Tatra Mt. Math. Publ. **44** (2009), 153–158
DOI: 10.2478/v10127-009-0055-0

TATRA MOUNTAINS
Mathematical Publications

A NOTE ON ϱ-UPPER CONTINUOUS FUNCTIONS

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ABSTRACT. In the present paper, we introduce the notion of classes of ϱ-upper continuous functions. We show that ϱ-upper continuous functions are Lebesgue measurable and, for $\varrho < \frac{1}{2}$, may not belong to Baire class 1. We also prove that a function with Denjoy property can be non-measurable.

1. Preliminaries

First, we shall collect some of the notions and definitions which appear frequently in the sequel. We apply standard symbols and notation. By $\mathbb{R}$ we denote the set of real numbers, by $\mathbb{N}$ we denote the set of positive integers. $\mathcal{B}_1$ denotes the set of all Baire class 1 functions. The symbol $|\cdot|$ stands for the Lebesgue measure on the real line and also for the absolute value of a real number. Throughout the paper we consider only real-valued functions defined on an open interval.

Let E be a measurable subset of $\mathbb{R}$ and let $x \in \mathbb{R}$. The numbers

$$\underline{d}^+(E,x) = \liminf_{t \to 0^+} \frac{|E \cap [x, x+t]|}{t}$$

and

$$\overline{d}^+(E,x) = \limsup_{t \to 0^+} \frac{|E \cap [x, x+t]|}{t}$$

are called the right lower density of E at x and right upper density of E at x. The left lower and upper densities of E at x are defined analogously. If

$$\underline{d}^+(E,x) = \overline{d}^+(E,x) \quad \text{and} \quad \underline{d}^-(E,x) = \overline{d}^-(E,x),$$

2000 Mathematics Subject Classification: 26A15, 54C30.
Keywords: density of a set at a point, continuous functions, measurable functions, path continuity, Denjoy property, Baire class 1.

then we call these numbers the right density and left density of E at x, respectively. The numbers

$$\overline{d}(E,x) = \limsup_{\substack{t\to 0^+\\ k\to 0^+\\ t+k\neq 0}} \frac{|E\cap [x-t,x+k]|}{k+t}$$

and

$$\underline{d}(E,x) = \liminf_{\substack{t\to 0^+\\ k\to 0^+\\ t+k\neq 0}} \frac{|E\cap [x-t,x+k]|}{k+t}$$

are called the upper and lower density of E at x, respectively. If $\overline{d}(E,x)=\underline{d}(E,x)$, we call this number the density of E at x and denote it by $d(E,x)$.

When $d(E,x)=1$, we say that x is a density point of E.

Definition 1.1. Let E be a measurable subset of $\mathbb{R}$. Let $x\in\mathbb{R}$ and $0<\varrho<1$. We say that the point x is a point of ϱ-type upper density of E if $\overline{d}(E,x)>\varrho$.

Definition 1.2. A real-valued function f defined on an open interval I is called ϱ-upper continuous at x provided that there is a measurable set $E\subset I$ such that x is a point of ϱ-type upper density of E, $x\in E$ and $f|E$ is continuous at x. If f is ϱ-upper continuous at every point of I, we say that f is ϱ-upper continuous.

We will denote the class of all ϱ-upper continuous functions defined on an open interval I by $\mathcal{UC}_\varrho$. The notion of ϱ-upper continuity is an example of so called path continuity, which was widely described in [2].

2. Main results

Theorem 2.1. *Let $0<\varrho<1$. If $f\in\mathcal{UC}_\varrho$, then f is measurable.*

Proof. Let $f\colon I\to\mathbb{R}$. Assume that $f\in\mathcal{UC}_\varrho$ and suppose that f is not measurable. Then there exists a number $a\in\mathbb{R}$ for which the set $\{x\in I\colon f(x)<a\}$ is non-measurable. Denote

$$A=\{x\in I\colon f(x)<a\},\quad B=\{x\in I\colon f(x)\geq a\}.$$

It is obvious that $B=I\setminus A$ is also non-measurable. Consider a measurable sets $A_1\subset A$, $B_1\subset B$ such that $A\setminus A_1$ and $B\setminus B_1$ do not contain a set of positive measure. Therefore, $A\setminus A_1$ and $B\setminus B_1$ are non-measurable sets. If

$$F=(A\setminus A_1)\cup(B\setminus B_1)=I\setminus(A_1\cup B_1),$$

then F is a measurable set of positive measure. Let $L(F)$ be a set of all density points of the set F. By the well-known Lebesgue Density Theorem [1], $|F\setminus L(F)|=0$. Therefore, there exists $x_0\in(A\setminus A_1)\cap L(F)$.

