Partial numberings and Precompleteness*

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Precompleteness is a powerful property of numberings. Most numberings commonly used in computability theory such as the Gödel numberings of the partial computable functions are precomplete. As is well known, exactly the precomplete numberings have the effective fixed point property. In this paper extensions of precompleteness to partial numberings are discussed. As is shown, most of the important properties shared by precomplete numberings carry over to the partial case.

1 Introduction

A numbering is a map from the natural numbers onto a given set. Numberings are a central tool in Russian style constructive mathematics allowing the transfer of computability concepts to abstract structures.

Apart from early investigations, e.g. by Mal'tsev [6, 5], in later studies mostly only totally defined numberings have been considered in the literature (cf. e.g. [1, 2]). Totality can always be assumed as long as purely algebraic structures are discussed. As was shown by the author [10], the situation changes if topological structures are studied: standard numberings of spaces without finite points are only partially defined, by necessity. Their domain of definition is Π^0_2-hard. Here, a point is called finite if its neighbourhood filter has a finite base.

A numbered set is a set together with a numbering of it. The natural numbers, e.g., form a numbered set with respect to the identity function as numbering. Morphisms between numbered sets are maps between the sets coming with a realizer, i.e. a computable function tracking names (indices) of the arguments to names of the corresponding objects under the map. In case of total numberings, realizers are total functions too.

A numbering is precomplete if every partial computable transformation of indices can be totalized relative to the numbering. A prominent example are the

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Gödel numberings of the partial computable functions, sometimes also called acceptable numberings. Precomplete numberings have important properties. Each of its fibers uniformly contains an infinite computably enumerable subset. Moreover, a numbering is precomplete, exactly if it has the effective fixed point property. As part of his work on numbered sets, Ershov [1] studied the category of (totally) numbered sets. This category has no nontrivial injective objects. If, however, we relativize the notion of an injective object by restricting universal quantification in its definition to those subobjects of the natural numbers that are effectively enumerable, a class of objects is obtained which turns out to contain exactly the precompletely numbered sets. This gives a nice category theoretic characterization of precompletely numbered sets.

In this paper we want to extend these results to the case of partial numberings. Realizers of morphisms need only be defined for names of objects in this case, not for all natural numbers. As has already been mentioned, in important special cases the logical complexity of the name set is much higher than that of computably enumerable sets. So, realizers will also be defined for numbers which are not names. They might even map such numbers to names of objects in the range of the morphism. The category of partially numbered sets is known to be Cartesian closed and equivalent to the category of modest sets which has been extensively studied in semantics of type systems [4]. Precompleteness can be extended to partial numberings in several ways. The notions differ e.g. in whether we demand that all values of a totalizer are names or not. Independently of how we decide, it may happen for some index that the value of a totalizer is a name while the value of the partial index function is not, though it exists. Correctly precomplete numberings satisfy the additional requirement that values of totalizers are names, just if the corresponding values of the given partial index functions are names. If totalizers are required to have only names as values, an analogue of Ershov's category theoretic characterization can be derived. The other properties of precomplete numberings mentioned above carry over to the partial case for all the extensions of precompleteness we study. Numberings of the respective kind are characterized by versions of the effective fixed point theorem, and their fibers uniformly contain infinite effectively enumerable sets. In case of correctly precomplete numberings we have in addition that also the set of non-names contains such a set.

The paper is organized as follows: Section 2 contains basic definitions. Then, in Section 3, various precompleteness notions are introduced and a category-theoretic characterization for one of these notions is derived. As we will see in Section 4, also in the partial case, for all the notions of precompleteness, exactly the numberings for which the effective fixed point theorem holds are precomplete. Further properties of precomplete numberings are discussed in Section 5.

2 Basic definitions

In what follows, let $\langle\ ,\ \rangle : \omega^2 \to \omega$ be a computable pairing function with corresponding projections π_1 and π_2 such that $\pi_i(\langle a_1, a_2\rangle) = a_i$. Let $P^{(n)}$ ($R^{(n)}$) denote the set of all n-ary partial (total) computable functions. For some Gödel numbering φ, we let $\varphi_i(a)\downarrow$ mean that the computation of $\varphi_i(a)$ stops, $\varphi_i(a)\downarrow \in C$ that it stops with value in C, and $\varphi_i(a)\downarrow_n$ that it stops within n steps. The complement of a set A is denoted by $\overline{A}$. Moreover, we write $F\colon A \rightharpoonup B$ to mean that F is a partial function from set A into set B with domain $\mathrm{dom}(F)$.

As is well known, a subset $C \subseteq \omega$ is decidable if there is a function $f \in R^{(1)}$ with $C = f^{-1}(\{0\})$. For $i \in \omega$ such that $\varphi_i \in R^{(1)}$ let $Z_i = \varphi_i^{-1}(\{0\})$. In any other case let Z be undefined. Then Z is a partial indexing of all decidable subsets of ω.

A *(partial) numbering* ν of a set S is a partial map $\nu\colon \omega \rightharpoonup S$ (onto). The value of ν at $n \in \mathrm{dom}(\nu)$ is denoted, interchangeably, by ν_n and $\nu(n)$. If $s \in S$ and $n \in \mathrm{dom}(\nu)$ with $\nu_n = s$, then n is said to be an *index* or a *name* of s. Numberings ν with $\mathrm{dom}(\nu) = \omega$, are called *total*. Let $\mathrm{Num}_p(S)$ be the set of all partial numberings of set S.

As was shown in [10], standard numberings of topological spaces like the computable real numbers with the Euclidean topology are only partially defined, by necessity. In these cases the domain of the numbering is Π^0_2-hard.

Definition 1. *For $S' \subseteq S$, $\nu' \in \mathrm{Num}_p(S')$ and $\nu, \kappa \in \mathrm{Num}_p(S)$,*

1. *ν' is m-*reducible *to ν, written $\nu' \leq_m \nu$, if there is some* reduction *function $g \in P^{(1)}$ with $\mathrm{dom}(\nu') \subseteq \mathrm{dom}(g)$, $g(\mathrm{dom}(\nu')) \subseteq \mathrm{dom}(\nu)$, and $\nu'_n = \nu_{g(n)}$, for all $n \in \mathrm{dom}(\nu')$.*

2. *ν is m-*equivalent *to κ, written $\nu \equiv_m \kappa$, if $\nu \leq_m \kappa$ and $\kappa \leq_m \nu$.*

This definition is due to Mal'cev [6].

