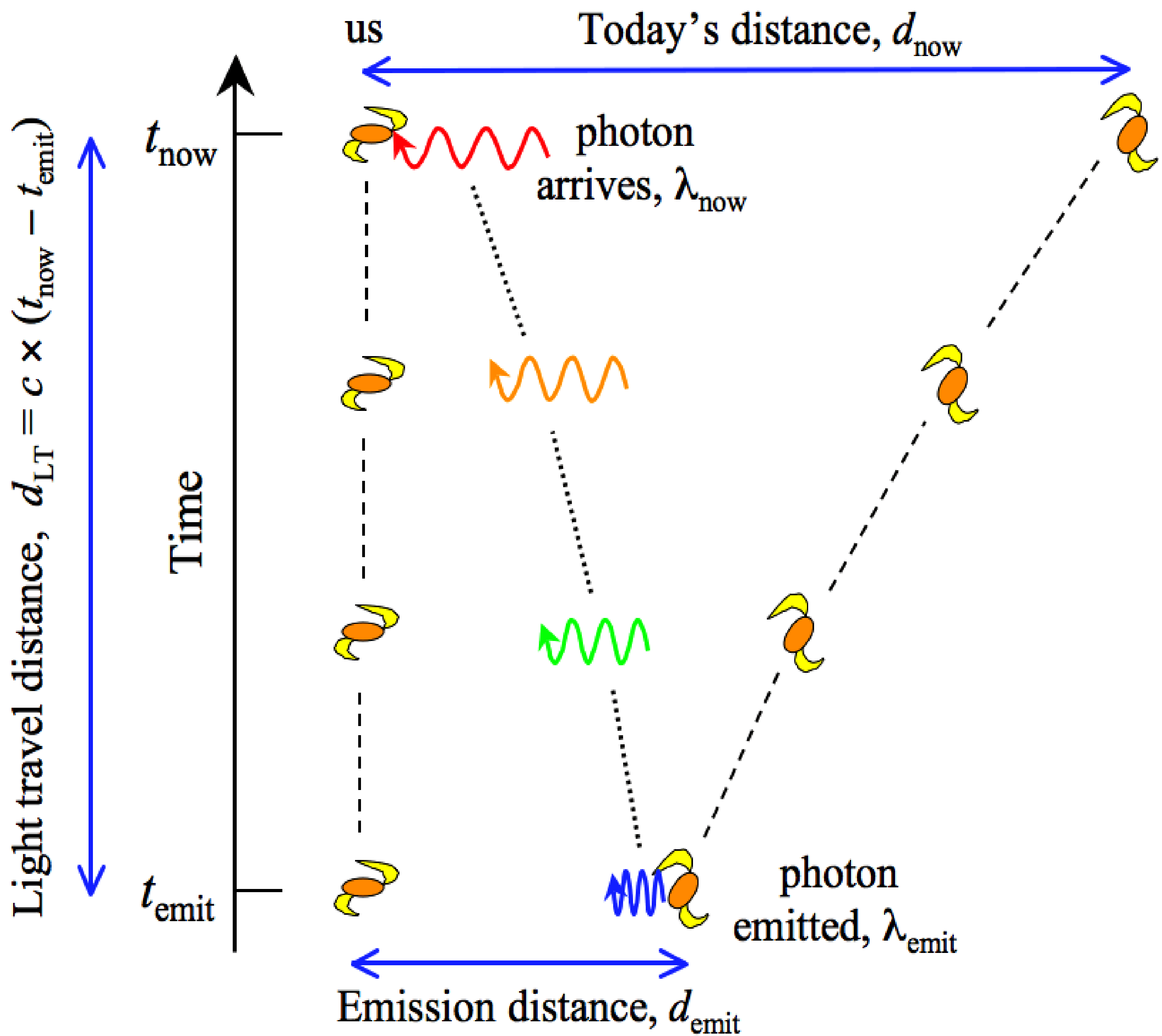
Distances in Cosmology



REVIEW OF “WORLD MODELS”

Simplify notation by adopting c=1, so that E=m. Friedmann's equation is then:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi}{3} G\rho$$

Substitute H(t)= $\dot{a}/a$ and recall that we can formally associate an energy density with the cosmological constant, i.e

$$\rho_\Lambda \equiv \frac{\Lambda}{8\pi G} \longrightarrow H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\left(\sum_i \rho_i + \rho_\Lambda\right)$$

