23 Feb '24

Descriptive Complexity
(Lectures 9 and 10)

In the last lecture, we saw:

1. Relativizations of the following notions & results to a given class of structures:
   - Satisfiability, unsatisfiability & validity
   - Compactness, Löwenheim-Skolem and Lindström's theorems
   - FO descriptions via:
     - sentences (definability)
     - theories (axiomatizability)

2. The various names for relativizations

3. Relationships between relativizations of the same notion / result to different classes of structures

④ Background for

Ehrenfeucht - Fraïssé (EF) games

- relational vocabulary
- substructure induced by a subset of the domain
- partial isomorphism between structures

⑤ The Ehrenfeucht - Fraïssé (EF) game

$\mathcal{G}(A, B, m)$

- The game
- Winning condition for players S & D
- Winning strategy for each player
- An example

Let's play!

First : revisiting the previous game.

① $\sigma = \{\}$;

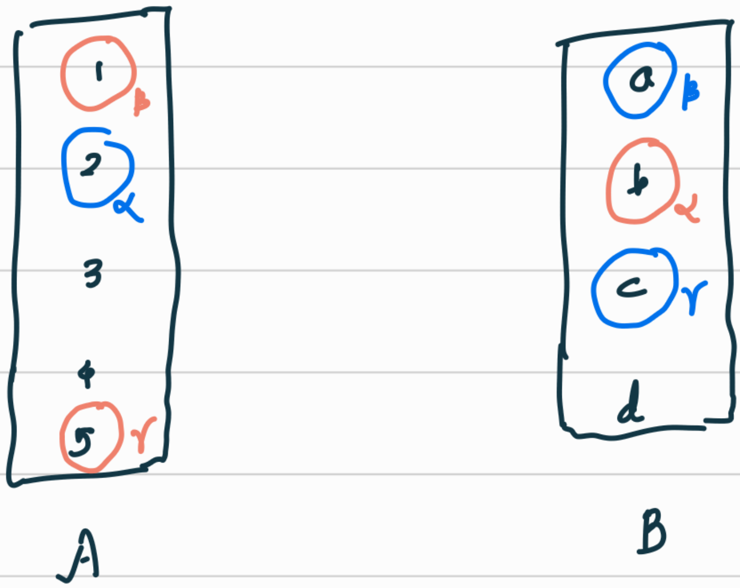
$A = \{1, 2, 3, 4, 5\}$; $B = \{a, b, c, d\}$

$m = 3$.

Play of $\mathcal{G}(A, B, 3)$

• – S (Abhisekh)

• – D (Pranshu)



$$\underset{\alpha}{2}, \underset{\beta}{1}, \underset{\gamma}{5}) \in A^3 \qquad (\underset{\alpha}{b}, \underset{\beta}{a}, \underset{\gamma}{c}) \in B^3$$

$$f : (2, 1, 5) \longrightarrow (b, a, c)$$

is a partial iso. between A & B.

So Pranshu (D) wins the game.

Indeed he has a winning strategy too.

Observe that if $m = 4$, D still always manages to win. That is D has a winning strategy as follows. Let $e_1 \dots e_i$ & $f_1 \dots f_i$ be the elements chosen in A & B resp. in rounds $1, \dots i$ for $i \in [m-1]$. Then in round $i+1$:

- If S picks e_j, then D picks f_j.
- If S picks f_j, then D picks e_j.
- Else S picks an element different from all e_js & f_js.

  Then D also picks from the structure not chosen by S, an element different from all e_js & f_js.

However, if $m=5$, S has a winning strategy: in round i, S chooses the structure A & element i.

Verify that this strategy is indeed winning.

Generalization: Fix $m \geq 0$. Let $\sigma = \{\}$.

① If $|A|$ & $|B|$ are both at least m, then player D has a winning strategy in $\mathcal{G}(A, B, m)$.

② If one of $|A|$ or $|B|$ is $< m$, then

ⓐ If $|A| = |B|$, then D has a winning strategy in $\mathcal{G}(A, B, m)$

ⓑ Else, S has a winning strategy in $\mathcal{G}(A, B, m)$.