Lemma 1. *Let S be nonempty and $\nu \in \mathrm{Num}_p(S)$ such that $\mathrm{dom}(\nu)$ is computably enumerable. Then S has a total numbering m-equivalent to ν.*

Proof. Since S is not empty, the same is true for $\mathrm{dom}(\nu)$. Therefore, there is some function $g \in R^{(1)}$ with $\mathrm{range}(g) = \mathrm{dom}(\nu)$. Set $g^*(j) = \mu i : g(i) = j$. Then $g^* \in P^{(1)}$ with $\mathrm{dom}(g^*) = \mathrm{range}(g)$. Moreover, for $j \in \mathrm{range}(g)$, $g(g^*(j)) = j$. Let $\bar{\nu} = \nu \circ g$. Then $\bar{\nu}$ is a total numbering of S. In addition, $\bar{\nu} \leq_m \nu$ via g and $\nu \leq_m \bar{\nu}$ via g^*. □

A somewhat stronger result has been shown by Mal'cev under the additional assumption that $\mathrm{dom}(\nu)$ is infinite [6, Theorem 2.2.1]. In this case g can be chosen as one-to-one.

A subset X of S is *completely enumerable*, if there is a computably enumerable set $C \subseteq \omega$ such that $\nu_i \in X$ if and only if $i \in C$, for all $i \in \mathrm{dom}(\nu)$. Thus, X is completely enumerable if we can enumerate all indices of elements of X and perhaps some numbers which are not used as names by numbering ν.

3 Precompleteness

Precompleteness is a powerful property shared by most numberings commonly used in computability theory: all Gödel numberings, e.g., as well as the numbering W of the computably enumerable sets are precomplete. Here, we will discuss extensions of this notion to partial numberings. We start with a definition due to Selivanov [9].

Definition 2. *A numbering* $\nu \in \mathrm{Num}_p(S)$ *is* precomplete*, if for any function* $p \in P^{(1)}$ *there is a function* $g \in R^{(1)}$ *with* $\mathrm{range}(g) \subseteq \mathrm{dom}(\nu)$ *such that for all* $i \in p^{-1}(\mathrm{dom}(\nu))$,

$$\nu_{p(i)} = \nu_{g(i)}.$$

Function g *is called* totalizer *of* p *or said to* totalize p.

Mal'tsev defined precomplete numberings as partial numberings with the effective fixed point property. In [6] they were called "complete". Later, when introducing the completeness notion for total numberings that we still use [7], he changed the notion into "precomplete". Mal'cev's first definition was rather strong as he could show that such numberings are necessarily totally defined [6, p. 182]. Ershov [1, 2] considered only total numberings and gave two new characterizations of precomplete numberings: in categorical terms and in terms of totalizers. The later one has become the standard definition of precompleteness in modern text books (cf. e.g. [12]). Both of Ershov's characterizations carry over to the partial case, thus showing the naturalness of the above definition.

If S is a set and $\nu \in \mathrm{Num}_p(S)$, (S, ν) is called *numbered set*. With respect to the identical numbering id_ω the set ω of all natural numbers is a numbered set which we denote by $\mathcal{N}$. In the case of total numberings morphisms between numbered sets are such that in particular every total numbering $\nu\colon \omega \to S$ is a morphism from $\mathcal{N}$ to (S, ν). Therefore, morphisms between partially numbered sets need not be total maps.

As has already been pointed out, in order to allow computability considerations for abstract objects, such objects are represented by natural numbers via numberings. So, the computability of a map between spaces of abstract objects can only mean that we have an algorithm tracking names of arguments into names of values under the given map.

Definition 3. *Let* (S,ν) *and* (S',ν') *be numbered sets. A partial map* $F\colon S \rightharpoonup S'$ *is* realizable *if it has a* realizer, *i.e. a function* $f \in P^{(1)}$ *with* $\mathrm{dom}(f) \supseteq \nu^{-1}(\mathrm{dom}(F))$ *and* $f(\nu^{-1}(\mathrm{dom}(F))) \subseteq \mathrm{dom}(\nu')$ *such that*

$$F(\nu_i) = \nu'_{f(i)},$$

for all $i \in \nu^{-1}(\mathrm{dom}(F))$.

Let $\mathbf{NUM_p}$ be the category of numbered sets with realizable partial maps as morphisms. For objects $\mathcal{S} = (S,\nu)$ and $\mathcal{S}' = (S',\nu')$ in $\mathbf{NUM_p}$, $(\mathcal{S}',F')$ is a *subobject* of $\mathcal{S}$ in $\mathbf{NUM_p}$, if $F'\colon S' \to S$ is a total one-to-one realizable map.

Definition 4. *Let* $\mathcal{S} = (S,\nu)$ *be a numbered set.*

1. *A numbering* ν' *of a subset* S' *of* S *is* principal, *if for every* $\kappa \in \mathrm{Num}_p(S')$,

$$\kappa \le_m \nu \Longrightarrow \kappa \le_m \nu'.$$

2. *A subobject* $((S'',\nu''),F'')$ *of* $\mathcal{S}$ *is a* c-subobject, *if* $F'' \circ \nu''$ *is a principal numbering of* $\mathrm{range}(F'')$.

Definitions 4(1) and (2) are due to Mal'cev [7] and Ershov [2], respectively.

Lemma 2. *Let* $\mathcal{S} = (S,\nu)$ *be a numbered set and* S' *a nonempty completely enumerable subset of* S. *Then* S' *has a principal numbering.*

Proof. The proof is a modification of the proof in [1]. Let $A \subseteq \omega$ witness that S' is completely enumerable and let $f \in R^{(1)}$ enumerate A. Define $\nu' = \nu \circ f$. If $f(n) \in \mathrm{dom}(\nu)$ then $\nu'_n \in S'$. Otherwise, ν'_n is undefined. Thus, $\mathrm{dom}(\nu') = f^{-1}(\mathrm{dom}(\nu))$, which means that $\nu' \in \mathrm{Num}_p(S')$.

