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On Equivalent Relations with the Helmholtz-Type Decomposition in a Variable Exponent Sobolev Space

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Abstract

In this paper, we consider an equivalence between the existence of a weak solution of Neumann problem to the Poisson equation and the Helmholtz decomposition of $\boldsymbol{L}^{p(\cdot)}(\Omega)$ which is a variable exponent Lebesgue space in a general domain Ω of $\mathbb{R}^d$. Furthermore we consider an equivalence between the existence of a weak solution to the Stokes problem and the Helmholtz-type decomposition of $W_0^{1,p(\cdot)}(\Omega)$ which is a variable exponent Sobolev space in a bounded domain Ω with a $C^{1,1}$-boundary. We use the equivalent relation with $W^{-m,p(\cdot)}$-version $(m \geq 0)$ of the J. L. Lions lemma in the author's previous paper.

Keywords. Neumann problem, Stokes problem, Helmholtz decomposition, J. L. Lions lemma.

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1. Introduction

Many mathematical and physical scientists are interested in the Helmholtz decomposition of the Lebesgue space $\boldsymbol{L}^p(\Omega)$ $(1 < p < \infty)$ into the direct sum of certain closed subspaces in theoretical hydrodynamics. More precisely, let $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ be a domain. Define

$$\boldsymbol{\mathcal{D}}(\Omega, \operatorname{div} 0) = \{\boldsymbol{v} \in \boldsymbol{C}_0^\infty(\Omega)\,;\, \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\}.$$

We denote

$$\boldsymbol{H}_p(\Omega) = \text{ the closure of } \boldsymbol{\mathcal{D}}(\Omega, \operatorname{div} 0) \text{ in } \boldsymbol{L}^p(\Omega)$$

and

$$\boldsymbol{G}_p(\Omega) = \{\boldsymbol{w} \in \boldsymbol{L}^p(\Omega); \boldsymbol{w} = \boldsymbol{\nabla}\pi \text{ for some } \pi \in W_{\text{loc}}^{1,p}(\Omega)\}.$$

We consider the validity of the Helmholtz decomposition

$$\boldsymbol{L}^p(\Omega) = \boldsymbol{H}_p(\Omega) \oplus \boldsymbol{G}_p(\Omega), \tag{1.1}$$

where $\oplus$ denotes the direct sum operation. In other words, an arbitrary vector $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$ can be uniquely expressed as the form

$$\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}, \quad \boldsymbol{v} \in \boldsymbol{H}_p(\Omega) \text{ and } \boldsymbol{w} \in \boldsymbol{G}_p(\Omega).$$

Galdi [14] showed that when Ω is a general domain of $\mathbb{R}^d$ $(d \geq 2)$, the validity of (1.1) is equivalent to the unique resolubility of a generalized Neumann problem for the Poisson equation in Ω, that is, for any given $\boldsymbol{u} \in \boldsymbol{L}^p(\Omega)$, to find a unique (up to an additive constant) function $\pi : \Omega \to \mathbb{R}$ such that

$$\begin{cases} \pi \in D^{1,p}(\Omega), \\ \int_\Omega (\boldsymbol{\nabla}\pi - \boldsymbol{u}) \cdot \boldsymbol{\nabla}\varphi dx = 0 \quad \text{for all } \varphi \in D^{1,p'}(\Omega), \end{cases} \tag{1.2}$$

where $D^{1,p}(\Omega) = \{\pi \in L^1_{\text{loc}}(\Omega); \boldsymbol{\nabla}\pi \in \boldsymbol{L}^p(\Omega)\}$ and p' is the conjugate exponent of p, that is, $p' = p/(p-1)$. If $p = 2$, employing the Hilbert structure of the space $\boldsymbol{L}^2$, one can prove (1.1) for any domain Ω (cf. [14, Theorem 1.1 in Chapter III]). So the generalized Neumann problem (1.2) with $p = 2$ has a unique (up to an additive constant) solution in an arbitrary domain. On the other hand, if $p \neq 2$, it is well-known that the solvability of the generalized Neumann problem (1.2) depends on the shape of Ω and the regularity of Ω. Therefore, the Helmholtz decomposition (1.1) also depends on the shape of Ω and the regularity of Ω. For smooth bounded domain Ω, the decomposition (1.1) holds, see Fujiwara and Morimoto [13], and if Ω is either a bounded or an exterior domain of C^1-class, the decomposition (1.1) holds, see Simader and Sohr [17] and Simader et al. [16].

The purposes of this paper is to derive the equivalence between the existence of a weak solution for the Neumann problem to the Poisson equation in a variable exponent Lebesgue-Sobolev space and the Helmholtz decomposition of a variable exponent Lebesgue space $\boldsymbol{L}^{p(\cdot)}(\Omega)$ in a general domain Ω of $\mathbb{R}^d$ $(d \geq 2)$. Furthermore, we show the equivalence between the existence of a unique weak solution for the homogeneous Stokes problem and the Helmholtz-type decomposition of a variable exponent Sobolev space $\boldsymbol{W}^{1,p(\cdot)}_0(\Omega)$ in a bounded domain Ω in $\mathbb{R}^d$ $(d \geq 2)$ with a $C^{1,1}$-boundary. Fortunately, since we know the equivalent conditions with the J. L. Lions lemma (Aramaki [4, 3]), we fully use these conditions. In this case, since we know the well-posedness of the Stokes problem, the Helmholtz-type decomposition is true.

The paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we show that the Helmholtz decomposition of $\boldsymbol{L}^{p(\cdot)}(\Omega)$ is equivalent to the unique (up to a constant) solvability of the Neumann problem for the Laplace operator in a general domain. In Section 4, we consider a relation between the unique solvability for the homogeneous Stokes problem and the Helmholtz-type decomposition of $\boldsymbol{W}^{1,p(\cdot)}_0(\Omega)$ in a bounded domain with a $C^{1,1}$-boundary. Section 5 is devoted to well-posedness of inhomogeneous Stokes problem using the result of Section 4.

2. Preliminaries

Throughout this paper, we only consider vector spaces of real valued functions over $\mathbb{R}$. For any normed space B, we denote B^d by the boldface character $\boldsymbol{B}$. Hereafter, we use this character to denote vectors and vector-valued functions, and we denote the standard inner product of vectors $\boldsymbol{a} = (a_1, \ldots, a_d)$ and $\boldsymbol{b} = (b_1, \ldots, b_d)$ in $\mathbb{R}^d$ by $\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^d a_i b_i$ and $|\boldsymbol{a}| = (\boldsymbol{a} \cdot \boldsymbol{a})^{1/2}$. Occasionally, we also use the same character for matrix values functions. Moreover, for the dual space B^* of B (resp. $\boldsymbol{B}^*$ of $\boldsymbol{B}$), we denote the duality bracket between B^* and B (resp. $\boldsymbol{B}^*$ and $\boldsymbol{B}$) by $\langle \cdot, \cdot \rangle_{B^*,B}$ (resp. $\langle \cdot, \cdot \rangle_{\boldsymbol{B}^*,\boldsymbol{B}}$).

In this section, we recall some well-known results on variable exponent Lebesgue-Sobolev spaces. See Diening et al. [8], Fan and Zhang [10], Kováčik and Rácosník [15] and references therein for more detail. Throughout this section, let Ω be a domain in $\mathbb{R}^d$ with a Lipschitz-continuous boundary $\Gamma = \partial\Omega$ and Ω is locally on the same side of $\partial\Omega$. For a real valued function $p \in C(\Omega)$, define

$$p^+ = \sup_{x\in\Omega} p(x) \text{ and } p^- = \inf_{x\in\Omega} p(x).$$

Let

$$C_+(\Omega) = \{p \in C(\Omega); 1 < p^- \le p^+ < \infty\}.$$

From now on, let $p \in C_+(\Omega)$. For any measurable function u on Ω, a modular $\rho_{p(\cdot)}$ is defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}dx.$$

The variable exponent Lebesgue space is defined by

$$L^{p(\cdot)}(\Omega) = \big\{u; u : \Omega \to \mathbb{R} \text{ is a measurable function satisfying } \rho_{p(\cdot)}(u) < \infty\big\}$$

equipped with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0; \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \le 1\right\}.$$

Then $L^{p(\cdot)}(\Omega)$ is a Banach space. We also define, for any integer $m \ge 0$,

$$W^{m,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega); \partial^\alpha u \in L^{p(\cdot)}(\Omega) \text{ for } |\alpha| \le m\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a multi-index, $|\alpha| = \sum_{i=1}^d \alpha_i$, $\partial^\alpha = \partial_1^{\alpha_1}\cdots\partial_d^{\alpha_d}$ and $\partial_i = \partial/\partial x_i$, endowed with the norm

$$\|u\|_{W^{m,p(\cdot)}(\Omega)} = \sum_{|\alpha|\le m} \|\partial^\alpha u\|_{L^{p(\cdot)}(\Omega)}.$$

Of course, $W^{0,p(\cdot)}(\Omega) = L^{p(\cdot)}(\Omega)$. Define

$$W_0^{m,p(\cdot)}(\Omega) = \text{ the closure of the set of } W^{m,p(\cdot)}(\Omega)\text{-functions with compact support in } \Omega.$$

The following three propositions are well known (see Fan et al. [11], Wei and Chen [18], Fan and Zhao [12], Zhao et al. [20], Yücedağ [19]).

Proposition 2.1. *Let $p \in C_+(\Omega)$ and let $u, u_n \in L^{p(\cdot)}(\Omega)$ $(n = 1, 2, \ldots)$. Then we have*
(i) $\|u\|_{L^{p(\cdot)}(\Omega)} < 1 (= 1, > 1) \Longleftrightarrow \rho_{p(\cdot)}(u) < 1 (= 1, > 1)$.
(ii) $\|u\|_{L^{p(\cdot)}(\Omega)} > 1 \Longrightarrow \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)} \le \rho_{p(\cdot)}(u) \le \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}$.
(iii) $\|u\|_{L^{p(\cdot)}(\Omega)} < 1 \Longrightarrow \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)} \le \rho_{p(\cdot)}(u) \le \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}$.
(iv) $\lim_{n\to\infty} \|u_n - u\|_{L^{p(\cdot)}(\Omega)} = 0 \Longleftrightarrow \lim_{n\to\infty} \rho_{p(\cdot)}(u_n - u) = 0$.
(v) $\|u_n\|_{L^{p(\cdot)}(\Omega)} \to \infty$ *as* $n \to \infty \Longleftrightarrow \rho_{p(\cdot)}(u_n) \to \infty$ *as* $n \to \infty$.

