a Hopf algebra.*

**Remark:**
If we explicitly write down the $L$-operator of the tensor product, it looks like

$$\begin{aligned}
a_{1\otimes 2,r}(z,z_1,z_2) &= a_{1,r}(z-z_1)\otimes a_{2,r}(z-z_2) + b_{1,r}(z-z_1)\otimes c_{2,r}(z-z_2), \\
b_{1\otimes 2,r}(z,z_1,z_2) &= a_{1,r}(z-z_1)\otimes b_{2,r}(z-z_2) + b_{1,r}(z-z_1)\otimes d_{2,r}(z-z_2), \\
c_{1\otimes 2,r}(z,z_1,z_2) &= c_{1,r}(z-z_1)\otimes a_{2,r}(z-z_2) + d_{1,r}(z-z_1)\otimes c_{2,r}(z-z_2), \\
d_{1\otimes 2,r}(z,z_1,z_2) &= c_{1,r}(z-z_1)\otimes b_{2,r}(z-z_2) + d_{1,r}(z-z_1)\otimes d_{2,r}(z-z_2).
\end{aligned}$$

**Proposition 6.6 (Examples)**

*a) The representation $(W = V, L_r(z) = R_r(z-z_0))$ is called the fundamental representation of $\mathcal{Y}(sl_2)$.*

b) *Let $V_\Lambda$ be an infinite dimensional complex vector space with basis $e_k, k \in \mathbb{Z}$. An action of $h$ shall be defined by $f(h)e_k = f(\Lambda - 2k)e_k$ for $f(h) \in End(V_\Lambda)$.*
*The pair $(W = V_\Lambda, L_r(z) = L_{\Lambda,r}(z - z_0))$ is called the evaluation Verma module $V_{\Lambda,r}(z_0)$ of $\mathcal{Y}(sl_2)$. $L_{\Lambda,r}(z-z_0)$ is defined as follows in terms of $a_{\Lambda,r}(z-z_0), b_{\Lambda,r}(z-z_0), c_{\Lambda,r}(z-z_0), d_{\Lambda,r}(z-z_0)$ :*

$$a_{\Lambda,r}(z - z_0)e_k = (z - z_0 + \eta + (\Lambda - 2k)\eta)e_k, \tag{102}$$
$$b_{\Lambda,r}(z - z_0)e_k = 2(\Lambda - k)\eta\, e_{k+1}, \tag{103}$$
$$c_{\Lambda,r}(z - z_0)e_k = 2k\eta\, e_{k-1}, \tag{104}$$
$$d_{\Lambda,r}(z - z_0)e_k = (z - z_0 - (\Lambda - 2k)\eta + \eta)e_k\,. \tag{105}$$

c) *If $\Lambda = n$, $n \in \mathbb{N}$, the representation $(V_\Lambda, L_{\Lambda,r}(z - z_0))$ has a finite dimensional quotient module of dimension $n+1$. This representation will be denoted $W_{\Lambda,r}(z_0)$.*

d) *If $n = 1$, this finite dimensional quotient module is isomorphic to the fundamental representation $(V, R_r(z - z_1))$.*

**Proof:**

a) In this case, the RLL-relations reduce to the rational Yang-Baxter-equation (cf. [46], p.21).

b) That these operators define a representation of $\mathcal{Y}(sl_2)$ is checked by a straightforward calculation of the RLL-relations.

c) This is shown by writing down the corresponding operators acting on a corresponding basis containing only finitely many elements, cf. [46], p.21.

d) This is checked by calculating the operators on $e_1, e_{-1} \in V_\Lambda$, suitably normalizing them and comparing to the $R$-matrix.

**Remark:**
In order to understand what follows, we need a further generalization of the notion of a representation of the Yangian $\mathcal{Y}(sl_2)$. This generalization will be provided by the notion of a functional representation of the Yangian. In order to understand the definition, we need the following spaces of functions.

**Definition 6.7 ($\mathcal{F}_n, \mathcal{F}_n^D$)** *Let $\Lambda_1, \dots, \Lambda_n \in \mathbb{N}$. Let $(z_1, \dots, z_n) \in \mathbb{C} -$ diag.*
*Let $S_i = \{-z_i - \Lambda_i\eta, -z_i - \Lambda_i\eta + 2\eta, \dots, \Lambda_i\eta - z_i\}$ with $S_i \cap S_j = \emptyset$ for all $i, j = 1, \dots, n$ with $i \neq j$. Let $D = \{(x_1, \dots, x_n) \,|\, x_i \in S_i$ for all $i = 1, \dots, n\}$. Then*

a) *$\mathcal{F}_n = \{f : \mathbb{C}^n \to \mathbb{C}, (x_1, \dots, x_n) \to f(x_1, \dots, x_n) \,|\, f$ holomorphic in $(x_1, \dots, x_n)\}$,*

b) *$\mathcal{F}_n^D = \mathcal{F}_n \,/\, \{f \in \mathcal{F}_n \,|\, f(x_1, \dots, x_n) = 0$ for all $(x_1, \dots, x_n) \in D\}$.*

**Definition 6.8 (Functional Representation)** *Let $\mathcal{F}_1$ be the vector space of all complex valued functions in $\mu$ (instead of $x_1$).*
*A functional representation of $\mathcal{Y}(sl_2)$ is a pair $(W, L_r^{\mathcal{F}}(z, \mu))$, where $W \subseteq \mathcal{F}_1$ and $L_r^{\mathcal{F}}(z, \mu)$ is a holomorphic function of $\mu$ and $z \in \mathbb{C}$ acting as a difference operator on $V \otimes W$ obeying the RLL-relations.*
*It commutes with $h \otimes 1 + 1 \otimes h$, where $h$ acts by multiplication with the continuous variable $\mu \in \mathbb{C}$ : $hv(\mu) = \mu v(\mu), v(\mu) \in W$.*

**Proposition 6.9 (Examples)**

a) *A functional representation of the Yangian is given by* $(W = \mathcal{F}_1, L^{\mathcal{F}}_{\Lambda,r}(z-z_0))$, *where* $L_{\Lambda,r}(z-z_0) \in End(\mathcal{F}_1), \Lambda, z_0 \in \mathbb{C}$, *is defined as follows*

$$\begin{aligned}
(\bar{a}_{\Lambda,r}(z,h)v)(\mu) &= a_{\Lambda,r}(z,h)v(\mu) = (z - z_0 + \mu\eta + \eta)v(\mu), \\
(\bar{b}_{\Lambda,r}(z,h)v)(\mu) &= b_{\Lambda,r}(z,h)T_h^{+2\eta}v(\mu) = (\Lambda + \mu)\eta\, v(\mu - 2), \\
(\bar{c}_{\Lambda,r}(z,h)v)(\mu) &= c_{\Lambda,r}(z,h)T_h^{+2\eta}v(\mu) = (\Lambda - \mu)\eta\, v(\mu + 2), \\
(\bar{d}_{\Lambda,r}(z,h)v)(\mu) &= d_{\Lambda,r}(z,h)v(\mu) = (z - z_0 - \mu\eta + \eta)v(\mu), \\
f(h)v(\mu) = f(\mu)v(\mu) &
\end{aligned}$$

*where* $v(\mu) \in \mathcal{F}_1$.
*It is called the functional Verma module* $V^{\mathcal{F}}_{\Lambda,r}(z_0)$.

b) *If we restrict the above representation to* $\mathcal{F}_1^R = \mathcal{F}_{1(\mu\in\{\Lambda-2k|k\in\mathbb{N}\})}$ *and set* $v(\Lambda - 2k) = e_k$, $e_k$ *defining the basis of an infinite dimensional vector space, we recover the evaluation Verma module* $V_{\Lambda,r}(z_0)$ *by means of the functional representation* $(\mathcal{F}_D, L^{\mathcal{F}_D}_{\Lambda,r}(z-z_0))$.
*The L-operator looks the same as one defined in a), but its action is restricted onto* $\mathcal{F}_1^R \subset \mathcal{F}_1$.

