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Some growth of composite entire and meromorphic functions with finite iterated order

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Abstract

The object of this paper is to investigate the growth properties of composite entire and meromorphic functions with finite iterated order. We have established some new results which are the improvement and extensions of the earliar results.

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1. Introduction, Definitions and Notations

We will assume that the reader is familiar with the standard notations and the fundamental results employed in the theory of entire and meromorphic functions, see [18] and [5].

Let f be a entire function with $M(r, f) = \max_{|z|=r} |f(z)|$. A well known theorem of Polya [13] asserts that: If f and g are entire functions, then the composite function $f \circ g$ is of infinite order unless (a) f is of finite order and g is a polynomial or (b) f is of order zero and g is of finite order. For the two transcendental entire functions $f(z)$ and $g(z)$, Clunie [4] showed that $\lim_{r\to\infty} \frac{T(r, f \circ g)}{T(r, f)} = \infty$ and $\lim_{r\to\infty} \frac{T(r, f \circ g)}{T(r, g)} = \infty$.

Since then, many authors {see, [1], [2], [7], [8], [11], [12], [15], [16], [19]} made closed investigation on growth properties of composition of two entire functions with finite order and achieved great results. In 2009, Jin Tu et.al [17] investigated the growth properties of two composite entire functions of finite iteration. In the paper, we investigate the comparative growth properties of composite entire or meromorphic functions with the finite iterated order.

In order to estimate the rate of growth of composite entire or meromorphic functions with finite iterated order more precisely, we recall the following definitions:

Definition 1.1. The order $\rho(f)$ and lower order $\lambda(f)$ of an entire function f are defined as

$$\rho(f) = \lim_{r\to\infty} \sup \frac{\log^{[2]} M(r, f)}{\log r} = \lim_{r\to\infty} \sup \frac{\log T(r, f)}{\log r}$$

and

$$\lambda(f) = \lim_{r\to\infty} \inf \frac{\log^{[2]} M(r, f)}{\log r} = \lim_{r\to\infty} \inf \frac{\log T(r, f)}{\log r},$$

where following Sato [14], we write $\log^{[0]} x = x$ and $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ for $k = 1, 2, 3, \ldots$

Furthermore, if $\rho(f) = 0$, we define $\rho^*(f)$ and $\lambda^*(f)$ as follows

$$\rho^*(f) = \lim_{r\to\infty} \sup \frac{\log^{[2]} M(r, f)}{\log\log r} = \lim_{r\to\infty} \sup \frac{\log T(r, f)}{\log\log r},$$

and

$$\lambda^*(f) = \lim_{r\to\infty} \inf \frac{\log^{[2]} M(r, f)}{\log\log r} = \lim_{r\to\infty} \inf \frac{\log T(r, f)}{\log\log r}.$$

Definition 1.2. ([3], [6]) The iterated i order $\rho_i(f)$ of an entire function f is defined as

$$\rho_i(f) = \lim_{r\to\infty} \sup \frac{\log^{[i+1]} M(r, f)}{\log r} = \lim_{r\to\infty} \sup \frac{\log^{[i]} T(r, f)}{\log r} \quad (i \in \mathbb{N}).$$

Similarly, we can define the iterated i lower order $\lambda_i(f)$ of an entire function f is defined as

$$\lambda_i(f) = \lim_{r\to\infty} \inf \frac{\log^{[i+1]} M(r, f)}{\log r} = \lim_{r\to\infty} \inf \frac{\log^{[i]} T(r, f)}{\log r} \quad (i \in \mathbb{N}).$$