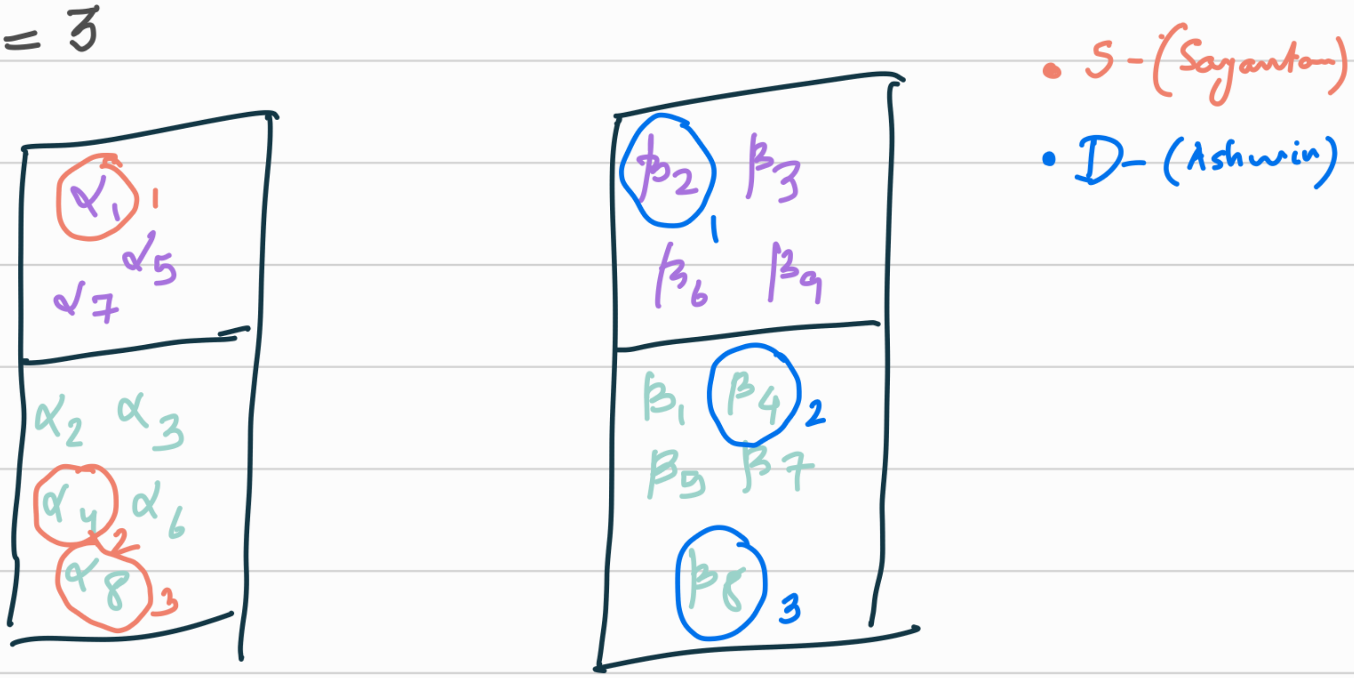
A new game!

(2) $\sigma = \{P\}$

(a) $A = \left(\mathrm{Dom}(A) = \{\alpha_1, \ldots \alpha_8\}, \; P^A = \{\alpha_1, \alpha_5, \alpha_7\} \right)$

$B = \left(\mathrm{Dom}(B) = \{\beta_1, \ldots \beta_9\}, \; P^B = \{\beta_2, \beta_3, \beta_6, \beta_9\} \right)$

$m = 3$



A B

$\bar{a} = (\alpha_1, \alpha_4, \alpha_8)$ $\bar{b} = (\beta_2, \beta_4, \beta_8)$