The following proposition is a generalized Hölder inequality.

Proposition 2.2. *Let $p \in C_+(\Omega)$. For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, we have*

$$\int_\Omega |uv|dx \le \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)} \le 2\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)},$$

where $p'(\cdot)$ is the conjugate exponent of $p(\cdot)$, that is, $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

Define

$$p^*(x) = \begin{cases} \frac{dp(x)}{d-p(x)} & \text{if } p(x) < d, \\ \infty & \text{if } p(x) \geq d. \end{cases}$$

Proposition 2.3. *Let $p \in C_+(\Omega)$ and $m \geq 0$ be an integer. Then we have the following.*

(i) *The spaces $L^{p(\cdot)}(\Omega)$ and $W^{m,p(\cdot)}(\Omega)$ are separable and reflexive Banach spaces.*

(ii) *If $q(\cdot) \in C_+(\Omega)$ and satisfies $q(x) \leq p(x)$ for all $x \in \Omega$, then $W^{m,p(\cdot)}(\Omega) \hookrightarrow W^{m,q(\cdot)}(\Omega)$, where $\hookrightarrow$ means that the embedding is continuous.*

(iii) *If $q(x) \in C_+(\Omega)$ satisfies that $q(x) < p^*(x)$ for all $x \in \Omega$, then the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is compact.*

We say that a real valued measurable function p belongs to $\mathcal{P}^{\log}(\Omega)$ if p has the log-Hölder continuity in Ω, that is, $p : \Omega \to \mathbb{R}$ satisfies that there exists a constant $C_{\log}(p) > 0$ such that

$$|p(x) - p(y)| \leq \frac{C_{\log}(p)}{\log(e + 1/|x-y|)} \text{ for all } x, y \in \Omega.$$

We also write $\mathcal{P}^{\log}_+(\Omega) = \{p \in \mathcal{P}^{\log}(\Omega)\,;\, 1 < p^- \leq p^+ < \infty\}$.

Proposition 2.4. *If $p \in \mathcal{P}^{\log}_+(\Omega)$ and $m \geq 0$ is an integer, then $\mathcal{D}(\Omega) := C_0^\infty(\Omega)$ is dense in $W_0^{m,p(\cdot)}(\Omega)$.*

For the proof, see [8, Corollary 11.2.4].

We denote the dual space of $W_0^{m,p(\cdot)}(\Omega)$ by $W^{-m,p'(\cdot)}(\Omega)$ and define

$$\boldsymbol{W}_0^{m,p(\cdot)}(\Omega, \operatorname{div} 0) = \{\boldsymbol{v} \in \boldsymbol{W}_0^{m,p(\cdot)}(\Omega)\,;\, \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega\},$$

which is clearly a closed subspace of $\boldsymbol{W}_0^{m,p(\cdot)}(\Omega)$.

Furthermore, we define

$$\dot{W}_0^{m,p(\cdot)}(\Omega) = \left\{f \in W_0^{m,p(\cdot)}(\Omega)\,;\, \int_\Omega f dx = 0\right\} \text{ if } m > 0 \text{ (integer)},$$

and if $m = 0$, $\dot{W}_0^{m,p(\cdot)}(\Omega) = \dot{L}^{p(\cdot)}(\Omega) = L_0^{p(\cdot)}(\Omega)$, where

$$L_0^{p(\cdot)}(\Omega) = \left\{f \in L^{p(\cdot)}(\Omega)\,;\, \int_\Omega f dx = 0\right\}.$$

We also define

$$\dot{\mathcal{D}}(\Omega) = \left\{f \in \mathcal{D}(\Omega)\,;\, \int_\Omega f dx = 0\right\}.$$

Next we consider the trace. Let Ω be a domain of $\mathbb{R}^d$ with a Lipschitz-continuous boundary Γ and $p \in \mathcal{P}^{\log}_+(\overline{\Omega})$. Since $W^{1,p(\cdot)}(\Omega) \subset W^{1,1}_{\mathrm{loc}}(\Omega)$, the trace $u\big|_\Gamma$ to Γ of any function u in $W^{1,p(\cdot)}(\Omega)$ is well defined as a function in $L^1_{\mathrm{loc}}(\Gamma)$. We define

$$\mathrm{Tr}(W^{1,p(\cdot)}(\Omega)) = \{f\,;\, f \text{ is the trace to } \Gamma \text{ of a function } F \in W^{1,p(\cdot)}(\Omega)\}$$

equipped with the norm

$$\|f\|_{\mathrm{Tr}(W^{1,p(\cdot)}(\Omega))} = \inf\{\|F\|_{W^{1,p(\cdot)}(\Omega)}\,;\, F \in W^{1,p(\cdot)}(\Omega) \text{ satisfying } F\big|_\Gamma = f \text{ on } \Gamma\}$$

for $f \in \mathrm{Tr}(W^{1,p(\cdot)}(\Omega))$. Then $\mathrm{Tr}(W^{1,p(\cdot)}(\Omega))$ is a Banach space. More precisely, see [8, Chapter 12]. We note that $W_0^{1,p(\cdot)}(\Omega) = \{F \in W^{1,p(\cdot)}(\Omega); F\big|_\Gamma = 0\}$, in the later we also write $F\big|_\Gamma = g$ by $F = g$ on Γ.

In the previous paper [4, 3], we derived $W^{-m,p(\cdot)}$-version of the J. L. Lions lemma and the equivalent relations.

Theorem 2.5. *Let Ω be a bounded domain of $\mathbb{R}^d$ with a Lipschitz-continuous boundary Γ and Ω be locally on the same side of Γ, and let $m \geq 0$ be a integer and $p \in \mathcal{P}_+^{\log}(\Omega)$. Then the following (a), (b), ... and (f) are equivalent.*

(a) *Classical J. L. Lions lemma: If $f \in W^{-m-1,p(\cdot)}(\Omega)$ satisfies $\boldsymbol{\nabla} f \in \boldsymbol{W}^{-m-1,p(\cdot)}(\Omega)$, then $f \in W^{-m,p(\cdot)}(\Omega)$.*

(b) *The Nečas inequality: there exists a constant $C_0 = C_0(m, p, \Omega)$ such that*

$$\|f\|_{W^{-m,p(\cdot)}(\Omega)} \leq C_0(\|f\|_{W^{-m-1,p(\cdot)}(\Omega)} + \|\boldsymbol{\nabla} f\|_{\boldsymbol{W}^{-m-1,p(\cdot)}(\Omega)}) \text{ for all } f \in W^{-m,p(\cdot)}(\Omega).$$

(c) *The operator grad has a closed range:* $\mathbf{grad}\,(W^{-m,p(\cdot)}(\Omega)/\mathbb{R})$ *is a closed subspace of* $\boldsymbol{W}^{-m-1,p(\cdot)}(\Omega)$.

(d) *A coarse version of the de Rham theorem: for any $\boldsymbol{h} \in \boldsymbol{W}^{-m-1,p(\cdot)}(\Omega)$, there exists a unique $[\pi] \in W^{-m,p(\cdot)}(\Omega)/\mathbb{R}$, where $[\pi]$ denotes the class in $W^{-m,p(\cdot)}(\Omega)/\mathbb{R}$ with the representative π, such that $\boldsymbol{h} = \boldsymbol{\nabla}\pi$ in $\boldsymbol{W}^{-m-1,p(\cdot)}(\Omega)$ if and only if*

$$\langle \boldsymbol{h}, \boldsymbol{v}\rangle_{\boldsymbol{W}^{-m-1,p(\cdot)}(\Omega),\boldsymbol{W}_0^{m+1,p'(\cdot)}(\Omega)} = 0 \text{ for all } \boldsymbol{v} \in \boldsymbol{W}_0^{m+1,p'(\cdot)}(\Omega, \mathrm{div}\, 0).$$

(e) *The operator* div *is surjective: the operator*

$$\mathrm{div}\, : \boldsymbol{W}_0^{m+1,p'(\cdot)}(\Omega) \to \dot{W}_0^{m,p'(\cdot)}(\Omega)$$

is continuous and surjective. In addition, if $f \in \dot{\mathcal{D}}(\Omega)$, then there exists $\boldsymbol{u}_f \in \boldsymbol{\mathcal{D}}(\Omega)$ such that $\mathrm{div}\, \boldsymbol{u}_f = f$ in Ω.

Consequently, for any $f \in \dot{W}_0^{m,p'(\cdot)}(\Omega)$, there exists a unique

$$[\boldsymbol{u}_f] \in \boldsymbol{W}_0^{m+1,p'(\cdot)}(\Omega)/\mathbf{Ker}\,\mathrm{div}$$

where $\mathbf{Ker}\,\mathrm{div} = \boldsymbol{W}_0^{m+1,p'(\cdot)}(\Omega, \mathrm{div}\, 0)$ and $[\boldsymbol{u}_f]$ denotes the class in $\boldsymbol{W}_0^{m+1,p'(\cdot)}(\Omega)/\mathbf{Ker}\,\mathrm{div}$ with the representative $\boldsymbol{u}_f$, such that $\mathrm{div}\, \boldsymbol{u}_f = f$ in Ω. Therefore, the operator

$$\mathrm{div}\, : \boldsymbol{W}_0^{m+1,p'(\cdot)}(\Omega)/\mathbf{Ker}\,\mathrm{div} \to \dot{W}_0^{m,p'(\cdot)}(\Omega)$$

is continuous and bijective. Hence, by the Banach open mapping theorem, there exists a constant $C_1 = C_1(m, p(\cdot), \Omega) > 0$ such that

$$\|[\boldsymbol{u}_f]\|_{\boldsymbol{W}_0^{m+1,p'(\cdot)}(\Omega)/\mathbf{Ker}\,\mathrm{div}} \leq C_1\|f\|_{W^{m,p'(\cdot)}(\Omega)} \text{ for all } f \in \dot{W}_0^{m,p'(\cdot)}(\Omega).$$

(f) *The J. L. Lions lemma: if $f \in \mathcal{D}'(\Omega)$ satisfies $\boldsymbol{\nabla} f \in \boldsymbol{W}^{-m-1,p(\cdot)}(\Omega)$, then we can find that $f \in W^{-m,p(\cdot)}(\Omega)$.*

Remark 2.6. When $p(\cdot) = \text{const.} = 2$ and $m = 0$, Amrouche et al. [1] derived this theorem in L^2-framework in the classical J. L. Lions lemma in the sense that $f \in H^{-1}(\Omega)$ and $\boldsymbol{\nabla} f \in \boldsymbol{H}^{-1}(\Omega)$ implies $f \in L^2(\Omega)$. Aramaki [7] derived an improvement to the case where $p(\cdot) = \text{const.} = p$ $(1 < p < \infty)$ and $m = 0$. Theorem 2.5 is an improvement of these works to the Sobolev space with a variable exponent which was derived by [4, 3].