The index i refers to the type of particle fluid under consideration, e.g. matter or radiation. If the Universe is flat (k=0):

$$\rho_{\rm tot} \equiv \sum_i \rho_i + \rho_\Lambda = \frac{3H^2}{8\pi G} \equiv \rho_{\rm crit}$$

Let us define the fraction of the critical density contributed by each component of the Universe :

$$\Omega_i \equiv \frac{\rho_i}{\rho_{\rm crit}}$$

So that we have Ω_m , Ω_r and Ω_Λ for matter, radiation and dark energy. These quantities are time-dependent, the values today are denoted as $\Omega_{m,0}$ We can rewrite the Friedmann eqn:

$$\frac{k}{a^2H^2} = \sum_i \Omega_i + \Omega_\Lambda - 1$$

If we define $\Omega_k = -k/(aH)^2$, we can write:

$$\sum_i \Omega_i + \Omega_\Lambda + \Omega_k = 1$$

Flat FRW Cosmologies

In the last lecture we showed that the density of matter evolves as:

$$\rho_{\mathrm{m}} = \rho_{\mathrm{m},0}\left(\frac{a}{a_0}\right)^{-3}$$

Set a_0=1. The Friedmann equation becomes:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_{\mathrm{m},0}a^{-3} + \frac{\Lambda}{3} \qquad \text{or} \qquad \dot{a}^2 = H_0^2\Omega_{\mathrm{m},0}a^{-1} + H_0^2\Omega_{\Lambda,0}a^2$$

In a flat Universe, $\Omega_{\Lambda,0} + \Omega_{m,0}=1$.

Case 1: $\Lambda>0$ Use the substitution: $u = \dfrac{2\Omega_{\Lambda,0}}{\Omega_{\mathrm{m},0}}a^3$

to obtain $\dot{u}^2 = 9H_0^2\Omega_{\Lambda,0}\left[2u + u^2\right] = 3\Lambda\left[2u + u^2\right]$

Take the positive root: $\displaystyle\int_0^u \frac{du}{(2u+u^2)^{1/2}} = \int_0^t (3\Lambda)^{1/2}\,dt = (3\Lambda)^{1/2}t$

This can be integrated by completing the square in the u-integral and with substitutions v = u + 1 and cosh w = v:

$$\int_0^u \frac{du}{\left[(u+1)^2-1\right]^{1/2}} = \int_1^v \frac{dv}{(v^2-1)^{1/2}} = \int_0^w \frac{\sinh w\, dw}{\left(\cosh^2 w - 1\right)^{1/2}} = \int_0^w dw = w$$

to yield the solution:

$$a^3 = \frac{\Omega_{\mathrm{m},0}}{2\Omega_{\Lambda,0}}\left[\cosh(3\Lambda)^{1/2}t - 1\right]$$

Case 2, $\Lambda < 0$: Introduce

$$u = -\frac{2\Omega_{\Lambda,0}}{\Omega_{\mathrm{m},0}}\, a^3$$

Solution:

$$a^3 = \frac{\Omega_{\mathrm{m},0}}{2(-\Omega_{\Lambda,0})}\left\{1 - \cos\left[3(-\Lambda)\right]^{1/2} t\right\}$$

Case 3, Λ=0 : This is now the Einstein-deSitter case which we have already encountered in the last lecture.