Furthermore, if $\rho_i(f) = 0$, then we define $\rho_i^*(f)$ and $\lambda_i^*(f)$ as follows

$$\rho_i^*(f) = \lim_{r\to\infty} \sup \frac{\log^{[i+1]} M(r, f)}{\log\log r} = \lim_{r\to\infty} \sup \frac{\log^{[i]} T(r, f)}{\log\log r} \quad (i \in \mathbb{N}),$$

and

$$\lambda_i^*(f) = \lim_{r\to\infty} \inf \frac{\log^{[i+1]} M(r, f)}{\log\log r} = \lim_{r\to\infty} \inf \frac{\log^{[i]} T(r, f)}{\log\log r} \quad (i \in \mathbb{N}).$$

Definition 1.3. ([3], [6]) The finiteness degree of the order of an entire functions $f(z)$ is defined by

$$i(f) = \begin{cases} 0 & \text{for } f \text{ polynomial.} \\ \min\{j \in \mathbb{N} : \rho_j(f) < \infty\} & \text{for } f \text{ transcendental with } \rho_j(f) < \infty \\ \infty & \text{for } f \text{ with } \rho_j(f) = \infty \text{ for all } j \in \mathbb{N}. \end{cases}$$

From the above definition, it is easy to see that $i(f)$ are positive integers.

Definition 1.4. A function $\rho_f(r)$ is called a proximate order of a meromorphic function $f(z)$ relative to $T(r, f)$ if

(i) $\rho_f(r)$ is real, continuous and piecewise differentiable for $r > r_0$, say;

(ii) $\lim_{r\to\infty} \rho_f(r) = \rho_g$,

(iii) $\lim_{r\to\infty} r\rho'_f(r) \log r = 0$, and

(iv) $\lim_{r\to\infty} \sup \dfrac{T(r, g)}{r^{\rho_f(r)}} = 1$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. (See [1].) If f is meromorphic and g is entire then for all large values of r

$$T(r, f \circ g) \leq \{1 + o(1)\} \frac{T(r, g)}{\log M(r, g)} T(M(r, g), f).$$

Lemma 2.2. (See [10].) Let f and g be two entire functions. Then for a sequence of values of r tending to infinity

$$T(r, f \circ g) \geq \frac{1}{3} \log M\left\{\frac{1}{8} M\left(\frac{r}{4}, g\right) + O(1), f\right\}.$$

Lemma 2.3. (See [4].) If $f(z)$ and $g(z)$ are entire functions, then for all sufficiently large values of r

$$M\left(\frac{1}{8} M\left(\frac{r}{2}, g\right) - |g(0)|, f\right) \leq M(r, f \circ g) \leq M(M(r, g), f).$$

Lemma 2.4. (See [9].) Let $g(z)$ be an integral function with $\lambda_g < \infty$, and assume that $a_i(z)$ $(i = 1, 2, \ldots, n;\ n \leq \infty)$ are entire functions satisfying $T(r, a_i(z)) = o\{T(r, g)\}$ and $\sum_{i=1}^{n} \delta(a_i(z), g) = 1$, then

$$\lim_{r\to\infty} \frac{T(r, g)}{\log M(r, g)} = \frac{1}{\pi}.$$

Lemma 2.5. Let $f(z)$ be a meromorphic function. Then for $\delta\,(>0)$ the function $r^{\rho_f+\delta-\rho_f(r)}$ and $r^{\lambda_f+\delta-\lambda_f(r)}$ are ultimately increasing functions of r.

Proof. Since

$$\frac{d}{dr}r^{\rho_f+\delta-\rho_f(r)}=\left\{\rho_f+\delta-\rho_f(r)-r\log r\rho_f'(r)\right\}r^{\rho_f+\delta-\rho_f(r)}>0$$

for all sufficiently large values or r, the lemma is proved. ■

3. Theorems

In this section we present the main results of the paper.

Theorem 3.1. Let $f(z)$ and $g(z)$ be two entire functions, then $\rho_p(f\circ g)=\infty$ if $\rho(g)=\infty$.