**Proof:**

a) The statement is proven by checking the rational $RLL$-relations.

b) This is proven by comparison.

**Remark:**

a) For a representation of the Yangian, we can define its operator algebra as the algebra generated by $\bar{a}_r(z,h), \bar{b}_r(z,h), \bar{c}_r(z,h), \bar{d}_r(z,h), h \in$ End $(W)$, where $W \subseteq \mathcal{F}$.

b) We can generalize the notion of a functional representation or operator algebra to operators depending on several weights $\mu_1, \ldots, \mu_n \in \mathbb{C}$, acting on the space of functions $\mathcal{F}_n$ which depend on the before-mentioned weights. The operators read $\bar{a}_r(z,h_1,\ldots,h_n), \bar{b}_r(z,h_1,\ldots,h_n), \bar{c}_r(z,h_1,\ldots,h_n), \bar{d}_r(z,h_1,\ldots,h_n), h_i \in$ End $(\mathcal{F}_n)$ with $h_i f(\mu_1,\ldots,\mu_n) = \mu_i f(\mu_1,\ldots,\mu_n)$ for every $f \in \mathcal{F}_n$ and $i = 1,\ldots,n$.

**Proposition 6.10 (Quantum determinant)**

a) *The following element of the operator algebra is a central element:*

$$\overline{Det}_r(z) = (\bar{a}_r(z-2\eta)\bar{d}_r(z) - \bar{c}_r(z-2\eta)\bar{b}_r(z))\,. \tag{106}$$

*It is called the quantum determinant.*

b) *If we have two finite dimensional irreducible representations of the Yangian named* $(V_1, L_1(z,h_1))$ *and* $(V_2, L_2(z,h_2))$ *with quantum determinants* $\bar{D}et_1(z) = Det_1(z)\mathbb{I}_{V_1}$ *and* $\bar{D}et_2(z) = Det_2(z)\mathbb{I}_{V_2}$, *where* $Det_1(z)$ *and* $Det_2(z)$ *are scalar functions and* $\mathbb{I}_{V_i}, i = 1,2$, *are the identity matrices on* $V_i$, *then the detrminant of the tensor product representation* $(V_1 \otimes V_2, L_{1\otimes 2}(z,h_1,h_2))$ *is given by* $\bar{D}et(z)_{1\otimes 2} = Det_1(z)Det_2(z)\cdot$ $\cdot\mathbb{I}_{V_1\otimes V_2}$, *where* $\mathbb{I}_{V_1\otimes V_2}$ *is the identity matrix on* $V_1 \otimes V_2$.

**Proof:**
This can be checked by explicitly commuting all the generators of $\mathcal{Y}(sl_2)$ with the quantum determinant. For the second part of the proposition, cf. [47], p.69.

**Proposition 6.11 ([46], pp.19 -20)** *Let* $(z_1, \dots, z_n) \in \mathbb{C}^n -$ *diag, and* $\Lambda_i \in \mathbb{N}, i = 1, \dots, n$.
*Let* $\mathcal{F}_n^D$ *be the space of functions defined before. Let*

$$\Delta_{n,r}^-(z) = \prod_{i=1}^n (z + z_i + \Lambda_i \eta) \text{ and } \Delta_{n,r}^+(z) = \prod_{i=1}^n (z + z_i - \Lambda_i \eta).$$

*Let the difference operators* $Y_{x_i}^{\pm} \in End(\mathcal{F}_n^D)$ *for* $i = 1, \dots, n$, *be given by*

$$\begin{aligned} (Y_{x_i}^{\pm} f)(x_1, \dots, x_n) &= (\Delta_{n,r}^{\pm}(x_i) T_{x_i}^{\pm 2\eta} f)(x_1, \dots, x_n) = \\ &= \Delta_{n,r}^{\pm}(x_i) f(x_1, \dots, x_i \pm 2\eta, \dots, x_n). \end{aligned}$$

*Then the operators*

$$\bar{a}_{aux,r}(z, \Lambda_1, \dots, \Lambda_n, z_1, \dots, z_n) = \prod_{i=1}^n (z + x_i), \quad (107)$$

$$\bar{b}_{aux,r}(z, \Lambda_1, \dots, \Lambda_n, z_1, \dots, z_n) = -\sum_{i=1}^n \prod_{j \neq i} \frac{z + x_j}{x_j - x_i} \Delta_{n,r}^+(x_i) T_{x_i}^{+2\eta}, \quad (108)$$

$$\bar{c}_{aux,r}(z, \Lambda_1, \dots, \Lambda_n, z_1, \dots, z_n) = \sum_{i=1}^n \prod_{j \neq i} \frac{z + x_j}{x_j - x_i} \Delta_{n,r}^-(x_i) T_{x_i}^{-2\eta}, \quad (109)$$

$$\begin{aligned} \overline{Det}_{aux,r}(z, \Lambda_1, \dots, \Lambda_n, z_1, \dots, z_n) &= \prod_{i=1}^n (z - z_i - \Lambda_i \eta - 2\eta) \times \\ &\times (z - z_i + \Lambda_i \eta) \end{aligned} \quad (110)$$

*define an operator algebra obeying the RLL-relations of the Yangian* $\mathcal{Y}(sl_2)$.
*The operator* $\bar{d}_r(z, \Lambda_1, \dots, \Lambda_n, z_1, \dots, z_n)$ *is defined implicitly by the quantum determinant.*

**Remark:**
Taken together as entries of a $2 \times 2$ matrix, the operators
$\bar{a}_{aux,r}(z, \Lambda_1, \dots, \Lambda_n, z_1, \dots, z_n), \dots, \bar{d}_{aux,r}(z, \Lambda_1, \dots, \Lambda_n, z_1, \dots, z_n)$
define the operator $\bar{L}_{aux,r}(z, \Lambda_1, \dots, \Lambda_n, z_1, \dots, z_n)$. It is a matrix on $V$ with entries in End $(\mathcal{F}_n^D)$.
The above defined representation coincides with the one given in [46] if we substitute $x_i = -y_i$ for every $i = 1, \dots, n$ and then consider the representation $L_{aux,r}(z) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

**Corollary 6.12** *Let* $n = 1, z' = z - \eta$ *and* $x_1 = -z_1 + h_1 \eta$.
*Then the operators*

$$\begin{aligned} \bar{a}_{1,r}(z') &= (z' - z_1 + h_1 \eta + \eta), \\ \bar{b}_{1,r}(z') &= (h_1 \eta + \Lambda \eta) T_h^{+2\eta}, \\ c_{1,r}(z') &= (h_1 \eta - \Lambda \eta) T_h^{-2\eta}, \\ \bar{d}_{1,r}(z') &= (z' - z_1 - h_1 \eta + \eta), \end{aligned}$$

*in End $(\mathcal{F}_1^D)$ are the operator algebra associated to the finite dimensional quotient module of the functional Verma module of Proposition 6.9 a).*

### 6.1.3 A special class of twisted representations

**Remark:**
This type of representation is needed to describe the case of non-periodic boundary conditions of the XXX chain as formulated by Sklyanin [47].
The simplest way to construct the wanted class of representations is to start with the following proposition and then make use of the Hopf algebra property of $\mathcal{Y}(sl_2)$.

**Proposition 6.13** *Let*

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{111}$$

*be an element of $GL(2, V)$. Then $(V, A^{(0)} \otimes \mathbb{I})$ is a representation of $\mathcal{Y}(sl_2)$.*

**Proof:**
The way to prove the statement is straightforward by checking the rational RLL-relations

$$R_r^{(12)}(z)(A \otimes \mathbb{I})(\mathbb{I} \otimes A) = (\mathbb{I} \otimes A)(A \otimes \mathbb{I})R_r^{(12)}(z)\,.$$

By writing out the left and right hand side explicitly, we see that they coincide.

