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THE SEASONAL VARIANCE AND COVARIANCE OF TIME SERIES

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In acestă lucrare vom introduce noțiunile de varianță sezonală şi covarianță sezonală , ceeace ne va permite definirea coeficienților sezonali al regresiei, similar coeficientilor regresiei şi determinarea condițiilor în care cele două noțiuni cioncid.

We will introduce the notions of seasonal variation *of a time series and* seasonal covariance *of two time series, that will permit us to obtain* the seasonal coefficient *of the regression, similar to the regression coefficient, and determine the cases when the two coefficients coincide.*

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1. Preliminary notions

In the following we will consider ST_n, the *vector space of n-time series* (the space $\mathbf{R}^n$ endowed with the canonical structure of **R**-vector space); an element of ST_n will be denoted by

$$\mathbf{X} = \{x_i\}_n = \{x_1, \ldots, x_n\},$$

and then

$$\mathbf{X} + \mathbf{Y} = \{x_i\}_n + \{y_i\}_n := \{x_i + y_i\}_n, \quad \alpha\mathbf{X} = \alpha\{x_i\}_n := \{\alpha x_i\}_n .$$

Particularly, each real number k defineds a constant time series:

$$\mathbf{K} = \{k\}_n = \{k, \ldots, k\}.$$

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1.1. Definition. a) If **X** and **Y** are two time series, we define their *product* by:

$$\mathbf{X}\cdot\mathbf{Y}=\{x_i\}_n\{y_i\}_n:=\{x_iy_i\}_n\,,$$

and their *inner product* by:

$$\langle \mathbf{X}\,|\,\mathbf{Y}\rangle := \frac{1}{n}\sum_{i=1}^{n} x_iy_i \quad .$$

The space ST_n of a *n*-time series endowed with this inner product is a Hilbert space.

b) We associate to a given time series $\mathbf{X}=\{x_i\}_n$:

- its *average*

$$\overline{\mathbf{X}}:=\langle \mathbf{X}\,|\,1\rangle=\frac{1}{n}\sum_{i=1}^{n}x_i \ ,$$

We observe that the inner product of two time series is the average of their product:

$$\langle \mathbf{X}|\mathbf{Y}\rangle=\overline{\mathbf{X}\cdot\mathbf{Y}}.$$

- its *canonical norm* (defined by the inner product)

$$\|\mathbf{X}\|=\langle \mathbf{X}\,|\,\mathbf{X}\rangle^{\frac{1}{2}}:=\sqrt{\mathbf{X}^2}=\sqrt{\frac{1}{n}\sum_{i=1}^{n}x_i^2}\ ,$$

- its *centered time series*:

$$\mathbf{nX}:=\mathbf{X}-\overline{\mathbf{X}}\ ,$$

- its *variance*:

$$\mathrm{Var}\ \mathbf{X}:=\|\,\mathbf{nX}\,\|^2=\|\,\mathbf{X}\,\|^2=(\overline{\mathbf{X}})^2=\|\,\mathbf{X}\,\|^2-\langle \mathbf{X}\,|\,1\rangle$$

c) Let **X** and **Y** be two time series; we define their *covariance* by:

$$\mathrm{Cov}\,(\mathbf{X}+\mathbf{Y})=\overline{\mathbf{X}\cdot\mathbf{Y}}-\overline{\mathbf{X}}\cdot\overline{\mathbf{Y}}=\langle \mathbf{X}\,|\,\mathbf{Y}\rangle-\langle \mathbf{X}\,|\,1\rangle\langle \mathbf{Y}\,|\,1\rangle=$$

$$=\langle \mathbf{X}\,|\,\mathbf{Y}\rangle=\langle \mathbf{X}\,|\,\langle \mathbf{Y}\,|\,1\rangle\rangle=\langle \mathbf{X}\,|\,\mathbf{nY}\rangle$$

