Parameterized partition relations on the set of real numbers

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Set theory and its neighbours
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Outline

1. Introduction
   - The Bernstein and Ramsey properties
   - Solovay models
   - Generic absoluteness
2. Strongly proper forcing
   - Weakly-Suslin partial orderings
   - Π^1_2-strongly proper forcing
3. Strong partition relations
   - The Bernstein property for products
   - The parameterized Bernstein property for products

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The Bernstein property

Recall:

A subset of $\mathbb{R}$, or of Baire space ω^ω, has the *Bernstein property* if it contains a perfect set or its complement contains a perfect set.

AC implies that there exists a *Bernstein set*, i.e., a set without the Bernstein property.

Bernstein sets do not have the Baire property. Hence no analytic set can be a Bernstein set.

However, in the constructible universe L there is a Δ^1_2 set without the Bernstein property.

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Thus, Shelah's model of *ZF* where every set of real numbers has the Baire property satisfies the Bernstein property, which shows that the consistency strength of *ZF* plus every set of reals has the Bernstein property is just *ZF*.

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Proposition

If M satisfies the Bernstein property, then M also satisfies that for every partition $g : \omega^\omega \to \omega$, there is a perfect set that lies in one piece of the partition.

The Ramsey property

Recall:

A subset A of $[\omega]^\omega := \{x \subseteq \omega : x \text{ is infinite}\}$ has the *Ramsey property* iff there exists $X \in [\omega]^\omega$ such that either $[X]^\omega \subseteq A$ or $[X]^\omega \cap A = \emptyset$.

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Every subset A of $[\omega]^\omega$ has the Ramsey property iff for every partition $g : [\omega]^\omega \to n$, there is $X \in [\omega]^\omega$ that lies in one piece of the partition.

However, for the infinite partition $g : [\omega]^\omega \to \omega$ given by $g(x) = x(0)$, there is no $X \in [\omega]^\omega$ such that $[X]^\omega$ is contained in one piece of the partition.

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The Bernstein property for products

If in M every set of reals has the property of Baire, then M also satisfies that for every partition $g : (\omega^\omega)^n \to m$ there are perfect subsets P_i of ω^ω, $i < n$, such that the product $\prod_{i<n} P_i$ lies in one piece of the partition (Galvin).

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If M satisfies the Ramsey property, then in M for every finite partition $g : (\omega^\omega)^\omega \to n$, there are perfect sets $P_i \subseteq \omega^\omega$, $i < \omega$, such that g is constant on $\prod_{i<\omega} P_i$.

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The Ramsey property for products fails, even without *AC*:

Let Z be the subset of $[\omega]^\omega \times [\omega]^\omega$ consisting of all pairs $\langle X_0, X_1 \rangle$ such that the first element of X_0 is smaller that the first element of X_1. Then there is no pair $X, Y \in [\omega]^\omega$ such that either $[X]^\omega \times [Y]^\omega$ is contained in, or disjoint from Z.

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Solovay models

Recall:

$L(\mathbb{R})$ is the smallest transitive model of *ZF* that contains all the ordinals and all the real numbers.

For κ an inaccessible cardinal, *Coll*$(\omega, < \kappa)$ is the *Levy-collapse* of κ.

Forcing with *Coll*$(\omega, < \kappa)$ makes all ordinals below κ countable, but it does not collapse κ. So κ becomes the new ω_1.

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If κ is an inaccessible cardinal in some model W, then the $L(\mathbb{R})$ of a model V obtained by Levy-collapsing κ to ω_1 over W has the following two properties:

1. For every $x \in \mathbb{R}$, ω_1 is an inaccessible cardinal in $W[x]$.
2. Every $x \in \mathbb{R}$ is *small-generic* over W. That is, there is a forcing notion $\mathbb{P}$ in W, which is countable in V, and there is, in V, a $\mathbb{P}$-generic filter g over W such that $x \in W[g]$.

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The converse is essentially true. Namely,

Theorem (Woodin)

Suppose that $W \subseteq V$ are models of ZF and V satisfies (1) and (2) above. Then one can force over V whithout adding new reals to create a $Coll(\omega, <\omega_1)$-generic filter C over W such that V and $W[C]$ have the same reals.

Definition

$L(\mathbb{R})$ is a *Solovay model over W* iff

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Solovay models and generic absoluteness

We can prove that the partition properties 1 and 2 hold in Solovay models by using *generic absoluteness*.

