6 Borel sets in the light of analytic sets

Analytic sets shed new light on Borel sets. Standard Borel spaces are somewhat similar to compact topological spaces.

6a Separation theorem

The first step toward deeper theory of Borel sets.

6a1 Theorem. [1] For every pair of disjoint analytic[2] subsets A, B of a countably separated measurable space $(X, \mathcal{A})$ there exists $C \in \mathcal{A}$ such that $A \subset C$ and $B \subset X \setminus C$.

Rather intriguing: (a) the Borel complexity of C cannot be bounded apriori; (b) the given A, B give no clue to any Borel complexity. How could it be proved?

We say that A is separated from B if $A \subset C$ and $B \subset X \setminus C$ for some $C \in \mathcal{A}$.

6a2 Core exercise. If A_n is separated from B for each $n = 1, 2, \dots$ then $A_1 \cup A_2 \cup \dots$ is separated from B.

Prove it.

6a3 Core exercise. If A_m is separated from B_n for all m, n then $A_1 \cup A_2 \cup \dots$ is separated from $B_1 \cup B_2 \cup \dots$

Prove it.

[1] "The first separation theorem for analytic sets", or "the Lusin separation theorem"; see Srivastava, Sect. 4.4 or Kechris, Sect. 14.B. To some extent, it is contained implicitly in the earlier Souslin's proof of Theorem 6a6.

[2] Recall 5d9.

6a4 Core exercise. It is sufficient to prove Theorem 6a1 for $X = \mathbb{R}$, $\mathcal{A} = \mathcal{B}(\mathbb{R})$.

Prove it.

Proof of Theorem 6a1. According to 6a4 we assume that $X = \mathbb{R}$, $\mathcal{A} = \mathcal{B}(\mathbb{R})$. By 5d10 we take Polish spaces Y, Z and continuous maps $f : Y \to \mathbb{R}$, $g : Z \to \mathbb{R}$ such that $A = f(Y)$, $B = g(Z)$. Similarly to the proof of 4c9 we choose a compatible metric on Y and a countable base $\mathcal{E} \subset 2^Y$ consisting of bounded sets; and similarly $\mathcal{F} \subset 2^Z$.

Assume the contrary: $f(Y) = A$ is not separated from $g(Z) = B$. Using 6a3 we find $U_1 \in \mathcal{E}$, $V_1 \in \mathcal{F}$ such that $f(U_1)$ is not separated from $g(V_1)$. Using 6a3 again we find $U_2 \in \mathcal{E}$, $V_2 \in \mathcal{F}$ such that $\overline{U}_2 \subset U_1$, $\operatorname{diam} U_2 \le 0.5 \operatorname{diam} U_1$, $\overline{V}_2 \subset V_1$, $\operatorname{diam} V_2 \le 0.5 \operatorname{diam} V_1$, and $f(U_2)$ is not separated from $g(V_2)$. And so on.

We get $\overline{U}_1 \supset U_1 \supset \overline{U}_2 \supset U_2 \supset \dots$ and $\operatorname{diam} U_n \to 0$; by completeness, $U_1 \cap U_2 \cap \dots = \{y\}$ for some $y \in Y$. Similarly, $V_1 \cap V_2 \cap \dots = \{z\}$ for some $z \in Z$. We note that $f(y) \ne g(z)$ (since $f(y) \in A$ and $g(z) \in B$) and take $\varepsilon > 0$ such that $\big(f(y) - \varepsilon, f(y) + \varepsilon\big) \cap \big(g(z) - \varepsilon, g(z) + \varepsilon\big) = \emptyset$. Using continuity we take n such that $f(U_n) \subset \big(f(y) - \varepsilon, f(y) + \varepsilon\big)$ and $g(V_n) \subset \big(g(z) - \varepsilon, g(z) + \varepsilon\big)$; then $f(U_n)$ is separated from $g(V_n)$, — a contradiction. □

6a5 Corollary. Let $(X, \mathcal{A})$ be a countably separated measurable space, and $A \subset X$ an analytic set. If $X \setminus A$ is also an analytic set then $A \in \mathcal{A}$.