It results that the covariance is a bilinear form that has the following properties:

$$\mathrm{Cov}\,(\mathbf{X},\mathbf{Y})=\langle \mathbf{nX}\,|\,\mathbf{Y}\rangle=\overline{\mathbf{n}\,\mathbf{X}\cdot\mathbf{Y}}$$

$$\mathrm{Cov}\,(\mathbf{X},\mathbf{nY})=\mathrm{Cov}\,(\mathbf{X},\mathbf{Y})$$

$$\mathrm{Cov}\,(\mathbf{X},\mathbf{X})=\mathrm{Var}\,\mathbf{X}$$

$$\mathrm{Var}\,(\mathbf{X}\pm\mathbf{Y})=\mathrm{Var}\,\mathbf{X}+\mathrm{Var}\,\mathbf{Y}\pm 2\,\mathrm{Cov}\,(\mathbf{X},\mathbf{Y}).$$

1.2. Definition. If $n=ks$; we will call *s the period* and *k the number of periods*. The time series **X** will be named a *periodical time series* if

$x_i = x_{s+i} = x_{2s+1} = \ldots = x_{(k-1)s+i}$, for every $i = 1,\ldots,s$; that means that a periodical time series is of the form:

$$\mathbf{C} = \{c_1,\ldots,c_s\}_n := \{c_1,\ldots,c_s,c_1,\ldots\ldots,c_s,\ldots,c_1,\ldots c_s\}.$$

The mean and the norm of such a series are:

$$\overline{\mathbf{C}} := \frac{1}{s}\sum_{i=1}^{s} c_i, \quad \|\mathbf{C}\|^2 := \frac{1}{s}\sum_{i=1}^{s} c_i^2.$$

A periodical time series with mean 0 will be named a *seasonal time series*.

1.3. Definition. Suppose that $n = ks$. At a given time series $\mathbf{X} \in \mathbf{ST}_n$ we will associate the following three time series:

a) its *periodical time series*:

$$\mathbf{cX} := \{c_1(\mathbf{X}),\ldots,c_s(\mathbf{X})\}_n = \{c_1(\mathbf{X}),\ldots,c_s(\mathbf{X}),\ldots\ldots,c_1(\mathbf{X}),\ldots c_s(\mathbf{X})\} \in \mathbf{ST}_n$$

where

$$c_i(\mathbf{X}) := \frac{1}{k}(x_i + x_{s+i} + x_{2s+i} + \ldots + x_{(k-1)s+i}) \ .$$

We observe that, generally, $\mathbf{c}(\mathbf{X})$ is not a seasonal time series (because $\overline{\mathbf{c}(\mathbf{X})} = \overline{\mathbf{X}}$).

b) its *seasonal time series*: the normed time series $\mathbf{n}(\mathbf{cX})$ of its periodical time series,

c) its *deseasoned time series*:

$$\mathbf{sX} := \mathbf{X} - \mathbf{cX} \ .$$

1.4. Proposition. a) The following functions are linear:

$$\mathbf{c} : \mathrm{ST}_n \to \mathrm{ST}_s \quad \mathbf{X} \mapsto \mathbf{cX},$$

$$\mathbf{s} : \mathrm{ST}_n \to \mathrm{ST}_s \quad \mathbf{X} \mapsto \mathbf{sX}.$$

b) The deseasoned time series **sX** and the periodical time series **cX** (associated to the time series **X**) are orthogonal:

$$\langle \mathbf{sX} \mid \mathbf{cX} \rangle = 0 \,.$$

Proof.

b) Indeed:

$$\langle \mathbf{sX} \mid \mathbf{cX} \rangle = \langle \mathbf{X} - \mathbf{cX} \mid \mathbf{cX} \rangle = \overline{\mathbf{X} \cdot \mathbf{cX}} - \overline{\mathbf{cX} \cdot \mathbf{cX}} = 0.$$

1.5. Proposition. Suppose that $n = ks$, and consider a time series $\mathbf{X} \in \mathrm{ST}_n$. Then:

a) its periodical time series $\mathbf{cX}$ has the following properties:

$$\overline{\mathbf{cX}} = \overline{\mathbf{X}} \qquad \mathbf{c}(\mathrm{n}\mathbf{X}) = \mathrm{n}(\mathbf{cX})$$

$$\|\mathbf{c}\mathrm{n}\mathbf{X}\|^2 = \mathrm{Var}(\mathbf{cX}) = \|\mathbf{cX}\|^2 - (\overline{\mathbf{X}})^2 \qquad \mathbf{c}(\mathbf{C}) = \mathbf{C}$$