$$a = \left(\frac{9}{4} H_0^2 t^2\right)^{1/3}$$

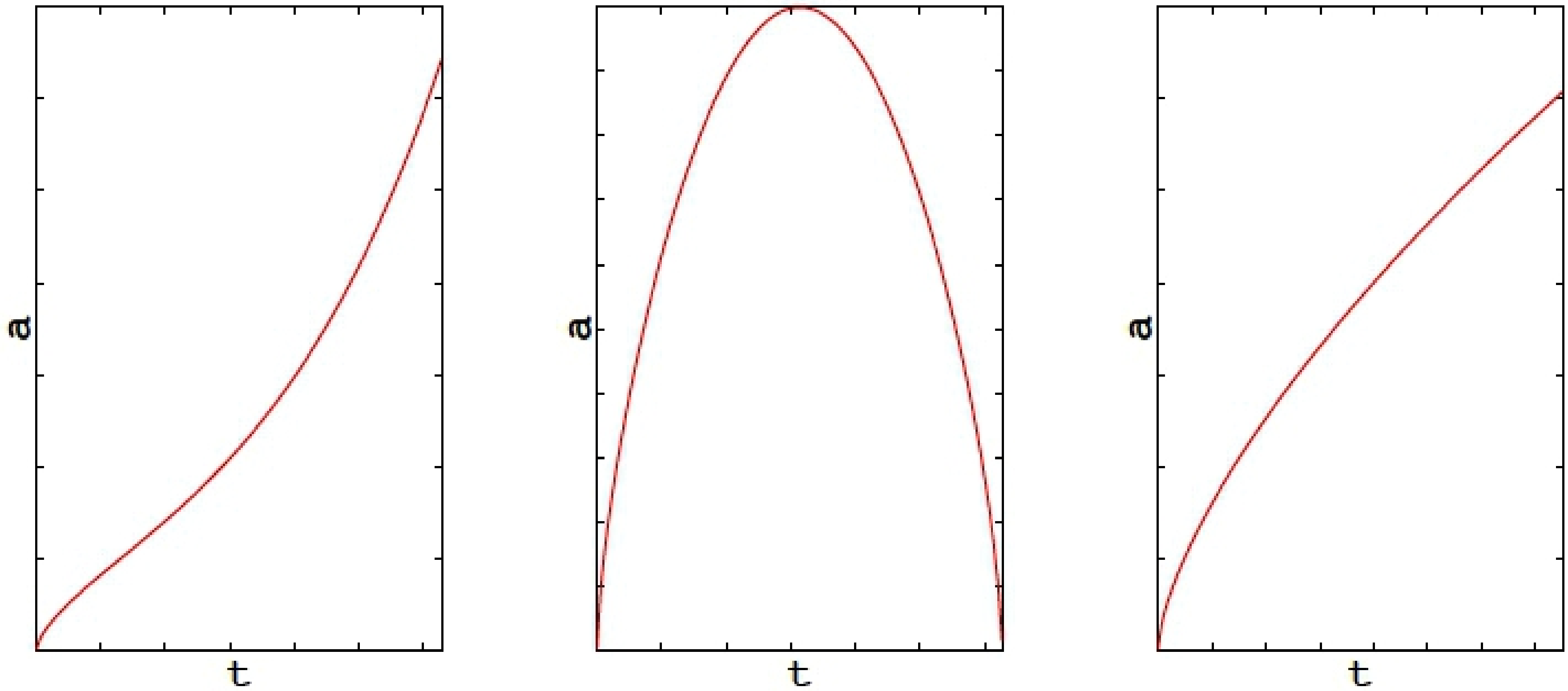


Figure 4.1: The three flat, pressureless cosmological models. On the left $\Lambda > 0$, in the middle $\Lambda < 0$ and on the right the Einstein de-Sitter model with $\Lambda = 0$.

A flat, pressureless universe with a small, but non-zero, cosmological constant initially evolves as if it were Einstein-deSitter.

Fo r $\Lambda > 0$, the second term on the right-hand side of these equations dominates at large values of t and the universe grows exponentially:

$$a \propto \exp\left[(\Lambda/3)^{1/2}\, t\right]$$

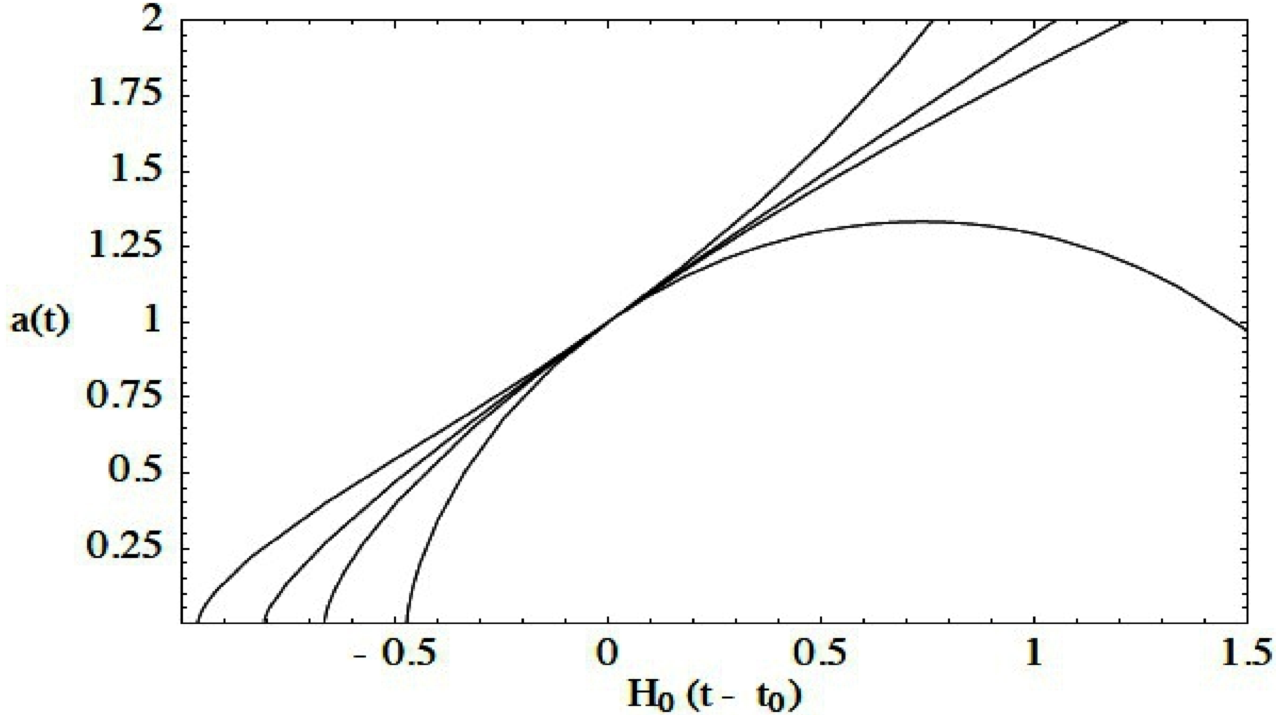
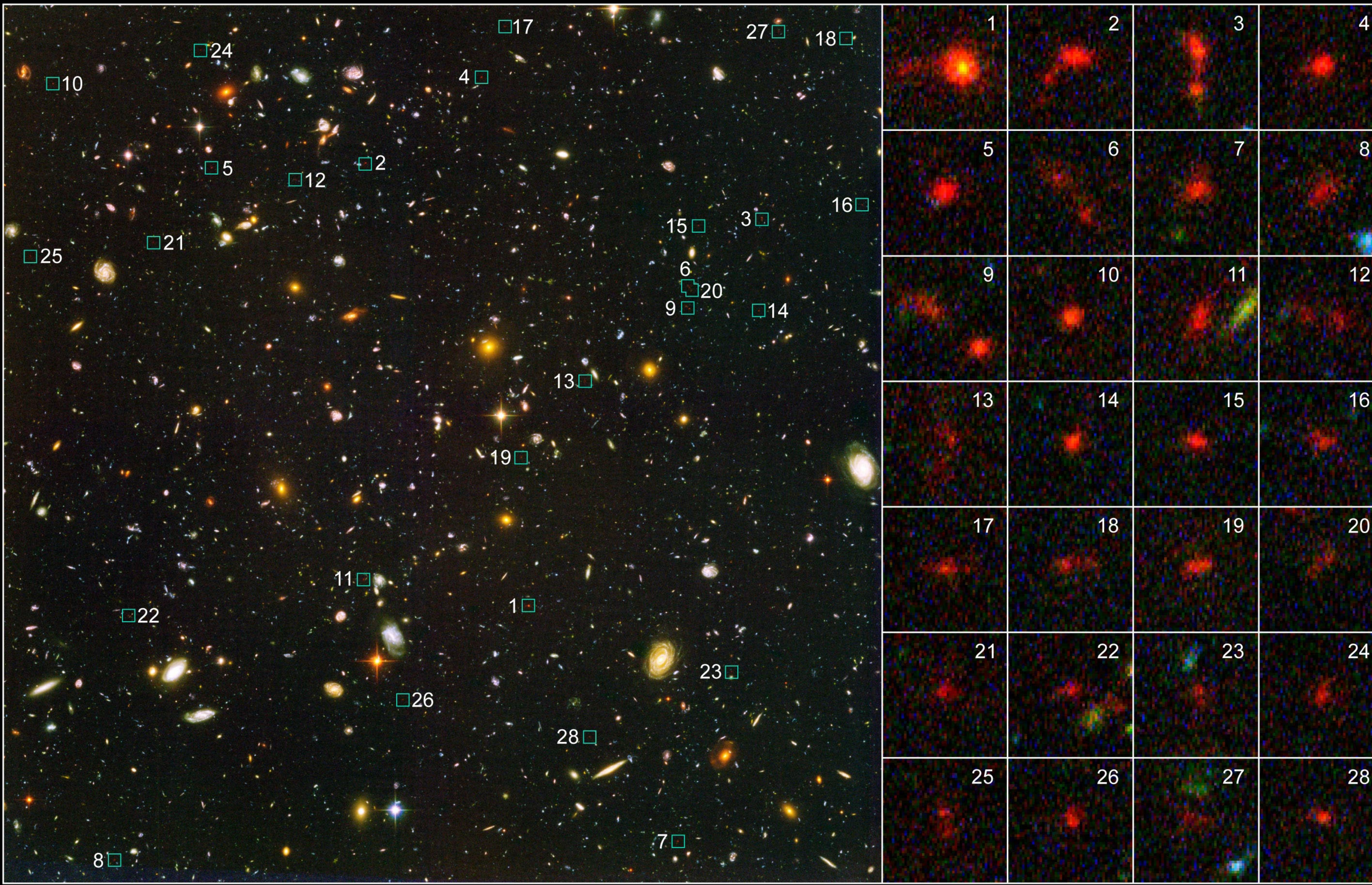