Now, let $\kappa \in \mathrm{Num}_p(S')$ with $\kappa \le_m \nu$ and let $g \in P^{(1)}$ be the corresponding reduction function. Then

$$g(\mathrm{dom}(\kappa)) \subseteq \nu^{-1}(S') \subseteq \mathrm{range}(f). \tag{1}$$

Set

$$h(n) = \begin{cases} \mu m : f(m) = g(n) & \text{if } g(n)\downarrow, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Then $h \in P^{(1)}$ with $\mathrm{dom}(h) \subseteq \mathrm{dom}(g)$. Moreover, $f(h(n)) = g(n)$, for $n \in \mathrm{dom}(h)$. Let $n \in \mathrm{dom}(\kappa)$. Then $n \in \mathrm{dom}(g)$, from which we obtain with (1) that $g(n) \in \mathrm{range}(f)$. Hence, $g(n) = f(h(n))$. Since $g(\mathrm{dom}(\kappa)) \subseteq \mathrm{dom}(\nu)$, it follows that $f(h(n)) \in \mathrm{dom}(\nu)$. We thus have that $\mathrm{dom}(\kappa) \subseteq \mathrm{dom}(h)$ and $h(\mathrm{dom}(\kappa)) \subseteq \mathrm{dom}(\nu')$. Moreover, $\kappa_n = \nu_{g(n)} = \nu_{f(h(n))} = \nu'_{h(n)}$. Therefore, $\kappa \le_m \nu'$. □

Definition 5. *A subobject* $((S', \nu'), F')$ *of a numbered set* $\mathcal{S} = (S, \nu)$ *is an* e-subobject *of* $\mathcal{S}$ *if* $((S', \nu'), F')$ *is a c-subobject of* $\mathcal{S}$ *and* $F'(S')$ *is a completely enumerable subset of* S.

Definition 5 is due to Ershov [2].

As we have seen in the preceding lemma, for each completely enumerable subset S' of S a numbering ν' can be defined so that (S', ν') is an e-subobject of $\mathcal{S}$ with respect to the identical embedding.

Theorem 3. *Let* $\mathcal{S} = (S, \nu)$ *be a numbered set. Then* ν *is precomplete if, and only if, for each e-subobject* $(\mathcal{T}, F)$ *of* $\mathcal{N}$ *and every realizable map* $H \colon T \rightharpoonup S$ *there is a total realizable map* $G \colon \omega \to S$ *such that*

$$H = G \circ F.$$

Proof. Let us first assume that ν is precomplete. Moreover, let $(\mathcal{T}, F)$ with $\mathcal{T} = (T, \kappa)$ be an e-subobject of $\mathcal{N}$ and $H \colon T \rightharpoonup S$ a realizable map. Suppose that F and H, respectively, are realized by $f, h \in P^{(1)}$. Then $F(T) = f(\mathrm{dom}(\kappa))$. By assumption $F(T)$ is computably enumerable. Thus, $f^{-1}(F(T))$ is computably enumerable as well. In addition, $f^{-1}(F(T)) \supseteq \mathrm{dom}(\kappa)$.

If T is empty, κ and F are empty maps. Let p be the nowhere defined function in this case. Otherwise, if T is not empty, the sets $F(T)$, $\mathrm{dom}(\kappa)$ and hence $f^{-1}(F(T))$ are nonempty as well. Let some enumeration of $f^{-1}(F(T))$ be fixed and for $n \in \omega$, set $k(n)$ to be the first m in this enumeration with $f(m) = n$. Then $k \in P^{(1)}$ and $f(k(n)) = n$, for $n \in F(T))$. Now, let $p = h \circ k$.

In both cases, $p \in P^{(1)}$. Let $g \in R^{(1)}$ with $\mathrm{range}(g) \subseteq \mathrm{dom}(\nu)$ be a totalizer of p, i.e., $\nu_{p(n)} = \nu_{g(n)}$, for $n \in p^{-1}(\mathrm{dom}(\nu))$. Set $G = \nu \circ g$. Since g is total, we have that $G \colon \omega \to S$ is a total map realized by g.

Suppose that T is not empty and let $x \in T$. Then $F(x) \in F(T)$. Thus, $k(F(x))$ is defined. Since $F(\kappa_{k(F(x))}) = f(k(F(x))) = F(x)$ and F is one-to-one, it follows that $\kappa_{k(F(x))} = x$, whereby we obtain that

$$H(x) = H(\kappa_{k(F(x))}) = \nu_{h(k(F(x)))} = \nu_{g(F(x))} = G(F(x)).$$

If T is empty, the last equation holds trivially.

For the converse implication let $p, h \in P^{(1)}$ with $\mathrm{range}(h) = \mathrm{dom}(p)$ and set $\mathcal{T} = (\mathrm{dom}(p), h)$. Moreover, set $F(x) = x$, for $x \in \mathrm{dom}(p)$. Then we have for $n \in \mathrm{dom}(h)$ that $F(h(n)) = h(n) = \mathrm{id}_\omega(h(n))$. Thus, $F \colon \mathrm{dom}(p) \to \omega$ is a one-to-one realizable total map and $(\mathcal{T}, F)$ is an e-subobject of $\mathcal{N}$.

Now, define $H(n) = \nu_{p(n)}$, for $n \in p^{-1}(\mathrm{dom}(\nu))$. Then we have for $n \in h^{-1}(p^{-1}(\mathrm{dom}(\nu)))$ that $H(h(n)) = \nu_{p(h(n))}$. Therefore, $H \colon T \rightharpoonup S$ is a realizable partial map. By assumption there is some total realizable map $G \colon \omega \to S$

with $H = G \circ F$. Let $g \in P^{(1)}$ realize G. Then we have that $\mathrm{dom}(g) \supseteq \omega$ and $\mathrm{range}(g) \subseteq \mathrm{dom}(\nu)$, which in particular means that $g \in R^{(1)}$. Moreover, we have for $n \in p^{-1}(\mathrm{dom}(\nu))$ that

$$\nu_{p(n)} = H(n) = G(F(n)) = G(n) = \nu_{g(n)}. \qquad \square$$

As follows from Definition 2, for a totalizer g of a function $p \in P^{(1)}$ we always have that $g(n) \in \mathrm{dom}(\nu)$, though there may be some $n \in \mathrm{dom}(p)$ with $p(n) \notin \mathrm{dom}(\nu)$. In the case of total numberings this problem will not occur. When dealing with partial numberings, however, there are situations in which a more symmetric notion of precompleteness is needed (cf. [11]).