$$\overline{\mathbf{X}\cdot\mathbf{cY}} = \overline{\mathbf{cX}\cdot\mathbf{Y}} = \overline{\mathbf{cX}\cdot\mathbf{cY}} \qquad \overline{\mathbf{X}\cdot\mathbf{cX}} = \overline{\mathbf{cX}\cdot\mathbf{cX}} = \|\mathbf{cX}\|^2$$

$$\mathrm{Cov}\,(\mathbf{X},\mathbf{cX}) = \overline{\mathbf{X}\cdot\mathbf{cX}} - \overline{\mathbf{X}}\cdot\overline{\mathbf{cX}} = \quad \|\mathbf{cX}\|^2 - (\overline{\mathbf{cX}})^2 = \mathrm{Var}\,(\mathbf{cX})$$

b) its deseasoned time series $\mathbf{sX}$ has the following properties:

$$\overline{\mathbf{sX}} = 0;\ \mathrm{n}(\mathbf{sX}) = \mathbf{sX}$$

$$\|\mathbf{sX}\|^2 = \mathrm{Var}(\mathbf{X}) - \mathrm{Var}(\mathrm{c}\mathbf{X}) = \|\mathbf{X}\|^2 - \|\mathrm{c}\mathbf{X}\|^2$$

$$\mathrm{Var}\,\mathbf{sX} = \|\mathbf{X}\|^2; \mathbf{s}(\mathrm{n}\mathbf{X}) = \mathrm{n}(\mathbf{sX}) = \mathbf{sX}$$

$$\overline{\mathbf{sX}\cdot\mathbf{sY}} = \overline{\mathbf{X}\cdot\mathbf{Y}} - \overline{\mathrm{c}\mathbf{X}\cdot\mathrm{c}\mathbf{Y}} = \mathrm{Cov}\,(\mathbf{X},\mathbf{Y}) - \mathrm{Cov}\,(\mathrm{c}\mathbf{X},\mathrm{c}\mathbf{Y}) \quad .$$

1.6. Lemma. Suppose that $n = ks$. A time series $\mathbf{X} \in \mathrm{ST}_n$ is a periodical (seasonal) time series if and only if it coincides with its periodical (seasonal) time series.

1.7. Proposition. a) For every time series $\mathbf{X} \in \mathrm{ST}_n$ we have:

$$\mathrm{Var}\,(\mathbf{sX}) = \|\mathbf{X} - \mathbf{cX}\|^2 - \overline{\mathbf{X}\cdot\mathbf{cX}}^2 = \|\mathbf{X}\|^2 + \|\mathbf{cX}\|^2 - 2\overline{\mathbf{X}\cdot\mathbf{cX}} =$$
$$= \|\mathbf{X}\|^2 - \|\mathbf{cX}\|^2 = \mathrm{Var}\,\mathbf{X} - \mathrm{Var}\,\mathbf{cX},$$

b) For every time series $\mathbf{X},\mathbf{Y} \in \mathrm{ST}_n$ we have:

$$\mathrm{Cov}\,(\mathbf{sX},\mathbf{sY}) = \mathrm{Cov}\,(\mathbf{sX},\mathbf{Y}) = \mathrm{Cov}\,(\mathbf{X},\mathbf{sY}) = \overline{\mathbf{sX}\cdot\mathbf{sY}} = \overline{\mathbf{X}\cdot\mathbf{Y}} - \overline{\mathbf{cX}\cdot\mathbf{cY}} =$$
$$= \mathrm{Cov}\,(\mathbf{X},\mathbf{Y}) - \mathrm{Cov}\,(\mathbf{cX},\mathbf{cY}).$$

2. The seasonal variation and covariance

2.1. Definition. Suppose that $n = ks$. We will define
a) the *seasonal variance* of a time series $\mathbf{X} \in \mathrm{ST}_n$ by

$$\mathrm{Var}_c\,\mathbf{X} := \mathrm{Var}\,\mathbf{sX},$$

b) the *seasonal covariance* of the time series $\mathbf{X}, \mathbf{Y} \in \mathrm{ST}_n$ by

$$\mathrm{Cov}_c\, \mathbf{X}, \mathbf{Y} := \mathrm{Cov}\,(\mathbf{sX}, \mathbf{sY}) \ .$$