Figure 4.3: Expansion histories for different values of $\Omega_{m,0}$, $\Omega_{\Lambda,0}$, and $\Omega_{k,0}$. From top to bottom, the curves describe $\Omega_{m,0}$, $\Omega_{\Lambda,0}$, $\Omega_{k,0} = (0.3, 0.7, 0.0)$, $(0.3, 0.0, 0.7)$, $(1.0, 0.0, 0.0)$, and $(4.0, 0.0, -3.0)$.

A wide variety of world models are conceivable, depending on the values of the parameters Ω. Observational cosmologists are interested in assessing which, if any, of these models is a valid description of the universe we live in. The measurements on which these tests are based generally involve the redshifts and radiant fluxes of distant sources.



Distant Galaxies in the Hubble Ultra Deep Field

Hubble Space Telescope • Advanced Camera for Surveys

NASA, ESA, R. Bouwens and G. Illingworth (University of California, Santa Cruz)

Cosmological Redshifts

We show that the redshift is directly related to the scale factor of the universe at the time the photons were emitted from the source as:

$$1 + z_{\mathrm{e}} = \frac{a_0}{a(t=e)}$$

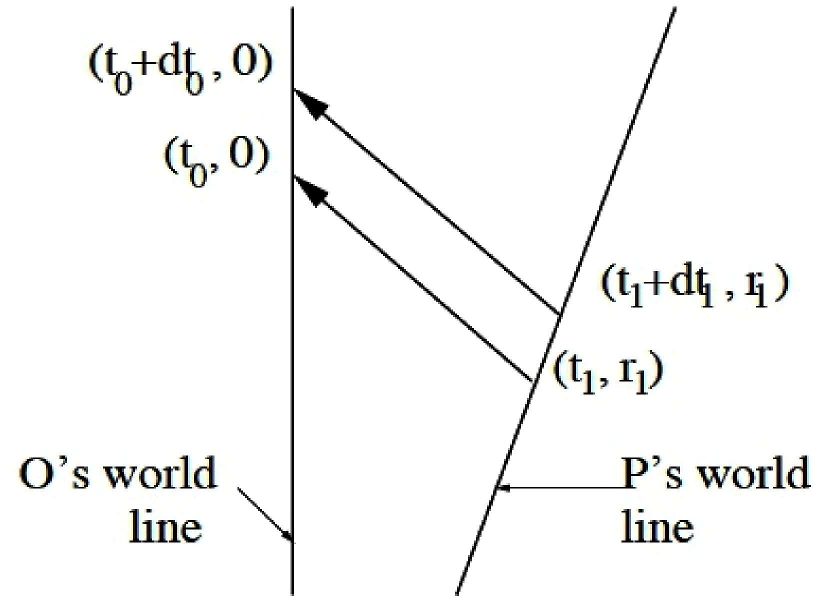
We begin with the Robertson-Walker metric:

$$(ds)^2 = (c\,dt)^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

In general relativity the propagation of light is along a null geodesic (ds = 0). With the observer at the origin (r = 0), we choose a radial null geodesic so that dθ = dφ = 0, so that

$$\frac{c\,dt}{a(t)} = \pm\frac{dr}{(1-kr^2)^{1/2}}$$

(+ is for emitted light ray, - is for a received one)