**Corollary 6.14** *Let $(W, L_r(z))$ be a representation of $\mathcal{Y}(sl_2)$. Let $A \in GL(2, V)$.*
*By means of the Hopf algebra property of $\mathcal{Y}(sl_2)$ $(W, A^{(0)} L_r^{(01)}(z))$ is a representation of $\mathcal{Y}(sl_2)$.*

**Remark:**
In Sklyanin [47], the matrix $A \in$ GL(2,$V$) was used to define the boundary conditions of the XXX chain, cf. the following section of this chapter.

## 6.2 The isomorphism to establish separation of variables for the XXX chain

**Synopsis:**
Here, we first write down the auxiliary representation of Definition 6.11 for $\Lambda_i = 1$ with $i = 1, \ldots, n$, since we want to compare this representation of the Yangian with the $n$fold tensor product of its fundamental representation (Definition 6.15).
Since the auxiliary representation is a functional representation we then have to define an isomorphism from the space of functions on which it acts to the space on which the $n$fold tensored fundamental representation acts. This is achieved in Proposition 6.16.
Then, since the isomorphism is – as in the $E_{\tau,\eta}(sl_2)$ case – constructed inductively, we formulate one inductive step in Proposition 6.19, thus connecting an auxiliary representation with $\Lambda_1 = \ldots = \Lambda_n = 1$ to a tensor product of a fundamental representation and an auxilary representation with $\Lambda_1 = \ldots = \Lambda_{n-1} = 1$, where the parameters $z_1, \ldots, z_n$ are fixed.
In Proposition 6.20, we show how to construct out of the isomorphism given in Proposition 6.19 an isomorphism with respect to which the $n$fold tensor product of fundamental

representations of the Yangian and the auxiliary representation of the Yangian with $\Lambda_i = 1$ for $i = 1, \dots, n$ are isomorphic.
In the quantum case, we want to find an isomorphism that maps the representation involved in constructing the XXX chain of order $n$ [47] - $L_r(z, z_1, \dots, z_n) \in$ End $(V^{\otimes(n+1)})$, which will be defined shortly - to the auxiliary representation of Proposition 6.11 with $\Lambda_i = 1$ for $i = 1, \dots, n$ $(\mathcal{F}_n^D, \bar{L}_{aux,r}(z, 1, \dots, 1, z_1, \dots, z_n)$
$= \bar{L}_{aux,r}(z, z_1, \dots, z_n))$. The auxiliary representation is characterized by the property that the operator $a_r^{aux}(z, z_1, \dots, z_n)$ is diagonal.
To construct such an isomorphism we first have to specify the results of Proposition 6.11.
**Remark (Auxiliary Representation):**
The definitions of Propopsition 6.11 being understood, let the auxiliary representation be given by the following operators

$$\bar{a}_{aux,r}(z, z_1, \dots, z_n) = \prod_{i=1}^{n} (z - z_i + \eta + x_i),$$

$$\bar{b}_{aux,r}(z, z_1, \dots, z_n) = \sum_{i=1}^{n} \prod_{j \neq i} \frac{z - z_j + \eta + x_j}{x_i - z_i - x_j + z_j} \prod_{j=1}^{n} (x_i - z_i + z_j - \eta) T_{x_i}^{+2\eta},$$

$$\bar{c}_{aux,r}(z, z_1, \dots, z_n) = \sum_{i=1}^{n} \prod_{j \neq i} \frac{z - z_j + \eta + x_j}{x_i - z_i - x_j + z_j} \prod_{j=1}^{n} (x_i - z_i + z_j + \eta) T_{x_i}^{-2\eta},$$

$$\bar{\text{Det}}_{aux,r}(z, z_1, \dots, z_n) = \prod_{i=1}^{n} (z - z_i - 2\eta)(z - z_i + 2\eta).$$

where the operator $d_{aux,r}(z, z_1, \dots, z_n)$ is defined implicitly, we put

$$o_{aux,r}(z, 1, \dots, 1, z_1, \dots, z_n) = o_{aux,r}(z, z_1, \dots, z_n)$$

for $o = \text{Det}, a, b, c, d,$ and the values of the operators $(x_i, \dots, x_n) \in D$
$= \{(x_1, \dots, x_n) | x_i \in \{-\eta, \eta\}$ for all $i = 1, \dots, n\}$.

**Definition 6.15 (L-operator)** *Let the L-operator*

$$L_r(z, z_1, \dots, z_n) \in \textit{End } (V^{\otimes(n+1)})$$

*be given by*

$$L_r(z, z_1, \dots, z_n)^{(01\dots n)} = R^{(01)}(z - z_1) \dots R^{(0i)}(z - z_i) \dots R^{(0n)}(z - z_n). \tag{112}$$

*To state the isomorphism between $L_r(z, z_1, \dots, z_n)$ and $L_{aux,r}(z, z_1, \dots, z_n)$, let us first state an isomorphism $I_{FC}$ that maps a basis of $\mathcal{F}_n^D$ to the standard tensor product basis of $V^{\otimes n}$.*

**Proposition 6.16** ($[r_{\sigma_1\ldots\sigma_n}]$, $I_{FC}$)

*a) A basis of* $\mathcal{F}_n^D$ *is given by*

$$\{[r_{\sigma_1\ldots\sigma_n}] = [\prod_{i=1}^n \delta_{\sigma_i\eta,x_i}]\,|\,\sigma_i \in \{-1,1\} \textit{ for all } i = 1,\ldots,n\},$$

*where by* $[\prod_{i=1}^n \delta_{\sigma_i\eta,x_i}]$ *we mean the equivalence class of functions which is one at* $(-z_1+\sigma_1\eta,\ldots,-z_n+\sigma_n\eta) \in D$ *and zero everywhere else on* $D$*. Note that we can find a meromorphic representant of this class.*

*b) The isomorphism* $I_{FC}: \mathcal{F}_n^D \to V^{\otimes n}$ *is given by* $I_{FC}[r_{\sigma_1\ldots\sigma_n}] = e[\sigma_1]\otimes\ldots\otimes e[\sigma_n]$ *for all possible combinations of* $\sigma_i \in \{-1,1\}$ *for* $i = 1,\ldots,n$*. Here,* $e[\sigma_1]\otimes\ldots\otimes e[\sigma_n]$ *is an element of the standard tensor product basis of* $V^{\otimes n}$*.*

**Proof:**

a) $[r_{\sigma_1\ldots\sigma_n}]$ is constructed to yield $[r_{\sigma_1\ldots\sigma_n}]$ has a value one at $(\sigma_1,\ldots,\sigma_n) \in D$ and $[r_{\sigma_1\ldots\sigma_n}]$ has value zero at all other points of $D$.
Thus, we can write every element $[f] \in \mathcal{F}_n^D$ as
$[f] = \sum_{i=1,\sigma_i\in\{-1,1\}}^n f(\sigma_1,\ldots,\sigma_n)[r_{\sigma_1\ldots\sigma_n}]$.

b) By construction.

**Remark** ($L^{\mathbb{C}}_{aux,r}(z,z_1,\ldots,z_n)$)**:**
By means of the isomorphism $I_{FC}$ defined above, we can define

$$L^{\mathbb{C}}_{aux,r}(z,z_1,\ldots,z_n) = (\mathbb{I}_2\otimes I_{FC})\bar{L}_{aux,r}(z,z_1,\ldots,z_n)(\mathbb{I}_2\otimes I_{FC}^{-1}),$$

where $\mathbb{I}_2 \in$ End $(V)$ is the identity matrix on $V$.