The absoluteness of the theory of a Solovay model under generic extensions follows from the fact that the property of $L(\mathbb{R})$ being a Solovay model over some model W is preserved under those extensions.

For example,

One can show that in a Solovay model every set of reals is Ramsey using generic absoluteness for Mathias forcing, which in turn follows from the fact that $L(\mathbb{R})$ being a Solovay model over some model W is preserved under Mathias forcing.

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Generic absoluteness

The preservation of the property of $L(\mathbb{R})$ being a Solovay model over some model W under forcing that does not collapse ω_1 implies a strong form of *generic absoluteness* for $L(\mathbb{R})$.

Lemma

Suppose that $L(\mathbb{R})^M$ and $L(\mathbb{R})^N$ are Solovay models over W such that $\mathbb{R}^M \subseteq \mathbb{R}^N$ and $\omega_1^M = \omega_1^N$. Then there exists an elementary embedding $j : L(\mathbb{R})^M \to L(\mathbb{R})^N$ which fixes all the ordinals.

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Π^1_2-strongly proper forcing

Definition

A partial ordering $\mathbb{P}$ *projective* if it is first-order definable, with parameters, in $H(\omega_1)$.

A projective partial ordering $\mathbb{P}$ is Π^1_2-*strongly proper* if for some $a \in H(\omega_1)$, for every countable transitive model N of some large-enough fragment of ZFC that contains a and the parameters of the definition of $\mathbb{P}$, and is Π^1_2-correct in V, and such that $(\mathbb{P}^N, \leq^N_{\mathbb{P}}, \perp^N_{\mathbb{P}}) \subseteq (\mathbb{P}, \leq_{\mathbb{P}}, \perp_{\mathbb{P}})$, and every $p \in \mathbb{P} \cap N$, there is $q \leq_{\mathbb{P}} p$ that is $(N, \mathbb{P})$-generic, i.e.,

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Strongly proper forcing

If $\mathbb{P}$ is projective, $N \preceq H(\lambda)$, and the parameters of the definition of $\mathbb{P}$ are in N, then a condition q is $(N, \mathbb{P})$-generic iff it is $(\bar{N}, \mathbb{P})$-generic, where $\bar{N}$ is the transitive collapse of N. Thus, a projective $\mathbf{\Pi}^1_2$-strongly proper poset is proper.

Strongly proper forcing

Examples

Every Suslin (i.e., analytic, or $\mathbf{\Sigma}^1_1$) ccc poset is $\mathbf{\Pi}^1_2$-strongly proper.

However,

It is consistent (modulo the existence of an inaccessible cardinal) that there is a Δ^1_3 ccc poset that is not $\mathbf{\Pi}^1_2$-strongly proper.

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Axiom A forcing

Definition (J. Baumgartner)

A partial ordering $\mathbb{P} = (\mathbb{P}, \leq)$ satisfies *Axiom A* if for every $n < \omega$ there exists a partial ordering $\leq_n$ of $\mathbb{P}$ such that:

1. $\leq_0 = \leq$ and for every $n < \omega$, $\leq_{n+1} \subseteq \leq_n$
2. For every maximal antichain A of $\mathbb{P}$, every $p \in \mathbb{P}$, and every $n < \omega$, there exists $q \leq_n p$ such that for some countable $A' \subseteq A$, A' is predense below q.
3. If $p_{n+1} \leq_n p_n$ for all $n < \omega$, then there exists q such that $q \leq p_n$, all $n < \omega$.

Weakly Suslin partial orderings

Definition

A forcing notion $\mathbb{P}$ is *weakly Suslin* if the set of conditions is an analytic set of reals and the ordering is an analytic relation.

Notice that if $\mathbb{P}$ is weakly Suslin, then the incompatibility relation $\perp_{\mathbb{P}}$ is Π^1_1. Hence, if A is a countable subset of $\mathbb{P}$ and $p \in \mathbb{P}$, then the statement "A is predense below p" is Π^1_2, with A and p as parameters.

Let us call a weakly Suslin partial ordering $\mathbb{P}$ *Axiom A weakly Suslin* if it satisfies Axiom A, witnessed by $(\leq_n)_n$, and all the $\leq_n$ are also analytic relations.

Weakly Suslin partial orderings

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Axiom A weakly Suslin posets are Π^1_2-strongly proper

Proposition

If $\mathbb{P}$ is, provably in ZFC, an Axiom A weakly Suslin partial ordering, then it is $\mathbf{\Pi}^1_2$-strongly proper.