Since f is ϱ-upper continuous at x_0, it follows that there exists a measurable set $E \subset I$ such that $x_0 \in E$, $\overline{d}(E, x_0) > \varrho$ and $f|E$ is continuous at x_0. As $x_0 \in A$, we have $f(x_0) < a$. Therefore, it is possible to find $\delta > 0$ such that

$$E \cap (x_0 - \delta, x_0 + \delta) \subset A.$$

Let

$$E' = E \cap (x_0 - \delta, x_0 + \delta).$$

Hence $x_0 \in E'$, $f|E'$ is continuous at x_0, $E' \subset A$ and

$$\overline{d}(E', x_0) = \overline{d}(E, x_0) > \varrho > 0. \tag{2.1}$$

We have

$$E' = (E' \cap A_1) \cup \big(E' \cap (A \setminus A_1)\big).$$

Since E' and $E' \cap A_1$ are measurable, $E' \cap (A \setminus A_1)$ is measurable, too. Hence, $|E' \cap (A \setminus A_1)| = 0$. Moreover,

$$\overline{d}(E' \cap A_1, x_0) = 1 - \underline{d}\big(I \setminus (E' \cap A_1), x_0\big) \leq 1 - \underline{d}(F, x_0) = 1 - 1 = 0,$$

because $(E' \cap A_1) \cap F = \emptyset$. It follows that

$$\overline{d}(E', x_0) \leq \overline{d}(E' \cap A_1, x_0) + \overline{d}\big(E' \cap (A \setminus A_1), x_0\big) = 0 + 0 = 0,$$

contradicting (2.1). Thus, the assumption that f may be non-measurable is false. □

Definition 2.1. We say that a real-valued function f defined on an open interval I has Denjoy property at $x_0 \in I$ if for each $\varepsilon > 0$ and $\delta > 0$ the set $\big\{x \in (x_0 - \delta, x_0 + \delta) \colon |f(x) - f(x_0)| < \varepsilon\big\}$ contains a measurable subset of positive measure. We say that f has Denjoy property if it has Denjoy property at each point $x \in I$.

Remark 2.1. Let $0 < \varrho < 1$. If $f \in \mathcal{UC}_\varrho$ then f has Denjoy property.

Theorem 2.2. *There exists a non-measurable function with Denjoy property.*

Proof. First, we will construct inductively a sequence $\{A_n \colon n \geq 1\}$ of measurable sets such that

(1) $A_i \cap A_j = \emptyset$ for each $i \neq j$,

(2) $|A_n| > 0$ for each $n \geq 1$,

(3) $\forall_{(a,b) \subset \mathbb{R}}\, \exists_{n \in \mathbb{N}}\, A_{3n-2} \cup A_{3n-1} \cup A_{3n} \subset (a, b)$.

Let $\big\{(a_n, b_n)\big\}_{n \in \mathbb{N}}$ be a sequence of all open intervals with rational endpoints. Consider (a_1, b_1). There exists a closed interval $[\alpha_1, \beta_1] \subset (a_1, b_1)$ and there exist pairwise disjoint closed subintervals I_1^1, I_2^1, I_3^1 of $[\alpha_1, \beta_1]$. Let $C_{I_j^1}$ be a perfect nowhere dense set with positive measure such that $C_{I_j^1} \subset I_j^1$ for $j = 1, 2, 3$. Let

$$A_1 = C_{I_1^1}, \quad A_2 = C_{I_2^1}, \quad A_3 = C_{I_3^1}.$$

Since the set $C_{I_1^1} \cup C_{I_2^1} \cup C_{I_3^1}$ is nowhere dense, there exists a closed interval $[\alpha_2, \beta_2]$ such that

$$\alpha_2 < \beta_2 \quad \text{and} \quad [\alpha_2, \beta_2] \subset (a_2, b_2) \setminus (C_{I_1^1} \cup C_{I_2^1} \cup C_{I_3^1}).$$

Moreover, one can find pairwise disjoint closed subintervals I_1^2, I_2^2, I_3^2 of $[\alpha_2, \beta_2]$. Let $C_{I_j^2}$ be a perfect nowhere dense set with positive measure such that $C_{I_j^2} \subset I_j^2$ for $j = 1, 2, 3$. Denote

$$A_4 = C_{I_1^2}, \quad A_5 = C_{I_2^2}, \quad A_6 = C_{I_3^2}.$$

Assume that sets the sets $A_1, A_2, \ldots, A_{3(n-1)}$ are already chosen and the conditions (1) and (2) are fulfilled. Also, assume that

$$A_{3i-2} \cup A_{3i-1} \cup A_{3i} \subset (a_i, b_i) \qquad \text{for every} \quad i \leq n-1.$$

Since $A_1 \cup \ldots \cup A_{3(n-1)}$ is a nowhere dense set, there exists $[\alpha_n, \beta_n]$ such that

$$\alpha_n < \beta_n \quad \text{and} \quad [\alpha_n, \beta_n] \subset (a_n, b_n) \setminus \big(A_1 \cup \ldots \cup A_{3(n-1)}\big).$$