Proof. Suppose that $\rho(g)=\infty$.By the Lemma 2.3 we have for all sufficiently large values of r

$$\begin{aligned}\rho_p(f\circ g) &= \limsup_{r\to\infty}\frac{\log^{[p+1]}M(r,f\circ g)}{\log r}\\ &\geq \limsup_{r\to\infty}\frac{\log^{[p+1]}M\left(\frac{1}{8}M\left(\frac{r}{2},g\right),f\right)}{\log r}\\ &= \limsup_{r\to\infty}\frac{\log^{[p+1]}M\left(\frac{1}{8}M\left(\frac{r}{2},g\right),f\right)}{\log\left(\frac{1}{8}M\left(\frac{r}{2},g\right)\right)}\frac{\log\left(\frac{1}{8}M\left(\frac{r}{2},g\right)\right)}{\log\frac{r}{2}}\frac{\log\frac{r}{2}}{\log r}\\ &= \lim_{r\to\infty}\sup\frac{\log^{[p+1]}M\left(\frac{1}{8}M\left(\frac{r}{2},g\right),f\right)}{\log\left(\frac{1}{8}M\left(\frac{r}{2},g\right)\right)}\cdot\left(\frac{\log M\left(\frac{r}{2},g\right)}{\log\frac{r}{2}}+\frac{\log\frac{1}{8}}{\log\frac{r}{2}}\right)\cdot\left(\frac{\log r}{\log r}-\frac{\log 2}{\log r}\right)\\ &= \rho_p(f)\,\rho(g).\end{aligned}$$

Hence $\rho_p(f\circ g)=\infty$ if $\rho(g)=\infty$. ■

Theorem 3.2. Let $f(z)$ be a meromorphic function and $g(z)$ be an entire function with finite iterated order then

$$\rho_p^*(f\circ g)\leq\rho_p^*(f)\,\rho^*(g)$$

Proof. By the Lemma 2.1, we have for all large values of r

$$T(r,f\circ g)\leq\{1+o(1)\}\frac{T(r,g)}{\log M(r,g)}T(M(r,g),f).$$

$$\text{i.e., }\log^{[p]}T(r,f\circ g)\leq\log^{[p]}T(r,g)-\log^{[p+1]}M(r,g)+\log^{[p]}T(M(r,g),f).$$

From the definition, we have

$$\begin{aligned}\rho_p^*(f\circ g) &= \limsup_{r\to\infty}\frac{\log^{[p]}T(r,f\circ g)}{\log\log r}\\ &\le \limsup_{r\to\infty}\frac{\log^{[p]}T(r,g)-\log^{[p+1]}M(r,g)+\log^{[p]}T(M(r,g),f)}{\log\log r}\\ &= \limsup_{r\to\infty}\frac{\log^{[p]}T(M(r,g),f)}{\log\log r}\\ &= \limsup_{r\to\infty}\frac{\log^{[p]}T(M(r,g),f)}{\log\log M(r,g)}\frac{\log\log M(r,g)}{\log\log r}\\ &= \rho_p^*(f)\,\rho^*(g).\end{aligned}$$

This proves the theorem. ■

Theorem 3.3. Let $f(z)$ and $g(z)$ be two non constant entire functions of finite iterated order then

$$\liminf_{r\to\infty}\frac{\log^{[p]}T(r,f\circ g)}{T(r,g)}\le 3\lambda_p(f)\,2^{\rho(g)}.$$

Proof. By the Lemma 2.3, we have for all sufficiently large values of r

$$M(r,f\circ g)\le M(M(r,g),f).$$

$$\text{or, }\log M(r,f\circ g)\le \log M(M(r,g),f).$$

Since

$$T(r,f\circ g)\le \log M(r,f\circ g)\le \log M(M(r,g),f),$$

$$\text{i.e., }\log^{[p]}T(r,f\circ g)\le \log^{[p+1]}M(M(r,g),f).$$

Therefore, for all sufficiently large values of r

$$\begin{aligned}\liminf_{r\to\infty}\frac{\log^{[p]}T(r,f\circ g)}{T(r,g)} &\le \liminf_{r\to\infty}\frac{\log^{[p+1]}M(M(r,g),f)}{\log M(r,g)}\frac{\log M(r,g)}{T(r,g)}\\ &= \lambda_p(f)\liminf_{r\to\infty}\frac{\log M(r,g)}{T(r,g)}.\end{aligned}\tag{1}$$