Remark 2.7. When $p(\cdot) = p = \text{const.}$, since we can prove that the classical Nečas inequality (b) (cf. [2, Theorem 2.3]) directly, consequently if Ω is a bounded domain with a Lipschitz-continuous boundary, then all of (a)-(f) are true in this case. For general integer $m \geq 0$ and $p = p(\cdot) \in \mathcal{P}_+^{\log}(\Omega)$, the author of [4] showed the above equivalence. For the case where $m = 0$ and $p \in \mathcal{P}_+^{\log}(\Omega)$, since the Nečas inequality holds (cf. [8, Theorem 14.3.18]), all of (a)-(f) are true for the case $m = 0$. Furthermore, the author of [3] proved directly that the J. L. Lions lemma (f) holds, so all of (a)-(f) are true for general integer $m \geq 0$.

3. Equivalence between the Helmholtz decomposition of $\boldsymbol{L}^{p(\cdot)}(\Omega)$ and the Neumann problem for the Poisson equation

In this section, we assume that Ω is a general domain of $\mathbb{R}^d$ $(d \geq 2)$ and $p \in \mathcal{P}_+^{\log}(\Omega)$.

3.1. The Helmholtz decomposition of $\boldsymbol{L}^{p(\cdot)}(\Omega)$.

Let

$$\boldsymbol{\mathcal{D}}(\Omega, \operatorname{div} 0) = \{\boldsymbol{u} \in \boldsymbol{C}_0^\infty(\Omega)\,;\, \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega\}$$

and define two spaces

$$\begin{aligned} \boldsymbol{H}_{p(\cdot)}(\Omega) &= \text{ the closure of } \boldsymbol{\mathcal{D}}(\Omega, \operatorname{div} 0) \text{ in } \boldsymbol{L}^{p(\cdot)}(\Omega), \\ \boldsymbol{G}_{p(\cdot)}(\Omega) &= \{\boldsymbol{w} \in \boldsymbol{L}^{p(\cdot)}(\Omega)\,;\, \boldsymbol{w} = \boldsymbol{\nabla}\pi \text{ for some } \pi \in W^{1,p(\cdot)}_{\mathrm{loc}}(\Omega)\}. \end{aligned}$$

Then we have the following lemma.

Lemma 3.1. (i) *The two spaces $\boldsymbol{H}_{p(\cdot)}(\Omega)$ and $\boldsymbol{G}_{p(\cdot)}(\Omega)$ are closed subspaces of $\boldsymbol{L}^{p(\cdot)}(\Omega)$.*

(ii) $\boldsymbol{H}_{p(\cdot)}(\Omega) = \boldsymbol{G}_{p'(\cdot)}(\Omega)^\perp$ *and so* $\boldsymbol{H}_{p(\cdot)}(\Omega)^\perp = \boldsymbol{G}_{p'(\cdot)}(\Omega)$. *Here, for any subspace B of a reflexive Banach space X, $B^\perp$ denotes the polar subspace, that is, $B^\perp = \{f \in X^*\,;\, \langle f, v\rangle_{X^*,X} = 0$ for all $v \in B\}$.*

Proof. (i) Since clearly $\boldsymbol{H}_{p(\cdot)}(\Omega)$ is a closed subspace of $\boldsymbol{L}^{p(\cdot)}(\Omega)$, it suffices to show that $\boldsymbol{G}_{p(\cdot)}(\Omega)$ is a closed subspaces of $\boldsymbol{L}^{p(\cdot)}(\Omega)$. Let $\boldsymbol{w}_n \in \boldsymbol{G}_{p(\cdot)}(\Omega)$ and $\boldsymbol{w}_n \to \boldsymbol{w}$ in $\boldsymbol{L}^{p(\cdot)}(\Omega)$ as $n \to \infty$. Then there exists $\pi_n \in W^{1,p(\cdot)}_{\mathrm{loc}}(\Omega)$ such that $\boldsymbol{w}_n = \boldsymbol{\nabla}\pi_n$ in Ω. We can choose a sequence of bounded domains $\{\Omega_k\}_{k=1}^\infty$ with Lipschitz-continuous boundaries such that $\Omega_1 \subset \Omega_2 \subset \cdots$, $\overline{\Omega_k} \subset \Omega$ and $\cup_{k=1}^\infty \Omega_k = \Omega$.

Fix Ω_1 and for every $n \in \mathbb{N}$, define

$$c_n^{(1)} = -\frac{1}{|\Omega_1|}\int_{\Omega_1} \pi_n dx.$$

Then we see that

$$\int_{\Omega_1} (\pi_n + c_n^{(1)})dx = 0.$$

By the Poincaré inequality (cf. [8, Theorem 8.2.4 (b)]),

$$\begin{aligned} \|(\pi_n + c_n^{(1)}) - (\pi_m + c_m^{(1)})\|_{L^{p(\cdot)}(\Omega_1)} &\leq C(\Omega_1)\|\boldsymbol{\nabla}\pi_n - \boldsymbol{\nabla}\pi_m\|_{\boldsymbol{L}^{p(\cdot)}(\Omega_1)} \\ &\leq C(\Omega_1)\|\boldsymbol{w}_n - \boldsymbol{w}_m\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \to 0 \text{ as } n, m \to \infty. \end{aligned}$$

Hence there exists $\pi^{(1)} \in L^{p(\cdot)}(\Omega)$ such that $\pi_n + c_n^{(1)} \to \pi^{(1)}$ in $L^{p(\cdot)}(\Omega_1)$. For any $\boldsymbol{\varphi} \in \boldsymbol{C}_0^\infty(\Omega_1)$, we have

$$\begin{aligned}\langle \boldsymbol{w}, \boldsymbol{\varphi}\rangle_{\boldsymbol{\mathcal{D}}'(\Omega),\boldsymbol{\mathcal{D}}(\Omega)} &= \int_{\Omega_1} \boldsymbol{w}\cdot\boldsymbol{\varphi}dx \\ &= \lim_{n\to\infty}\int_{\Omega_1} \boldsymbol{\nabla}\pi_n \cdot \boldsymbol{\varphi}dx \\ &= -\lim_{n\to\infty}\int_{\Omega_1}(\pi_n + c_n^{(1)})\operatorname{div}\boldsymbol{\varphi}dx \\ &= -\int_{\Omega_1}\pi^{(1)}\operatorname{div}\boldsymbol{\varphi}dx \\ &= \langle \boldsymbol{\nabla}\pi^{(1)}, \boldsymbol{\varphi}\rangle_{\boldsymbol{\mathcal{D}}'(\Omega),\boldsymbol{\mathcal{D}}(\Omega)}.\end{aligned}$$

Thereby $\boldsymbol{w} = \boldsymbol{\nabla}\pi^{(1)}$ a.e. in Ω_1.

For Ω_2, similarly if we define a constant $c_n^{(2)}$ by $\int_{\Omega_2}(\pi_n + c_n^{(2)})dx = 0$, then there exists $\pi^{(2)} \in L^{p(\cdot)}(\Omega_2)$ such that $\boldsymbol{w} = \boldsymbol{\nabla}\pi^{(2)}$ a.e. in Ω_2. Hence $\boldsymbol{\nabla}(\pi^{(1)} - \pi^{(2)}) = 0$ in Ω_1, so $\pi^{(2)} = \pi^{(1)} + c$ a.e. in Ω_1 with a constant $c = c(\Omega_1, \Omega_2)$. If we redefine $\pi^{(2)}$ by $\pi^{(2)} - c$, then we can see that $\pi^{(2)} \in L^{p(\cdot)}(\Omega_2)$ and write $\pi^{(2)} = \pi^{(1)}$ a.e. in Ω_1.

Repeating this procedure, we may assume that for any $k \in \mathbb{N}$, there exists $\pi^{(k)} \in L^{p(\cdot)}(\Omega_k)$ such that $\boldsymbol{w} = \boldsymbol{\nabla}\pi^{(k)}$ a.e. in Ω_k and $\pi^k = \pi^{k+l}$ a.e. in Ω_k for any $l \in \mathbb{N}$. For a.e. $x \in \Omega$, define $\pi(x) = \pi^{(k)}(x)$ for $x \in \Omega_k$. Then the function π is well-defined in Ω and $\pi \in L^{p(\cdot)}_{\mathrm{loc}}(\Omega)$. Since $\boldsymbol{\nabla}\pi = \boldsymbol{w} \in \boldsymbol{L}^{p(\cdot)}(\Omega)$, we see that $\pi \in W^{1,p(\cdot)}_{\mathrm{loc}}(\Omega)$. Therefore, $\boldsymbol{w} \in \boldsymbol{G}_{p(\cdot)}(\Omega)$.

(ii) First we show that $\boldsymbol{H}_{p(\cdot)}(\Omega)^\perp \subset \boldsymbol{G}_{p'(\cdot)}(\Omega)$. Let $\boldsymbol{u} \in \boldsymbol{H}_{p(\cdot)}(\Omega)^\perp$. Then $\boldsymbol{u} \in \boldsymbol{L}^{p'(\cdot)}(\Omega)(\subset \boldsymbol{L}^1_{\mathrm{loc}}(\Omega))$ and

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle_{\boldsymbol{L}^{p'(\cdot)}(\Omega),\boldsymbol{L}^{p(\cdot)}(\Omega)} = \int_\Omega \boldsymbol{u}\cdot\boldsymbol{v}dx = 0 \text{ for all } \boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\Omega, \operatorname{div} 0).$$

By [14, Lemma 1.1, p. 105], there exists a function $\pi \in W^{1,1}_{\mathrm{loc}}(\Omega)$ such that $\boldsymbol{u} = \boldsymbol{\nabla}\pi$. By the Poincaré inequality (cf. [8, Corollary 8.2.6]), we see that $\pi \in W^{1,p'(\cdot)}_{\mathrm{loc}}(\Omega)$, so $\boldsymbol{u} \in \boldsymbol{G}_{p'(\cdot)}(\Omega)$.