Proof. Follows immediately from Theorem 6a1. □

6a6 Theorem. (Souslin) Let $(X, \mathcal{A})$ be a standard Borel space. The following two conditions on a set $A \subset X$ are equivalent:[1]

(a) $A \in \mathcal{A}$;

(b) both A and $X \setminus A$ are analytic.

Proof. (b)$\Longrightarrow$(a): by 6a5; (a)$\Longrightarrow$(b): by 5d9 and 2b11(a). □

6b Borel bijections

An invertible homomorphism is an isomorphism, which is trivial. An invertible Borel map is a Borel isomorphism, which is highly nontrivial.

6b1 Core exercise. Let $(X, \mathcal{A})$ be a standard Borel space, $(Y, \mathcal{B})$ a countably separated measurable space, and $f : X \to Y$ a measurable bijection. Then f is an isomorphism (that is, f^{-1} is also measurable).[2]

[1] See also Footnote 1 on page 70.

[2] A topological counterpart: a continuous bijection from a compact space to a Hausdorff space is a homeomorphism (that is, the inverse map is also continuous).

Prove it.

6b2 Corollary. A measurable bijection between standard Borel spaces is an isomorphism.

6b3 Corollary. Let $(X, \mathcal{A})$ be a standard Borel space and $\mathcal{B} \subset \mathcal{A}$ a countably separated sub-σ-algebra; then $\mathcal{B} = \mathcal{A}$.[1] [2]

Thus, standard σ-algebras are never comparable.[3]

6b4 Core exercise. Let R_1, R_2 be Polish topologies on X.

(a) If R_2 is stronger than R_1 then $\mathcal{B}(X, R_1) = \mathcal{B}(X, R_2)$ (that is, the corresponding Borel σ-algebras are equal).

(b) If R_1 and R_2 are stronger than some metrizable (not necessarily Polish) topology then $\mathcal{B}(X, R_1) = \mathcal{B}(X, R_2)$.

Prove it.

A lot of comparable Polish topologies appeared in Sect. 3c. Now we see that the corresponding Borel σ-algebras must be equal. Another example: the strong and weak topologies on the unit ball of a separable infinite-dimensional Hilbert space. This is instructive: the structure of a standard Borel space is considerably more robust than a Polish topology.

In particular, we upgrade Theorem 3c12 (as well as 3c15 and 3d1).

6b5 Theorem. For every Borel subset B of the Cantor set X there exists a Polish topology R on X, stronger than the usual topology on X, such that B is clopen in (X, R), and $\mathcal{B}(X, R)$ is the usual $\mathcal{B}(X)$.

Here is another useful fact.

6b6 Core exercise. Let $(X, \mathcal{A})$ be a standard Borel space. The following two conditions on $A_1, A_2, \dots \in \mathcal{A}$ are equivalent:

(a) the sets $A_1, A_2, \dots$ generate $\mathcal{A}$;

(b) the sets $A_1, A_2, \dots$ separate points.

Prove it.

The graph of a map $f : X \to Y$ is a subset $\{(x, f(x)) : x \in X\}$ of $X \times Y$. Is measurability of the graph equivalent to measurability of f?

6b7 Proposition. Let $(X, \mathcal{A}), (Y, \mathcal{B})$ be measurable spaces, $(Y, \mathcal{B})$ countably separated, and $f : X \to Y$ measurable; then the graph of f is measurable.

---

[1] A topological counterpart: if a Hausdorff topology is weaker than a compact topology then these two topologies are equal.

[2] See also the footnote to 5d7.

[3] Similarly to compact Hausdorff topologies.

6b8 Core exercise. It is sufficient to prove Prop. 6b7 for $Y = \mathbb{R}$, $\mathcal{B} = \mathcal{B}(\mathbb{R})$.
Prove it.

Proof of Prop. 6b7. According to 6b8 we assume that $Y = \mathbb{R}$, $\mathcal{B} = \mathcal{B}(\mathbb{R})$. The map

$$X \times \mathbb{R} \ni (x, y) \mapsto f(x) - y \in \mathbb{R}$$

is measurable (since the map $\mathbb{R} \times \mathbb{R} \ni (z, y) \mapsto z - y \in \mathbb{R}$ is). Thus, $\{(x, y) : f(x) - y = 0\}$ is measurable. $\square$

Here is another proof, not using 6b8.