Let $\rho_g(r)$ be a proximate order of $g(z)$ relative to $T(r,g)$. Since by the definition $\limsup_{r\to\infty}\frac{T(r,g)}{r^{\rho_g(r)}}=1$, it follows that if $\varepsilon\ (0<\varepsilon<1)$ is an arbitrary then for all large values of r

$$T(r,g)<(1+\varepsilon)\,r^{\rho_g(r)},\tag{2}$$

and for a sequence of values of r tending to infinity

$$T(r,g)>(1-\varepsilon)\,r^{\rho_g(r)}.\tag{3}$$

We know that

$$\begin{aligned} \log M(r,g) &\le 3T(2r,g) < 3(1+\varepsilon)(2r)^{\rho_g(2r)}, \quad \text{by (2)} \\ \text{or, } \log M(r,g) &< 3(1+\varepsilon)\frac{(2r)^{\rho_g+\delta}}{(2r)^{\rho_g+\delta-\rho_g(2r)}}, \text{ where } \delta\,(>0) \text{ is arbitrary.} \end{aligned} \tag{4}$$

Since $r^{\rho_g+\delta-\rho_g(r)}$ is ultimately increasing function of r so for $r < 2r$

$$\begin{aligned} r^{\rho_g+\delta-\rho_g(r)} &< (2r)^{\rho_g+\delta-\rho_g(2r)} \\ \text{or, } \frac{1}{r^{\rho_g+\delta-\rho_g(r)}} &> \frac{1}{(2r)^{\rho_g+\delta-\rho_g(2r)}} \end{aligned} \tag{5}$$

From (4) and (5) , we get for all large values of r

$$\begin{aligned} \log M(r,g) &< 3(1+\varepsilon)\frac{2^{\rho_g+\delta}r^{\rho_g+\delta}}{r^{\rho_g+\delta-\rho_g(r)}} \\ \text{or, } \log M(r,g) &< 3(1+\varepsilon)\,2^{\rho_g+\delta}r^{\rho_g(r)}. \end{aligned} \tag{6}$$

Again from (3) and (6) we have

$$\liminf_{r\to\infty}\frac{\log M(r,g)}{T(r,g)} < 3\frac{(1+\varepsilon)}{(1-\varepsilon)}2^{\rho_g+\delta}.$$

Since $\varepsilon\,(>0)$ and $\delta\,(>0)$ are arbitrary,

$$\liminf_{r\to\infty}\frac{\log M(r,g)}{T(r,g)} \le 3\cdot 2^{\rho_g}. \tag{7}$$

Hence the theorem follows from (1) and (7) . ■

Corollary 3.4. In the line of Theorem 3.3, we can also prove

$$\liminf_{r\to\infty}\frac{\log M(r,g)}{T(r,g)} \le 3\lambda_p(f)\,2^{\lambda(g)}.$$

Proof. Let $\lambda_g(r)$ be a proximate order of $g(z)$ relative to $T(r,g)$, then

$$\liminf_{r\to\infty}\frac{T(r,g)}{r^{\lambda_g(r)}} = 1.$$

It follows that if ε $(0 < \varepsilon < 1)$ is an arbitrary, then for all large values of r

$$T(r,g) < (1+\varepsilon)\,r^{\lambda_g(r)} \tag{8}$$

and for a sequence of values of r tending to infinity

$$T(r,g) > (1-\varepsilon)\,r^{\lambda_g(r)}. \tag{9}$$

Again

$$\log M(r,g) \leq 3T(2r,g) < 3(1+\varepsilon)(2r)^{\lambda_g(2r)}, \text{ by } (8)$$
$$\log M(r,g) < 3(1+\varepsilon)\frac{(2r)^{\lambda_g+\delta}}{(2r)^{\lambda_g+\delta-\lambda_g(2r)}}, \text{ where } \delta\,(>0) \text{ is arbitrary.} \quad (10)$$