Since $\boldsymbol{H}_{p(\cdot)}(\Omega)$ is a closed subspace of a reflexive Banach space $\boldsymbol{L}^{p(\cdot)}(\Omega)$, we see that $\boldsymbol{G}_{p'(\cdot)}(\Omega)^\perp \subset \boldsymbol{H}_{p(\cdot)}(\Omega)^{\perp\perp} = \boldsymbol{H}_{p(\cdot)}(\Omega)$.

Conversely, let $\boldsymbol{u} \in \boldsymbol{H}_{p(\cdot)}(\Omega)$. Then there exists a sequence $\{\boldsymbol{u}_j\} \subset \boldsymbol{\mathcal{D}}(\Omega, \operatorname{div} 0)$ such that $\boldsymbol{u}_j \to \boldsymbol{u}$ in $\boldsymbol{L}^{p(\cdot)}(\Omega)$. For any $\boldsymbol{h} \in \boldsymbol{G}_{p'(\cdot)}(\Omega)$, there exists a function $\pi \in W^{1,p'(\cdot)}_{\mathrm{loc}}(\Omega)$ such that $\boldsymbol{h} = \boldsymbol{\nabla}\pi$ in Ω. Hence

$$\int_\Omega \boldsymbol{u}_j \cdot \boldsymbol{h}dx = \int_\Omega \boldsymbol{u}_j \cdot \boldsymbol{\nabla}\pi dx = -\int_\Omega (\operatorname{div}\boldsymbol{u}_j)\pi dx = 0.$$

Letting $j \to \infty$, we have

$$\int_\Omega \boldsymbol{u}\cdot\boldsymbol{h}dx = 0 \text{ for all } \boldsymbol{h} \in \boldsymbol{G}_{p'(\cdot)}(\Omega),$$

that is, $\boldsymbol{u} \in \boldsymbol{G}_{p'(\cdot)}(\Omega)^\perp$, so $\boldsymbol{H}_{p(\cdot)}(\Omega) \subset \boldsymbol{G}_{p'(\cdot)}(\Omega)^\perp$. Hence $\boldsymbol{H}_{p(\cdot)}(\Omega) = \boldsymbol{G}_{p'(\cdot)}(\Omega)^\perp$. This implies the conclusions of (ii). □

Definition 3.2. We say the Helmholtz decomposition of $\boldsymbol{L}^{p(\cdot)}(\Omega)$ holds if

$$\boldsymbol{L}^{p(\cdot)}(\Omega) = \boldsymbol{H}_{p(\cdot)}(\Omega) \oplus \boldsymbol{G}_{p(\cdot)}(\Omega) \text{ (direct sum)}, \tag{3.1}$$

that is, any $\boldsymbol{u} \in \boldsymbol{L}^{p(\cdot)}(\Omega)$ is uniquely written by $\boldsymbol{u} = \boldsymbol{w} + \boldsymbol{v}$, where $\boldsymbol{w} \in \boldsymbol{H}_{p(\cdot)}(\Omega)$ and $\boldsymbol{v} \in \boldsymbol{G}_{p(\cdot)}(\Omega)$.

3.2. **The Neumann problem for the Poisson equation.** Define a space

$$D^{1,p(\cdot)}(\Omega) = \{\pi \in L^1_{\mathrm{loc}}(\Omega); \nabla\pi \in \boldsymbol{L}^{p(\cdot)}(\Omega)\}$$

equipped with the semi-norm $\|\pi\|_{D^{1,p(\cdot)}(\Omega)} = \|\nabla\pi\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)}$ (cf. [8, Definition 12.2.1]).

We consider the following Neumann problem for the Poisson equation: for given $\boldsymbol{u} \in \boldsymbol{L}^{p(\cdot)}(\Omega)$, to find a unique (up to an additive constant) function $\pi : \Omega \to \mathbb{R}$ such that $\pi \in D^{1,p(\cdot)}(\Omega)$ and

$$\int_\Omega (\nabla\pi - \boldsymbol{u}) \cdot \nabla\varphi dx = 0 \text{ for all } \varphi \in D^{1,p'(\cdot)}(\Omega). \tag{3.2}$$

Remark 3.3. If Ω and $\boldsymbol{u}$ are regular, then the definition (3.2) means that

$$\begin{cases} \Delta\pi = \operatorname{div}\boldsymbol{u} & \text{in } \Omega, \\ \frac{\partial\pi}{\partial\boldsymbol{n}} = \boldsymbol{u}\cdot\boldsymbol{n} & \text{on } \partial\Omega, \end{cases}$$

where $\boldsymbol{n}$ is the unit outer normal vector to Γ.

We have the following theorem.

Theorem 3.4. *Let Ω be a general domain of $\mathbb{R}^d$ $(d \geq 2)$ and $p \in \mathcal{P}^{\log}_+(\Omega)$. Then the Helmholtz decomposition (3.1) holds if and only if the Neumann problem (3.2) is uniquely solvable (up to an additive constant) for any $\boldsymbol{u} \in \boldsymbol{L}^{p(\cdot)}(\Omega)$.*

Proof. Step 1. We show that if the Neumann problem (3.2) is solvable uniquely (up to an additive constant) for any $\boldsymbol{u} \in \boldsymbol{L}^{p(\cdot)}(\Omega)$, then the Helmholtz decomposition (3.1) holds. Let $\boldsymbol{u} \in \boldsymbol{L}^{p(\cdot)}(\Omega)$. By the hypothesis, there exists a unique (up to an additive constant) $\pi \in D^{1,p(\cdot)}(\Omega)$ such that

$$\int_\Omega (\nabla\pi - \boldsymbol{u}) \cdot \nabla\varphi dx = 0 \text{ for all } \varphi \in D^{1,p'(\cdot)}(\Omega). \tag{3.3}$$

Define $\boldsymbol{w} = \boldsymbol{u} - \nabla\pi$. For any $\boldsymbol{v} \in \boldsymbol{G}_{p'(\cdot)}(\Omega)$, there exists $\pi' \in W^{1,p'(\cdot)}_{\mathrm{loc}}(\Omega)$ with $\boldsymbol{v} = \nabla\pi' \in \boldsymbol{L}^{p'(\cdot)}(\Omega)$, so $\pi' \in D^{1,p'(\cdot)}(\Omega)$. Hence it follows from (3.3) that

$$\int_\Omega \boldsymbol{w} \cdot \boldsymbol{v} dx = 0 \text{ for all } \boldsymbol{v} \in \boldsymbol{G}_{p'(\cdot)}(\Omega),$$

so $\boldsymbol{w} \in \boldsymbol{G}_{p'(\cdot)}(\Omega)^\perp$. By Lemma 3.1 (ii), we see that $\boldsymbol{w} \in \boldsymbol{H}_{p(\cdot)}(\Omega)$. Thus we can write $\boldsymbol{u} = \boldsymbol{w} + \nabla\pi$, where $\boldsymbol{w} \in \boldsymbol{H}_{p(.}(\Omega)$, $\nabla\pi \in \boldsymbol{G}_{p(\cdot)}(\Omega)$.

For the uniqueness of representation, let $\boldsymbol{w} = \nabla\pi$, where $\boldsymbol{w} \in \boldsymbol{H}_{p(\cdot)}(\Omega)$ and $\pi \in W^{1,p(\cdot)}_{\mathrm{loc}}(\Omega)$. Since $\boldsymbol{G}_{p'(\cdot)}(\Omega)^\perp = \boldsymbol{H}_{p(\cdot)}(\Omega)$,

$$\int_\Omega \nabla\pi \cdot \nabla\varphi dx = \int_\Omega \boldsymbol{w} \cdot \nabla\varphi dx = 0 \text{ for all } \varphi \in D^{1,p'(\cdot)}(\Omega).$$

By the uniqueness (up to an additive constant) of the solution for the Neumann problem (3.2), we have $\pi = \text{const.}$, so $\boldsymbol{w} = \boldsymbol{0}$.

Step 2. Conversely we assume that the Helmholtz decomposition (3.1) of $\boldsymbol{L}^{p(\cdot)}(\Omega)$ holds. Let $\boldsymbol{u} \in \boldsymbol{L}^{p(\cdot)}(\Omega)$. Then it follows from (3.1) that $\boldsymbol{u} = \boldsymbol{w}_1 + \boldsymbol{w}_2$, where $\boldsymbol{w}_1 \in \boldsymbol{G}_{p(\cdot)}(\Omega)$ and $\boldsymbol{w}_2 \in \boldsymbol{H}_{p(\cdot)}(\Omega)$. By the definition of $\boldsymbol{G}_{p(\cdot)}(\Omega)$, we can write $\boldsymbol{w}_1 = \boldsymbol{\nabla}\pi$ for some $\pi \in D^{1,p(\cdot)}(\Omega)$.

Since $\boldsymbol{w}_2 \in \boldsymbol{H}_{p(\cdot)}(\Omega) = \boldsymbol{G}_{p'(\cdot)}(\Omega)^{\perp}$, we have

$$\int_\Omega \boldsymbol{w}_2 \cdot \boldsymbol{\nabla}\varphi dx = 0 \text{ for all } \varphi \in D^{1,p'(\cdot)}(\Omega).$$

Hence

$$\int_\Omega (\boldsymbol{\nabla}\pi - \boldsymbol{u}) \cdot \boldsymbol{\nabla}\varphi dx = -\int_\Omega \boldsymbol{w}_2 \cdot \boldsymbol{\nabla}\varphi dx = 0 \text{ for all } \varphi \in D^{1,p'(\cdot)}(\Omega).$$

For $\boldsymbol{u} \in \boldsymbol{L}^{p(\cdot)}(\Omega)$, $\boldsymbol{w}_1 = \boldsymbol{\nabla}\pi$ is determined uniquely, so π is unique (up to an additive constant). □

Remark 3.5. When $p(\cdot) = p =$ const., Theorem 3.4 was proved by [14, Lemma 1.2]. If Ω is a bounded domain with a C^1-boundary and $p \in \mathcal{P}_+^{\log}(\Omega)$, it follows from Aramaki [5] that the Helmholtz decomposition (3.1) holds. Therefore, when Ω is a bounded domain with a C^1-boundary, for any $\boldsymbol{u} \in \boldsymbol{L}^{p(\cdot)}(\Omega)$, the Neumann problem (3.2) has a unique (up to an additive constant) solution $\pi \in D^{1,p(\cdot)}(\Omega)$. This is an extension of the result of Diening et al. [9, Theorem 4.2] in which the authors assumed that Ω is a bounded domain with a $C^{1,1}$-boundary.