Products of Π^1_2-strongly proper forcing

Proposition

Suppose that $\mathbb{P}$ is projective and, provably in ZFC, equal to the ω-product with full support of projective forcing notions $\mathbb{P}_n$, $n < \omega$, and each $\mathbb{P}_n$ is $\mathbf{\Pi}^1_2$-strongly proper, then $\mathbb{P}$ is $\mathbf{\Pi}^1_2$-strongly proper.

Examples of Π^1_2-strongly proper forcing

Examples

Axiom A forcing notions with perfect sets as conditions, such as *Laver forcing*, *Mathias forcing*, *Miller forcing*, *Sacks forcing*, *Amoeba-Sacks*, and *Silver forcing*, as well as their finite or countable infinite products with full support, are weakly Suslin and $\mathbf{\Pi}^1_2$-strongly proper.

Π^1_2-strongly proper forcing and Solovay models

Theorem

Suppose that $L(\mathbb{R})$ is a Solovay model over some model W and $\mathbb{P}$ is weakly-Suslin and $\mathbf{\Pi}^1_2$-strongly proper (actually, $\mathbb{P}$ being $\mathbf{\Sigma}^1_3$ is enough). Then the $L(\mathbb{R})$ of any $\mathbb{P}$-generic extension is also a Solovay model over W.

Generic absoluteness under Π^1_2-strongly proper forcing

Theorem

Suppose $L(\mathbb{R})^V$ *is a Solovay model over some model* $W \subseteq V$*, and* G *and* H *are generic over* V *for weakly Suslin* $\mathbf{\Pi}^1_2$*-strongly proper forcing notions. Let* $\mathbb{R}_G$ *and* $\mathbb{R}_H$ *be the sets of reals of* $V[G]$ *and* $V[H]$*, respectively, and suppose* $\mathbb{R}_G \subseteq \mathbb{R}_H$*. Then there is an elementary embedding*

$$j : L(\mathbb{R}_G) \to L(\mathbb{R}_H)$$

that is the identity on the ordinals and the reals. In particular, there is such an embedding from $L(\mathbb{R})$ *into* $L(\mathbb{R}_G)$*.*

The Bernstein property for products

Theorem

If $L(\mathbb{R})$ is a Solovay model over some model W, then $L(\mathbb{R})$ satisfies that for every partition $g : (\omega^\omega)^n \to Or$, there are perfect sets P_i, $i < n$, such that the product $\prod_i P_i$ lies in one piece of the partition.

The same argument, but using just Shoenfield's absoluteness, also shows, in *ZFC*, that this partition property for products holds for Borel partitions into countably-many pieces.

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Recall:

Amoeba-Sacks forcing consists of pairs (m, P), where $m \in \omega$ and P is a perfect subtree of $2^{<\omega}$. The ordering is given by: $(m, P) \leq (n, Q)$ iff $m \geq n$, $P \subseteq Q$, and $P \cap 2^n = Q \cap 2^n$.

If G is Amoeba-Sacks generic over some model V, then $\bigcap\{P : (n, P) \in G, \text{ some } n\}$ is a perfect tree whose branches form a perfect set of Sacks reals over V.

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Lemma

Suppose $\prod_{i<\omega} G_i$ is generic over V for the ω-product with full support of Amoeba-Sacks forcing. For each i, let g_i be the filter on Sacks forcing generated by some infinite branch of the perfect set added by G_i. Then for every $n < \omega$, the product $\prod_{i<n} g_i$ is generic over V for the n-product of Sacks forcing.

The parameterized Bernstein property

Theorem (Di Prisco, 1993)

If $L(\mathbb{R})$ is a Solovay model over some model W, then in $L(\mathbb{R})$ for every partition $g : [\omega]^\omega \times \omega^\omega \to n$, there is a set $X \in [\omega]^\omega$ and a perfect set $P \subseteq \omega^\omega$ such that the product $[X]^\omega \times P$ lies in one piece of the partition.

The parameterized Bernstein property for finite products

Theorem

If $L(\mathbb{R})$ is a Solovay model over some model W, then $L(\mathbb{R})$ satisfies that for every partition $g : [\omega]^\omega \times (\omega^\omega)^n \to m$, there are $X \in [\omega]^\omega$ and perfect sets P_i, $i < n$, such that the product $[X]^\omega \times \prod_{i<n} P_i$ lies in one piece of the partition.

The argument also shows, using Shoenfield's absoluteness, that in *ZFC* the parameterized Bernstein property for products holds for Borel partitions.

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