Since $r^{\lambda_g+\delta-\lambda_g(r)}$ is ultimately increasing function of r so for $r < 2r$

$$r^{\lambda_g+\delta-\lambda_g(r)} < (2r)^{\lambda_g+\delta-\lambda_g(2r)}$$
$$\text{or, } \frac{1}{r^{\lambda_g+\delta-\lambda_g(r)}} > \frac{1}{(2r)^{\lambda_g+\delta-\lambda_g(2r)}} \quad (11)$$

From (10) and (11) we obtained

$$\log M(r,g) < 3(1+\varepsilon)\frac{(2r)^{\lambda_g+\delta}}{r^{\lambda_g+\delta-\lambda_g(r)}} = 3(1+\varepsilon)2^{\lambda_g+\delta}r^{\lambda_g(r)} \quad (12)$$

Thus, from (9) and (12) for a sequence of values of r tending to infinity, we get

$$\frac{\log M(r,g)}{T(r,g)} < 3\frac{(1+\varepsilon)}{(1-\varepsilon)}2^{\lambda_g+\delta}$$

$$\liminf_{r\to\infty}\frac{\log M(r,g)}{T(r,g)} \leq 3\frac{(1+\varepsilon)}{(1-\varepsilon)}2^{\lambda_g+\delta}$$

Since $\varepsilon\,(>0)$ and $\delta\,(>0)$ are arbitrary,

$$\liminf_{r\to\infty}\frac{\log M(r,g)}{T(r,g)} \leq 3\cdot 2^{\lambda_g} \quad (13)$$

From (1) and (13), it follows that

$$\liminf_{r\to\infty}\frac{\log M(r,g)}{T(r,g)} \leq 3\lambda_p(f)\,2^{\lambda(g)}.$$

■

Theorem 3.5. Let $f(z)$ and $g(z)$ be nonconstant meromorphic and entire functions respectively such that $\rho_p(f)=0$, then

$$\limsup_{r\to\infty}\frac{\log^{[p]}T(r,f\circ g)}{\log T(r,f)} \leq \frac{\rho_p^*(f)\,\rho(g)}{\rho(f)}$$

and

$$\liminf_{r\to\infty}\frac{\log^{[p]}T(r,f\circ g)}{\log T(r,f)} \leq \frac{\lambda_p^*(f)\,\lambda(g)}{\lambda(f)}.$$

Proof. Since

$$\limsup_{r\to\infty}\frac{\log^{[p]} T(r, f\circ g)}{\log T(r, f)} \le \limsup_{r\to\infty}\left\{\frac{\log^{[p+1]} M(M(r,g), f)}{\log\log M(r,g)}\cdot\frac{\log\log M(r,g)}{\log r}\cdot\frac{\log r}{\log T(r,f)}\right\} = \frac{\rho_p^*(f)\,\rho(g)}{\rho(f)}.$$

Again

$$\liminf_{r\to\infty}\frac{\log^{[p]} T(r, f\circ g)}{\log T(r, f)} \le \liminf_{r\to\infty}\left\{\frac{\log^{[p+1]} M(M(r,g), f)}{\log\log M(r,g)}\cdot\frac{\log\log M(r,g)}{\log r}\cdot\frac{\log r}{\log T(r,f)}\right\} = \frac{\lambda_p^*(f)\,\lambda(g)}{\lambda(f)}.$$