4. A relation between the Stokes problem and the Helmholtz-type decomposition of $\boldsymbol{W}_0^{1,p(\cdot)}(\Omega)$

In this section, when Ω is a bounded domain of $\mathbb{R}^d$ ($d \geq 2$) with a $C^{1,1}$-boundary Γ and $p \in \mathcal{P}_+^{\log}(\overline{\Omega})$, we can derive that if the homogeneous Stokes problem is well-posed, then the Helmholtz-type decomposition of $\boldsymbol{W}_0^{1,p(\cdot)}(\Omega)$ holds. We prove this by the method of functional analysis. Conversely, when Ω is bounded domain of $\mathbb{R}^d$ ($d \geq 2$) with a C^1-boundary Γ, if we assume the Helmholtz-type decomposition of $\boldsymbol{W}_0^{1,p(\cdot)}(\Omega)$ holds, then we can prove the well-posedness of the homogeneous Stokes problem.

4.1. **The Stokes problem.** Assume that Ω is a bounded domain of $\mathbb{R}^d$ ($d \geq 2$) with a C^1-boundary Γ and $p \in \mathcal{P}_+^{\log}(\overline{\Omega})$. For given $\boldsymbol{f} \in \boldsymbol{W}^{-1,p(\cdot)}(\Omega)$, $g \in L^{p(\cdot)}(\Omega)$ and $\boldsymbol{h} \in \mathrm{Tr}(\boldsymbol{W}^{1,p(\cdot)}(\Omega))$, we consider the following inhomogeneous Stokes problem: to find $(\boldsymbol{u}, \pi) \in \boldsymbol{W}^{1,p(\cdot)}(\Omega) \times L_0^{p(\cdot)}(\Omega)$ such that

$$\begin{cases} -\Delta \boldsymbol{u} + \boldsymbol{\nabla}\pi = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = g & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{h} & \text{on } \Gamma. \end{cases} \tag{4.1}$$

The compatibility condition becomes

$$\int_\Omega g dx = \int_\Gamma \boldsymbol{h} \cdot \boldsymbol{n} d\sigma, \tag{4.2}$$

where $d\sigma$ is the surface measure on Γ induced by the Lebesgue measure dx.

Definition 4.1. We say that $(\boldsymbol{u},\pi)\in \boldsymbol{W}^{1,p(\cdot)}(\Omega)\times L^{p(\cdot)}_0(\Omega)$ is a weak solution of (4.1) if $\boldsymbol{u}$ satisfies that $\operatorname{div}\boldsymbol{u}=g$ in Ω, $\boldsymbol{u}=\boldsymbol{h}$ on Γ and

$$\int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{w}dx-\int_\Omega \pi\operatorname{div}\boldsymbol{w}dx=\langle \boldsymbol{f},\boldsymbol{w}\rangle_{\boldsymbol{W}^{-1,p(\cdot)}(\Omega),\boldsymbol{W}^{1,p'(\cdot)}_0(\Omega)} \tag{4.3}$$

for all $\boldsymbol{w}\in \boldsymbol{W}^{1,p'(\cdot)}_0(\Omega)$.

In particular, when $g=0$ and $\boldsymbol{h}=\boldsymbol{0}$ in (4.1), we call the problem (4.1) a homogeneous Stokes problem.

The authors of [8, Theorem 14.2.2 and Remark 14.2.28] derived the following theorem.

Theorem 4.2. *Let Ω be a bounded domain of $\mathbb{R}^d$ $(d\ge 2)$ with a $C^{1,1}$-boundary Γ, and let $p\in\mathcal{P}^{\log}_+(\overline{\Omega})$. If $\boldsymbol{f}\in \boldsymbol{W}^{-1,p(\cdot)}(\Omega)$, $g\in L^{p(\cdot)}(\Omega)$ and $\boldsymbol{h}\in \operatorname{Tr}(\boldsymbol{W}^{1,p(\cdot)}(\Omega))$ satisfy the compatibility condition (4.2), then the problem (4.1) has a unique weak solution $(\boldsymbol{u},\pi)\in \boldsymbol{W}^{1,p(\cdot)}(\Omega)\times L^{p(\cdot)}_0(\Omega)$, and there exists a constant $C>0$ depending only on p,d and Ω such that*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p(\cdot)}(\Omega)}+\|\pi\|_{L^{p(\cdot)}(\Omega)}\le C(\|\boldsymbol{f}\|_{\boldsymbol{W}^{-1,p(\cdot)}(\Omega)}+\|g\|_{L^{p(\cdot)}(\Omega)}+\|\boldsymbol{h}\|_{\operatorname{Tr}(\boldsymbol{W}^{1,p(\cdot)}(\Omega))}). \tag{4.4}$$

We call that the problem (4.1) is well-posed if the conclusions of Theorem 4.2 hold.

4.2. **The Helmholtz-type decomposition of $\boldsymbol{W}^{1,p(\cdot)}_0(\Omega)$.** Assume that Ω is a bounded domain of $\mathbb{R}^d$ $(d\ge 2)$ with a C^1-boundary Γ and $p\in\mathcal{P}^{\log}_+(\overline{\Omega})$. Define two spaces.

$$\begin{aligned}\boldsymbol{V}_{1,p(\cdot)}(\Omega)&=\boldsymbol{W}^{1,p(\cdot)}_0(\Omega,\operatorname{div}0)=\{\boldsymbol{v}\in \boldsymbol{W}^{1,p(\cdot)}_0(\Omega)\,;\,\operatorname{div}\boldsymbol{v}=0\text{ in }\Omega\},\\ \boldsymbol{G}_{1,p(\cdot)}(\Omega)&=\{\boldsymbol{v}=(-\Delta)^{-1}\nabla q\,;\,q\in L^{p(\cdot)}(\Omega)\}.\end{aligned}$$

Here $\boldsymbol{v}=(-\Delta)^{-1}\nabla q$ means that $\boldsymbol{v}$ is a unique weak solution for the Poisson equation with the Dirichlet boundary condition

$$\begin{cases}-\Delta\boldsymbol{v}=\nabla q & \text{in }\Omega,\\ \boldsymbol{v}=\boldsymbol{0} & \text{on }\Gamma.\end{cases}$$

Actually, for given $\boldsymbol{f}\in \boldsymbol{W}^{-1,p(\cdot)}(\Omega)$, the problem

$$\begin{cases}-\Delta\boldsymbol{v}=\boldsymbol{f} & \text{in }\Omega,\\ \boldsymbol{v}=\boldsymbol{0} & \text{on }\Gamma.\end{cases} \tag{4.5}$$

has a unique weak solution $\boldsymbol{v}\in \boldsymbol{W}^{1,p(\cdot)}_0(\Omega)$, that is, $\boldsymbol{v}$ satisfies that

$$\int_\Omega \nabla\boldsymbol{v}\cdot\nabla\boldsymbol{w}dx-\langle \boldsymbol{f},\boldsymbol{w}\rangle_{\boldsymbol{W}^{-1,p(\cdot)}(\Omega),\boldsymbol{W}^{1,p'(\cdot)}_0(\Omega)}=0\text{ for all }\boldsymbol{w}\in \boldsymbol{W}^{1,p'(\cdot)}_0(\Omega),$$

and there exists s positive constant C depending only on p,d and Ω such that

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p(\cdot)}(\Omega)}\le C\|\boldsymbol{f}\|_{\boldsymbol{W}^{-1,p(\cdot)}(\Omega)}. \tag{4.6}$$

For the proof, see Aramaki [6, Theorem 6.1] in which the author uses a variational inequality, or [8, Remark 14.1.23] in which the authors use the Newton potential.

It follows from Theorem 4.2 that we can derive the following Helmholtz-type decomposition of $\boldsymbol{W}^{1,p(\cdot)}_0(\Omega)$.

Theorem 4.3. *Assume that Ω is a bounded domain of $\mathbb{R}^d$ $(d \geq 2)$ with a $C^{1,1}$-boundary Γ and $p \in \mathcal{P}_+^{\log}(\overline{\Omega})$. If the homogeneous Stokes problem (4.1) with $g = 0$ and $\boldsymbol{h} = \boldsymbol{0}$ is well-posed, then the following Helmholtz-type decomposition of $\boldsymbol{W}_0^{1,p(\cdot)}(\Omega)$ holds:*

$$\boldsymbol{W}_0^{1,p(\cdot)}(\Omega) = \boldsymbol{V}_{1,p(\cdot)}(\Omega) \oplus \boldsymbol{G}_{1,p(\cdot)}(\Omega) \quad (\textit{direct sum}). \tag{4.7}$$

Proof. Step 1. We can see that the spaces $\boldsymbol{V}_{1,p(\cdot)}(\Omega)$ and $\boldsymbol{G}_{1,p(\cdot)}(\Omega)$ are closed subspaces of $\boldsymbol{W}_0^{1,p(\cdot)}(\Omega)$ and $\boldsymbol{V}_{1,p(\cdot)}(\Omega) \cap \boldsymbol{G}_{1,p(\cdot)}(\Omega) = \{\boldsymbol{0}\}$. Indeed, clearly $\boldsymbol{V}_{1,p(\cdot)}(\Omega)$ is a closed subspace of $\boldsymbol{W}_0^{1,p(\cdot)}(\Omega)$. We show that $\boldsymbol{G}_{1,p(\cdot)}(\Omega)$ is a closed subspace of $\boldsymbol{W}_0^{1,p(\cdot)}(\Omega)$. Let $\boldsymbol{v}_n = (-\Delta)^{-1}\boldsymbol{\nabla} q_n$ with $q_n \in L^{p(\cdot)}(\Omega)$, and $\boldsymbol{v}_n \to \boldsymbol{v}$ in $\boldsymbol{W}_0^{1,p(\cdot)}(\Omega)$ as $n \to \infty$. Then $-\Delta \boldsymbol{v}_n = \boldsymbol{\nabla} q_n \to -\Delta \boldsymbol{v}$ in $\boldsymbol{W}^{-1,p(\cdot)}(\Omega)$, so $\{\boldsymbol{\nabla} q_n\}$ is a Cauchy sequence in $\boldsymbol{W}^{-1,p(\cdot)}(\Omega)$. Hence there exists $\boldsymbol{g} \in \boldsymbol{W}^{-1,p(\cdot)}(\Omega)$ such that $\boldsymbol{\nabla} q_n \to \boldsymbol{g}$ in $\boldsymbol{W}^{-1,p(\cdot)}(\Omega)$. From Theorem 2.5 (c) with $m = 0$, $\boldsymbol{\nabla}(L^{p(\cdot)}(\Omega)/\mathbb{R})$ is a closed subspace of $\boldsymbol{W}^{-1,p(\cdot)}(\Omega)$. Thus there exists $q \in L_0^{p(\cdot)}(\Omega)$ such that $\boldsymbol{g} = \boldsymbol{\nabla} q$. Therefore, $-\Delta \boldsymbol{v} = \boldsymbol{\nabla} q$ in $\boldsymbol{W}^{-1,p(\cdot)}(\Omega)$ and $\boldsymbol{v} = \boldsymbol{0}$ on Γ, that is, $\boldsymbol{v} = (-\Delta)^{-1}\boldsymbol{\nabla} q$.