Definition 6. *A numbering $\nu \in \mathrm{Num}_p(S)$ is*

1. faintly precomplete*, if for any function $p \in P^{(1)}$ there is a function $g \in R^{(1)}$ such the following two conditions hold, for all $i \in \mathrm{dom}(p)$,*

   (a) $p(i) \in \mathrm{dom}(\nu) \Longrightarrow g(i) \in \mathrm{dom}(\nu)$,

   (b) $p(i) \in \mathrm{dom}(\nu) \Longrightarrow \nu_{p(i)} = \nu_{g(i)}$.

   Function g is called faint totalizer *of p or said to* faintly totalize *p.*

2. correctly precomplete*, if for any function $p \in P^{(1)}$ there is a function $g \in R^{(1)}$ such the following two conditions hold, for all $i \in \mathrm{dom}(p)$,*

   (a) $p(i) \in \mathrm{dom}(\nu) \Longleftrightarrow g(i) \in \mathrm{dom}(\nu)$,

   (b) $p(i) \in \mathrm{dom}(\nu) \Longrightarrow \nu_{p(i)} = \nu_{g(i)}$.

   In this case g is called correct totalizer *of p or said to* correctly totalize *p.*

Note that in case of a correctly precomplete numbering ν and a correct totalizer g of some computable partial function p, for an argument $i \notin \mathrm{dom}(p)$ the value $g(i)$ is not determined to be in $\mathrm{dom}(\nu)$, or not to be in $\mathrm{dom}(\nu)$, respectively, by the above definition.

Obviously, precompleteness in the sense of Definition 2 and correct precompleteness both imply faint precompleteness. In the case of total numberings all three notions coincide. As follows from the next results, however, the first two concepts are incomparable in the case of proper partial numberings.

Lemma 4. *If $\nu \in \mathrm{Num}_p(S)$ is not total, there is some $p \in P^{(1)}$ such that no $g \in R^{(1)}$ both totalizes and correctly totalizes p.*

Proof. Since ν is not total, there is some $\bar{a} \in \overline{\mathrm{dom}(\nu)}$. Set $p(\bar{a}) = \bar{a}$ and let p be undefined, otherwise. Moreover, assume that $g \in R^{(1)}$ totalizes as well as correctly totalizes p. Then

$$\mathrm{range}\, g \subseteq \mathrm{dom}(\nu) \tag{2}$$

and for $a \in \mathrm{dom}(p)$,

$$p(a) \in \mathrm{dom}(\nu) \Longleftrightarrow g(a) \in \mathrm{dom}(\nu). \tag{3}$$

By the choice of $\bar{a}$ and p, it follows from (3) that $g(\bar{a}) \notin \mathrm{dom}(\nu)$, in contradiction to (2). □

Let φ be a Gödel numbering of $P^{(1)}$ and $\hat{\varphi}$ be the co-restriction of φ to $R^{(1)}$.

Lemma 5. *$\hat{\varphi}$ is correctly precomplete, but not precomplete.*

Proof. We first show that $\hat{\varphi}$ is correctly precomplete. Let to this end $p \in P^{(1)}$. Since Gödel numberings are precomplete [12], there is some $g \in R^{(1)}$ with $\varphi_{p(i)} = \varphi_{g(i)}$, for $i \in \mathrm{dom}(p)$. It follows that g correctly totalizes p with respect to $\hat{\varphi}$.

Let us next assume that $\hat{\varphi}$ is also precomplete. Define $q \in P^{(1)}$ by $q(i) = i$, for $i \in \omega$. Then there is some $f \in R^{(1)}$ with $\mathrm{range}(f) \subseteq \mathrm{dom}(\hat{\varphi})$ and $\hat{\varphi}_{q(i)} = \hat{\varphi}_{f(i)}$, for $i \in q^{-1}(\mathrm{dom}(\hat{\varphi}))$. Thus, $\hat{\varphi}_{f(i)} = \hat{\varphi}_i$, for $i \in \mathrm{dom}(\hat{\varphi})$. It follows that $\lambda i.\varphi_{f(i)}$ is an enumeration of $R^{(1)}$ with computable universal function. By [12, p. 116, Theorem 13], however, such enumerations do not exist. □

Next, let K be the halting set. Set $\nu_i = 0$, if $i \in K$, and let ν be undefined, otherwise. Then $\nu \in \mathrm{Num}_p(\{0\})$.

Lemma 6. *ν is precomplete, but not correctly precomplete.*

Proof. Let $p \in P^{(1)}$ and $a_0 \in K$. Set $g(a) = a_0$, for $a \in \omega$. Then $g \in R^{(1)}$ with $\mathrm{range}(g) \subseteq \mathrm{dom}(\nu)$ and $\nu_{p(a)} = \nu_{g(a)}$, for $a \in p^{-1}(\mathrm{dom}(\nu))$. Thus, g totalizes p.

In order to see that ν is not correctly precomplete, let $a_1 \in \overline{K}$ and set $q(a) = a_1$, for $a \in K$. In any other case, let q be undefined. Then $q \in P^{(1)}$. As is well known [8, p. 81, Theorem II(a)], K is not m-reducible to its complement, i.e., $h(K)$ intersects K, for all $h \in R^{(1)}$. Let $a_h \in K$ with $h(a_h) \in K$. Then $a_h \in \mathrm{dom}(q)$ such that $h(a_h) \in \mathrm{dom}(\nu)$, but $q(a_h) \notin \mathrm{dom}(\nu)$. It follows that q cannot be correctly totalized. □

4 The effective fixed point property

As we will see next, also in the partial case and for all precompleteness notions just introduced, precomplete numberings are exactly those numberings that have an effective fixed point property. We start with faint precompleteness.