**Corollary 6.17** *By Corollary 6.12 and Proposition 6.16* $L^{\mathbb{C}}_{aux,r}(z,z_1)$ *is equal to* $R_r(z-z_1)$ *as an operator in End* $(V^{\otimes 2})$*.*

**Definition 6.18** ($\mathcal{A}_{n,r}(z_1,\ldots,z_n), A_{n,r}(z_1,\ldots,z_n)$)

*a) Let* $\mathbb{I}_{n-1} \in$ *End* $(V^{\otimes(n-1)})$ *be the identity matrix on* $V^{\otimes(n-1)}$*. Let* $\pi_- = \pi_-(z_1-2\eta) = \prod_{i=2}^n(z_1-z_i-2\eta)$*. Let us put* $o^{\mathbb{C}}_{aux,r}(z,z_1,\ldots,z_n) = o(z)$ *for* $o = a,b,c,d$*. Then the matrix* $\mathcal{A}_{n,r}(z_1,\ldots,z_n) \in$ *End* $(V^{\otimes n})$ *is given by*

$$\mathcal{A}_{n,r} = \begin{pmatrix} \mathbb{I}_{n-1} & 0 \\ a^{-1}(z_1-2\eta)c(z_1-2\eta) & \pi_- a^{-1}(z_1-2\eta) \end{pmatrix}. \tag{113}$$

*b) The matrix* $A_{n,r}(z_1,\ldots,z_n) \in$ *End* $(V^{\otimes n})$ *is given by*

$$A_{n,r}(z_1,\ldots,z_n) = \mathcal{A}_{2,r}^{(n-1n)}(z_{n-1},z_n)\ldots\times$$
$$\times\mathcal{A}_{i,r}^{(n-i+1\ldots n)}(z_{n-i+1},\ldots,z_n)\ldots\mathcal{A}_{n,r}^{(1\ldots n)}(z_1,\ldots,z_n). \tag{114}$$

**Proposition 6.19** *Let $\mathcal{A}_{n,r}(z_1, \dots, z_n)$ be given by the above definition. Let $\mathbb{I}_2$ be the identity matrix on $V$.*
*Then,*

$$(\mathbb{I}_2 \otimes \mathcal{A}_{n,r}^{-1}(z_1, \dots, z_n)) R_r^{(01)}(z - z_1)(L_{aux,r}^{\mathbb{C}})^{(02\dots n)}(\mathbb{I}_2 \otimes \mathcal{A}_{n,r}(z, z_2, \dots, z_n)) \\ = L_{aux,r}^{\mathbb{C}}(z, z_1, \dots, z_n). \tag{115}$$

**Remark:**
Instead of writing down one identity involving L-operators, we can also write down the following four identities involving the entries of L-operators:

$$\mathcal{A}_{n,r}^{-1}(z_1, \dots, z_n)(a_r(z - z_1) \otimes a_{aux,r}^{\mathbb{C}}(z, z_2, \dots, z_n) + \\ b_r(z - z_1) \otimes c_{aux,r}^{\mathbb{C}}(z, z_2, \dots, z_n))\mathcal{A}_{n,r}(z_1, \dots, z_n) = a_{aux,r}^{\mathbb{C}}(z, z_1, \dots, z_n),$$

$$\mathcal{A}_{n,r}^{-1}(z_1, \dots, z_n)(a_r(z - z_1) \otimes b_{aux,r}^{\mathbb{C}}(z, z_2, \dots, z_n) + \\ b_r(z - z_1) \otimes d_{aux,r}^{\mathbb{C}}(z, z_2, \dots, z_n))\mathcal{A}_{n,r}(z_1, \dots, z_n) = b_{aux,r}^{\mathbb{C}}(z, z_1, \dots, z_n),$$

$$\mathcal{A}_{n,r}^{-1}(z_1, \dots, z_n)(c_r(z - z_1) \otimes a_{aux,r}^{\mathbb{C}}(z, z_2, \dots, z_n) + \\ d_r(z - z_1) \otimes c_{aux,r}^{\mathbb{C}}(z, z_2, \dots, z_n))\mathcal{A}_{n,r}(z_1, \dots, z_n) = c_{aux,r}^{\mathbb{C}}(z, z_1, \dots, z_n),$$

$$\mathcal{A}_{n,r}^{-1}(z_1, \dots, z_n)(c_r(z - z_1) \otimes b_{aux,r}^{\mathbb{C}}(z, z_2, \dots, z_n) + \\ d_r(z - z_1) \otimes d_{aux,r}^{\mathbb{C}}(z, z_2, \dots, z_n))\mathcal{A}_{n,r}(z_1, \dots, z_n) = d_{aux,r}^{\mathbb{C}}(z, z_1, \dots, z_n).$$

**Proof:**
Let us put $\mathcal{A}_{n,r}(z_1, \dots, z_n) = \mathcal{A}_r$ throughout the proof, since $n$ stays fixed. For the sake of simplicity, we put $o_{aux,r}^{\mathbb{C}}(z, z_2, \dots, z_n) \equiv o(z)$ for $o = a, b, c, d$.
We have to check that the $L$-operator of $R_r^{(01)}(z - z_1)(L_{aux,r}^{\mathbb{C}})^{(02\dots n)}(z, z_2, \dots, z_n)$ $= R_{\otimes}^{(01\dots n)}(z, z_1, z_2, \dots, z_n)$ when conjugated by $\mathcal{A}_{n,r}(z_1, \dots, z_n)$ coincides with the $L$-operator $(L_{aux,r}^{\mathbb{C}})^{(01\dots n)}(z, z_1, \dots, z_n)$. This is checked by checking the corresponding identity for each entry of the L-operator $L_{aux,r}^{\mathbb{C}}(z, z_1, \dots, z_n)$ separately, as was formulated in the remark.
For the entry $a_{\otimes}^{\mathbb{C}}(z, z_1, \dots, z_n)$ conjugation with $\mathcal{A}_r$ yields the following:

$$\mathcal{A}_r^{-1} a_{\otimes}^{\mathbb{C}}(z, z_1, \dots, z_n)\mathcal{A}_r = \\ \begin{pmatrix} 1 & 0 \\ -(\pi_-)^{-1}c(z_1 - 2\eta) & (\pi_-)^{-1}a(z_1 - 2\eta) \end{pmatrix} \begin{pmatrix} (z - z_1 + 2\eta)a(z) & 0 \\ 2\eta c(z) & (z - z_1)a(z) \end{pmatrix} \times \\ \times \begin{pmatrix} 1 & 0 \\ a^{-1}(z_1 - 2\eta)c(z_1 - 2\eta) & (\pi_-)a^{-1}(z_1 - 2\eta) \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 \\ -(\pi_-)^{-1}c(z_1 - 2\eta) & (\pi_-)^{-1}a(z_1 - 2\eta) \end{pmatrix} \times \\ \times \begin{pmatrix} (z - z_1 + 2\eta)a(z) & 0 \\ 2\eta c(z) + (z - z_1)a(z)(a^{-1}c)(z_1 - 2\eta) & (\pi_-)(z - z_1)a(z)a^{-1}(z_1 - 2\eta) \end{pmatrix} = \\ \begin{pmatrix} (z - z_1 + 2\eta)a(z) & 0 \\ a_{21}(z) & (z - z_1)(\pi_-)^{-1}\pi_- a(z_1 - 2\eta)a(z)a^{-1}(z_1 - 2\eta) = a_{22}(z) \end{pmatrix},$$

where the entry $a_{21}(z)$ is given and can be simplified in the following manner

$$a_{21}(z) = (\pi_-)^{-1}\left(-(z - z_1 + 2\eta)c(z_1 - 2\eta)a(z)\right. \\ (\pi_-)^{-1} + 2\eta a(z_1 - 2\eta)c(z) + (z - z_1)a(z_1 - 2\eta)a(z)(a^{-1}c)(z_1 - 2\eta)) = \\ (\pi_-)^{-1}\left(-(z - z_1 + 2\eta)c(z_1 - 2\eta)a(z) + 2\eta a(z_1 - 2\eta)c(z) + (z - z_1)a(z)c(z_1 - 2\eta)\right),$$

where the second line was obtained by using the first of the sixteen relations of the Yangian $\mathcal{Y}(sl_2)$. By the ninth of the relation - with $z' = z_1 - 2\eta, w' = z$ - the last line equals zero. The entry $a_{22}(z)$ may by the first relation also be further simplified to yield

$$a_{22}(z) = (z - z_1)a(z).$$

Hence, the conjugated matrix reads

$$\mathcal{A}_r^{-1} a_{\otimes}^{\mathbb{C}}(z, z_1, \dots, z_n)\mathcal{A}_r = \begin{pmatrix} (z - z_1 + 2\eta)a_{aux,r}^{\mathbb{C}}(z, z_2, \dots, z_n) & 0 \\ 0 & (z - z_1)a_{aux,r}^{\mathbb{C}}(z, z_2, \dots, z_n) \end{pmatrix}.$$