Imagine now that one crest of the light wave was emitted at time t_e at distance r_e , and received at the origin $r_0 = 0$ at t_0 , and that the next wave crest was emitted at $t_e+\Delta t_e$ and received at $t_0+\Delta t_0$. The two waves satisfy the relations:

$$\int_{t_e}^{t_0} \frac{dt}{a(t)} = -\frac{1}{c}\int_{r_e}^{0} \frac{dr}{\sqrt{1-kr^2}} \quad \text{and} \quad \int_{t_e+\Delta t_e}^{t_0+\Delta t_0} \frac{dt}{a(t)} = -\frac{1}{c}\int_{r_e}^{0} \frac{dr}{\sqrt{1-kr^2}}$$

Subtract the two equations:

$$\int_{t_e+\Delta t_e}^{t_0+\Delta t_0} \frac{dt}{a(t)} - \int_{t_e}^{t_0} \frac{dt}{a(t)} = 0$$

Expand $$\int_{t_e+\Delta t_e}^{t_0+\Delta t_0} \frac{dt}{a(t)} = \int_{t_e}^{t_0} \frac{dt}{a(t)} + \int_{t_0}^{t_0+\Delta t_0} \frac{dt}{a(t)} - \int_{t_e}^{t_e+\Delta t_e} \frac{dt}{a(t)}$$

to obtain: $$\int_{t_0}^{t_0+\Delta t_0} \frac{dt}{a(t)} = \int_{t_e}^{t_e+\Delta t_e} \frac{dt}{a(t)}$$

Any change in a(t) during the time intervals between successive wave crests can be safely neglected, so that a(t) is a constant with respect to the time integration. Consequently,

$$\frac{\Delta t_e}{a(t_e)} = \frac{\Delta t_0}{a(t_0)}; \qquad \frac{\Delta t_e}{\Delta t_0} = \frac{a(t_e)}{a(t_0)}$$

The time interval between successive wave crests is the inverse of the frequency of the light wave, related to its wavelength by the relation

$c = \lambda \cdot \nu$, so that

$$\frac{\lambda_0}{\lambda_e} = 1 + z = \frac{a(t_0)}{a(t_e)}$$

Time Evolution of the Hubble Parameter

In a flat Universe, $\rho_{\rm m} = \rho_{\rm m,0} \left(\frac{a}{a_0}\right)^{-3}$ so that we can write the Friedmann equation as:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_{\rm m,0} a^{-3} + \frac{\Lambda}{3} \quad \text{or} \quad \dot{a}^2 = H_0^2 \Omega_{\rm m,0} a^{-1} + H_0^2 \Omega_{\Lambda,0} a^2$$

We can thus write $$\left(\frac{H(z)}{H_0}\right)^2 = \Omega_{\rm m,0} \cdot (1+z)^3 + \Omega_{\lambda,0}$$

The right-hand side is often referred to as $E(z)$, so that $H(z) = H_0 \, E(z)^{1/2}$

We can derive a relationship between time t and redshift z by considering the following:

$$a = \frac{a_0}{1+z}; \qquad da = -\frac{a_0}{(1+z)^2}dz$$

and
$$\frac{\dot{a}}{a} = H(z) = \frac{da}{dz}\frac{dz}{dt}\frac{(1+z)}{a_0}$$

so that
$$\int_{t1}^{t2} dt = -\frac{1}{H_0}\int_{z1}^{z2}\frac{dz}{(1+z)E(z)^{1/2}}$$

The age of the Universe is
$$t_0 = \int_0^{t_0} = \frac{1}{H_0}\int_0^{\infty}\frac{dz}{(1+z)E(z)^{1/2}}$$

In Einstein-de Sitter cosmology, $\Omega_m=1$, $\Omega_\lambda=0$:

$$t_0 = \frac{1}{H_0}\int_0^{\infty}\frac{dz}{(1+z)^{5/2}} = \frac{2}{3}H_0^{-1}(1+z)^{-3/2}\Big|_{\infty}^{0} = \frac{2}{3}H_0^{-1}$$

Cosmological Distances

1. **Proper Distances**

We define a proper distance, as the distance between two events, A and B, in a reference frame for which they occur simultaneously ($t_A = t_B$).