Proof of Prop. 6b7 (again). We take $B_1, B_2, \dots \in \mathcal{B}$ that separate points and note that

$$y = f(x) \iff (x, y) \in \bigcap_n \Big(\big(f^{-1}(B_n) \times B_n \big) \cup \big((X \setminus f^{-1}(B_n)) \times (Y \setminus B_n) \big) \Big)$$

since $y = f(x)$ if and only if $\forall n \, \big(\, y \in B_n \iff f(x) \in B_n \, \big)$. $\square$

6b9 Extra exercise. If a measurable space $(Y, \mathcal{B})$ is not countably separated then there exist a measurable space $(X, \mathcal{A})$ and a measurable map $f : X \to Y$ whose graph is not measurable.
Prove it.

6b10 Proposition. Let $(X, \mathcal{A}), (Y, \mathcal{B})$ be standard Borel spaces and $f : X \to Y$ a function. If the graph of f is measurable then f is measurable.

Proof. The graph $G \subset X \times Y$ is itself a standard Borel space by 2b11. The projection $g : G \to X$, $g(x, y) = x$, is a measurable bijection. By 6b2, g is an isomorphism.Thus, $f^{-1}(B) = g\big(G \cap (X \times B)\big) \in \mathcal{A}$ for $B \in \mathcal{B}$. $\square$

Here is a stronger result.

6b11 Proposition. Let $(X, \mathcal{A}), (Y, \mathcal{B})$ be countably separated measurable spaces and $f : X \to Y$ a function. If the graph of f is analytic[1] then f is measurable.

Proof. Denote the graph by G. Let $B \in \mathcal{B}$, then $G \cap (X \times B)$ is analytic (think, why), therefore its projection $f^{-1}(B)$ is analytic. Similarly, $f^{-1}(Y \setminus B)$ is analytic. We note that $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$, apply 6a5 and get $f^{-1}(B) \in \mathcal{A}$. $\square$

6b12 Extra exercise. Give an example of a nonmeasurable function with measurable graph, between countably separated measurable spaces.

[1]As defined by 5d9, taking into account that $X \times Y$ is countably separated by 1d24.

6c A non-Borel analytic set of trees

An example, at last...

We adapt the notion of a tree to our needs as follows.

6c1 Definition. (a) A *tree* consists of an at most countable set T of "nodes", a node 0_T called "the root", and a binary relation "$\rightsquigarrow$" on T such that for every $s \in T$ there exists one and only one finite sequence $(s_0, \dots, s_n) \in T \cup T^2 \cup T^3 \cup \dots$ such that $0_T = s_0 \rightsquigarrow s_1 \rightsquigarrow \dots \rightsquigarrow s_{n-1} \rightsquigarrow s_n = s$.

(b) An *infinite branch* of a tree T is an infinite sequence $(s_0, s_1, \dots) \in T^\infty$ such that $0_T = s_0 \rightsquigarrow s_1 \rightsquigarrow \dots$; the set $[T]$ of all infinite branches is called the *body* of T.

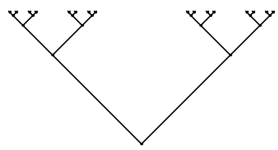
(c) A tree T is *pruned* if every node belongs to some (at least one) infinite branch. (Or equivalently, $\forall s \in T \; \exists t \in T \; s \rightsquigarrow t$.)

We endow the body $[T]$ with a metrizable topology, compatible with the metric

$$\rho\big((s_n)_n, (t_n)_n\big) = 2^{-\inf\{n : s_n \ne t_n\}} \, .$$

The metric is separable and complete (think, why); thus, $[T]$ is Polish.

6c2 Example. The full binary tree $\{0,1\}^{<\infty} = \bigcup_{n=0,1,2,\dots} \{0,1\}^n$:



Its body is homeomorphic to the Cantor set $\{0,1\}^\infty$.