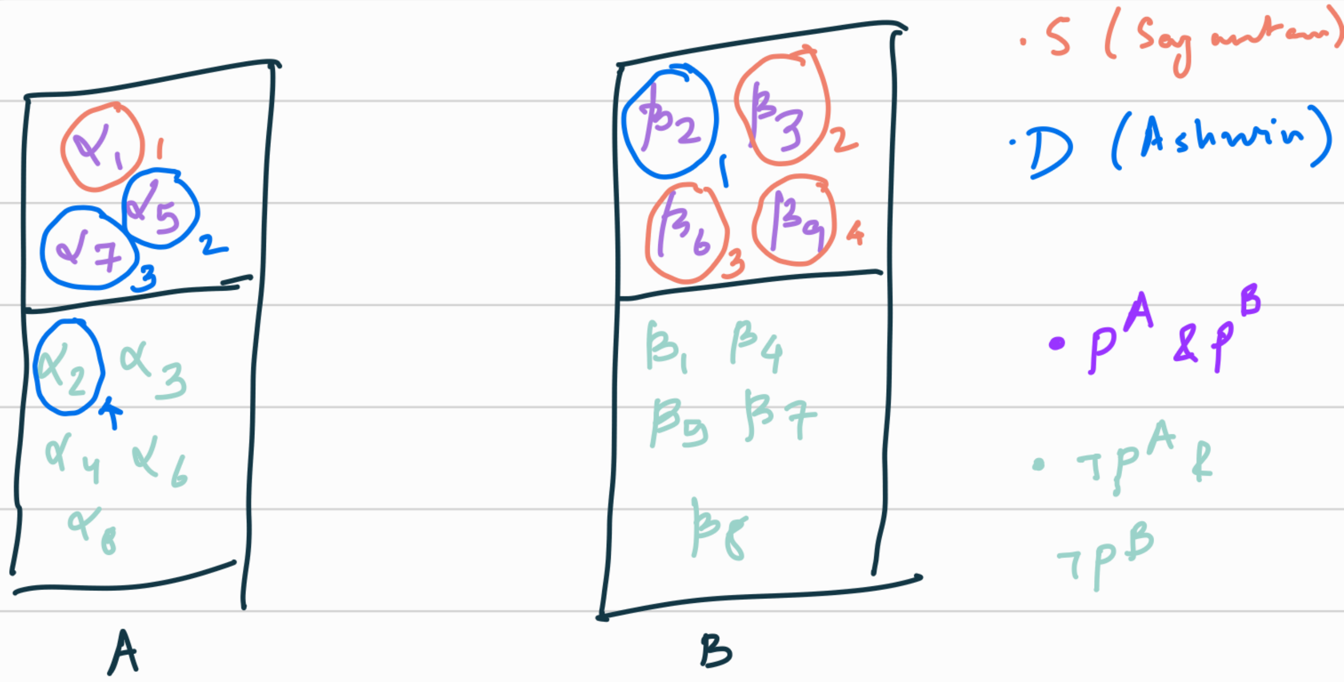
$f : \bar{a} \to \bar{b}$

is a p.i. b/w A & B.

Hence D (Ashwin) wins the play.

Indeed he has a winning strategy too.

Same arena but $m = 4$:



$$\bar{a} = (\alpha_1, \alpha_5, \alpha_7, \alpha_2) \;;\; \bar{b} = (\beta_2, \beta_3, \beta_6, \beta_9)$$

$$f : \bar{a} \longrightarrow \bar{b}$$

is <u>not</u> a p.i. b/w A & B.

$$\left(\because \alpha_2 \notin P^A \text{ but } \beta_9 \in P^B \right)$$

Hence S (Sayantan) wins the play.
Indeed he has a winning strategy too.

Generalization: Fix $m \geq 0$.

Let $\sigma = \{P_1 \ldots P_k\}$, all P_is unary

For $c \in \{0,1\}^k$, let

$$A_c = \{ a \in A \mid (a \in P_i^A \longleftrightarrow c[i] = 1)\ \forall i \in [k] \}$$

Define B_c similarly.

① If $|A_c|$ & $|B_c|$ are both $\geq m$ $\forall c \in \{0,1\}^k$ then player $\mathcal{D}$ has a winning strategy in $\mathcal{G}(A, B, m)$.

② $\exists c \in \{0,1\}^k$ s.t. atleast one of $|A_c|$ or $|B_c|$ is $< m$ and $|A_c| \neq |B_c|$, then player $\mathcal{S}$ has a winning strategy in $\mathcal{G}(A, B, m)$.

③ ($\forall c \in \{0,1\}^k$ either $|A_c| = |B_c| < m$ or both $|A_c|$ & $|B_c|$ are $\geq m$,) and ($\exists c \in \{0,1\}^k$ s.t. either $|A_c|$ or $|B_c|$ is $< m$)

then

player $\mathcal{D}$ has a winning strategy in $\mathcal{G}(A, B, m)$.