2.2. Proposition. a) The function defined by the seasonal variance: $\mathrm{Var}_c : \mathrm{ST}_n \to \mathbf{R}$, $(\mathbf{X} \mapsto \mathrm{Var}_c(\mathbf{X}))$ verifies:

$$\mathrm{Var}_c(\mathbf{X}+\mathbf{Y}) := \mathrm{Var}_c(\mathbf{X}) + \mathrm{Var}_c(\mathbf{Y}) + 2\,\mathrm{Cov}_c(\mathbf{X}, \mathbf{Y})$$

b) The function defined by the seasonal covariance:

$$\mathrm{Cov}_c\, \mathrm{ST}_n \times \mathrm{ST}_n \to \mathbf{R}, \quad ((\mathbf{X}, \mathbf{Y}) \mapsto \mathrm{Cov}\,(\mathbf{X}, \mathbf{Y}))$$

is bilinear.

Proof. a) Indeed:

$$\mathrm{Var}_c(\mathbf{X}+\mathbf{Y}) = \mathrm{Var}(\mathbf{X}+\mathbf{Y}) - \mathrm{Var}(\mathbf{cX}+\mathbf{cY}) =$$
$$= \mathrm{Var}\,\mathbf{X} + \mathrm{Var}\,\mathbf{Y} + 2\,\mathrm{Cov}(\mathbf{X}, \mathbf{Y}) - (\mathrm{Var}\,\mathbf{cX} + \mathrm{Var}\,\mathbf{cY} + 2\,\mathrm{Cov}(\mathbf{cX}, \mathbf{cY})) =$$
$$= \mathrm{Var}_c(\mathbf{X}) + \mathrm{Var}_c(\mathbf{Y}) + 2\,\mathrm{Cov}_c(\mathbf{X}, \mathbf{Y}),$$

b) It results from the bilinearity of the covariance and from the linearity of the function **s**.

2.3. Remark. a) Every time series **X** verifies:

$$\mathrm{Var}\,\mathbf{X} = \mathrm{Var}\,\mathbf{cX} + \mathrm{Var}_c(\mathbf{X})$$
$$\mathrm{Var}_c(\mathbf{X}) = \| \mathbf{sX} \|^2 .$$

b) For every time series **X**, the following statements are equivalent:

$$\mathrm{Var}_c(\mathbf{X}) = \mathrm{Var}\,\mathbf{X},$$
$$\mathrm{Var}\,\mathbf{cX} = 0,$$
$$\| \mathbf{cX} \|^2 = \overline{\mathbf{X}}^2 .$$

c) For every time series **X** and **Y** we have:

$$\mathrm{Cov}(\mathbf{X}, \mathbf{Y}) = \mathrm{Cov}\,(\mathbf{cX}, \mathbf{cY}) + \mathrm{Cov}_c(\mathbf{X}, \mathbf{Y})$$
$$\mathrm{Cov}_c(\mathbf{X}, \mathbf{Y}) = \overline{\mathbf{X} \cdot \mathbf{Y}} - \overline{\mathbf{cX} \cdot \mathbf{cY}}$$

d) For every time series **X** and **Y** the following statements are equivalent:

$$\mathrm{Cov}_c(\mathbf{X}, \mathbf{Y}) = \mathrm{Cov}\,(\mathbf{cX}, \mathbf{cY}),$$
$$\mathrm{Cov}\,(\mathbf{cX}, \mathbf{cY}) = 0,$$
$$\overline{\mathbf{cX} \cdot \mathbf{cY}} = \overline{\mathbf{cX}} \cdot \overline{\mathbf{cY}} \ ,$$

e) The seasonal variance of a time series **X** coincides with the seasonal covariance of **X** with itself:

$$\mathrm{Var}_c(\mathbf{X}) = \mathrm{Cov}_c(\mathbf{X}, \mathbf{X}) .$$

2.4. Remark. **X** is a periodical time series if and only if its seasonal variance vanishes.

Proof. It is evidently that if **X** is a periodical time series then $\mathbf{X}=\mathbf{cX}$ and it results that $\mathrm{Var}_c(\mathbf{X})=\mathrm{Var}\,\mathbf{X}-\mathrm{Var}\,\mathbf{cX}=0$.