Theorem 7. *Let $\nu \in \mathrm{Num}_p(S)$. Then the following four statements are equivalent:*

1. *Numbering ν is faintly precomplete.*
2. *There is some function $h \in R^{(1)}$ such that the subsequent two requirements hold, for all $i \in \omega$,*
   - *(a)* $\varphi_i(h(i))\downarrow \in \mathrm{dom}(\nu) \Longrightarrow h(i) \in \mathrm{dom}(\nu)$,
   - *(b)* $\varphi_i(h(i))\downarrow \in \mathrm{dom}(\nu) \Longrightarrow \nu_{\varphi_i(h(i))} = \nu_{h(i)}$.
3. *There is some function $h \in R^{(1)}$ such that the subsequent two requirements hold, for all $i \in \omega$ with $\varphi_i \in R^{(1)}$,*
   - *(a)* $\varphi_i(h(i)) \in \mathrm{dom}(\nu) \Longrightarrow h(i) \in \mathrm{dom}(\nu)$,
   - *(b)* $\varphi_i(h(i)) \in \mathrm{dom}(\nu) \Longrightarrow \nu_{\varphi_i(h(i))} = \nu_{h(i)}$.
4. *There is some function $h \in R^{(1)}$ such that the subsequent two requirements hold, for all $i \in \omega$ with $\varphi_i \in R^{(1)}$ and* $\mathrm{range}(\varphi_i) \subseteq \mathrm{dom}(\nu)$,
   - *(a)* $h(i) \in \mathrm{dom}(\nu)$,
   - *(b)* $\nu_{\varphi_i(h(i))} = \nu_{h(i)}$.

Proof. The proof follows the one for total numberings [12]. We first show that (1) implies (2). Let $p \in P^{(1)}$ be defined by $p(n) = \varphi_n(n)$ and $g \in R^{(1)}$ faintly totalize p. Moreover, let $q \in R^{(1)}$ such that $\varphi_{q(i)} = \varphi_i \circ g$ and set $h = g \circ q$. Then $h \in R^{(1)}$. In addition, we have for $i \in \omega$ with $\varphi_i(h(i))$ being defined that $\varphi_i(h(i)) = \varphi_i(g(q(i))) = \varphi_{q(i)}(q(i)) = p(q(i))$. By definition, $h(i) = g(q(i))$. If $\varphi_i(h(i)) \in \mathrm{dom}(\nu)$, it follows that $p(q(i)) \in \mathrm{dom}(\nu)$ and hence that $g(q(i)) \in \mathrm{dom}(\nu)$, as g faintly totalizes p. Furthermore, $\nu_{p(q(i))} = \nu_{g(q(i))}$. Thus, $\nu_{\varphi_i(h(i))} = \nu_{h(i)}$.

Obviously, (3) is a special case of (2) and (4) a special case of (3). So, it remains to show that (4) entails (1). Assume that $p \in P^{(1)}$ and let $q \in R^{(1)}$ with $\varphi_{q(n)}(m) = p(n)$. Set $g = h \circ q$. Then $g \in R^{(1)}$. Moreover, we have for $i \in \mathrm{dom}(p)$ with $p(i) \in \mathrm{dom}(\nu)$ that $\varphi_{q(i)} \in R^{(1)}$ and $\mathrm{range}(\varphi_{q(i)}) \subseteq \mathrm{dom}(\nu)$. Hence, $h(q(i)) \in \mathrm{dom}(\nu)$ as well and $\nu_{\varphi_{q(i)}(h(q(i)))} = \nu_{h(q(i))}$, i.e., $g(i) \in \mathrm{dom}(\nu)$ and $\nu_{p(i)} = \nu_{g(i)}$. □

The corresponding result for precomplete numberings is just a consequence of this proof.

Theorem 8. *Let* $\nu \in \mathrm{Num}_p(S)$. *Then the following four statements are equivalent:*

1. *Numbering* ν *is precomplete.*
2. *There is some function* $h \in R^{(1)}$ *with* $\mathrm{range}(h) \subseteq \mathrm{dom}(\nu)$ *such that for all* $i \in \omega$ *with* $\varphi_i(h(i))\downarrow \in \mathrm{dom}(\nu)$,
$$\nu_{\varphi_i(h(i))} = \nu_{h(i)}.$$
3. *There is some function* $h \in R^{(1)}$ *with* $\mathrm{range}(h) \subseteq \mathrm{dom}(\nu)$ *such that for all* $i \in \omega$ *with* $\varphi_i \in R^{(1)}$ *and* $\varphi_i(h(i)) \in \mathrm{dom}(\nu)$,
$$\nu_{\varphi_i(h(i))} = \nu_{h(i)}.$$
4. *There is some function* $h \in R^{(1)}$ *with* $\mathrm{range}(h) \subseteq \mathrm{dom}(\nu)$ *such that for all* $i \in \omega$ *with* $\varphi_i \in R^{(1)}$ *and* $\mathrm{range}(\varphi_i) \subseteq \mathrm{dom}(\nu)$,
$$\nu_{\varphi_i(h(i))} = \nu_{h(i)}.$$

In the preceding theorems it may happen that $\varphi_i(h(i))\downarrow \notin \mathrm{dom}(\nu)$, but $h(i) \in \mathrm{dom}(\nu)$. A similar asymmetry was found in the definitions of precompleteness and faint precompleteness. It motivated us to introduce correct precompeteness. As we will see now, in this case we also have a symmetric version of the effective fixed point theorem.

Theorem 9. *Let* $\nu \in \mathrm{Num}_p(S)$. *Then the following three statements are equivalent:*

1. *Numbering* ν *is correctly precomplete.*
2. *There is some function* $h \in R^{(1)}$ *such that the subsequent two requirements hold, for all* $i \in \omega$ *for which* $\varphi_i(h(i))$ *is defined,*
   (a) $\varphi_i(h(i)) \in \mathrm{dom}(\nu) \Longleftrightarrow h(i) \in \mathrm{dom}(\nu)$,
   (b) $\varphi_i(h(i)) \in \mathrm{dom}(\nu) \Longrightarrow \nu_{\varphi_i(h(i))} = \nu_{h(i)}$.
3. *There is some function* $h \in R^{(1)}$ *such that the subsequent two requirements hold, for all* $i \in \omega$ *with* $\varphi_i \in R^{(1)}$,
   (a) $\varphi_i(h(i)) \in \mathrm{dom}(\nu) \Longleftrightarrow h(i) \in \mathrm{dom}(\nu)$,

(b) $\varphi_i(h(i)) \in \mathrm{dom}(\nu) \Longrightarrow \nu_{\varphi_i(h(i))} = \nu_{h(i)}$.