Let us compare this to what we expect by writing $a_{aux,r}^{\mathbb{C}}(z, z_1, \dots, z_n)$, writing it as a $2 \times 2$ matrix with entries in $\mathrm{End}(V^{\otimes(n-1)})$.

$$a_{aux,r}^{\mathbb{C}}(z, z_1, \dots, z_n) = \begin{pmatrix} (z - z_1 + 2\eta)a_{aux,r}^{\mathbb{C}}(z, z_2, \dots, z_n) & 0 \\ 0 & (z - z_1)a_{aux,r}^{\mathbb{C}}(z, z_2, \dots, z_n). \end{pmatrix}$$

This coincides with what we calculated.
Let us do the same calculation with the operator $b_{\otimes}^{\mathbb{C}}(z, z_1, \dots, z_n)$.

$$\begin{aligned} &\mathcal{A}_r^{-1} b_{\otimes}^{\mathbb{C}}(z, z_1, \dots, z_n)\mathcal{A}_r = \\ &\begin{pmatrix} 1 & 0 \\ -(\pi_-)^{-1}c(z_1 - 2\eta) & (\pi_-)^{-1}a(z_1 - 2\eta) \end{pmatrix} \begin{pmatrix} (z - z_1 + 2\eta)b(z) & 0 \\ 2\eta d(z) & (z - z_1)b(z) \end{pmatrix} \\ &\begin{pmatrix} 1 & 0 \\ (a^{-1}c)(z_1 - 2\eta) & \pi_- a^{-1}(z_1 - 2\eta) \end{pmatrix} = \\ &\begin{pmatrix} (z - z_1 + 2\eta)b(z) & 0 \\ b_{21}(z) & (z - z_1)a(z_1 - 2\eta)b(z)a^{-1}(z_1 - 2\eta) \end{pmatrix}, \end{aligned}$$

where the entry $b_{21}(z)$ and its subsequent simplification are given by

$$\begin{aligned} &b_{21}(z) = \\ &(\pi_-)^{-1}\left(-(z - z_1 + 2\eta)c(z_1 - 2\eta)b(z) + 2\eta a(z_1 - 2\eta)d(z)\right. \\ &\left.+(z - z_1)a(z_1 - 2\eta)b(z)a^{-1}(z_1 - 2\eta)c(z_1 - 2\eta)\right) = \\ &(\pi_-)^{-1}\left(-(z - z_1 + 2\eta)c(z_1 - 2\eta)b(z) + 2\eta a(z_1 - 2\eta)d(z) + \right. \\ &\left.(z - z_1 + 2\eta)b(z)a(z_1 - 2\eta)a^{-1}(z_1 - 2\eta)c(z_1 - 2\eta) - 2\eta a(z)b(z_1 - 2\eta)a^{-1}(z_1 - 2\eta)c(z_1 - 2\eta)\right) = \\ &(\pi_-)^{-1}(2\eta a(z)d(z_1 - 2\eta) - 2\eta a(z)b(z_1 - 2\eta)a^{-1}(z_1 - 2\eta)c(z_1 - 2\eta)) = \\ &(\pi_-)^{-1}2\eta a(z)(d(z_1 - 2\eta)a(z_1) - b(z_1 - 2\eta)a^{-1}(z_1 - 2\eta)c(z_1 - 2\eta)a(z_1))a^{-1}(z_1) = \\ &(\pi_-)^{-1}2\eta a(z)(d(z_1 - 2\eta)a(z_1) - b(z_1 - 2\eta)c(z_1))a^{-1}(z_1) = \\ &2\eta(\pi_-)^{-1} Det_{aux,r}^{\mathbb{C}}(z_1, z_2, \dots, z_n)a(z)a^{-1}(z_1) = 2\eta(\pi_+)a(z)a^{-1}(z_1), \end{aligned}$$

where $\pi_+$ is given by

$$\pi_+ = \prod_{i=1}^{n}(z_1 - z_i + 2\eta).$$

Here we used the second relation and the tenth relation for $z' = z_1 - 2\eta, w' = z$ and the ninth relation with $z' = z_1 - 2\eta, w' = z_1$.
The result is that the conjugated matrix of $b_{\otimes}^{\mathbb{C}}(z, z_1, \dots, z_n)$ reads

$$\mathcal{A}_r^{-1} b_{\otimes}^{\mathbb{C}}(z, z_1, \dots, z_n)\mathcal{A}_r = \begin{pmatrix} (z - z_1 + 2\eta)b(z) & 0 \\ 2\eta(\pi_+)a(z)a^{-1}(z_1) & (z - z_1)a(z_1 - 2\eta)b(z)a^{-1}(z_1 - 2\eta) \end{pmatrix}$$

Let us compare this with $b^{\mathbb{C}}_{aux,r}(z,z_1,\dots,z_n)$ which we write the same way as the matrix $a^{\mathbb{C}}_{aux,r}(z,z_1,\dots,z_n)$ before. This yields - writing each matrix element separately -

$$\begin{aligned}
I_{FC}^{-1}(b^{\mathbb{C}}_{aux,r})_{11}(z,z_1,\dots,z_n)I_{FC} &= (z-z_1+2\eta)\sum_{i=2}^{n}\prod_{j\neq i,j=2}^{n}\frac{z-z_j+x_j+\eta}{x_i-z_i-x_j+z_j}\\
&\times \prod_{j=2}^{n}(x_i-z_i+z_j-\eta)\frac{z_1-z_i-2\eta}{z_1-z_i-2\eta}T^{+2\eta}_{x_i},\\
I_{FC}^{-1}(b^{\mathbb{C}}_{aux,r})_{12}(z,z_1,\dots,z_n)I_{FC} &= 0,\\
I_{FC}^{-1}(b^{\mathbb{C}}_{aux,r})_{21}(z,z_1,\dots,z_n)I_{FC} &= \prod_{j=2}^{n}\frac{z-z_j+x_j+\eta}{\eta+z_1-z_j+x_j}\prod_{j=1}^{n}(z_1-z_j+2\eta),\\
I_{FC}^{-1}(b^{\mathbb{C}}_{aux,r})_{22}(z,z_1,\dots,z_n)I_{FC} &= (z-z_1)\sum_{i=2}^{n}\prod_{j\neq i,j=2}^{n}\frac{z-z_j+x_j+\eta}{x_i-z_i-x_j+z_j}\times\\
&\times \prod_{j=2}^{n}(x_i-z_i+z_j-\eta)\frac{z_1-z_i-2\eta}{z_1-z_i}T^{+2\eta}_{x_i},
\end{aligned}$$

or – written in terms of the operators $o^{\mathbb{C}}_{aux,r}(z,z_2,\dots,z_n)$ and inverting the isomorphism $I_{FC}$ again –

$$\begin{aligned}
(b^{\mathbb{C}}_{aux,r})_{11}(z,z_1,\dots,z_n) &= (z-z_1+2\eta)b^{\mathbb{C}}_{aux,r}(z,z_2,\dots,z_n),\\
(b^{\mathbb{C}}_{aux,r})_{12}(z,z_1,\dots,z_n) &= 0,\\
(b^{\mathbb{C}}_{aux,r})_{21}(z,z_1,\dots,z_n) &= 2\eta(\pi_+)a^{\mathbb{C}}_{aux,r}(z,z_2,\dots,z_n){a^{\mathbb{C}}_{aux,r}}^{-1}(z_1,z_2,\dots,z_n),\\
(b^{\mathbb{C}}_{aux,r})_{22}(z,z_1,\dots,z_n) &= (z-z_1)a^{\mathbb{C}}_{aux,r}(z_1-2\eta,z_2,\dots,z_n)\times\\
&\times\ b^{\mathbb{C}}_{aux,r}(z,z_2,\dots,z_n)(a^{\mathbb{C}}_{aux,r})^{-1}(z_1-2\eta,z_2,\dots,z_n).
\end{aligned}$$

