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A NOTE ON AXIAL FUNCTIONS ON THE PLANE

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ABSTRACT. A function from the plane to the plane is axial if it does not change one coordinate. We show that every function is a superposition of three axial functions both measurable and with the Baire property. Every Borel function is a composition of three Borel axial functions.

Definition 1. Let X, Y be any (nonempty) sets. A function $f : X\times Y \to X\times Y$ is axial if

$$f(x,y) = \big(x, g(x,y)\big) \qquad \text{for some} \quad g : X\times Y \to Y$$

or

$$f(x,y) = \big(g(x,y), y\big) \qquad \text{for some} \quad g : X\times Y \to X.$$

Banach (see [S]) asked whether every permutation of $\mathbb{N}\times\mathbb{N}$ is a composition of the finite number of axial permutations. This was answered by Nosarzewska [N] who proved that 5 axial permutations are sufficient. Later, Ehrenfeucht and Grzegorek [EG], [G] proved that every permutation of $X\times Y$ is a composition of 4 axial permutations.

In the same paper they showed that every function $f : X\times Y \to X\times Y$, where X and Y are arbitrary sets provided that at least one of them is infinite, is a composition of only three axial functions (not necessarily $1-1$ or onto). We show that in case $X = Y = \mathbb{R}$ we may also require the axial functions to be measurable and to have the Baire property. If the original function f is Borel they can be chosen Borel. An interesting result is obtained in [AK], where it is proven that every permutation of the plane is a composition of very special axial permutations (translations on every horizontal or vertical line).

Recall that a function (from $\mathbb{R}$ or $\mathbb{R}^2$ to $\mathbb{R}$ or $\mathbb{R}^2$) is measurable (has the Baire property, is Borel) if preimages of open sets are measurable (have the Baire property, are Borel sets). A set A has the Baire property if $A = (G\cup P_1)\setminus P_2$, where G is open, P_1 and P_2 are of first category.

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We list some known facts and a (folklore) lemma. The composition of measurable (with the Baire property) functions need not be measurable (nor have the Baire property). But the composition of Borel functions is Borel. If a Borel function f (from a metric space to a metric space) is $1-1$, then the image of its domain is a Borel set and the inverse function f^{-1} is Borel (e.g., [K]). If $f=(f_1,f_2):\mathbb{R}^2\to\mathbb{R}^2$ is Borel, then so are its coordinate functions $f_i:\mathbb{R}^2\to\mathbb{R}$. The following lemma is a special case of the *Borel Isomorphism Theorem* (see, e.g., [K])

Lemma 2. *Let $\mathcal{C}$ be a Cantor ternary set. There is a Borel (even Baire 1) function $g:\mathbb{R}^2\to\mathcal{C}$ which is $1-1$ and onto.*

Lemma 3. *Let $f:X\to Y$, $g:Y\to Z$ be any functions and $g\circ f:X\to Z$ is Borel. If f (or g) is a Borel 1–1 function, then g (f) is Borel (not necessary 1–1).*

Proof. Assume f is Borel, then f^{-1} is Borel and $g=(g\circ f)\circ f^{-1}$ is Borel as a composition of Borel functions. Analogously, if g is Borel, then $f=g^{-1}\circ(g\circ f)$ is Borel, too. $\square$

Theorem 4. *Any function $f:\mathbb{R}^2\to\mathbb{R}^2$ is a compositon of three axial functions that are measurable and have the Baire property. Every Borel function*

$$f:\mathbb{R}^2\to\mathbb{R}^2$$

is a composition of three axial Borel functions.

Proof. We prove the second part of the theorem. The construction for the first part is the same and we only give the appropriate comments in brackets.

Let

$$f=(f_x,f_y):\mathbb{R}^2\to\mathbb{R}^2$$

be a Borel function ($f_x,f_y:\mathbb{R}^2\to\mathbb{R}$). Functions f_x and f_y are also Borel.

We define our first axial function

$$h_1(x,y)=\big(x,g(x,y)\big),$$

where g is from Lemma 2 (i.e., g is a Borel bijection from $\mathbb{R}^2$ to $\mathcal{C}$). This way we move all points of the plane into different y-level. Clearly h_1 is Borel, h_1 is not onto but $h_1(\mathbb{R}^2)$ is a Borel subset on the plane.
(Note that the image of h_1 is contained in the set $\mathbb{R}\times\mathcal{C}$ of both measure zero and first category.)

We define a vertical function $h_2:\mathbb{R}^2\to\mathbb{R}^2$ by putting on $h_1(\mathbb{R}^2)$

$$h_2\big(h_1(x,y)\big)=h_2\big(x,g(x,y)\big)=\big(f_x(x,y),g(x,y)\big)$$

and identity on the rest of the plane (i.e., on $\mathbb{R}^2 \setminus h_1(\mathbb{R}^2)$). We note that $(f_x, g) = h_2 \circ h_1 : \mathbb{R}^2 \to \mathbb{R}^2$ is a Borel function and h_1 is Borel and 1–1 so by Lemma 3, the function $h_2|_{h_1(\mathbb{R}^2)}$ is Borel and the entire h_2 is also Borel.
(If f_x is an arbitrary function, then h_2 is both measurable and has the Baire property as identity on a full measure and residual set $\mathbb{R}^2 \setminus h_1(\mathbb{R}^2) \subset \mathbb{R}^2 \setminus \mathbb{R} \times \mathcal{C}$. Moreover, $h_2(h_1(\mathbb{R}^2)) \subset \mathbb{R} \times \mathcal{C}$ is of measure zero and first category.)

Finally, we define $h_3 : \mathbb{R}^2 \to \mathbb{R}^2$ — a horizontal function by

$$h_3\big(f_x(x,y), g(x,y)\big) = \big(f_x(x,y), f_y(x,y)\big) \quad on\ h_2 \circ h_1(\mathbb{R}^2)$$

and identity elsewhere. We can see that

$$h_3 \circ h_2 \circ h_1 = h_3\big(f_x(x,y),\, g(x,y)\big) = \big(f_x(x,y),\, f_y(x,y)\big) = f.$$

Again, since $h_3 \circ h_2 \circ h_1$ is Borel and $h_2 \circ h_1$ is 1–1 (on $h_2 \circ h_1(\mathbb{R}^2)$), then h_3 is Borel (by Lemma 3).
(The function h_3 being identity on the set $\mathbb{R}^2 \setminus \mathbb{R} \times \mathcal{C}$ is measurable and has the Baire property for any function f.) $\square$

Eggleston [E] gave an example of a continuous homeomorphism of $\mathbb{R}^2$ that is not a composition of any number of axial homeomorphisms. He also showed (answering the question put by Ulam [S]) that every homeomorphism of a unit square can be approximated in supremum metric by a compositions of the finite number of axial homeomorphisms (but the number of these axial homeomorphisms cannot be bounded).
We do not know if the same is true for continuous functions instead of homeomorphisms.

Question. Can every continuous function $f : [0,1]^2 \to [0,1]^2$ be approximated (in supremum norm) by compositions of axial continuous functions?

(The author conjectures the answer is 'no'.)

We also do not know whether Theorem 4 is true if we require h_i to be bijections provided that f is a bijection.

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