Proof. Because of Theorem 7 we only have to show that the additional implications hold. Assume first that ν is correctly precomplete and let h be as in the proof of 7((1)⇒(2)). Suppose that $h(i) \in \mathrm{dom}(\nu)$, i.e., $g(q(i)) \in \mathrm{dom}(\nu)$. Since g correctly totalizes p, it follows that $p(q(i)) \in \mathrm{dom}(\nu)$, which means that $\varphi_i(h(i)) \in \mathrm{dom}(\nu)$ as well.

Next, assume that Statement (3) holds and $p \in P^{(1)}$. Let $g \in R^{(1)}$ be the totalizer constructed in the proof of 7((4)⇒(1)) and suppose that $g(i) \in \mathrm{dom}(\nu)$, for $i \in \mathrm{dom}(p)$. Then $h(q(i)) \in \mathrm{dom}(\nu)$ and hence $\varphi_{q(i)}(h(q(i))) \in \mathrm{dom}(\nu)$, i.e., $p(i) \in \mathrm{dom}(\nu)$. □

5 Further properties

The subsequent result shows that each fiber of a faintly precomplete numbering is uniformly undecidable.

Lemma 10. *Let S contain at least two elements and $\nu \in \mathrm{Num}_p(S)$ be faintly precomplete. Then there is some function $r \in P^{(1)}$ such the following properties hold, for all $j \in \mathrm{dom}(Z)$ and $s \in S$,*

1. $r(j){\downarrow}$,
2. $\emptyset \neq Z_j \subseteq \nu^{-1}(\{s\}) \Longrightarrow r(j) \in \nu^{-1}(\{s\}) \setminus Z_j$.

If ν is even correctly precomplete, r has the additional property that for $j \in \mathrm{dom}(Z)$,

3. $\emptyset \neq Z_j \subseteq \overline{\mathrm{dom}(\nu)} \Longrightarrow r(j) \in \overline{\mathrm{dom}(\nu)} \setminus Z_j$.

Proof. The proof is again a refinement of the one for total numberings [12]. Since S has at least two elements there are indices $a, b \in \mathrm{dom}(\nu)$ with $\nu_a \neq \nu_b$. Define $f, g \in P^{(1)}$ as follows:

$$f(\langle j,n\rangle) = \begin{cases} a & \text{if } \varphi_j(n){\downarrow} \text{ with } \varphi_j(n) = 0,\\ \pi_1(\mu\langle d,e\rangle : [\varphi_j(d){\downarrow}_e \wedge \varphi_j(d) = 0]) & \text{if } \varphi_j(n){\downarrow} \text{ with } \varphi_j(n) \neq 0 \\ & \quad \text{and for some } d \in \omega,\ \varphi_j(d){\downarrow} \text{ with } \varphi_j(d) = 0,\\ \text{undefined} & \text{otherwise,} \end{cases}$$

$$g(\langle j,n\rangle) = \begin{cases} b & \text{if } \varphi_j(n){\downarrow} \text{ with } \varphi_j(n) = 0,\\ a & \text{if } \varphi_j(n){\downarrow} \text{ with } \varphi_j(n) \neq 0,\\ \text{undefined} & \text{otherwise.} \end{cases}$$

Since ν is faintly precomplete there are functions $f', g' \in R^{(1)}$ such that for all $i \in \mathrm{dom}(f)$,

$$f(i) \in \mathrm{dom}(\nu) \Longrightarrow f'(i) \in \mathrm{dom}(\nu) \tag{4}$$

$$\nu_{f'(i)} = \nu_{f(i)}, \text{ if } f(i) \in \mathrm{dom}(\nu), \tag{5}$$

and for all $i \in \mathrm{dom}(g)$,

$$g(i) \in \mathrm{dom}(\nu) \Longrightarrow g'(i) \in \mathrm{dom}(\nu) \tag{6}$$

$$\nu_{g'(i)} = \nu_{g(i)}, \text{ if } g(i) \in \mathrm{dom}(\nu).$$

As ν has the effective fixed point property, there are further on functions $h, k \in R^{(1)}$ so that for all $j \in \omega$,

$$f'(\langle j, h(j)\rangle) \in \mathrm{dom}(\nu) \Longrightarrow h(j) \in \mathrm{dom}(\nu) \tag{7}$$

$$\nu_{f'(\langle j,h(j)\rangle)} = \nu_{h(j)}, \text{ if } f'(\langle j, h(j)\rangle) \in \mathrm{dom}(\nu), \tag{8}$$

and

$$g'(\langle j, k(j)\rangle) \in \mathrm{dom}(\nu) \Longrightarrow k(j) \in \mathrm{dom}(\nu) \tag{9}$$

$$\nu_{g'(\langle j,k(j)\rangle)} = \nu_{k(j)}, \text{ if } g'(\langle j, k(j)\rangle) \in \mathrm{dom}(\nu).$$

Now, define $r \in P^{(1)}$ by

$$r(j) = \begin{cases} h(j) & \text{if } \varphi_j(h(j))\downarrow \text{ with } \varphi_j(h(j)) \neq 0, \\ k(j) & \text{if } \varphi_j(h(j))\downarrow \text{ with } \varphi_j(h(j)) = 0, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

It remains to show that r has the desired properties. Let to this end $j \in \mathrm{dom}(Z)$ with Z_j being nonempty. Then φ_j is total. Since h and k are total functions as well, we obtain that $r(j)$ is defined. Moreover, it follows that there is a smallest number $\langle d, e\rangle$ so that $\varphi_j(d)\downarrow_e$ with $\varphi_j(d) = 0$.

Now assume that $s \in S$ such that $Z_j \subseteq \nu^{-1}(\{s\})$. Then $d \in Z_j \subseteq \nu^{-1}(\{s\}) \subseteq \mathrm{dom}(\nu)$. Thus, $f(\langle j, h(j)\rangle)\downarrow \in \mathrm{dom}(\nu)$, from which we obtain with (4) and (7) that $f'(\langle j, h(j)\rangle), h(j) \in \mathrm{dom}(\nu)$ as well.

As $\mathrm{range}(\lambda n.g(\langle j, n\rangle)) \subseteq \mathrm{dom}(\nu)$, it follows with (6) and (9) that $g(\langle j, k(j)\rangle)$, $g'(\langle j, k(j)\rangle)$, $k(j) \in \mathrm{dom}(\nu)$.