If we compare this to the conjugated matrix, we see that both coincide.
It remains to check the operator $c^{\mathbb{C}}_{\otimes}(z,z_1,\dots,z_n)$. Its conjugation yields

$$\begin{aligned}
&\mathcal{A}_r^{-1}c^{\mathbb{C}}_{\otimes}(z,z_1,\dots,z_n)\mathcal{A}_r =\\
&\begin{pmatrix} 1 & 0\\ -(\pi_-)^{-1}c(z_1-2\eta) & (\pi_-)^{-1}a(z_1-2\eta)\end{pmatrix}\begin{pmatrix}(z-z_1)c(z) & 2\eta a(z)\\ 0 & (z-z_1+2\eta)c(z)\end{pmatrix}\times\\
&\times\begin{pmatrix} 1 & 0\\ (a^{-1}c)(z_1-2\eta) & (\pi_-)a^{-1}(z_1-2\eta)\end{pmatrix}=\dots=\\
&\begin{pmatrix} c_{11}(z) & (\pi_-)2\eta a(z)a(z_1-2\eta)\\ c_{21}(z) & c_{22}(z)\end{pmatrix},
\end{aligned}$$

where the corresponding coefficients and their simplifications are given below.

$$\begin{aligned}
&c_{11}(z) =\\
&2\eta a(z)(a^{-1}c)(z_1-2\eta)+(z-z_1)c(z) =\\
&(2\eta a(z)(a^{-1}c)(z_1-2\eta)a(z_1)+(z-z_1)c(z)a(z_1))a^{-1}(z_1) =\\
&(2\eta a(z)c(z_1)+(z-z_1)c(z)a(z_1))a^{-1}(z_1) =\\
&(z-z_1+2\eta)a(z_1)c(z)a^{-1}(z_1).
\end{aligned}$$

Here we used the ninth relation first with $z' = z_1, w' = z_1 - 2\eta$, then with $z' = z, w' = z_1$.

$$\begin{aligned}(\pi_-)c_{21}(z)(-2\eta c(z_1 - 2\eta)a(z) - (z - z_1)c(z)a(z_1 - 2\eta) = \\ +(z - z_1 + 2\eta)a(z_1 - 2\eta)c(z))a^{-1}(z_1 - 2\eta)c(z_1 - 2\eta) = 0\end{aligned}$$

due to the fifth relation with $z' = z_1 - 2\eta, w' = z$.

$$\begin{aligned}c_{22}(z) = (-2\eta c(z_1 - 2\eta)a(z) + (z - z_1 + 2\eta)a(z_1 - 2\eta)c(z))a^{-1}(z_1 - 2\eta) = \\ (z - z_1)c(z)a(z_1 - 2\eta)a^{-1}(z_1 - 2\eta) = (z - z_1)c(z),\end{aligned}$$

where we used the fifth relation with $z' = z_1 - 2\eta, w' = z$.
So the conjugated matrix looks like

$$\mathcal{A}_r^{-1}c_{\otimes}^{\mathbb{C}}(z, z_1, \ldots, z_n)\mathcal{A}_r = \begin{pmatrix} (z - z_1 + 2\eta)a(z_1)c(z)a^{-1}(z_1) & 2\eta(\pi_-)a(z)a^{-1}(z_1 - 2\eta) \\ 0 & (z - z_1)c(z) \end{pmatrix}.$$

The entries of the matrix $c_{aux,r}^{\mathbb{C}}(z, z_1, \ldots, z_n)$ yield

$$\begin{aligned}(c_{aux,r}^{\mathbb{C}})_{11}(z, z_1, \ldots, z_n) &= I_{FC}\sum_{i=2}^{n}\prod_{j\neq i, j=2}^{n}\frac{z - z_j + x_j + \eta}{x_i - z_i - x_j + z_j}(z - z_1 + 2\eta) \times \\ &\times \frac{z_1 - z_i + 2\eta}{z_1 - z_i}\prod_{j=2}^{n}(2\eta - z_i + z_j)T_{x_i}^{-2\eta}I_{FC}^{-1}, \\ (c_{aux,r}^{\mathbb{C}})_{12}(z, z_1, \ldots, z_n) &= I_{FC}(2\eta\prod_{j=2}^{n}\frac{z - z_j + x_j + \eta}{\eta - z_1 + z_j - x_j}\prod_{j=2}^{n}(z_1 - z_i - 2\eta))I_{FC}^{-1}, \\ (c_{aux,r}^{\mathbb{C}})_{21}(z, z_1, \ldots, z_n) &= 0, \\ (c_{aux,r}^{\mathbb{C}})_{22}(z, z_1, \ldots, z_n) &= I_{FC}(\sum_{i=2}^{n}\prod_{j\neq i, j=2}^{n}\frac{z - z_j + x_j + \eta}{x_i - z_i - x_j + z_j}\prod_{j=2}^{n}(2\eta - z_i + z_j) \\ &\times (z - z_1)\frac{z_1 - z_i + 2\eta}{z_1 - z_i + 2\eta}T_{x_i}^{-2\eta})I_{FC}^{-1}.\end{aligned}$$

If we rewrite this in terms of operators $o_{aux,r}^{\mathbb{C}}(z, z_2, \ldots, z_n)$, we get

$$\begin{aligned}(c_{aux,r}^{\mathbb{C}})_{11}(z, z_1, \ldots, z_n) &= (z - z_1 + 2\eta)a_{aux,r}^{\mathbb{C}}(z_1, z_2, \ldots, z_n) \times \\ &\times c_{aux,r}^{\mathbb{C}}(z, z_2, \ldots, z_n)(a_{aux,r}^{\mathbb{C}})^{-1}(z_1, z_2, \ldots, z_n), \\ (c_{aux,r}^{\mathbb{C}})_{12}(z, z_1, \ldots, z_n) &= 2\eta(\pi_-)a_{aux,r}^{\mathbb{C}}(z, z_2, \ldots, z_n) \times \\ &\times (a_{aux,r}^{\mathbb{C}})^{-1}(z_1 - 2\eta, z_2, \ldots, z_n) \\ (c_{aux,r}^{\mathbb{C}})_{21}(z, z_1, \ldots, z_n) &= 0, \\ (c_{aux,r}^{\mathbb{C}})_{22}(z, z_1, \ldots, z_n) &= (z - z_1)c_{aux,r}^{\mathbb{C}}(z, z_2, \ldots, z_n).\end{aligned}$$

These are the same entries as appearing in $\mathcal{A}_r^{-1}c_{\otimes}^{\mathbb{C}}(z, z_1, \ldots, z_n)\mathcal{A}_r$.
Since the quantum determinants were shown to be multiplicative in Proposition 6.10 and $d_{aux,r}^{\mathbb{C}}(z, z_1, \ldots, z_n)$ is defined implicitly by means of the quantum determinant, this completes the proof.