This proves the theorem. ∎

Theorem 3.6. Let $f(z)$ and $g(z)$ be nonconstant meromorphic and entire functions respectively. Then

$$\limsup_{r\to\infty}\frac{\log^{[p]} T(r, f\circ g)}{\log T(r, g)} \le \rho_p^*(f)$$

and

$$\liminf_{r\to\infty}\frac{\log^{[p]} T(r, f\circ g)}{\log T(r, g)} \le \lambda_p^*(f).$$

Proof. Since

$$\limsup_{r\to\infty}\frac{\log\log M(r,g)}{\log T(r,g)} \le 1.$$

Therefore,

$$\limsup_{r\to\infty}\frac{\log^{[p]} T(r, f\circ g)}{\log T(r, g)} \le \limsup_{r\to\infty}\left\{\frac{\log^{[p+1]} M(M(r,g), f)}{\log\log M(r,g)}\cdot\frac{\log\log M(r,g)}{\log T(r,g)}\right\} \le \rho_p^*(f).$$

Again

$$\liminf_{r\to\infty}\frac{\log^{[p]} T(r, f\circ g)}{\log T(r, g)} \le \liminf_{r\to\infty}\left\{\frac{\log^{[p+1]} M(M(r,g), f)}{\log\log M(r,g)}\cdot\frac{\log\log M(r,g)}{\log T(r,g)}\right\} \le \lambda_p^*(f).$$

This completes the theorem. ∎

Theorem 3.7. Let $f(z)$ and $g(z)$ be two nonconstant entire functions such that such that $\rho_p(f)$ and $\lambda(g)$ are finite. Also suppose that $a_i(z)$ $(i = 1, 2, \ldots, n;\ n \le \infty)$ are entire functions satisfying $T(r, a_i(z)) = o\{T(r,g)\}$ as $r\to\infty$ and $\sum_{i=1}^{n}\delta(a_i(z), g) = 1$.Then

$$\frac{\pi\rho_p(f)}{4^{\lambda(g)}} \le \limsup_{r\to\infty}\frac{\log^{[p]} T(r, f\circ g)}{T(r,g)} \le \pi\rho_p(f).$$

Proof. By the Lemma 2.2, we have for a sequence of values of r tending to infinity

$$T(r, f\circ g) \geq \frac{1}{3}\log M\left\{\frac{1}{8}M\left(\frac{r}{4}, g\right) + O(1), f\right\}$$

$$\text{or,}\quad \log^{[p]} T(r, f\circ g) \geq \log^{[p+1]} M\left\{\frac{1}{9}M\left(\frac{r}{4}, g\right), f\right\} + O(1).$$

Therefore

$$\begin{aligned}
\frac{\log^{[p]} T(r, f\circ g)}{T(r,g)} &\geq \frac{\log^{[p+1]} M\left\{\frac{1}{9}M\left(\frac{r}{4}, g\right), f\right\}}{T(r,g)} + \frac{O(1)}{T(r,g)} \\
&= \frac{\log^{[p+1]} M\left\{\frac{1}{9}M\left(\frac{r}{4}, g\right), f\right\}}{\log\left\{\frac{1}{9}M\left(\frac{r}{4}, g\right)\right\}} \cdot \frac{\log\left\{\frac{1}{9}M\left(\frac{r}{4}, g\right)\right\}}{T\left(\frac{r}{4}, g\right)} \cdot \frac{T\left(\frac{r}{4}, g\right)}{T(r,g)} + \frac{O(1)}{T(r,g)} \\
&= \frac{\log^{[p+1]} M\left\{\frac{1}{9}M\left(\frac{r}{4}, g\right), f\right\}}{\log\left\{\frac{1}{9}M\left(\frac{r}{4}, g\right)\right\}} \cdot \frac{\log M\left(\frac{r}{4}, g\right)}{T\left(\frac{r}{4}, g\right)} \cdot \frac{T\left(\frac{r}{4}, g\right)}{T(r,g)} + \frac{O(1)}{T(r,g)} + \frac{O(1)}{T\left(\frac{r}{4}, g\right)}.
\end{aligned}$$