Conversely, suppose that

$$\mathrm{Var}_c(\mathbf{X})=\mathrm{Var}\,\mathbf{sX}=\|\,\mathbf{cX}\,\|^2=0 \ ,$$

then $\mathbf{sX}=\mathbf{X}-\mathbf{cX}=0$ and it results that **X** is a periodical time series.

2.5. Corollary. The time series **X** is *not periodical* if

$$\mathrm{Var}_c(\mathbf{X})=\mathrm{Var}\,\mathbf{sX}\neq 0 \ ,$$

or equivalently if and only if

$$\mathrm{Var}\,\mathbf{X}\neq\mathrm{Var}\,\mathbf{cX}.$$

2.6. Definition. Suppose that $n=ks$. We will say that the time series **X** and **Y** are *seasonal correlated*, if their seasonal covariance is zero:

$$\mathrm{Cov}_c(\mathbf{X},\mathbf{Y})=0$$

or equivalently if

$$\mathrm{Cov}(\mathbf{X},\mathbf{Y})=\mathrm{Cov}(\mathbf{cX},\mathbf{cY}).$$

2.7. Proposition. Let $n=ks$. For every time series $\mathbf{X},\mathbf{Y}$ the following statements are equivalent:

1) **X** and **Y** are seasonal correlated: $\mathrm{Cov}_c(\mathbf{X},\mathbf{Y})=0$,

2) $\mathrm{Cov}_c(\mathbf{X},\mathbf{Y})=\mathrm{Cov}(\mathbf{X},\mathbf{Y})$,

3) $\mathrm{Cov}(\mathbf{cX},\mathbf{cY})=0$,

4) $\overline{\mathbf{cX}\cdot\mathbf{cY}}=\overline{\mathbf{X}\cdot\mathbf{Y}}$.

3. The regression with trend and seasonal component

We know (see [2]) that for the time series with a trend **U** and a random component $\boldsymbol{\varepsilon}$:

$$\mathbf{X}=\mathbf{a}+\mathbf{bU}+\boldsymbol{\varepsilon} \ ,$$

the coefficients given by the regression are:

$$\hat{\mathbf{b}}_{\mathbf{U}}=\frac{\mathrm{Cov}(\mathbf{U},\mathbf{X})}{\mathrm{Var}(\mathbf{U})}, \quad \hat{\mathbf{a}}_{\mathbf{U}}=\mathbf{X}-\hat{\mathbf{b}}_{\mathbf{U}}\overline{\mathbf{U}} \ .$$

The following Proposition will give us similar formulas in the case when the time series has also a seasonal component:

$$\mathbf{X}=\mathbf{a}+\mathbf{bU}+\mathbf{C}+\boldsymbol{\varepsilon}.$$

3.1. Proposition. If $\mathbf{X} = \mathbf{a} + \mathbf{b}\mathbf{U} + \mathbf{C} + \boldsymbol{\varepsilon}$, ($\mathbf{U}$ being nonconstant and nonperiodical ($\mathbf{U} \neq \mathbf{cU}$) time series), then

$$\hat{\mathbf{a}}_{\mathbf{U+C}} = \overline{\mathbf{X}} - \hat{\mathbf{b}}_{\mathbf{U+C}}\overline{\mathbf{U}}, \quad \hat{\mathbf{b}}_{\mathbf{U+C}} = \frac{\mathrm{Cov}_c(\mathbf{U},\mathbf{X})}{\mathrm{Var}_c(\mathbf{U})},$$
$$\hat{\mathbf{C}}_{\mathbf{C+U}} = \mathbf{nc}(\mathbf{X} - \hat{\mathbf{b}}_{\mathbf{U+C}}\mathbf{U}),$$

and the regressor of $\mathbf{X}$ is

$$\hat{\mathbf{X}}_{\mathbf{C+U}} = \hat{\mathbf{b}}_{\mathbf{U+C}}\mathbf{sU} + \mathbf{cX} .$$

Proof. From $\overline{\mathbf{X}} = \mathbf{a} + \mathbf{b}\overline{\mathbf{U}}$ it results $\mathbf{a} = \overline{\mathbf{X}} - \mathbf{b}\overline{\mathbf{U}}$ and then the time series $\mathbf{Y}$ verifies:

$$\mathbf{Y} = \mathbf{nX} = \mathbf{bnU} + \mathbf{C} + \boldsymbol{\varepsilon}.$$