6c3 Example. The full infinitely splitting tree: $\{1,2,\dots\}^{<\infty}$. Its body is homeomorphic to $\{1,2,\dots\}^\infty$, as well as to $[0,1] \setminus \mathbb{Q}$ (the space of irrational numbers), since these two spaces are homeomorphic:

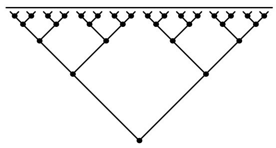
$$\{1,2,\dots\}^\infty \ni (k_1, k_2, \dots) \mapsto \cfrac{1}{k_1 + \cfrac{1}{k_2 + \dots}}$$

Let T be a tree and $T_1 \subset T$ a nonempty subset such that $\forall s \in T \; \forall t \in T_1 (s \rightsquigarrow t \implies s \in T_1)$. Then T_1 is itself a tree, — a subtreee of T. Clearly, $[T_1] \subset [T]$ is a closed subset.

6c4 Definition. (a) A *regular scheme* on a set X is a family $(A_s)_{s \in T}$ of subsets of X indexed by a tree T, satisfying $A_s \supset A_t$ whenever $s \rightsquigarrow t$.

(b) A regular scheme $(A_s)_{s \in T}$ on a metric space X, indexed by a pruned tree T, has *vanishing diameter* if $\operatorname{diam}(A_{s_n}) \to 0$ (as $n \to \infty$) for every $(s_n)_n \in [T]$.

6c5 Example. Dyadic intervals $[\frac{k}{2^n}, \frac{k+1}{2^n}] \subset [0,1]$, naturally indexed by the full binary tree, are a vanishing diameter scheme.



For every $(s_n)_n \in [T]$ we have $A_{s_n} \downarrow \{x\}$ for some $x \in [0,1]$, which gives a continuous map from the Cantor set onto $[0,1]$. Note that the map is not one-to-one.

Let $(A_s)_{s\in T}$ be a regular scheme on X, and $x \in X$. The set $T_x = \{s \in T : A_s \ni x\}$, if not empty, is a subtree of T. The following two conditions on the scheme are equivalent (think, why):

(6c6a) $$T_x \text{ is a pruned tree for every } x \in X;$$

(6c6b) $$A_{0_T} = X\,, \quad \text{and} \quad A_s = \bigcup_{t:s\rightsquigarrow t} A_t \text{ for all } s \in T\,.$$

Let X be a complete metric space, and $(F_s)_{s\in T}$ a vanishing diameter scheme of closed sets on X. Then for every $(s_n)_n \in [T]$ we have $F_{s_n} \downarrow \{x\}$ for some $x \in X$. We define the *associated map* $f : [T] \to X$ by $F_{s_n} \downarrow \{f\big((s_n)_n\big)\}$. This map is continuous (think, why), and $f^{-1}(\{x\}) = [T_x]$ for all $x \in X$ (look again at 6c5); here, if $x \notin F_{0_T}$ then $T_x = \emptyset$, and we put $[\emptyset] = \emptyset$.

If the scheme satisfies (6c6) then $f([T]) = X$ (since $[T_x] \neq \emptyset$ for all x).

6c7 Core exercise. On every compact metric space there exists a vanishing diameter scheme of closed sets, satisfying (6c6), indexed by a finitely splitting tree (that is, the set $\{t : s \rightsquigarrow t\}$ is finite for every s).

Prove it.

It follows easily that every compact metrizable space is a continuous image of the Cantor set.

6c8 Core exercise. On every complete separable metric space there exists a vanishing diameter scheme of closed sets, satisfying (6c6).

Prove it.

It follows easily that every Polish space is a continuous image of the space of irrational numbers. And therefore, every analytic set (in a Polish space) is also a continuous image of the space of irrational numbers!

An analytic set A in a Polish space Y is the image of some Polish space X under some continuous map $\varphi : X \to Y$;

$$A = \varphi(X) \subset Y .$$

We choose complete metrics on X and Y. According to 6c8 we take on X a vanishing diameter scheme of closed sets $(F_s)_{s\in T}$ satisfying (6c6).