We show that $\boldsymbol{V}_{1,p(\cdot)}(\Omega) \cap \boldsymbol{G}_{1,p(\cdot)}(\Omega) = \{\boldsymbol{0}\}$. If $\boldsymbol{v} = (-\Delta)^{-1}\boldsymbol{\nabla} q \in \boldsymbol{V}_{1,p(\cdot)}(\Omega) \cap \boldsymbol{G}_{1,p(\cdot)}(\Omega)$, then $\operatorname{div} \boldsymbol{v} = 0$ in Ω. Hence $(\boldsymbol{v}, -q)$ satisfies

$$\begin{cases} -\Delta \boldsymbol{v} + \boldsymbol{\nabla}(-q) = \boldsymbol{0} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{v} = 0 & \text{in } \Omega, \\ \boldsymbol{v} = \boldsymbol{0} & \text{on } \Gamma, \end{cases}$$

that is, $(\boldsymbol{v}, -q)$ is a weak solution of (4.1) with $\boldsymbol{f} = \boldsymbol{0}, g = 0$ and $\boldsymbol{h} = \boldsymbol{0}$. By the uniqueness of solution (Theorem 4.2), we have $\boldsymbol{v} = \boldsymbol{0}$. Hence, we can get $\boldsymbol{V}_{1,p'(\cdot)}(\Omega) \cap \boldsymbol{G}_{1,p'(\cdot)}(\Omega) = \{\boldsymbol{0}\}$.

Step 2. We claim that $\boldsymbol{V}_{1,p(\cdot)}(\Omega) \oplus \boldsymbol{G}_{1,p(\cdot)}(\Omega)$ is dense in $\boldsymbol{W}_0^{1,p(\cdot)}(\Omega)$. It suffices to show that if $\boldsymbol{L} \in \boldsymbol{W}^{-1,p'(\cdot)}(\Omega)$ (i.e., $\boldsymbol{L}$ is a continuous linear functional on $\boldsymbol{W}_0^{1,p(\cdot)}(\Omega)$) satisfies

$$\langle \boldsymbol{L}, \boldsymbol{w} \rangle_{\boldsymbol{W}^{-1,p'(\cdot)}(\Omega), \boldsymbol{W}_0^{1,p(\cdot)}(\Omega)} = 0 \text{ for all } \boldsymbol{w} \in \boldsymbol{V}_{1,p(\cdot)}(\Omega) \oplus \boldsymbol{G}_{1,p(\cdot)}(\Omega), \tag{4.8}$$

then $\boldsymbol{L} = \boldsymbol{0}$. From a coarse version of the de Rham theorem (Theorem 2.5 (d) with $m = 0$), there exists $\pi \in L_0^{p'(\cdot)}(\Omega)$ such that $\boldsymbol{L} = \boldsymbol{\nabla}\pi$ in $\boldsymbol{W}^{-1,p'(\cdot)}(\Omega)$. If we define $\boldsymbol{v} = (-\Delta)^{-1}\boldsymbol{\nabla}\pi$, then $\boldsymbol{v} \in \boldsymbol{W}_0^{1,p'(\cdot)}(\Omega)$ and $-\Delta \boldsymbol{v} = \boldsymbol{\nabla}\pi$ in $\boldsymbol{W}^{-1,p'(\cdot)}(\Omega)$. For any $q \in L^{p(\cdot)}(\Omega)$, since $\boldsymbol{w} := (-\Delta)^{-1}\boldsymbol{\nabla} q \in \boldsymbol{G}_{1,p(\cdot)}$, that is, $\boldsymbol{w} \in \boldsymbol{W}_0^{1,p(\cdot)}(\Omega)$ and $-\Delta \boldsymbol{w} = \boldsymbol{\nabla} q$, we have

$$\begin{aligned} 0 &= \langle \boldsymbol{L}, \boldsymbol{w} \rangle_{\boldsymbol{W}^{-1,p'(\cdot)}(\Omega), \boldsymbol{W}_0^{1,p(\cdot)}(\Omega)} \\ &= \langle \boldsymbol{\nabla}\pi, \boldsymbol{w} \rangle_{\boldsymbol{W}^{-1,p'(\cdot)}(\Omega), \boldsymbol{W}_0^{1,p(\cdot)}(\Omega)} \\ &= \langle -\Delta \boldsymbol{v}, \boldsymbol{w} \rangle_{\boldsymbol{W}^{-1,p'(\cdot)}(\Omega), \boldsymbol{W}_0^{1,p(\cdot)}(\Omega)} \\ &= \langle \boldsymbol{v}, -\Delta \boldsymbol{w} \rangle_{\boldsymbol{W}_0^{1,p'(\cdot)}(\Omega), \boldsymbol{W}^{-1,p(\cdot)}(\Omega)} \\ &= \langle \boldsymbol{v}, \boldsymbol{\nabla} q \rangle_{\boldsymbol{W}_0^{1,p'(\cdot)}(\Omega), \boldsymbol{W}^{-1,p(\cdot)}(\Omega)} \\ &= -\langle \operatorname{div} \boldsymbol{v}, q \rangle_{L^{p'(\cdot)}(\Omega), L^{p(\cdot)}(\Omega)}. \end{aligned}$$

Since q is an arbitrary function in $L^{p(\cdot)}(\Omega)$, we have $\operatorname{div} \boldsymbol{v} = 0$ in Ω, so $\boldsymbol{v} \in \boldsymbol{V}_{1,p'(\cdot)}(\Omega) \cap \boldsymbol{G}_{1,p'(\cdot)}(\Omega) = \{\boldsymbol{0}\}$ by Step 1. Thereby, we have $\boldsymbol{v} = \boldsymbol{0}$, so $\boldsymbol{L} = \boldsymbol{\nabla}\pi = \boldsymbol{0}$.

Step 3. $\boldsymbol{V}_{1,p(\cdot)}(\Omega) \oplus \boldsymbol{G}_{1,p(\cdot)}(\Omega)$ is closed in $\boldsymbol{W}_0^{1,p(\cdot)}(\Omega)$. Indeed, let $\boldsymbol{u}_n = \boldsymbol{v}_n + \boldsymbol{w}_n, \boldsymbol{v}_n \in \boldsymbol{V}_{1,p(\cdot)}(\Omega), \boldsymbol{w}_n \in \boldsymbol{G}_{1,p(\cdot)}(\Omega)$ and $\boldsymbol{u}_n \to \boldsymbol{u}$ in $\boldsymbol{W}_0^{1,p(\cdot)}(\Omega)$. We show that $\boldsymbol{u} \in \boldsymbol{V}_{1,p(\cdot)}(\Omega) \oplus$

$\boldsymbol{G}_{1,p(\cdot)}(\Omega)$. Since $\boldsymbol{w}_n \in \boldsymbol{G}_{1,p(\cdot)}(\Omega)$, we can write $-\Delta \boldsymbol{w}_n = \boldsymbol{\nabla} q_n$ for some $q_n \in \boldsymbol{L}_0^{p(\cdot)}(\Omega)$. Put $\boldsymbol{f}_n = -\Delta \boldsymbol{u}_n \in \boldsymbol{W}^{-1,p(\cdot)}(\Omega)$. Then $(\boldsymbol{v}_n, q_n) \in \boldsymbol{W}_0^{1,p(\cdot)}(\Omega) \times L_0^{p(\cdot)}(\Omega)$ is a weak solution of

$$\begin{cases} -\Delta \boldsymbol{v}_n + \boldsymbol{\nabla} q_n = \boldsymbol{f}_n \text{ in } \Omega, \\ \operatorname{div} \boldsymbol{v}_n = 0 \text{ in } \Omega, \\ \boldsymbol{v}_n = \boldsymbol{0} \text{ on } \Gamma. \end{cases}$$

By (4.4), we have

$$\|\boldsymbol{v}_n\|_{\boldsymbol{W}^{1,p(\cdot)}(\Omega)} + \|q_n\|_{L^{p(\cdot)}(\Omega)} \le C\|\boldsymbol{f}_n\|_{\boldsymbol{W}^{-1,p(\cdot)}(\Omega)}.$$

Since $\boldsymbol{u}_n \to \boldsymbol{u}$ in $\boldsymbol{W}_0^{1,p(\cdot)}(\Omega)$, we see that $\{\boldsymbol{f}_n\}$ is bounded in $\boldsymbol{W}^{-1,p(\cdot)}(\Omega)$, so $\{\boldsymbol{v}_n\}$ is bounded in $\boldsymbol{W}^{1,p(\cdot)}(\Omega)$ and $\{q_n\}$ is bounded in $L^{p(\cdot)}(\Omega)$. Since $\{\boldsymbol{\nabla} q_n = \boldsymbol{f}_n + \Delta \boldsymbol{v}_n\}$ is bounded in $\boldsymbol{W}^{-1,p(\cdot)}(\Omega)$, we see that $\{\boldsymbol{w}_n = (-\Delta)^{-1}\boldsymbol{\nabla} q_n\}$ is also bounded in $\boldsymbol{W}_0^{1,p(\cdot)}(\Omega)$. Passing to a subsequence, we may assume that $\boldsymbol{v}_n \to \boldsymbol{v}, \boldsymbol{w}_n \to \boldsymbol{w}$ weakly in $\boldsymbol{W}_0^{1,p(\cdot)}(\Omega)$, $q_n \to q$ weakly in $L^{p(\cdot)}(\Omega)$ and $\boldsymbol{\nabla} q_n \to \boldsymbol{\nabla} q$ weakly in $\boldsymbol{W}^{-1,p(\cdot)}(\Omega)$. Since $L_0^{p(\cdot)}(\Omega)$ is a closed subspace of $L^{p(\cdot)}(\Omega)$, it is weakly closed, so $q \in L_0^{p(\cdot)}(\Omega)$. Since $\operatorname{div} \boldsymbol{v}_n = 0$ in Ω, we have $\operatorname{div} \boldsymbol{v} = 0$ in Ω, so $\boldsymbol{v} \in \boldsymbol{V}_{1,p(\cdot)}(\Omega)$. Since $\boldsymbol{w}_n = (-\Delta)^{-1}\boldsymbol{\nabla} q_n \to \boldsymbol{w} = (-\Delta)^{-1}\boldsymbol{\nabla} q$ weakly in $\boldsymbol{W}_0^{1,p(\cdot)}(\Omega)$, we have $\boldsymbol{w} \in \boldsymbol{G}_{1,;p(\cdot)}(\Omega)$. Therefore, we have $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w} \in \boldsymbol{V}_{1,p(\cdot)}(\Omega) \oplus \boldsymbol{G}_{1,p(\cdot)}(\Omega)$.