Thus for a sequence of values of r tending to infinity

$$\begin{aligned}
\limsup_{r\to\infty}\frac{\log^{[p]} T(r, f\circ g)}{T(r,g)} &\geq \limsup_{r\to\infty}\left\{\frac{\log^{[p+1]} M\left\{\frac{1}{9}M\left(\frac{r}{4}, g\right), f\right\}}{\log\left\{\frac{1}{9}M\left(\frac{r}{4}, g\right)\right\}} \cdot \frac{\log M\left(\frac{r}{4}, g\right)}{T\left(\frac{r}{4}, g\right)} \cdot \frac{T\left(\frac{r}{4}, g\right)}{T(r,g)}\right\} \\
&= \rho_p(f)\cdot\limsup_{r\to\infty}\frac{\log M\left(\frac{r}{4}, g\right)}{T\left(\frac{r}{4}, g\right)}\cdot\limsup_{r\to\infty}\frac{T\left(\frac{r}{4}, g\right)}{T(r,g)} \\
&= \pi\rho_p(f)\limsup_{r\to\infty}\frac{T\left(\frac{r}{4}, g\right)}{T(r,g)}, \qquad \text{by the Lemma 2.4.} \qquad (14)
\end{aligned}$$

From (8) and (9), we have

$$\begin{aligned}
\frac{T\left(\frac{r}{4}, g\right)}{T(r,g)} &> \frac{(1-\varepsilon)}{(1+\varepsilon)}\cdot\frac{\left(\frac{r}{4}\right)^{\lambda_g\left(\frac{r}{4}\right)}}{r^{\lambda_g(r)}} = \frac{(1-\varepsilon)}{(1+\varepsilon)}\cdot\frac{\left(\frac{r}{4}\right)^{\lambda_g+\delta}}{\left(\frac{r}{4}\right)^{\lambda_g+\delta-\lambda_g\left(\frac{r}{4}\right)}}\cdot\frac{1}{r^{\lambda_g(r)}} \\
&> \frac{(1-\varepsilon)}{(1+\varepsilon)}\cdot\frac{\left(\frac{r}{4}\right)^{\lambda_g+\delta}}{r^{\lambda_g+\delta-\lambda_g(r)}}\cdot\frac{1}{r^{\lambda_g(r)}} = \frac{(1-\varepsilon)}{(1+\varepsilon)}\cdot\frac{1}{4^{\lambda_g+\delta}}.
\end{aligned}$$

Since $\varepsilon\ (>0)$ and $\delta\ (>0)$ are arbitrary, we get

$$\lim_{r\to\infty}\frac{T\left(\frac{r}{4}, g\right)}{T(r,g)} \geq \frac{1}{4^{\lambda_g}}. \qquad (15)$$

Thus from (14) and (15) we have

$$\limsup_{r\to\infty}\frac{\log^{[p]} T(r, f\circ g)}{T(r,g)} \geq \frac{\pi\rho_p(f)}{4^{\lambda(g)}}. \qquad (16)$$

Also, we have for all sufficiently large values of r

$$\limsup_{r\to\infty}\frac{\log^{[p]} T(r, f\circ g)}{T(r,g)} \leq \limsup_{r\to\infty}\left\{\frac{\log^{[p+1]} M(M(r,g), f)}{\log M(r,g)}\cdot\frac{\log M(r,g)}{T(r,g)}\right\}$$

$$\text{i.e., } \limsup_{r\to\infty}\frac{\log^{[p]} T(r, f\circ g)}{T(r,g)} \leq \pi\rho_p(f), \quad \text{by the Lemma 2.4.} \tag{17}$$

Hence, from (16) and (17)

$$\frac{\pi\rho_p(f)}{4^{\lambda(g)}} \leq \limsup_{r\to\infty}\frac{\log^{[p]} T(r, f\circ g)}{T(r,g)} \leq \pi\rho_p(f).$$

■

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