# Chapter 10: Introduction to Intuitionistic Logic

## PART 1: INTRODUCTION

**The intuitionistic logic** has developed as a result of certain philosophical views on the foundation of mathematics, known as *intuitionism*.

**Intuitionism** was originated by L. E. J. Brouwer in 1908.

**The first Hilbert proof** system (Hilber style formalization) of the intuitionistic logic is due to A. Heyting (1930).

**We present** here a Hilbert style proof system developed by Rasiowa in 1959 that is equivalent to the Heyting's original formalization.

**We discuss the relationship** between intuitionistic and classical logic.

**We also present** the original version of Gentzen work (1935).

**Gentzen was the first** who formulated a first syntactically decidable formalization for classical and intuitionistic logic and proved its equivalence with the Heyting's original Hilbert style formalization (famous Gentzen's Hauptsatz).

**We present first,** as it has happened historically, the intutionistic proof systems called also formalizations of the intuitionistic logic.

**The semantics** for the intuitionistic logic will be presented in a seperate chapter.

**Intuitionistic semantics** was fist defined by Tarski in 1937, and Tarski and Stone in 1938 in terms of pseudo-boolean algebras, called also Heyting algebras to memorize Heying first proof system.

**An intuitionistic tautology** is a formula that is true in all pseudo-boolean algebras.

**Pseudo-boolean** algebras are called algebraic models for the intuitionistic logic.

**An uniform theory** and presentation of algebraic models for classical, intuitionistic, modal and many other logics was given by Rasiowa and Sikorski in 1964, and Rasiowa in 1978.

**Alternative semantics** is given in terms of Kripke models.

**Kripke models** were invented by Kripke in 1964. They provide semantics for not only the intuitionistic logic, but also for all known modal logics, believe logics, and many others.

**Both semantics** algebraic and Kripke models are equivalent for the intuitionistic logic.

**Motivation** for intuitionistic approach.

**The basic difference** between classical and intuitionists perspective lies in the interpretation of the word *exists*.

**For example,** let $A(x)$ be a statement in the arithmetic of natural numbers. For the mathematicians the sentence

$$\exists x A(x)$$

is true if it is a theorem of arithmetic, i.e. if it can be *deduced* from the axioms of arithmetic by means of classical logic.

**When a mathematician** proves sentence $\exists x A(x)$, this does not mean that he/she is able to indicate a *method of construction* of a natural number $n$ such that $A(n)$ holds.

**For an intuitionist** the sentence

$$\exists x A(x)$$

is true only he is able to provide a constructive method of finding a number $n$ such that $A(n)$ is true.

**Moreover,** mathematicians often obtain a proof of existential sentence

$$\exists x A(x)$$

by proving a logically equivalent sentence

$$\neg \forall x \ \neg A(x).$$

**Next they use** the classical logical equivalence

$$\neg \forall x \ \neg A(x)) \equiv \exists x A(x)$$

(and Modus Ponens twice) and say that they have proved $\exists x A(x))$.

**For the intuitionist** such method is not acceptable, for it does not give any *method of constructing* a number $n$ such that $A(n)$ holds.

**The same argument** applies to the following proof by contradiction.

**To prove** a statement

$$\exists x A(x)$$

we assume, $\neg\exists x A(x)$, and hence, by de-Morgan Law, we have assumed

$$\forall x \neg A(x).$$

If a contradiction follows,

$$\exists x A(x))$$

has been proven.

**For these reasons** the intuitionist do not accept the classical tautologies

$$(\neg\forall x\ \neg A(x) \Rightarrow \exists x A(x)),$$

$$(\forall x \neg A(x) \Rightarrow \neg\exists x A(x))$$

as as intuitionistically provable sentences, or consequently by intuitionistic Completeness Theorem, as intuitionistic tautologies.

**The intuitionists** interpret differently then clas-sicists not only quantifiers but also the propo-sitional connectives.

**Intuitive ideas** are as follows.

**Intuitionistic implication** $(A \Rightarrow B)$ is con-sidered by to be true if there exists a method by which a *proof of* $B$ can be deduced from the proof of $A$.

**In the case** of the classical implication

$$(\neg\forall x\ \neg A(x) \Rightarrow \exists x A(x))$$

there is no general method which, from a proof of the sentence

$$\neg\forall x\neg A(x),$$

permits is to obtain proof of the sentence

$$\exists x A(x).$$

**Hence, the intuitionists** can't accept it as an intuitionistically provable formula, or intuitionistic tautology.

**Intuitionistic negation** of a statement $A$, $\neg A$, is considered intuitionistically true if the acceptance of the sentence $A$ leads to absurdity.

**As a result intuitionistic** understanding of negation and implication we have that in the intuitionistic proof system $I$, called intuitionistic logic $I$

$$\vdash_I (A \Rightarrow \neg\neg A),$$

but

$$\not\vdash_I (\neg\neg A \Rightarrow A).$$

**Consequently,** any intuitionistic semantics $I$ must be such that,

$$\models_I (A \Rightarrow \neg\neg A)$$

and

$$\not\models_I (\neg\neg A \Rightarrow A).$$

**Intuitionistic disjunction** $(A \cup B)$ is true if one of the sentences $A, B$ is true and there is a method by which it is possible to find out which of them is true.

**As a consequence** classical law of excluded middle

$$(A \cup \neg A)$$

is not acceptable by the intuitionists since there is no general method of finding out, for any given sentence $A$, whether $A$ or $\neg A$ is true.

**Hence, the intuitionistic** proof system $I$, or logic for short, must be such that

$$\not\vdash_I (A \cup \neg A).$$

**The intutionistic semantics** $I$ must be such that

$$\models_I (A \cup \neg A).$$

**Intuitionists' view** of the concept of infinite set also differs from that which is generally accepted in mathematics.

**Intuitionists reject** the idea of infinite set as a closed whole.

**They look upon an infinite set** as something which is constantly in a state of formation.

**For example,** the set of all natural numbers is infinite in the sense that for any given finite set of natural numbers it is always possible to add one more natural number.

**The notion** of the set of all subsets of the set of all natural numbers is not regarded meaningful.

**Intuitionists reject** the general idea of a set as defined by a modern set theory.

**An exact, formal exposition** of the basic ideas of intuitionism is outside the range of our investigations.

**Our goal is** to give, in this chapter, a presentation of the intuitionistic logic formulated as a proof system and discuss the relationship between classical and intuitionistic logics.