From Step 2 and Step 3, the conclusion Theorem 4.3 holds. □

We consider the converse of Theorem 4.3.

Theorem 4.4. *Assume that Ω is a bounded domain with a C^1-boundary Γ and suppose that the Helmholtz-type decomposition (4.7) holds. Then for $\boldsymbol{f} \in \boldsymbol{W}^{-1,p(\cdot)}(\Omega)$, there exists a unique weak solution $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p(\cdot)}(\Omega) \times L_0^{p(\cdot)}(\Omega)$ for the homogeneous Stokes problem (4.1) with $g = 0$ and $\boldsymbol{h} = \boldsymbol{0}$. Furthermore, there exists a constant $C > 0$ depending only on p, d and Ω such that*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p(\cdot)}(\Omega)} + \|\pi\|_{L^{p(\cdot)}(\Omega)} \le C\|\boldsymbol{f}\|_{\boldsymbol{W}^{-1,p(\cdot)}(\Omega)}. \tag{4.9}$$

Proof. For $\boldsymbol{f} \in \boldsymbol{W}^{-1,p(\cdot)}(\Omega)$, the problem (4.5) has a unique weak solution $\boldsymbol{v} \in \boldsymbol{W}_0^{1,p(\cdot)}(\Omega)$ and there exists a constant $C > 0$ depending only on p, d and Ω such that the estimate (4.6) holds. By the Helmholtz-type decomposition (4.7), we can uniquely write

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w} \text{ with } \boldsymbol{u} \in \boldsymbol{V}_{1,p(\cdot)}(\Omega), \boldsymbol{w} \in \boldsymbol{G}_{1,p(\cdot)}(\Omega),$$

and

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p(\cdot)}(\Omega)} \le C\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p(\cdot)}(\Omega)}. \tag{4.10}$$

Since $\boldsymbol{w} \in \boldsymbol{G}_{1,p(\cdot)}(\Omega)$, there exists $\pi \in L_0^{p(\cdot)}(\Omega)$ such that $-\Delta \boldsymbol{w} = \boldsymbol{\nabla}\pi$ in $\boldsymbol{W}^{-1,p(\cdot)}(\Omega)$. Hence we have

$$\begin{cases} -\Delta \boldsymbol{u} + \boldsymbol{\nabla}\pi = -\Delta \boldsymbol{v} = \boldsymbol{f} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \Gamma. \end{cases}$$

Thus $(\boldsymbol{u}, \pi)$ is a weak solution for the homogeneous Stokes problem (4.1) with $g = 0$ and $\boldsymbol{h} = \boldsymbol{0}$.

If $\boldsymbol{f} = \boldsymbol{0}$, then $-\Delta \boldsymbol{u} = \boldsymbol{\nabla}(-\pi)$. Hence $\boldsymbol{u} = (-\Delta)^{-1}\boldsymbol{\nabla}(-\pi) \in \boldsymbol{G}_{1,p(\cdot)}(\Omega)$. Since $\boldsymbol{u} \in \boldsymbol{W}_0^{1,p(\cdot)}(\Omega)$ satisfies $\operatorname{div} \boldsymbol{u} = 0$ in Ω, we see that $\boldsymbol{u} \in \boldsymbol{V}_{1,p(\cdot)}(\Omega)$, so $\boldsymbol{u} = \boldsymbol{0}$ in Ω from (4.7). Thus $\boldsymbol{\nabla}\pi = \boldsymbol{0}$. Since $\pi \in L_0^{p(\cdot)}(\Omega)$, we have $\pi = 0$ in Ω. This implies the uniqueness of a weak solution.

We show the estimate (4.9). If $(\boldsymbol{u}, \pi) \in \boldsymbol{W}_0^{1,p(\cdot)}(\Omega) \times L_0^{p(\cdot)}(\Omega)$ is a weak solution of the homogeneous Stokes problem (4.1) with $g = 0$ and $\boldsymbol{h} = \boldsymbol{0}$, then

$$\int_\Omega \boldsymbol{\nabla u} \cdot \boldsymbol{\nabla \varphi} dx - \int_\Omega \pi \operatorname{div} \boldsymbol{\varphi} dx = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_{\boldsymbol{W}^{-1,p(\cdot)}(\Omega), \boldsymbol{W}_0^{1,p'(\cdot)}(\Omega)} \tag{4.11}$$

for all $\boldsymbol{\varphi} \in \boldsymbol{W}_0^{1,p'(\cdot)}(\Omega)$. Since the projection $\boldsymbol{W}_0^{1,p(\cdot)}(\Omega) \to \boldsymbol{V}_{1,p(\cdot)}(\Omega)$ is linear and bounded, it follows from (4.6) and (4.10) that

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_0^{1,p(\cdot)}(\Omega)} \le C\|\boldsymbol{v}\|_{\boldsymbol{W}_0^{1,p(\cdot)}(\Omega)} \le C_1\|\boldsymbol{f}\|_{\boldsymbol{W}^{-1,p(\cdot)}(\Omega)}. \tag{4.12}$$

From (4.11), we can write

$$\int_\Omega \pi \operatorname{div} \boldsymbol{\varphi} dx = \int_\Omega \boldsymbol{\nabla u} \cdot \boldsymbol{\nabla \varphi} dx - \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_{\boldsymbol{W}^{-1,p(\cdot)}(\Omega), \boldsymbol{W}_0^{1,p'(\cdot)}(\Omega)} \text{ for all } \boldsymbol{\varphi} \in \boldsymbol{W}_0^{1,p'(\cdot)}(\Omega).$$

For any $\psi \in L^{p'(\cdot)}(\Omega)$, $\psi - c_\psi \in L_0^{p'(\cdot)}(\Omega)$, where

$$c_\psi = \frac{1}{|\Omega|} \int_\Omega \psi dx.$$

By Theorem 2.5 (e) with $m = 0$, the divergence operator div : $\boldsymbol{W}_0^{1,p'(\cdot)}(\Omega)/\boldsymbol{V}_{1,p'(\cdot)}(\Omega) \to L_0^{p'(\cdot)}(\Omega)$ is a topological bijection. Hence there exists $\boldsymbol{\varphi} \in \boldsymbol{W}_0^{1,p'(\cdot)}(\Omega)$ such that $\operatorname{div} \boldsymbol{\varphi} = \psi - c_\psi$ and there exists a constant $C_2 > 0$ such that

$$\|[\boldsymbol{\varphi}]\|_{\boldsymbol{W}_0^{1,p'(\cdot)}(\Omega)/\boldsymbol{V}_{1,p'(\cdot)}} \le C_2\|\psi - c_\psi\|_{L^{p'(\cdot)}(\Omega)}. \tag{4.13}$$

Since

$$\|[\boldsymbol{\varphi}]\|_{\boldsymbol{W}_0^{1,p'(\cdot)}(\Omega)/\boldsymbol{V}_{1,p'(\cdot)}} = \inf\{\|\boldsymbol{\varphi} + \boldsymbol{v}\|_{\boldsymbol{W}_0^{1,p'(\cdot)}(\Omega)} \,;\, \boldsymbol{v} \in \boldsymbol{V}_{1,p'(\cdot)}(\Omega)\}$$

is achieved, we can replace the left-hand side of (4.13) with $\|\boldsymbol{\varphi}\|_{\boldsymbol{W}_0^{1,p'(\cdot)}(\Omega)}$. Therefore, using the Hölder inequality (Proposition 2.2), we have

$$\begin{aligned}
\|\boldsymbol{\varphi}\|_{\boldsymbol{W}_0^{1,p'(\cdot)}(\Omega)} &\le C_2\left(\|\psi\|_{L^{p'(\cdot)}(\Omega)} + \frac{1}{|\Omega|}\int_\Omega |\psi| dx \|1\|_{L^{p'((\cdot)}(\Omega)}\right) \\
&\le C_2(\|\psi\|_{L^{p'(\cdot)}(\Omega)} + \frac{2}{|\Omega|}\|\psi\|_{L^{p'(\cdot)}(\Omega)}\|1\|_{L^{p(\cdot)}(\Omega)}\|1\|_{L^{p'((\cdot)}(\Omega)}) \\
&\le C_3\|\psi\|_{L^{p'(\cdot)}(\Omega)}.
\end{aligned}$$

Since $\pi \in L_0^{p(\cdot)}(\Omega)$, we see that

$$\int_\Omega \pi\psi dx = \int_\Omega \pi(\psi - c_\psi) dx = \int_\Omega \pi \operatorname{div} \boldsymbol{\varphi} dx.$$

Hence using the Hölder inequality, the duality and (4.12), we have

$$\begin{aligned}\left|\int_\Omega \pi\psi dx\right| &\le \left|\int_\Omega \boldsymbol{\nabla u}\cdot\boldsymbol{\nabla\varphi} dx\right| + |\langle \boldsymbol{f},\boldsymbol{\varphi}\rangle_{\boldsymbol{W}^{-1,p(\cdot)}(\Omega),\boldsymbol{W}_0^{1,p'(\cdot)}(\Omega)}| \\ &\le 2\|\boldsymbol{\nabla u}\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)}\|\boldsymbol{\nabla\varphi}\|_{\boldsymbol{L}^{p'(\cdot)}(\Omega)} + \|\boldsymbol{f}\|_{\boldsymbol{W}^{-1,p(\cdot)}(\Omega)}\|\boldsymbol{\varphi}\|_{\boldsymbol{W}_0^{1,p'(\cdot)}(\Omega)} \\ &\le C_4\|\boldsymbol{f}\|_{\boldsymbol{W}^{-1,p(\cdot)}(\Omega)}\|\boldsymbol{\varphi}\|_{\boldsymbol{W}_0^{1,p'(\cdot)}(\Omega)} \\ &\le C_3C_4\|\boldsymbol{f}\|_{\boldsymbol{W}^{-1,p(\cdot)}(\Omega)}\|\psi\|_{L^{p'(\cdot)}(\Omega)}.\end{aligned}$$

Thus we have

$$\begin{aligned}\|\pi\|_{L^{p(\cdot)}(\Omega)} &= \sup\left\{\left|\int_\Omega \pi\psi dx\right| ;\ \psi\in L^{p'(\cdot)}(\Omega), \|\psi\|_{L^{p'(\cdot)}(\Omega)}\le 1\right\} \\ &\le C_3C_4\|\boldsymbol{f}\|_{\boldsymbol{W}^{-1,p(\cdot)}(\Omega)}.\end{aligned}$$

So we get the estimate (4.9). □

From Theorem 4.3 and Theorem 4.4, we get the following corollary.