Let us now consider the cases that $\varphi_j(h(j)) \neq 0$ or $\varphi_j(h(j)) = 0$.

Case $\varphi_j(h(j)) \neq 0$: In this case we have that $r(j) = h(j)$. In addition, $h(j) \notin Z_j$. Thus, $r(j) \in \mathrm{dom}(\nu) \setminus Z_j$. Because of (5) and (8) it follows further on that

$$\nu_{r(j)} = \nu_{h(j)} = \nu_{f'(\langle j,h(j)\rangle)} = \nu_{f(\langle j,h(j)\rangle)} = \nu_d = s.$$

Consequently, $r(j) \in \nu^{-1}(\{s\}) \setminus Z_j$.

Case $\varphi_j(h(j)) = 0$: Now, $h(j) \in Z_j \subseteq \nu^{-1}(\{s\})$ and therefore

$$s = \nu_{h(j)} = \nu_{f'(\langle j,h(j)\rangle)} = \nu_{f(\langle j,h(j)\rangle)} = \nu_a.$$

Assume that $\varphi_j(k(j)) = 0$. Then $k(j) \in Z_j \subseteq \nu^{-1}(\{s\})$. It follows that

$$\nu_a = s = \nu_{k(j)} = \nu_{g'(\langle j,k(j)\rangle)} = \nu_{g(\langle j,k(j)\rangle)} = \nu_b,$$

contradicting our choice of a and b. Hence, $\varphi_j(k(j)) \neq 0$, which means that $r(j) \notin Z_j$. Moreover, we have that

$$\nu_{r(j)} = \nu_{k(j)} = \nu_{g'(\langle j,k(j)\rangle)} = \nu_{g(\langle j,k(j)\rangle)} = \nu_a.$$

This shows that $r(j) \in \nu^{-1}(\{s\}) \setminus Z_j$ in the second case as well.

Next, we assume that ν is even correctly precomplete and deal with the case that $Z_j \subseteq \overline{\mathrm{dom}(\nu)}$. Note that now also the reverse implications hold in (4), (6), (7) and (9). Again we consider the two cases:

Case $\varphi_j(h(j)) \neq 0$: Now, $h(j) \notin Z_j$, $r(j) = h(j)$, and $f(\langle j, h(j)\rangle) = d$. Since $d \in Z_j \subseteq \overline{\mathrm{dom}(\nu)}$, we obtain with the reverse implications in (4) and (7) that $h(j) \in \overline{\mathrm{dom}(\nu)}$. Thus, $r(j) \in \overline{\mathrm{dom}(\nu)} \setminus Z_j$.

Case $\varphi_j(h(j)) = 0$: In this case $h(j) \in Z_j \subseteq \overline{\mathrm{dom}(\nu)}$. On the other hand, $f(\langle j, h(j)\rangle) = a \in \mathrm{dom}(\nu)$, contradicting Properties (4) and (7). Thus, this case will not appear. □

As a consequence we obtain that each fiber of a faintly precomplete numbering uniformly contains an infinite computably enumerable set.

Proposition 11. *Let S contain at least two elements and $\nu \in \mathrm{Num}_p(S)$ be faintly precomplete. Then there is a one-to-one function $g \in R^{(2)}$ such that for all $m \in \mathrm{dom}(\nu)$ and $n \in \omega$,*

1. $g(m,n) \in \mathrm{dom}(\nu)$,
2. $\nu_{g(m,n)} = \nu_m$.

If ν is even correctly precomplete, g has the additional property that for $m \in \overline{\mathrm{dom}(\nu)}$,

3. $g(m,n) \in \overline{\mathrm{dom}(\nu)}$.

Proof. Let $q \in R^{(2)}$ with

$$\varphi_{q(i,a)}(n) = \begin{cases} \varphi_i(n) & \text{if } n \neq a, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathrm{range}(\lambda a.q(i,a)) \subseteq \mathrm{dom}(Z)$, for $i \in \mathrm{dom}(Z)$. Moreover, $Z_{q(i,a)} = Z_i \cup \{a\}$ in this case. Let a_0 be a Z-index of the empty set and define

$$\begin{aligned} p(m,0) &= q(a_0,m), \\ p(m,n+1) &= q(p(m,n), r(p(m,n))), \end{aligned}$$

where the function r is as in Lemma 10. Then it follows by an easy induction on n that for all $m,n \in \omega$,

$$\begin{gathered} p(m,n)\downarrow \in \mathrm{dom}(Z), \\ r(p(m,n)) \in Z_{p(m,n+1)} \setminus Z_{p(m,n)}, \\ m \in \mathrm{dom}(\nu) \Longrightarrow r(p(m,n)) \in \mathrm{dom}(\nu), \\ \nu_{r(p(m,n))} = \nu_m, \text{ if } m \in \mathrm{dom}(\nu). \end{gathered} \tag{10}$$

In particular, we have that $r(p(m,n)) \neq r(p(m,n'))$, for all $n,n' \in \omega$ with $n \neq n'$. If ν is correctly precomplete, also the reverse implication in (10) holds. Therefore, the function $g'(m,n) = r(p(m,n))$ nearly has the desired properties. It remains to turn it into a one-to-one function. Let to this end $\langle\!\langle m,n\rangle\!\rangle = (m+n)(m+n+1)/2+n$ be the well known Cantor pairing function and set

$$\begin{aligned} &g(0,0 = g'(0,0), \\ &g(m,n) = g'(m, \mu a : g'(m,a) \notin \{\, g(m',n') \mid \langle\!\langle m',n'\rangle\!\rangle < \langle\!\langle m,n\rangle\!\rangle \,\}), \end{aligned}$$

for $m,n \in \omega$ with $\langle\!\langle m,n\rangle\!\rangle > 0$. Then g is as desired. □

An easy conclusion of the last result is that a numbering is faintly precomplete if every partial computable function has a uniform infinite family of faint totalizers, and similarly in the precomplete and correctly precomplete case.