On

$$(ds)^2 = (c\,dt)^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

and set $d\theta = d\varphi = 0$ and $dt=0$, so that

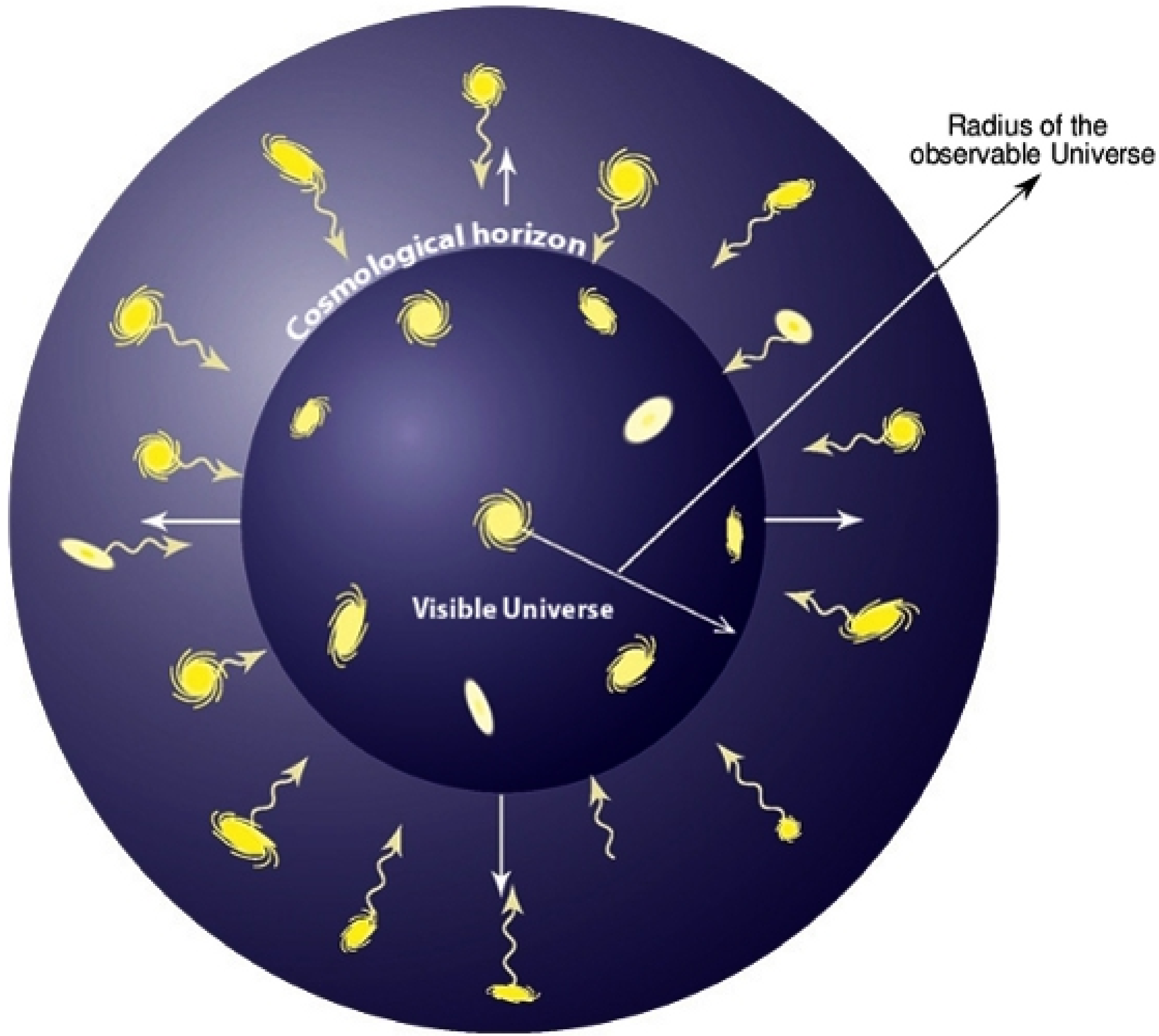
$$s(t) = \int_0^s ds' = a(t)\int_0^r \frac{dr}{(1-kr^2)^{1/2}}$$