Corollary 4.5. *Assume that Ω is a bounded domain of $\mathbb{R}^d$ $(d\ge 2)$ with a $C^{1,1}$-boundary Γ and $p\in\mathcal{P}_+^{\log}(\Omega)$. Then the homogeneous Stokes problem (4.1) with $g=0$ and $\boldsymbol{h}=\boldsymbol{0}$ is well-posed if and only if the Helmholtz-type decomposition (4.7) of $\boldsymbol{W}_0^{1,p(\cdot)}(\Omega)$ holds.*

5. Inhomogeneous Stokes problem

In this section, assume that Ω is a bounded domain of $\mathbb{R}^d$ $(d\ge 2)$ with a C^1-boundary Γ. We consider the inhomogeneous Stokes problem (4.1), where $\boldsymbol{f}\in\boldsymbol{W}^{-1,p(\cdot)}(\Omega)$, $g\in L^{p(\cdot)}(\Omega)$ and $\boldsymbol{h}\in\mathrm{Tr}(\boldsymbol{W}^{1,p(\cdot)}(\Omega))$ satisfy the compatibility condition (4.2).

Lemma 5.1. *Let Ω be a bounded domain of $\mathbb{R}^d$ $(d\ge 2)$ with a Lipschitz-continuous boundary Γ and $p\in\mathcal{P}_+^{\log}(\overline{\Omega})$. Assume that $g\in L^{p(\cdot)}(\Omega)$ and $\boldsymbol{h}\in\mathrm{Tr}(\boldsymbol{W}^{1,p(\cdot)}(\Omega))$ satisfy the compatibility condition (4.2). Then there exists $\boldsymbol{w}\in\boldsymbol{W}^{1,p(\cdot)}(\Omega)$ such that*

$$\begin{cases}\operatorname{div}\boldsymbol{w}=g & \text{in } \Omega,\\ \boldsymbol{w}=\boldsymbol{h} & \text{on } \Gamma.\end{cases} \tag{5.1}$$

Furthermore, there exists a constant $C>0$ depending only on p, d and Ω such that

$$\|\boldsymbol{w}\|_{\boldsymbol{W}^{1,p(\cdot)}(\Omega)}\le C(\|g\|_{L^{p(\cdot)}(\Omega)}+\|\boldsymbol{h}\|_{\mathrm{Tr}(\boldsymbol{W}^{1,p(\cdot)}(\Omega))}). \tag{5.2}$$

Proof. By definition of $\mathrm{Tr}(\boldsymbol{W}^{1,p(\cdot)}(\Omega))$, there exists $\boldsymbol{w}_0\in\boldsymbol{W}^{1,p(\cdot)}(\Omega)$ such that $\boldsymbol{w}_0=\boldsymbol{h}$ on Γ and $\|\boldsymbol{w}_0\|_{\boldsymbol{W}^{1,p(\cdot)}(\Omega)}\le C\|\boldsymbol{h}\|_{\mathrm{Tr}(\boldsymbol{W}^{1,p(\cdot)}(\Omega))}$. It follows from the Green theorem and the compatibility condition (4.2) that

$$\int_\Omega \operatorname{div}\boldsymbol{w}_0 dx=\int_\Gamma \boldsymbol{w}_0\cdot\boldsymbol{n}d\sigma=\int_\Gamma \boldsymbol{h}\cdot\boldsymbol{n}d\sigma=\int_\Omega g dx.$$

Hence $\operatorname{div}\boldsymbol{w}_0-g\in L_0^{p(\cdot)}(\Omega)$. From Theorem 2.5 (e) with $m=0$, we see that div : $\boldsymbol{W}_0^{1,p(\cdot)}(\Omega)/\boldsymbol{V}_{1,p(\cdot)}(\Omega)\to L_0^{p(\cdot)}(\Omega)$ is a topological bijection. So there exists $\boldsymbol{w}_1\in\boldsymbol{W}_0^{1,p(\cdot)}(\Omega)$ such that $\operatorname{div}\boldsymbol{w}_1=\operatorname{div}\boldsymbol{w}_0-g$ in Ω and there exists a constant $C_1>0$ such that

$$\begin{aligned}\|[\boldsymbol{w}_1]\|_{\boldsymbol{W}_0^{1,p(\cdot)}(\Omega)/\boldsymbol{V}_{1,p(\cdot)}(\Omega)} &\le C_1\|\operatorname{div}\boldsymbol{w}_0-g\|_{L^{p(\cdot)}(\Omega)} \\ &\le C_2(\|g\|_{L^{p(\cdot)}(\Omega)}+\|\boldsymbol{h}\|_{\mathrm{Tr}(\boldsymbol{W}^{1,p(\cdot)}(\Omega))}).\end{aligned}$$

Since $\|[\boldsymbol{w}_1]\|_{\boldsymbol{W}_0^{1,p(\cdot)}(\Omega)/\boldsymbol{V}_{1,p(\cdot)}(\Omega)} = \inf\{\|\boldsymbol{w}_1+\boldsymbol{w}\|_{\boldsymbol{W}^{1,p(\cdot)}(\Omega)}; \boldsymbol{w} \in \boldsymbol{V}_{1,p(\cdot)}(\Omega)\}$ is achieved, we can assume that

$$\|\boldsymbol{w}_1\|_{\boldsymbol{W}_0^{1,p(\cdot)}(\Omega)} \le C_2(\|g\|_{L^{p(\cdot)}(\Omega)} + \|\boldsymbol{h}\|_{\mathrm{Tr}(\boldsymbol{W}^{1,p(\cdot)}(\Omega))}).$$

Put $\boldsymbol{w} = \boldsymbol{w}_0 - \boldsymbol{w}_1$. Then we see that $\operatorname{div}\boldsymbol{w} = g$ in Ω and $\boldsymbol{w} = \boldsymbol{w}_0 = \boldsymbol{h}$ on Γ, and

$$\|\boldsymbol{w}\|_{\boldsymbol{W}_0^{1,p(\cdot)}(\Omega)} \le \|\boldsymbol{w}_0\|_{\boldsymbol{W}^{1,p(\cdot)}(\Omega)} + \|\boldsymbol{w}_1\|_{\boldsymbol{W}^{1,p(\cdot)}(\Omega)} \le C(\|g\|_{L^{p(\cdot)}(\Omega)} + \|\boldsymbol{h}\|_{\mathrm{Tr}(\boldsymbol{W}^{1,p(\cdot)}(\Omega))}).$$

This completes the proof. □

Finally we have the following theorem.

Theorem 5.2. *Let Ω be a bounded domain of $\mathbb{R}^d$ ($d \ge 2$) with a C^1-boundary Γ, and let $p \in \mathcal{P}_+^{\log}(\overline{\Omega})$. Assume that the Helmholtz-type decomposition (4.7) holds. Then for $\boldsymbol{f} \in \boldsymbol{W}^{-1,p(\cdot)}(\Omega)$, $g \in L^{p(\cdot)}(\Omega)$ and $\boldsymbol{h} \in \mathrm{Tr}(\boldsymbol{W}^{1,p(\cdot)}(\Omega))$ satisfying the compatibility condition (4.2), then the inhomogeneous Stokes problem (4.1) has a unique weak solution $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p(\cdot)}(\Omega) \times L_0^{p(\cdot)}(\Omega)$. Furthermore, the estimate (4.4) holds.*

Proof. By Lemma 5.1, there exists $\boldsymbol{w} \in \boldsymbol{W}^{1,p(\cdot)}(\Omega)$ such that $\operatorname{div}\boldsymbol{w} = g$ in Ω and $\boldsymbol{w} = \boldsymbol{h}$ on Γ, and the estimate (5.2) holds. If we put $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{w}$, we can see that the problem (4.1) is reduced to

$$\begin{cases} -\Delta \boldsymbol{v} + \boldsymbol{\nabla}\pi = \boldsymbol{f} + \Delta\boldsymbol{w} & \text{in } \Omega, \\ \operatorname{div}\boldsymbol{v} = 0 & \text{in } \Omega, \\ \boldsymbol{v} = \boldsymbol{0} & \text{on } \Gamma. \end{cases} \tag{5.3}$$

Since $\boldsymbol{f} + \Delta\boldsymbol{w} \in \boldsymbol{W}^{-1,p(\cdot)}(\Omega)$, it follows from Theorem 4.4 that the problem (5.3) has a unique weak solution $(\boldsymbol{v},\pi) \in \boldsymbol{W}_0^{1,p(\cdot)}(\Omega) \times L_0^{p(\cdot)}(\Omega)$, and

$$\begin{aligned} \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p(\cdot)}(\Omega)} + \|\pi\|_{L^{p(\cdot)}(\Omega)} &\le C\|\boldsymbol{f} + \Delta\boldsymbol{w}\|_{\boldsymbol{W}^{-1,p(\cdot)}(\Omega)} \\ &\le C_1(\|\boldsymbol{f}\|_{\boldsymbol{W}^{-1,p(\cdot)}(\Omega)} + \|\boldsymbol{w}\|_{\boldsymbol{W}^{1,p(\cdot)}(\Omega)}). \end{aligned} \tag{5.4}$$

If we put $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$, we can see that $(\boldsymbol{u},\pi) \in \boldsymbol{W}^{1,p(\cdot)}(\Omega) \times L_0^{p(\cdot)}(\Omega)$ is a unique weak solution of (4.1). The estimate (4.4) follows form (5.2) and (5.4). □

Remark 5.3. Theorem 5.2 insists that when Ω is bounded domain with a C^1-boundary (weaker than the regularity of Theorem 4.2), if we further assume that the Helmholtz-type decomposition (4.7) holds, then the same conclusion as Theorem 4.2 holds. By our recognition, this result is new.

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