6c9 Core exercise. The family $\big(\overline{\varphi(F_s)}\big)_{s\in T}$ is a vanishing diameter scheme of closed sets on Y. (Here $\overline{\varphi(F_s)}$ is the closure of the image.)

Prove it.

We consider the associated maps $f : [T] \to X$ and $g : [T] \to Y$. Clearly, $\varphi \circ f = g$, $f([T]) = X$, and therefore $g([T]) = A$. We conclude.

6c10 Proposition. For every analytic set A in a Polish space X there exists a vanishing diameter scheme $(F_s)_{s\in T}$ of closed sets on X whose associated map f satisfies $f([T]) = A$.

And further...

6c11 Proposition. A subset A of a Polish space X is analytic if and only if

$$A = \bigcup_{(s_n)_n\in[T]} \bigcap_n F_{s_n}$$

for some regular scheme $(F_s)_{s\in T}$ of closed (or Borel) sets $F_s \subset X$ indexed by a pruned tree T.[1]

Proof. "Only if": follows from 6c10.

"If": A is the projection of the Borel set of pairs $\big((s_n)_n, x\big) \in [T] \times X$ satisfying $x \in F_{s_n}$ for all n. □

We return to 6c10. The relation $f([T]) = A$, in combination with $f^{-1}(\{x\}) = [T_x]$, gives $A = \{x : [T_x] \ne \emptyset\}$, that is,

$$A = \{x : T_x \in \mathrm{IF}(T)\}$$

where $\mathrm{IF}(T)$ is the set of all subtrees of T that have (at least one) infinite branch. (Such trees are called *ill-founded.*) Thus, every analytic set $A \subset X$ is the inverse image of $\mathrm{IF}(T)$ under the map $x \mapsto T_x$ for some regular scheme of closed sets.

The set $\mathrm{Tr}(T)$ of all subtrees of T (plus the empty set) is a closed subset of the space 2^T homeomorphic to the Cantor set (unless T is finite). Thus $\mathrm{IF}(T) \subset \mathrm{Tr}(T)$ is a subset of a compact metrizable space.

[1] In other words: a set is analytic if and only if it can be obtained from closed sets by the so-called Souslin operation; see Srivastava, Sect. 1.12 or Kechris, Sect. 25.C.

6c12 Core exercise. $\mathrm{IF}(T)$ is an analytic subset of $\mathrm{Tr}(T)$.
Prove it.

We return to a regular scheme of closed sets and the corresponding map

$$X \ni x \mapsto T_x \in \mathrm{Tr}(T)\,.$$

6c13 Core exercise. Let $B \subset 2^T$ and $A = \{x : T_x \in B\} \subset X$.
(a) If B is clopen then A belongs to $\Pi_2 \cap \Sigma_2$ (that is, both G_δ and F_σ).
(b) If $B \in \Pi_n$ then $A \in \Pi_{n+2}$. If $B \in \Sigma_n$ then $A \in \Sigma_{n+2}$. (Here $n = 1, 2, \dots$)
Prove it.

6c14 Proposition. Let $T = \{1, 2, \dots\}^{<\infty}$ be the full infinitely splitting tree. Then the subset $\mathrm{IF}(T)$ of $\mathrm{Tr}(T)$ does not belong to the algebra $\cup_n \Sigma_n$.

Proof. By the hierarchy theorem (see Sect. 1c), there exists a Borel subset A of the Cantor set such that $A \notin \cup_n \Sigma_n$. By Prop. 3e2, A is analytic. By Prop. 6c10, $A = \{x : T_x \in \mathrm{IF}(T_1)\}$ for some tree T_1 and some scheme. Applying 6c13 to $B = \mathrm{IF}(T_1)$ we get $\mathrm{IF}(T_1) \notin \cup_n \Sigma_n$ in 2^T, therefore in $\mathrm{Tr}(T)$. It remains to embed T_1 into T. $\square$