**Proposition 6.20** *Let $A_{n,r}(z_1,\dots,z_n)$ the matrix defined before and let $\mathbb{I}_2 \in$ End $(V)$ be the identity matrix on $V$.*
*Then*

$$(\mathbb{I}_2 \otimes A_{n,r}^{-1}(z_1,\dots,z_n))L_r(z,z_1,\dots,z_n)(\mathbb{I}_2 \otimes A_{n,r}(z_1,\dots,z_n)) \\ = L^{\mathbb{C}}_{aux,r}(z,z_1,\dots,z_n). \tag{116}$$

**Remark:**
Written down in the components of both L-operators we get the following four identities:

$$A_{n,r}^{-1}(z_1,\dots,z_n)a_r(z,z_1,\dots,z_n)A_{n,r}(z_1,\dots,z_n) = a^{\mathbb{C}}_{aux,r}(z,z_1,\dots,z_n),$$
$$A_{n,r}^{-1}(z_1,\dots,z_n)b_r(z,z_1,\dots,z_n)A_{n,r}(z_1,\dots,z_n) = b^{\mathbb{C}}_{aux,r}(z,z_1,\dots,z_n),$$
$$A_{n,r}^{-1}(z_1,\dots,z_n)c_r(z,z_1,\dots,z_n)A_{n,r}(z_1,\dots,z_n) = c^{\mathbb{C}}_{aux,r}(z,z_1,\dots,z_n),$$
$$A_{n,r}^{-1}(z_1,\dots,z_n)d_r(z,z_1,\dots,z_n)A_{n,r}(z_1,\dots,z_n) = d^{\mathbb{C}}_{aux,r}(z,z_1,\dots,z_n).$$

**Proof:**
Let us proof the four identities we wrote down in the remark instead of proving the identity involving the L-operators. The proof is by induction. Let us start with $n=2$. By definition $\mathcal{A}_{2,r}(z_1,z_2) = A_{2,r}(z_1,z_2)$. Let us prove just one identity, e.g. the one which reads $A_{2,r}^{-1}(z_1,z_2)c_r(z,z_1,z_2)A_{2,r}(z_1,z_2) = c^{\mathbb{C}}_{aux,r}(z,z_1,z_2)$, since the other identities are shown in a similar manner.

$$A_{2,r}^{-1}(z_1,z_2)c_r(z,z_1,z_2)A_{2,r}(z_1,z_2) =$$
$$\mathcal{A}_{2,r}^{-1}(z_1,z_2)c_r(z,z_1,z_2)\mathcal{A}_{2,r}(z_1,z_2) =$$
$$\mathcal{A}_{2,r}^{-1}(z_1,z_2)(c_r(z-z_1)\otimes a_r(z-z_2) + d_r(z-z_1)\otimes c_r(z-z_2))\mathcal{A}_{2,r}(z_1,z_2) =$$
$$\mathcal{A}_{2,r}^{-1}(z_1,z_2)(c_r(z-z_1)\otimes a^{\mathbb{C}}_{aux,r}(z,z_2) + d_r(z-z_1)\otimes c^{\mathbb{C}}_{aux,r}(z,z_2))\mathcal{A}_{2,r}(z_1,z_2) =$$
$$c^{\mathbb{C}}_{aux,r}(z,z_1,z_2),$$

where we used the definition of the $L_r(z,z_1,z_2)$, the identity of $R_r(z-z_1)$ and $L^{\mathbb{C}}_{aux,r}(z,z_1)$ and the preceding proposition.
Let us now assume that

$$(A_{n,r}^{-1})^{(2\dots n+1)}(z_2,\dots,z_{n+1})o_r(z,z_2,\dots,z_{n+1})A_{n,r}^{(2\dots n+1)}(z_2,\dots,z_{n+1}) \\ = o^{\mathbb{C}}_{aux,r}(z,z_2,\dots,z_{n+1})$$

holds true for some fixed $n$ for $o = a,b,c,d$.
We claim that under these circumstances it follows that

$$A_{n+1,r}^{-1}(z_1,\dots,z_{n+1})o_r(z,z_1,\dots,z_{n+1})A_{n,r}(z_1,\dots,z_{n+1}) \\ = o^{\mathbb{C}}_{aux,r}(z,z_1,\dots,z_{n+1})$$

for $o = a,b,c,d$.
Let us show it for $d^{\mathbb{C}}_{aux,r}(z,z_1,\dots,z_{n+1})$, since the proofs of the identities involving the other operators are strucuturally completely similar. First note that by definition $A_{n,r}^{(2\dots n+1)}(z_2,\dots,z_{n+1})\mathcal{A}_{n+1,r}^{(1\dots n+1)}(z_1,\dots,z_{n+1}) = A_{n+1,r}^{(1\dots n+1)}(z_1,\dots,z_{n+1})$. Hence,

$$d^{\mathbb{C}}_{aux,r}(z,z_1,\dots,z_{n+1}) =$$
$$\mathcal{A}_{n+1,r}^{-1}(z_1,\dots,z_{n+1})(c_r(z-z_1)\otimes c^{\mathbb{C}}_{aux,r}(z,z_2,\dots,z_{n+1})+$$

$$+d_r(z-z_1)\otimes d^{\mathbb{C}}_{aux,r}(z,z_2,\dots,z_{n+1}))\mathcal{A}_{n+1,r}(z_1,\dots,z_{n+1}) =$$
$$\mathcal{A}^{-1}_{n+1,r}(z_1,\dots,z_{n+1})(A^{-1}_{n,r})^{(2\dots n+1)}(z_2,\dots,z_{n+1})(c_r(z-z_1)\otimes b_r(z,z_2,\dots,z_{n+1})+$$
$$d_r(z-z_1)\otimes d_r(z,z_2,\dots,z_{n+1}))A^{(2\dots n+1)}_{n,r}(z_2,\dots,z_{n+1})\mathcal{A}_{n+1,r}(z_1,\dots,z_{n+1}) =$$
$$A^{-1}_{n+1,r}(z_1,\dots,z_{n+1})d_r(z,z_1,\dots,z_{n+1})A_{n+1,r}(z_1,\dots,z_{n+1}),$$

where we used the preceding proposition, the assumption on the operators denoted $o^{\mathbb{C}}_{aux,r}(z,z_2,\dots,z_{n+1})$, the definition of $L_r(z,z_1,\dots,z_{n+1})$ and of $A_{n+1,r}(z_1,\dots,z_{n+1})$.
**Remark:**
The last corollary states the isomorphism between the representation of the XXX-chain with arbitrary boundary conditions and the auxiliary representation.

**Corollary 6.21** *Let $A\in GL(2,V)$ and $\mathbb{I}_2$ be the identity matrix on $V$.*
*Then, for $\bar{L}_{aux,r}(z,z_1,\dots,z_n)\in End\ (V^{(0)}\otimes\mathcal{F}^D_n)$ and $L_r(z,z_1,\dots,z_n)\in\ End\ (V^{(0)}\otimes V^{\otimes n})$*

$$A^{(0)}L_{aux,r}(z,z_1,\dots,z_n) = (\mathbb{I}_2\otimes I^{-1}_{FC}) \tag{117}$$
$$(A\otimes A^{-1}_{n,r}(z_1,\dots,z_n))L_r(z,z_1,\dots,z_n)(\mathbb{I}_2\otimes A_{n,r}(z_1,\dots,z_n))(\mathbb{I}_2\otimes I_{FC}).$$

# 7 Appendix 2: Spaces of elliptic polynomials

**Definition:**

a) Let $\Im(\tau) > 0$. Let $\Gamma = \mathbb{Z} + \tau\mathbb{Z}$ and $\Gamma^* \simeq (\mathbb{C}^\times)^2$ the group of group homomorphisms $\Gamma \to \mathbb{C}^\times$. Let $\chi \in \Gamma^*$.
Then we define the homomorphism $\phi : \Gamma^* \to E_\tau$ by $\phi : \chi \mapsto \frac{1}{2\pi i}(\ln \chi(\tau) - \tau \ln \chi(1))$ – where 1 and $\tau$ are an oriented basis of $\Gamma$.