Fix any three closed pairwise disjoint nondegenerate intervals I_1^n, I_2^n, $I_3^n \subset [\alpha_n, \beta_n]$. Let$C_{I_j^n}$ be a Cantor set with positive measure such that $C_{I_j^n} \subset I_j^n$ for $j = 1, 2, 3$. Let

$$A_{3n-2} = C_{I_1^n}, \quad A_{3n-1} = C_{I_2^n}, \quad A_{3n} = C_{I_3^n}.$$

So, by recursion, we can construct a sequence $\{A_n \colon n \geq 1\}$ of measurable sets such that $A_i \cap A_j = \emptyset$ for each $i \neq j$, $|A_n| > 0$ for each $n \geq 1$ and

$$A_{3n-2} \cup A_{3n-1} \cup A_{3n} \subset (a_n, b_n) \qquad \text{for every} \quad n \in \mathbb{N}.$$

Choose any interval $(a, b) \subset \mathbb{R}$. There exists n_0 such that $(a_{n_0}, b_{n_0}) \subset (a, b)$. Hence,

$$A_{3n_0-2} \cup A_{3n_0-1} \cup A_{3n_0} \subset (a_{n_0}, b_{n_0}) \subset (a, b).$$

Let $B \subset \bigcup_{n=1}^{\infty} A_{3n-2}$ be any non-measurable set. Define $E = B \cup \bigcup_{n=1}^{\infty} A_{3n-1}$ and let $f \colon \mathbb{R} \to \mathbb{R}$ be a characteristic function of the set E.

Let $x \in R$. We consider two cases.

If $x \notin E$, then $f(x) = 0$. Then for each $\varepsilon > 0$ we have

$$\big\{t \colon |f(t) - f(x)| < \varepsilon\big\} \supset \bigcup_{n=1}^{\infty} A_{3n}.$$

Since

$$\left|(a, b) \cap \bigcup_{n=1}^{\infty} A_{3n}\right| > 0$$

for each $(a, b) \subset \mathbb{R}$, we have that for each $\varepsilon > 0$ and $\delta > 0$ the set $\big\{t \in (x - \delta, x + \delta) \colon |f(t) - f(x)| < \varepsilon\big\}$ contains a measurable set of positive measure.

If $x \in E$, then $f(x) = 1$. Then for each $\varepsilon > 0$ we have

$$\{t\colon |f(t) - f(x)| < \varepsilon\} \supset \bigcup_{n=1}^{\infty} A_{3n-1}.$$

Since

$$\left|(a,b) \cap \bigcup_{n=1}^{\infty} A_{3n-1}\right| > 0$$

for each $(a,b) \subset \mathbb{R}$, we have that for each $\varepsilon > 0$ and $\delta > 0$ the set $\{t \in (x-\delta, x+\delta)\colon |f(t) - f(x)| < \varepsilon\}$ contains a measurable set of positive measure.

Hence, f has Denjoy property at x. Since x was arbitrary, we conclude that f has Denjoy property.

Certainly, the set E is non-measurable and f is non-measurable. □

Theorem 2.3. *$\mathcal{UC}_\varrho \not\subset \mathcal{B}_1$ for every $\varrho < \frac{1}{2}$.*

P r o o f. We will construct a function f such that $f \in \mathcal{UC}_\varrho$ for every $\varrho < \frac{1}{2}$ and f is not in Baire class 1.

Let $C \subset [0,1]$ be a perfect nowhere dense set with positive measure such that for arbitrary interval (a,b), if $C \cap (a,b) \neq \emptyset$, then $|C \cap (a,b)| > 0$. Let

$$A = \left\{x \in C\colon \underline{d}(C,x) > \frac{1}{2}\right\}.$$

From Lebesgue Density Theorem, $|C \setminus A| = 0$. In particular, A is dense in C. On the other hand, ends of ambiguous intervals do not belong to A. Hence, the set $C \setminus A$ is dense in C, too. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a characteristic function of the set A, $f = \chi_A$. Since A and $C \setminus A$ are dense in C, we deduce that f is discontinuous at each point $x \in C$. Hence, f is not Baire 1 function.

We will show that $f \in \mathcal{UC}_\varrho$ for every $\varrho < \frac{1}{2}$.

If $x \in A$, then $f|A$ is constant, so it is continuous at x and

$$\overline{d}(A,x) = \overline{d}(C,x) \geq \underline{d}(C,x) \geq \frac{1}{2} > \varrho$$

because $|C \setminus A| = 0$.

If $x \in \mathbb{R}\setminus C$, then $f|(\mathbb{R}\setminus C)$ is continuous at x and $\mathbb{R}\setminus C$ is an open subset of $\mathbb{R}$. Hence,

$$\overline{d}(\mathbb{R} \setminus C, x) = 1 > \varrho.$$

If $x \in C \setminus A$, then $f|(\mathbb{R} \setminus A)$ is continuous at x. Since $x \notin A$, we have that $\underline{d}(C,x) \leq \frac{1}{2}$. It follows that

$$\overline{d}(\mathbb{R} \setminus A, x) = 1 - \underline{d}(A,x) = 1 - \underline{d}(C,x) \geq 1 - \frac{1}{2} = \frac{1}{2} > \varrho.$$

Hence, $f \in \mathcal{UC}_\varrho$ for every $\varrho < \frac{1}{2}$. □

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Received December 19, 2008

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