Theorem 12. *Let S contain at least two elements and $\nu \in \mathrm{Num}_p(S)$. Then the following statements hold:*

1. *Numbering ν is faintly precomplete if, and only if, for every function $p \in P^{(1)}$ there is a one-to-one function $h \in R^{(2)}$ such that for all $m \in p^{-1}(\mathrm{dom}(\nu))$ and all $n \in \omega$, $h(m,n) \in \mathrm{dom}(\nu)$ with*

$$\nu_{p(m)} = \nu_{h(m,n)}.$$

2. *Numbering ν is precomplete if, and only if, for every function $p \in P^{(1)}$ there is a one-to-one function $h \in R^{(2)}$ with $\mathrm{range}(h) \subseteq \mathrm{dom}(\nu)$ such that for all $m \in p^{-1}(\mathrm{dom}(\nu))$ and all $n \in \omega$,*

$$\nu_{p(m)} = \nu_{h(m,n)}.$$

3. *Numbering ν is correctly precomplete if, and only if, for every function $p \in P^{(1)}$ there is a one-to-one function $h \in R^{(2)}$ such that for all $m, n \in \omega$,*
   (a) $p(m) \in \mathrm{dom}(\nu) \Longleftrightarrow h(m,n) \in \mathrm{dom}(\nu)$,
   (b) $p(m) \in \mathrm{dom}(\nu) \Longrightarrow \nu_{p(m)} = \nu_{h(m,n)}$.

Proof. In all three cases the "if" part is obvious. For the converse direction let $g \in R^{(2)}$ be as in the preceding proposition. Moreover, for a given function $p \in P^{(1)}$, let $f \in R^{(1)}$ be a (correct, faint) totalizer. Then set $h(m,n) = g(f(m),n)$. As is easily verified, h has the desired properties. □

Definition 7. *A numbering $\nu \in \mathrm{Num}_p(S)$ is* (faintly, correctly) 1-precomplete *if every function $p \in P^{(1)}$ has a (faint, correct) one-to-one totalizer.*

Corollary 13. *Let S have at least two elements. Then every numbering $\nu \in \mathrm{Num}_p(S)$ is (faintly, correctly) precomplete, exactly if it is (faintly, correctly) 1-precomplete.*

Proof. Again the "if" direction is obvious. For the other direction let $p \in P^{(1)}$ and $h \in R^{(2)}$ as in Theorem 12. Then the function g with $g(m) = h(m,m)$ is a (faint, correct) one-to-one totalizer of p. □

A further consequence of Theorem 12 is that a numbering is (faintly) precomplete, exactly if each computable total index transformation with values in $\mathrm{dom}(\nu)$ has a uniform infinite family of fixed points; similarly in the correctly precomplete case.

Theorem 14. *Let S have at least two elements and $\nu \in \mathrm{Num}_p(S)$. Then the following statements hold:*

1. *Numbering ν is faintly precomplete if, and only if, there is some function $h \in R^{(2)}$ which is one-to-one in the second argument such that for all $i \in \omega$ with $\varphi_i \in R^{(1)}$ and* $\mathrm{range}(\varphi_i) \subseteq \mathrm{dom}(\nu)$*, as well as all $n \in \omega$, $h(i,n) \in \mathrm{dom}(\nu)$ and*
$$\nu_{\varphi_i(h(i,n))} = \nu_{h(i,n)}.$$

2. *Numbering ν is precomplete if, and only if, there is some function $h \in R^{(2)}$ which is one-to-one in the second argument such that* $\mathrm{range}(h) \subseteq \mathrm{dom}(\nu)$ *and for all $i \in \omega$ with $\varphi_i \in R^{(1)}$ and* $\mathrm{range}(\varphi_i) \subseteq \mathrm{dom}(\nu)$*, as well as all $n \in \omega$,*
$$\nu_{\varphi_i(h(i,n))} = \nu_{h(i,n)}.$$

3. *Numbering ν is correctly precomplete if, and only if, there is some function $h \in R^{(2)}$ which is one-to-one in the second argument such that for all $i, n \in \omega$ for which $\varphi_i(h(i,n))$ is defined,*

   (a) $\varphi_i(h(i,n)) \in \mathrm{dom}(\nu) \Longleftrightarrow h(i,n) \in \mathrm{dom}(\nu)$,

   (b) $\varphi_i(h(i,n)) \in \mathrm{dom}(\nu) \Longrightarrow \nu_{\varphi_i(h(i,n))} = \nu_{h(i,n)}$.

The proof proceeds in the same way as for Theorems 7- 9. Most conclusions in this section extend results in [3] to partial numberings.

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References

[1] Yu.L. Ershov, Theorie der Numerierungen I, *Zeitschrift für mathematische Logik Grundlagen der Mathematik* 19 (1973) 289–388.

[2] Yu.L. Ershov, Theory of numberings, in: E.R. Griffor, ed., *Handbook of Computability Theory*, Elsevier Science B.V., Amsterdam, 1999, pp. 473–503.

[3] A. Kanda and A.H. Lachlan, Alternative characterizations of precomplete numberings, *Zeitschrift für mathematische Logik Grundlagen der Mathematik* 33 (1987) 97–100.

[4] G. Longo and E. Moggi, Constructive natural deduction and its "omega-set" interpretation, *Mathematical Structures in Computer Science* 1 (2) (1991) 215–253.

[5] A.I. Mal'tsev, *Algorithms and Recursive Functions*. Wolters-Noordhoff Publishing, Groningen, 1965.

[6] A.I. Mal'tsev, Constructive algebras I, in: H.G. Wells, III, ed., *The Metamathematics of Algebraic Systems. Collected papers: 1936–1967*, North-Holland, Amsterdam, 1971, pp. 148–214.

[7] A.I. Mal'tsev, Sets with complete numberings, in: H.G. Wells, III, ed., *The Metamathematics of Algebraic Systems. Collected papers: 1936–1967*, North-Holland, Amsterdam, 1971, pp. 287–312.

[8] H. Rogers, Jr., *Theory of Effective Functions and Effective Computability*. McGraw-Hill Book Company, New York, 1967.

[9] V.L. Selivanov, private communication.

[10] D. Spreen, On some decision problems in programming, *Information and Computation* 122 (1995) 120–139; Corrigendum 148 (1999) 241–244.

[11] D. Spreen, An isomorphism theorem for partial numberings, *This Volume*, pp. **??**–**??**.

[12] K. Weihrauch, *Computability*, Springer-Verlag, Berlin, 1987.