b) For $\chi \in \Gamma^*$ let $\Theta_k(\chi)$ be the space of entire holomorphic functions $f(z)$ of level $k$ obeying the following property: $f(z + r + s\tau) = e^{-\pi i k(\tau s^2 + 2sz)}\chi(r + s\tau)f(z)$ for all $r + s\tau \in \Gamma$. Hence

$$\Theta_k(\chi) = \{f(z) \text{ holomorphic, entire } | \\ f(z + r + s\tau) = e^{-\pi i k(\tau s^2 + 2sz)}\chi(r + s\tau)f(z) \text{ for all } r + s\tau \in \Gamma\}.$$

The dimension of $\Theta_k(\chi)$ is 0 if $k < 0$, $k$ if $k \geq 1$, for $k = 0$ it is one if $\phi(\chi) = 0$ and 0 otherwise.
For the elements of these function spaces, we obtain the following result:
**Proposition E.1:**
The function of $z \in \mathbb{C}$

$$f(a, w_1, \ldots, w_n, , z) = e^{az} \prod_{i=1}^{n} \theta(z + w_i)$$

belongs to $\Theta_n(\chi)$ with $\chi(r + s\tau) = (-1)^{(r+s)n} e^{ra + s(a\tau - 2\pi i \sum_j w_j)}$.
Every function in $\Theta_n(\chi)$ is of the form $C \cdot f(a, w_1, \ldots, w_n, z)$ for some constant $C$. This representation is unique up to permutation of the $(w_1, \ldots, w_n)$ if one requires the $w_i$ to be in the fundamental domain $F = \{u + v\tau | u, v \in [0, 1)\}$.
**Proof of Proposition E.1:**
It follows from the transformation properties of theta functions that the number of zeroes $\int_{\partial F} d \ln g$, counted with multiplicities, of $g \in \Theta(\chi)_n$ in $F$ is $n$. If $w_1, \ldots, w_n$ denote the zeroes of $g$ then $g(z)/f(z, w_1, \ldots, w_n, a)$ is doubly periodic (since $\chi_f(r + s\tau)$ and $\chi_g(r + s\tau)$ do not depend on $z$) and regular, thus constant. Uniqueness follows as $a$ is uniquely determined by the $w_i, i = 1, \ldots, n$, and $\chi_g$.
**Corollary E.2:**
Let $E_\tau$ be the elliptic curve determined by $\tau$ and, for $k > 0$, let $S^k(E) = E/S_k$ its $k^{\text{th}}$ symmetric power. The map $\mathbb{P}$: $(\Theta_k(\chi)) \to S^k(E)$, sending an element of $\Theta_k(\chi)$ to the set of its zeroes mod $\Gamma$, is injective (i.e. to a given set of zeroes $[w_1', \ldots, w_k'] \in S^k(E)$ there corresponds at most one element of $\Theta_k(\chi)$). Its image consists of classes $[w_1, \ldots, w_k] \in S^k(E)$ subject to the condition that $\sum_{j=1}^k w_j = \phi(\chi) + k\delta$, $\delta$ being the image of $(1 + \tau)/2$ in $E$.
**Theorem E.3:**
Let $z_1, \ldots, z_n \in \mathbb{C}$ be pairwise distinct modulo $\Gamma$ and $\chi \in \Gamma^*$ such that $\sum_{i=1}^n z_i \neq \phi(\chi) + k\delta$ mod $\Gamma$. Then for any $f_1, \ldots, f_n \in \mathbb{C}$ there exists a unique function $f \in \Theta_n(\chi)$ such that $f(z_i) = f_i, i = 1, \ldots, n$.

The interpolation formula is given by

$$f(z) = \sum_{i=1}^{n} f_i e^{2\pi i a(z-z_j)} \frac{\theta(z-z_j+b)}{\theta(b)} \prod_{j=1, j\neq i}^{n} \frac{\theta(z-z_j)}{\theta(z_i-z_j)},$$

with

$$a = \frac{1}{2\pi i}(\ln \chi(1) - \frac{n}{2})$$
$$b = -\phi(\chi) - n\delta + \sum_{i=1}^{n} z_i - \kappa\frac{1+\tau}{2}.$$

**Proof of Theorem E.3:**
The function $f(z)$ has the desired transformation properties. The condition on the sum of evaluation points ensures that the appearing denominator does not vanish identically. The function is unique since the difference of any two such functions is a theta function vanishing at $n$ points $z_1, \dots, z_n$. By Corollary E.2, since $\sum_{i=1}^{n} z_i \neq \phi(\chi) + k\delta$, it vanishes identically.
Let us now turn our interest to special classes of difference equations involving coefficients that are doubly periodic functions:

$$A_(z)Q(z-2\eta) + A_+(z)Q(z+2\eta)\epsilon(z)Q(z),$$

with $A_\pm(z) \in \Theta_n(e^{\mp 2\pi i s n\eta - 2\pi i \sum_{i=1}^{n} z_i s}(-1)^{r+s})$.
**Proposition E.4:**
Suppose that $A_\pm(z) \in \Theta_n(e^{\mp 2\pi i s n\eta - 2\pi i \sum_{i=1}^{n} z_i s}(-1)^{r+s})$ with $n$ even.
To obtain a non-trivial solution of the above difference equation,
$Q(z) = e^{az} \prod_{j=1}^{\frac{n}{2}} \theta(z+w_j) \in \Theta_{\frac{n}{2}}(\chi)$ and $\epsilon(z) \in \Theta_n(e^{2\pi i \sum_{i=1}^{n} z_i s})$.
The character of $Q(z)$ is fixed up to one parameter by the Bethe Ansatz equations

$$A_+(-w_i) \prod_{j=1, j\neq i}^{\frac{n}{2}} \theta(-w_i + w_j - 2\eta) = e^{4\eta a} A_-(-w_i) \prod_{j=1, j\neq i}^{\frac{n}{2}} \theta(-w_i + w_j + 2\eta),$$

for $i = 1, \dots, n$ and $w_i \neq w_j \mod \Gamma$, for $i \neq j$.
An explicit formula for $\epsilon(z)$ is given by

$$\epsilon(z) = \frac{A_+(z)Q(z+2\eta) + A_-(z)Q(z-2\eta)}{Q(z)}.$$

$(Q(z), \epsilon(z))$ form an elliptic polynomial solution. Conversely, if $(\epsilon(z), Q(z))$ is an elliptic polynomial solution of the above difference equation, then there exists a solution $a, w_1, \dots, w_{\frac{n}{2}}$ of the Bethe Ansatz equations such that $Q(z)$ is of the above written form up to a constant $C$ and $\epsilon(z)$ is also of the above written form.
**Proof of Proposition E.4:**
A necessary condition of the above difference equation having a non-trivial solution $Q(z)$ is that all terms are theta functions with the same character. So the character of $\epsilon(z)$ has to be $e^{2\pi i \sum_{i=1}^{n} z_i}$.

Let $Q(z)$ be the above written function. The formula for $\epsilon(z)$ transforms as requested, but may be singular at the zeroes of $Q(z)$. This is precisely prevented by the system of Bethe Ansatz equations, ensuring that all possible residues of $\epsilon(z)$ vanish. Thus, $\epsilon(z)$ is regular everywhere, leading to its being an elliptic polynomial solution. Hence, $(\epsilon(z), Q(z))$ is an elliptic polynomial solution of the difference equation.
Suppose now, that we have an elliptic polynomial solution $(\epsilon(z), Q(z))$ of the difference equation. Since we know that $\epsilon(z) \in \Theta_{\frac{n}{2}}$ we know that by Proposition E.1, it can be written - up to a constant $C$ - the way we write it in the Proposition. The points $w_i, i = 1, \dots, \frac{n}{2}$, are the zeroes of $Q(z)$, so the right nad side of the difference equation vanishes at these points, causing also the left hand side to vanish: this yields the $w_i, i = 1, \dots, n$ to obey the Bethe Ansatz equations.

**Lebenslauf**

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