We have to minimize the function:

$$\Phi = \| \mathbf{Y} - \mathbf{bnU} - \mathbf{C} \|^2 - \lambda\overline{\mathbf{C}} = \| \mathbf{Y} \|^2 + \mathbf{b}^2 \,\mathrm{Var}\,\mathbf{U} + \| \mathbf{C} \|^2 - 2\mathbf{b}\,\mathrm{Cov}(\mathbf{U},\mathbf{Y}) -$$
$$-2\overline{\mathbf{C}\cdot\mathbf{Y}} + 2\mathbf{b}\,\mathrm{Cov}(\mathbf{U},\mathbf{C}) - \lambda\overline{\mathbf{C}} = \| \mathbf{Y} \|^2 + \mathbf{b}^2 \,\mathrm{Var}\,\mathbf{U} + \frac{1}{s}\sum_{i=1}^{s} c_i^2 -$$
$$-2\mathbf{b}\,\mathrm{Cov}(\mathbf{U},\mathbf{Y}) - \frac{2}{s}\sum_{i=1}^{s} c_i c_i(\mathbf{Y}) + 2\mathbf{b}\left(\frac{1}{s}\sum_{i=1}^{s} c_i c_i(\mathbf{U}) - \frac{1}{s}\overline{\mathbf{U}}\sum_{i=1}^{s} c_i\right) - \frac{\lambda}{s}\sum_{i=1}^{s} c_i$$

when $\overline{\mathbf{C}} = 0$.

Because:

$$\begin{cases} \dfrac{\partial\Phi}{\partial \mathrm{b}} = 2\mathbf{bYC}\,\mathrm{Var}\,\mathbf{U} - 2\,\mathrm{Cov}\,(\mathbf{U},\mathrm{Y}\mathbf{Y}) + 2\,\mathrm{Cov}\,(\mathbf{U},\mathrm{C}) \\ \dfrac{\partial\Phi}{\partial c_i} = \dfrac{2}{s}c_i - \dfrac{2}{s}c_i(\mathbf{Y}) + \dfrac{2\mathbf{b}}{s}(c_i(\mathbf{U}) - \overline{\mathbf{U}}) - \dfrac{\lambda}{s} \end{cases}$$

we have to solve the system:

$$\begin{cases} 2\mathbf{b}\,\mathrm{Var}\,\mathbf{U} - 2\,\mathrm{Cov}\,(\mathbf{U},\mathbf{Y}) + 2\,\mathrm{Cov}\,(\mathbf{U},\mathbf{C}) = 0 \\ \dfrac{2}{s}c_i - \dfrac{2}{s}c_i(\mathbf{Y}) + \dfrac{2\mathbf{b}}{s}(c_i(\mathbf{U}) - \overline{\mathbf{U}}) - \dfrac{\lambda}{s} = 0, \quad (i = 1,\ldots,s), \\ \dfrac{1}{s}\sum_{i=1}^{s} c_i = 0. \end{cases}$$

From the first equation we obtain

$$\hat{\mathbf{b}}_{\mathbf{U+C}}\,\mathrm{Var}\,\mathbf{U} = \mathrm{Cov}\,(\mathbf{U},\mathbf{Y}) - \mathrm{Cov}\,(\mathbf{U},\hat{\mathbf{C}}_{\mathbf{U+C}})$$

and from the second

$$\hat{\mathbf{C}}_{\mathbf{U+C}} - \mathbf{c}(\mathbf{Y}) + \hat{\mathbf{b}}_{\mathbf{U+C}}\mathbf{ncU} = \frac{\hat{\lambda}}{2} \ .$$

Then $\frac{\hat{\lambda}}{2} = -\overline{\mathbf{c}(\mathbf{Y})}$ and $\hat{\mathbf{C}}_{\mathbf{U+C}} = \mathbf{nc}(\mathbf{Y} - \hat{\mathbf{b}}_{\mathbf{U+C}}\mathbf{U}) = \mathbf{nc}(\mathbf{X} - \hat{\mathbf{b}}_{\mathbf{U+C}}\mathbf{U})$.