Observe that the above generalization can be stated equivalently as below.

Generalization (Equivalent form of above).

Fix $m \geq 0$. Let $\sigma = \{P_1, \ldots, P_k\}$, all P_is unary. Let A_c & B_c be defined as above for $c \in \{0,1\}^k$.

① Player $\mathcal{D}$ has a winning strategy in $\mathcal{G}(A, B, m)$

iff

$\forall c \in \{0,1\}^k$ either $|A_c| = |B_c|$ or both $|A_c|$ & $|B_c|$ are $\geq m$.

② Player $\mathcal{S}$ has a winning strategy precisely when Player $\mathcal{D}$ does not have a winning strategy.

Revisiting $\mathcal{G}(\mathcal{A}, \mathcal{B}, m)$.

From the definition of the winning strategy for the players S and D, we can see an induction / recursion that is involved.

Towards this, we first observe that given a σ-structure A & an r-tuple $\bar{a}$ from A, the expansion $(A, \bar{a})$ can be seen as a structure $\mathbb{A}$ over the vocabulary $\sigma_r = \sigma \cup \{c_1, \ldots c_r\}$ where $c_1 \ldots c_r$ are fresh constants that do not appear in σ.

Specifically:

- The domain of $\mathbb{A}$ = the domain of A
- The symbols of σ are interpreted in $\mathbb{A}$ precisely as they are in A.

(P.T.O.)

– The symbol c_i in $\sigma_r \setminus \sigma$ for $i \in [r]$ is interpreted in $\mathbb{A}$ as a_i.

Here $\bar{a} = (a_1, \ldots, a_r)$. where $a_i \in Dom(A)$ $\forall i \in [r]$

By the same token, a formula $\varphi(\bar{x})$ over σ where $\bar{x} = (x_1, \ldots, x_r)$ can be seen as a sentence Φ over σ_r $\left(= \sigma \cup \{c_1 \ldots c_r\}\right)$ given by:

$$\Phi = \varphi[x_i \mapsto c_i]$$

That is, Φ is obtained from φ by replacing every occurrence of x_i with c_i, for each $i \in [r]$.

The following is now easy to see.

<u>Lemma 1</u>:

$$\mathbb{A} \models \varphi(\bar{a}) \text{ iff } \mathbb{A} \models \Phi$$

We can now make explicit the induction/recursion inherent in the definition of winning strategy for S & D in $\mathcal{G}(A, B, m)$, as follows. We state this for S; the statement for D is similar.

<u>Lemma 2</u>:

Let σ be a relational vocabulary (so possibly containing constant symbols). Let A, B be σ-structures, and $m \geq 1$ be a given number. Let $\bar{a}, \bar{b}$ be r-tuples from A & B resp. for $r \geq 0$.

Then Player S has a winning strategy in $\mathcal{G}((A, \bar{a}), (B, \bar{b}), m)$* iff any one of the following holds:

(* Here $(A, \bar{a})$ & $B, \bar{b}$ are seen as σ_r-structures.)

(P.T.O.)

(a) There exists $a_{r+1} \in A$ such that for all $b_{r+1} \in B$, if $\bar{a}' = \bar{a} \cdot a_{r+1}$ and $\bar{b}' = \bar{b} \cdot b_{r+1}$ are the $(r+1)$-tuples obtained by expanding $\bar{a}$ with a_{r+1} and $\bar{b}$ with b_{r+1}, then Player S has a winning strategy in $\mathcal{G}((A, \bar{a}'), (B, \bar{b}'), m-1)$.

(b) There exists $b_{r+1} \in B$ such that for all $a_{r+1} \in A$, if $\bar{a}' = \bar{a} \cdot a_{r+1}$ and $\bar{b}' = \bar{b} \cdot b_{r+1}$ are the $(r+1)$-tuples obtained by expanding $\bar{a}$ with a_{r+1} and $\bar{b}$ with b_{r+1}, then Player S has a winning strategy in $\mathcal{G}((A, \bar{a}'), (B, \bar{b}'), m-1)$.