A similar argument applied to the transfinite Borel hierarchy shows that $\mathrm{IF}(T)$ is a non-Borel subset of $\mathrm{Tr}(T)$. Thus, $\mathrm{Tr}(T)$ contains a non-Borel analytic set. The same holds for the Cantor set (since $\mathrm{Tr}(T)$ embeds into 2^T) and for $[0, 1]$ (since the Cantor set embeds into $[0, 1]$).[1]

6c15 Extra exercise. Taking for granted that $\mathrm{IF}(T)$ is not a Borel set (for $T = \{1, 2, \dots\}^{<\infty}$), prove that the real numbers of the form

$$\cfrac{1}{k_1 + \cfrac{1}{k_2 + \dots}}$$

such that some infinite subsequence $(k_{i_1}, k_{i_2}, \dots)$ of the sequence $(k_1, k_2, \dots)$ satisfies the condition: each element is a divisor of the next element, are a non-Borel analytic subset of $\mathbb{R}$.[2]

---

[1] In fact, the same holds for all uncountable Polish spaces, as well as all uncountable standard Borel spaces (these are mutually isomorphic).

[2] Lusin 1927.

6d Borel injections

The second step toward deeper theory of Borel sets.

6d1 Theorem. [1] Let X, Y be Polish spaces and $f : X \to Y$ a continuous map. If f is one-to-one then $f(X)$ is Borel measurable.

If a tree has an infinite branch then, of course, this tree is infinite and moreover, of infinite height (that is, for every n there exists an n-element branch). The converse does not hold in general (think, why), but holds for finitely splitting trees ("König's lemma"). In general the condition $T_x \in \operatorname{IF}(T)$ (that is, $[T_x] \ne \emptyset$) cannot be rewritten in the form $\forall n \; T_x \cap R_n \ne \emptyset$ (for some $R_n \subset T$), since in this case the set $\{x : \forall n \; T_x \cap R_n \ne \emptyset\} = \{x : \forall n \, \exists s \in R_n \;\; s \in T_x\} = \cap_n \cup_{s \in R_n} F_s$ must be an $F_{\sigma\delta}$-set (given a regular scheme of closed sets), while the set $\{x : T_x \in \operatorname{IF}(T)\} = A$, being just analytic, need not be $F_{\sigma\delta}$. But if each T_x is finitely splitting then König's lemma applies and so, A is Borel measurable (given a regular scheme of closed sets). In particular, this is the case if each T_x does not split at all, that is, is a branch! (Thus, we need only the trivial case of König's lemma.) In terms of the scheme $(B_s)_{s \in T}$ it means that

$$B_{t_1} \cap B_{t_2} = \emptyset \quad \text{whenever } s \rightsquigarrow t_1, \; s \rightsquigarrow t_2, \; t_1 \ne t_2 \,. \tag{6d2}$$

We conclude.

6d3 Lemma. Let $(B_s)_{s \in T}$ be a regular scheme of Borel sets satisfying (6d2). Then the following set is Borel measurable:

$$B = \{x : T_x \in \operatorname{IF}(T)\} = \bigcup_{(s_n)_n \in [T]} \bigcap_n B_{s_n} \,.$$

Indeed,

$$B = \bigcap_n \bigcup_{s \in R_n} B_s \,,$$

where R_n is the n-th level of T (that is, s_n in $0_T = s_0 \rightsquigarrow s_1 \rightsquigarrow \dots \rightsquigarrow s_n$ runs over R_n).

6d4 Core exercise. For every regular scheme $(A_s)_{s \in T}$ of Borel sets satisfying (6c6) there exists a regular scheme $(B_s)_{s \in T}$ of Borel sets $B_s \subset A_s$, satisfying (6c6) and (6d2).

Prove it.

[1] Lusin-Souslin; see Srivastava, Th. 4.5.4 or Kechris, Th. (15.1).

We combine it with 6c8.

6d5 Lemma. On every complete separable metric space there exists a vanishing diameter scheme of Borel sets, satisfying (6c6) and (6d2).