Because

$$\mathrm{Cov}\,(\mathbf{U},\hat{\mathbf{C}}_{\mathbf{U+C}}) = \overline{(\mathbf{nc}(\mathbf{Y} - \hat{\mathbf{b}}_{\mathbf{U+C}}\mathbf{U}))\cdot\mathbf{U}} = \mathrm{Cov}\,(\mathbf{U},\mathbf{cY}) - \hat{\mathbf{b}}_{\mathbf{U+C}}(\mathbf{X})\,\mathrm{Var}\,(\mathbf{cU}),$$

and

$$\hat{\mathbf{b}}_{\mathbf{U+C}}[\mathrm{Var}\,\mathbf{U} - \mathrm{Var}\,(\mathbf{cU})] = \mathrm{Cov}\,(\mathbf{U},\mathbf{Y}) - \mathrm{Cov}\,(\mathbf{U},\mathbf{cY}) = \mathrm{Cov}\,(\mathbf{sU},\mathbf{sY}),$$

we have

$$\hat{\mathbf{b}}_{\mathbf{U+C}} = \frac{\mathrm{Cov}\,(\mathbf{sU},\mathbf{sX})}{\mathrm{Var}\,(\mathbf{sU})} = \frac{\mathrm{Cov}_c\,(\mathbf{U},\mathbf{X})}{\mathrm{Var}_c\,(\mathbf{U})} \ .$$

4. Example

We will consider the classical time series of the International Airline Passengers (1949-1960) (see [1]):

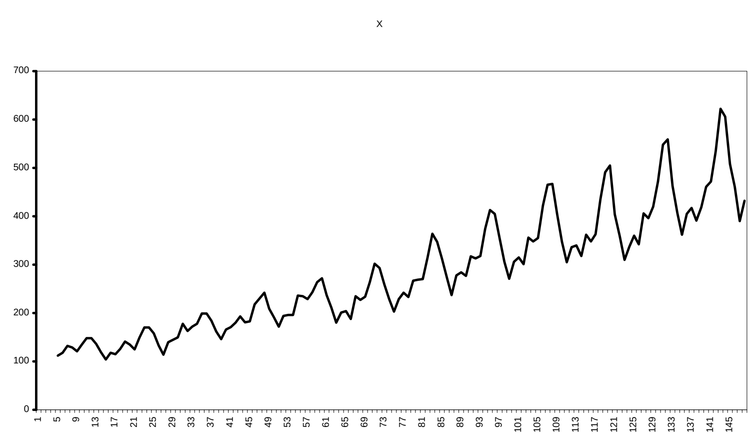

Fig. 1

and will consider that it is of the form:

$$\mathbf{X = U \cdot C \cdot \varepsilon},$$

where **U** is a trend.
Let **R** be the associated time series:

$$\mathbf{R} = \ln \mathbf{X} = \ln \mathbf{U} + \ln \mathbf{C} + \ln \boldsymbol{\varepsilon} .$$

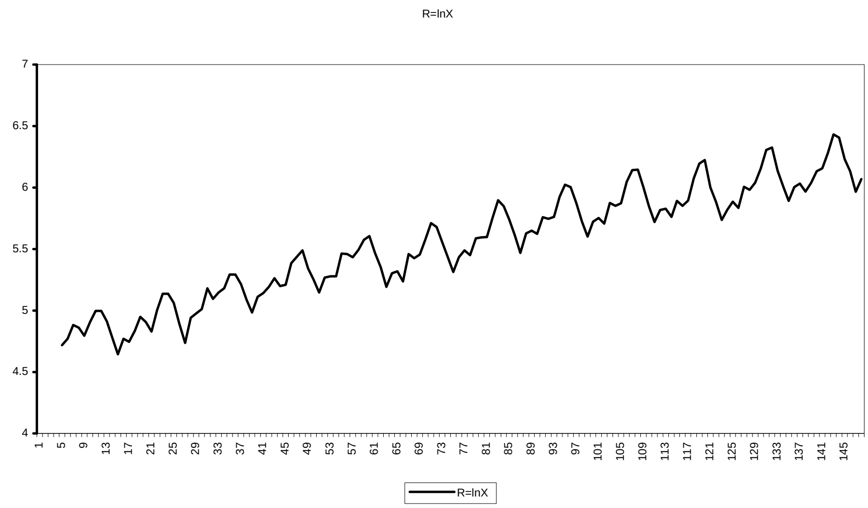

Fig. 2

We will suppose that

$$\mathbf{U}' = \ln \mathbf{U} = \mathbf{a} + \mathbf{bN}, \quad ,$$

where $\mathbf{N} = \{1,2,3,\ldots,n\}$ is a line-trend, and will denote $\mathbf{C}' = \ln \mathbf{C}$. Then we obtain:

$$\hat{\mathrm{b}}_{\mathrm{U'+C'}} = \frac{\mathrm{Cov}_c(\mathbf{U}',\mathbf{N})}{\mathrm{Var}_c(\mathbf{N})} = \frac{\mathrm{Cov}_c(\mathrm{s}\mathbf{U}',\mathrm{s}\mathbf{N})}{\mathrm{Var}_c(\mathrm{s}\mathbf{N})} = \frac{17.28}{1716} = 0.01, \mathbf{a} = \overline{\mathbf{R}} - \mathbf{b}\overline{\mathbf{N}} = \text{-}4.81.,$$

$$\mathbf{C}' = \mathbf{nc(R - bN)} =$$

$$= \{-0.09; -0.11; 0.02; -0.01; -0.01; 0.11; 0.22; 0.21; 0.06; -0.08; -0.22; -0.11\}.$$

The graphic of $\hat{\mathbf{R}}_{\mathbf{U'+C'}} = \hat{\mathbf{b}}_{\mathbf{U'+C'}}\mathbf{sN} + \mathbf{cR}$ versus **R**, the line $\mathbf{a} + \mathbf{bN}$, the seasonal component $\mathbf{C}'$ and the error are shown in Figure 3, and the graphic of $\mathbf{X}\hat{}$ versus

X and the errors shown are in Fig. 4.

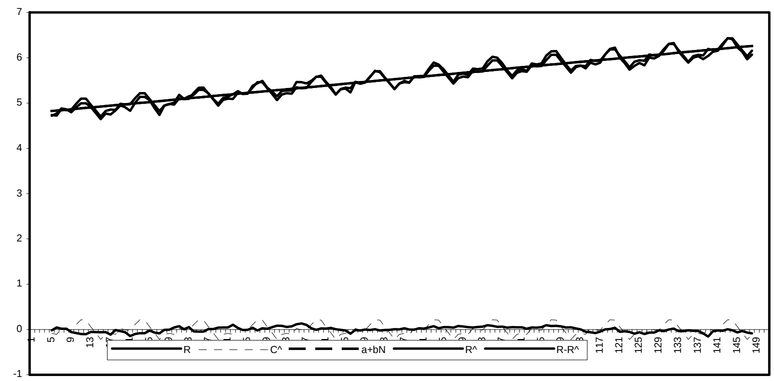

Fig. 3

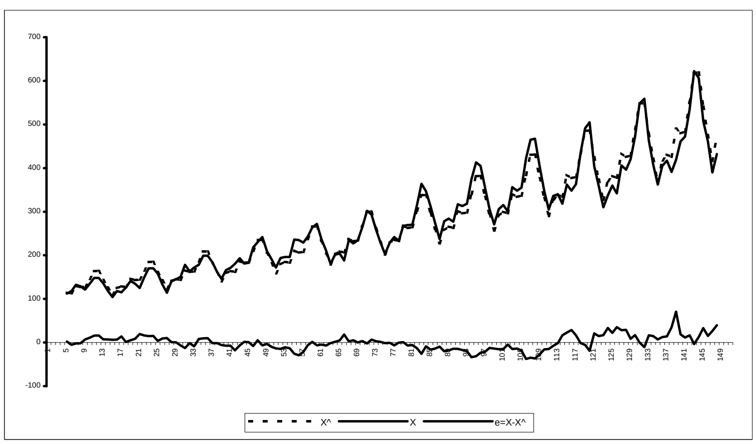

Fig. 4

R E F E R E N C E S

1. *G.E.P. Box and G.C. Jenkins*, Time Series Analysis: Forecasting and Control, Holden Day, San Francisco (2-nd edition), 1976.
2. *P.J. Brockwell and R.A. Davis*, Time Series: Theory and Methods, Springer-Verlag-New York-Berlin-Heidelberg, 1983.