Lemma 3:

Let σ, A, B, $\bar{a}$, $\bar{b}$, r be as in Lemma 2. Then Player S has a winning strategy in $\mathcal{G}((A,\bar{a}),(B,\bar{b}),m)$ for $m=0$

iff

the function

$$g : \bar{a} \longrightarrow \bar{b}$$

defined as

$$g(a_i) = b_i \qquad \forall i \in [r]$$

is not an isomorphism between $A[\bar{a}]$ & $B[\bar{b}]$.

Revisiting the plays!

We saw earlier that player $\mathcal{S}$ has a winning strategy in $\mathcal{G}(A, B, m)$ when:

$\sigma = \{\}$

$A = (\mathrm{Dom}(A) = \{\alpha_1, \ldots, \alpha_5\})$

$B = (\mathrm{Dom}(B) = \{\beta_1, \ldots, \beta_4\})$

$m = 5$

Specifically, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (1, 2, 3, 4, 5)$ & $(\beta_1, \beta_2, \beta_3, \beta_4) = (a, b, c, d)$

A winning strategy for $\mathcal{S}$ is one in which in round $i \in [5]$, $\mathcal{S}$ chooses the element α_i in A (irrespective of the moves of $\mathcal{D}$ in the previous rounds).

This winning strategy can be represented as a tree as described below.

The nodes of the tree at the $(2i)^{th}$ level represent the possible positions of the game at the end of i rounds when player S plays according to his winning strategy.

The nodes of the tree at the $(2i-1)^{th}$ level represent "intermediate" positions of the game after the $(i-1)$ th and before the end of the ith rounds. These intermediate positions augment the positions of the game at the end of $(i-1)$ rounds, with the moves of player S in the ith round.

So the first 2 levels of the strategy tree are as below.

$$\begin{tikzcd}
 & & ((),()) \arrow[d, "S"] & & \\
 & & ((\alpha_1),()) \arrow[dll, "D"'] \arrow[dl, "D"] \arrow[dr, "D"] \arrow[drr, "D"] & & \\
((\alpha_1),(\beta_1)) & ((\alpha_1),(\beta_2)) & & ((\alpha_1),(\beta_3)) & ((\alpha_1),(\beta_4))
\end{tikzcd}$$

Suppose after $r \geq 1$ rounds, the position of the game is $(\bar{a}_r, \bar{b}_r)$.

Then after $(r+1)$ rounds, the possible positions of the game are $(\bar{a}_{r+1}, \bar{b}_{r+1})$ where:

- $\bar{a}_{r+1} = \bar{a}_r \cdot \alpha_{r+1}$
- $\bar{b}_{r+1} \in \{\bar{b}^1_{r+1}, \bar{b}^2_{r+1}, \bar{b}^3_{r+1}, \bar{b}^4_{r+1}\}$

  where:

  $$\bar{b}^j_{r+1} = \bar{b}_r \cdot \beta_j$$

Then the corresponding portion of the strategy tree is as below:

$$
\begin{tikzcd}
 & & (\bar{a}_r, \bar{b}_r) \arrow[d, "S"] & & \\
 & & (\bar{a}_{r+1}, \bar{b}_r) \arrow[dll, "D"'] \arrow[dl, "D"] \arrow[dr, "D"] \arrow[drr, "D"] & & \\
(\bar{a}_{r+1}, \bar{b}^1_{r+1}) & (\bar{a}_{r+1}, \bar{b}^2_{r+1}) & & (\bar{a}_{r+1}, \bar{b}^3_{r+1}) & (\bar{a}_{r+1}, \bar{b}^4_{r+1})
\end{tikzcd}
$$

At the end of $m = 5$ rounds, the position of the game (in any play of the game) is of the form

$$(\bar{a}_5, \bar{b}_5) = (\bar{a}^*, \bar{b}^*)$$

where:

$$\bar{a}^* = (\alpha_1, \alpha_2, \ldots, \alpha_5)$$

$$\bar{b}^* = (\beta_{i_1}, \beta_{i_2}, \ldots \beta_{i_5})$$

and $\beta_{i_1}, \ldots, \beta_{i_5} \in Dom(B)$

That $\bar{a}^*$ is of the form above is because of the winning strategy of player S.

For the same reason (namely the winning strategy of S)

$$f : \bar{a}^* \longrightarrow \bar{b}^*$$

is not a partial isomorphism (p. i.) from A to B.