Given X, Y, f as in 6d1, we take $(B_s)_{s\in T}$ on X according to 6d5, introduce $A_s = f(B_s) \subset Y$ and get

$$\bigcup_{(s_n)_n\in[T]} \bigcap_n A_{s_n} = \bigcup_{(s_n)_n\in[T]} \bigcap_n f(B_{s_n}) = f\Big(\bigcup_{(s_n)_n\in[T]} \bigcap_n B_{s_n} \Big) = f(X)\,,$$

$(A_s)_{s\in T}$ being a vanishing diameter scheme on Y satisfying (6d2) (think, why). However, are A_s Borel sets? For now we only know that they are analytic.

In spite of the vanishing diameter, it may happen that $\cap_n \overline{A_{s_n}} \ne \cap_n A_{s_n}$ (since $\cap_n A_{s_n}$ may be empty); nevertheless,

$$\bigcup_{(s_n)_n\in[T]} \bigcap_n \overline{A_{s_n}} = \bigcup_{(s_n)_n\in[T]} \bigcap_n A_{s_n} = f(X)\,, \tag{6d6}$$

since (for some $x \in X$) $\cap_n \overline{A_{s_n}} = \cap_n \overline{f(B_{s_n})} \supset \cap_n f(\overline{B_{s_n}}) = f\big(\cap_n \overline{B_{s_n}}\big) = f(\{x\}) = \{f(x)\} \subset f(X)$. (Then necessarily $\cap_n \overline{A_{s_n}} = \cap_n A_{s'_n}$ for another branch $(s'_n)_n \in [T]$.) However, $(\overline{A_s})_{s\in T}$ need not satisfy (6d2).

By (6d6) and 6d3, Theorem 6d1 is reduced to the following.

6d7 Lemma. For every regular scheme $(A_s)_{s\in T}$ of analytic sets, satisfying (6d2), there exists a regular scheme $(B_s)_{s\in T}$ of Borel sets, satisfying (6d2) and such that

$$A_s \subset B_s \subset \overline{A_s} \quad \text{for all } s \in T\,.$$

6d8 Core exercise. Let $A_1, A_2, \dots$ be disjoint analytic sets. Then there exist disjoint Borel sets $B_1, B_2, \dots$ such that $A_n \subset B_n$ for all n.

Prove it.

We can get more: $A_n \subset B_n \subset \overline{A_n}$ for all n (just by replacing B_n with $B_n \cap \overline{A_n}$).

Proof of Lemma 6d7. First, we use 6d8 for constructing B_s for $s \in R_1$ (the first level of T), that is, $0_T \rightsquigarrow s$. Then, for every $s_1 \in R_1$, we do the same for s such that $s_1 \rightsquigarrow s$ (staying within B_{s_1}); thus we get B_s for $s \in R_2$. And so on. □

Theorem 6d1 is thus proved.

6d9 Core exercise. If $(X, \mathcal{A})$ is a standard Borel space, $(Y, \mathcal{B})$ a countably separated measurable space, and $f : X \to Y$ a measurable one-to-one map then $f(X) \in \mathcal{B}$.[1]

Prove it.

6d10 Corollary. If a subset of a countably separated measurable space is itself a standard Borel space then it is a measurable subset.[2] [3]

6d11 Corollary. A subset of a standard Borel space is itself a standard Borel space if and only if it is Borel measurable.

[1]The topological counterpart is not quite similar: a continuous image of a compact topological space in a Hausdorff topological space is closed, even if the map is not one-to-one.

[2]A topological counterpart: if a subset of a Hausdorff topological space is itself a compact topological space then it is a closed subset.

[3]See also the footnotes to 4c12, 4d10 and 5d7.

Hints to exercises

6a4: recall the proof of 5d11.

6b1: use 6a5 and 6a6.

6b4: use 6b3, 4d7 and 3c6.

6b6: use 6b1 and 1d32.

6b8: similar to 6a4.

6c9: be careful: φ need not be uniformly continuous.

6c12: recall the proof of 6c11.

6c13: recall 1b, 1c.

6d4: $A_1 \cup A_2 \cup \dots = A_1 \uplus (A_2 \setminus A_1) \uplus \dots$

6d8: first, apply Theorem 6a1 to A_1 and $A_2 \cup A_3 \cup \dots$

6d9: similar to 6a4.

Index