Stat 331 Selected Homework 8 Solutions

Problem 3.58

Let $X_i = \begin{cases} 1 & \text{If } i^{th} \text{ couple remains} \\ 0 & \text{Otherwise} \end{cases}$ where i = 1, 2, . . . , N

Because we have 2N individuals and n deaths:

$$E(X_i) = P\{X_i = 1\} = \frac{\binom{2N-2}{n}}{\binom{2N}{n}}$$

We want $E\left(\sum_{i=1}^{N} X_i\right) = N \cdot E(X_i) = \frac{(2N-n)(2N-n-1)}{2(2N-1)}$

Problem 3.79

Let N: = Number of Rallies until the next point is scored
Let A: = Event that Ann wins next rally
Let B: = Event that Ann loses next rally and wins that one after that

Then μ = E(N| A)P(A) + E(N|B)P(A) = $1 \cdot p + (1+\mu)\cdot(1-p)$, solving for μ, $\mu = \frac{1}{p}$

Problem 3.83

Let E(n) : = Expected wait for n consecutive Heads. To get n consecutive heads, you must get n – 1 consecutive heads. After n – 1 consecutive heads the following can happen:

- You can get another head with probability ½
- You can get a tail with probability ½ and then start over

Thus we can write:

E(n) = ½(E(n-1) + 1) + ½(E(n-1) + 1 + E(n))

Solving For E(n) we get E(n) = 2 E(n-1) +2

Proof by induction that $E(n) = 2^{n+1} - 2$:

First Show it's true for n = 1, E(1) = 2, this is true because the expected wait time for an event with probability p is 1/p, thus E(1) = 1/(1/2) = 2.

Assume that the above holds for n and show that it is true for n = k+1:

$$\begin{aligned} E(k+1) &= 2E(k)+2 \\ &= 2(2^{k+1}-2)+2 \\ &= 2^{k+2}-4+2 \\ &= 2^{(k+1)+1}-2 \end{aligned}$$

Therefore $E(n) = 2^{n+1} - 2$

Extra Problems

Problem 3

From Class $\overline{X} \mid H_0 \sim N\left(0, \sigma^2/n\right)$ and $\overline{X} \mid H_1 \sim N\left(A, \sigma^2/n\right)$

a) $P_F = P(\overline{X} > \gamma \mid H_1) = 1 - P(\overline{X} \le \gamma \mid H_0) = 1 - \Phi\left(\frac{\gamma}{\sigma/\sqrt{n}}\right) \Rightarrow \gamma = \frac{\sigma}{\sqrt{n}}\Phi^{-1}(1-\alpha)$

b) $P_M = P(\overline{X} < \gamma \mid H_1) = \Phi\left(\frac{\gamma - A}{\sigma/\sqrt{n}}\right) = \Phi\left(\Phi^{-1}(1-\alpha) - \frac{A\sqrt{n}}{\sigma}\right)$

c) $P_M = P(\overline{X} < \gamma \mid H_1) = \Phi\left(\Phi^{-1}(1-\alpha) - \sqrt{SNR}\right)$, thus higher SNR means smaller P_M. The name is appropriate since larger SNR $\Leftrightarrow$ larger n, A or smaller σ^2.