The last fact above can be captured in logic :

We express the isomorphism type of $\bar{a}^*$ in FO.

Let $\varphi_{\bar{a}^*}(x_1, \ldots x_5) := \bigwedge_{\substack{\ell, k \in [5] \\ \ell \neq k}} \neg(x_\ell = x_k)$

Then $(A, \bar{a}^*) \models \varphi_{\bar{a}^*}(x_1, \ldots, x_5)$

but $(B, \bar{b}^*) \not\models \varphi_{\bar{a}^*}(x_1, \ldots, x_5)$

Then $\varphi_{\bar{a}^*}(\bar{x})$ distinguishes $(A, \bar{a}^*)$ from $(B, \bar{b}^*)$

Observe that $\varphi_{\bar{a}^*}(\bar{x})$ is quantifier-free.

We could alternatively also do the following:

$\boxed{\text{We express the isomorphism type of } \bar{b}^* \text{ in FO.}}$

For example if

$$\bar{b}^* = (\beta_2, \beta_3, \beta_1, \beta_2, \beta_4)$$

then

$$\varphi_{\bar{b}^*}(x_1, \ldots x_5) :=$$

$$\left(\bigwedge_{\substack{\ell \in \{2,3,5\}, \\ k \in [5], \\ \ell \neq k}} \neg(x_\ell = x_k) \right) \wedge (x_1 = x_4)$$

Then

$$(A, \bar{a}^*) \not\models \varphi_{\bar{b}^*}(x_1, \ldots, x_5)$$

but $(B, \bar{b}^*) \models \varphi_{\bar{b}^*}(x_1 \ldots x_5)$

so that

$\neg \varphi_{\bar{b}^*}(x_1 \ldots x_5)$ distinguishes

$(A, \bar{a}^*)$ from $(B, \bar{b}^*)$.

Observe again that $\neg \varphi_{\bar{b}^*}(x_1, \ldots, x_5)$ is quantifier-free.

Lets now look at a position $(\bar{a}_4, \bar{b}_4)$ of the game at the end of 4 rounds backwards from the positions of the game at the end of 5 rounds.

- $\bar{a}^* = \bar{a}_5 = \bar{a}_4 \cdot \alpha_5$
- $\bar{b}^* = \bar{b}_5 \in \{\bar{b}_5^1, \bar{b}_5^2, \bar{b}_5^3, \bar{b}_5^4\}$

where:

$$\bar{b}_5^j = \bar{b}_4 \cdot \beta_j$$

As seen above, for each $j \in [4]$, there is some quantifier-free FO formula $\varphi_5^j(x_1, \ldots, x_5)$ that distinguishes $(A, \bar{a}^*)$ from $(B, \bar{b}_5^j)$. That is.

$$(A, \bar{a}^*) \models \varphi_5^j(x_1, \ldots x_5)$$

but $(B, \bar{b}_5^j) \not\models \varphi_5^j(x_1, \ldots x_5)$.

(For instance, $\varphi_5^j(\bar{x})$ describes the isomorphism type of I_5^j.)

We can now "aggregate" the φ_5^j's to get an FO formula $\varphi_4(x_1, \dots x_4)$ that distinguishes $(A, \bar{a}_4)$ from $(B, \bar{b}_4)$, as depicted below.

Distinguished in A & B resp. by

$$\varphi_4(x_1 \dots x_4) := \exists x_5 \begin{bmatrix} \varphi_5^1(x_1, \dots, x_5) \wedge \\ \varphi_5^2(x_1, \dots, x_5) \wedge \\ \varphi_5^3(x_1, \dots, x_5) \wedge \\ \varphi_5^4(x_1, \dots, x_5) \end{bmatrix}$$

$$\begin{tikzcd}
 & & \overbrace{(\bar{a}_4, \bar{b}_4)} \arrow[d, "S"] & & \\
 & & (\bar{a}^*, \bar{b}_4) \arrow[dll, "D"'] \arrow[dl, "D"] \arrow[dr, "D"] \arrow[drr, "D"] & & \\
 \underbrace{(\bar{a}^*, \bar{b}_5^1)} & \underbrace{(\bar{a}^*, \bar{b}_5^2)} & & \underbrace{(\bar{a}^*, \bar{b}_5^3)} & \underbrace{(\bar{a}^*, \bar{b}_5^4)}
\end{tikzcd}$$

distinguished in A & B resp. by $\varphi_5^1(\bar{x})$

distinguished in A & B resp. by $\varphi_5^2(\bar{x})$

distinguished in A & B resp. by $\varphi_5^3(\bar{x})$

distinguished in A & B resp. by $\varphi_5^4(\bar{x})$

Claim:

(a) $(A, \bar{a}_4) \models \varphi_4(x_1, \ldots x_4)$

(b) $(B, \bar{b}_4) \not\models \varphi_4(x_1, \ldots x_4)$

Proof Sketch:

(a) Choose $x_5 = \alpha_5$ in A.

Since $\varphi_5^j(x_1, \ldots x_5)$ distinguishes $(A, \bar{a}^*)$ from $(B, \bar{b}_5^j)$, we have

$$(A, \bar{a}^*) \models \varphi_5^j(x_1 \ldots x_5) \quad \forall j \in [4]$$

Then $(A, \bar{a}_4) \models \varphi_4(x_1 \ldots x_4)$.

(b) Suppose $(B, \bar{b}_4) \models \varphi_4(x_1, \ldots, x_4)$

Then for some $\beta \in \mathrm{Dom}(B)$

$$(B, \bar{b}_4 \cdot \beta) \models \bigwedge_{j \in [4]} \varphi_5^j(x_1, \ldots, x_5)$$

We argue for the case when $\beta = \beta_1$.

If $\beta = \beta_1$, then $\bar{b}_4 \cdot \beta = \bar{b}_5^1$

Then $(B, \bar{b}_5^1) \models \bigwedge_{j \in [4]} \varphi_5^j(x_1, \ldots, x_5)$

This contradicts the fact that

$$(B, \bar{b}_5^1) \not\models \varphi_5^1(x_1, \ldots, x_5).$$

Reasoning similarly for the cases when $\beta = \beta_2$, $\beta = \beta_3$ & $\beta = \beta_4$, we conclude that

$$(B, \bar{b}_4) \not\models \varphi_4(x_1, \ldots, x_4)$$

$\square$

Observe that $\bar{b}_4 = \bar{b}_3 \cdot \beta_j$ for some $j \in [4]$. So $\varphi_4(x_1 \ldots x_4)$ above can be denoted as $\varphi_4^j(x_1 \ldots x_4)$.

Working out similarly for positions $(\bar{a}_3, \bar{b}_3)$, $(\bar{a}_2, \bar{b}_2)$, $(\bar{a}_1, \bar{b}_1)$ and $((), ())$ of the game at the end of $m=3$, $m=2$ & $m=1$ & $m=0$ rounds, we obtain that:

- $\varphi_3(x_1, x_2, x_3) := \exists x_4 \bigwedge_{j \in [4]} \varphi_4^j(x_1, \ldots x_4)$

  distinguishes $(A, \bar{a}_3)$ from $(B, \bar{b}_3)$

- $\varphi_2(x_1, x_2) := \exists x_3 \bigwedge_{j \in [4]} \varphi_3^j(x_1, \ldots x_3)$

  distinguishes $(A, \bar{a}_2)$ from $(B, \bar{b}_2)$

- $\varphi_1(x_1) := \exists x_2 \bigwedge_{j \in [4]} \varphi_2^j(x_1, x_2)$

  distinguishes $(A, \bar{a}_1)$ from $(B, \bar{b}_1)$

&

- $\varphi_0 := \exists x_1 \bigwedge_{j \in [4]} \varphi_1^j(x_1)$

  distinguishes A from B

In summary,

The winning strategy of player S in $\mathcal{G}(A, B, m)$ can be transformed into an FO sentence φ_0 that distinguishes A & B.

More generally,

the winning strategy of player S in $\mathcal{G}((A, \bar{a}_i), (B, \bar{b}_i), m-i)$ can be transformed into an FO formula $\varphi_i(x_1, \ldots x_i)$ that distinguishes $(A, \bar{a}_i)$ & $(B, \bar{b}_i)$.

Observe further that

- φ_0 above has quantifier nesting depth 5 $(= m)$ and 0 free variables.

& more generally

- φ_i above has quantifier nesting depth $m-i$ and i free variables.