This has solutions:

$$s(t) = a(t) \cdot \begin{cases} \frac{1}{\sqrt{k}} \sin^{-1}(r\sqrt{k}) & \text{for } k > 0 \\ r & \text{for } k = 0 \\ \frac{1}{\sqrt{|k|}} \sinh^{-1}(r\sqrt{|k|}) & \text{for } k < 0 \end{cases}$$

In a flat universe, the proper distance to an object is just its coordinate distance, s(t) = a(t) · r. Because $\sin^{-1}(x) > x$ and $\sinh^{-1}(x) < x$, in a closed universe (k **>** 0) the proper distance to an object is greater than its coordinate distance, while in an open universe (k **<** 0) the proper distance to an object is less than its coordinate distance.

The Horizon


Radius of the
observable Universe
Cosmological horizon
Visible Universe

As the universe expands and ages, an observer at any point is able to see increasingly distant objects as the light from them has time to arrive. This means that, as time progresses, increasingly larger regions of the universe come into causal contact with the observer. The proper distance to the furthest observable point—the particle horizon— at time t is the horizon distance, $s_h(t)$.

Again we return to the Robertson-Walker metric, placing an observer at the origin (r = 0) and let the particle horizon for this observer at time t be located at radial coordinate distance r_{hor}. This means that a photon emitted at t = 0 at r_{hor} will reach our observer at the origin at time t.

Since photons move along null geodesics, ds = 0. Considering only radially traveling photons (dθ = dφ = 0), we find

$$\int_0^t \frac{dt}{a(t)} = \frac{1}{c} \int_0^{r_{\rm hor}} \frac{dr}{(1 - kr^2)^{1/2}}$$

$$r_{\text{hor}} = \begin{cases} \sin\left(c\int_0^t \frac{dt}{a(t)}\right) & \text{for } k=1 \\ c\int_0^t \frac{dt}{a(t)} & \text{for } k=0 \\ \sinh\left(c\int_0^t \frac{dt}{a(t)}\right) & \text{for } k=-1 \end{cases}$$

If the scale factor evolves with time as $a(t) = t^{\alpha}$, we can see that the above time integral diverges as we approach $t = 0$, if $\alpha>1$. This
would imply that the whole universe in is causal contact. However,
$\alpha=1/2$ and $2/3$ in the radiation and matter-dominated regime, so there
is a horizon.

The proper distance from the origin to r_{hor} is given by:

$$s_{\rm hor}(t) = a(t)\int_0^{r_{\rm hor}} \frac{dr}{(1-kr^2)^{1/2}} = a(t)\int_0^t \frac{cdt}{a(t)} \qquad \text{for } k=0$$

So $s_{hor}(t)=2ct$ in the radiation-dominated era and $s_{hor}(t)=3ct$ in the matter-dominated era.

Notice that these distances are larger than ct, the distance travelled by a photon in time t. How could this be? The reason lies in our definition of proper distance, as the distance between two events measured in a frame of reference where those two events happen at the same time.

To understand this, consider a photon in emitted at comoving radial coordinate r_{hor} at time $t = 0$. We want to know what is the proper distance of that photon from our position, at $r = 0$, at a later time t.The coordinate of the photon at time t may be found by integrating

$$\int_0^t \frac{c\,dt}{a(t)} = \int_r^{r_{\rm hor}} \frac{dr}{(1-kr^2)^{1/2}}$$

As before, we consider zero curvature models.Substituting for a(t) we obtain:

$$r = r_{\rm hor} - \frac{2c}{H_0}\left(\frac{t}{t_0}\right)^{1/3}$$

where $t_0 = 2t_H/3$ is the present age of the universe. Recalling that $r_{hor} = 2c/H_0$, and that the proper distance in a flat universe is just $s(t) = a(t) \cdot r$, we find that the proper distance of the photon from Earth as a function of time is

$$s(t) = \frac{2c}{H_0}\left[\left(\frac{t}{t_0}\right)^{2/3} - \left(\frac{t}{t_0}\right)\right] \qquad (\text{for } k = 0)$$

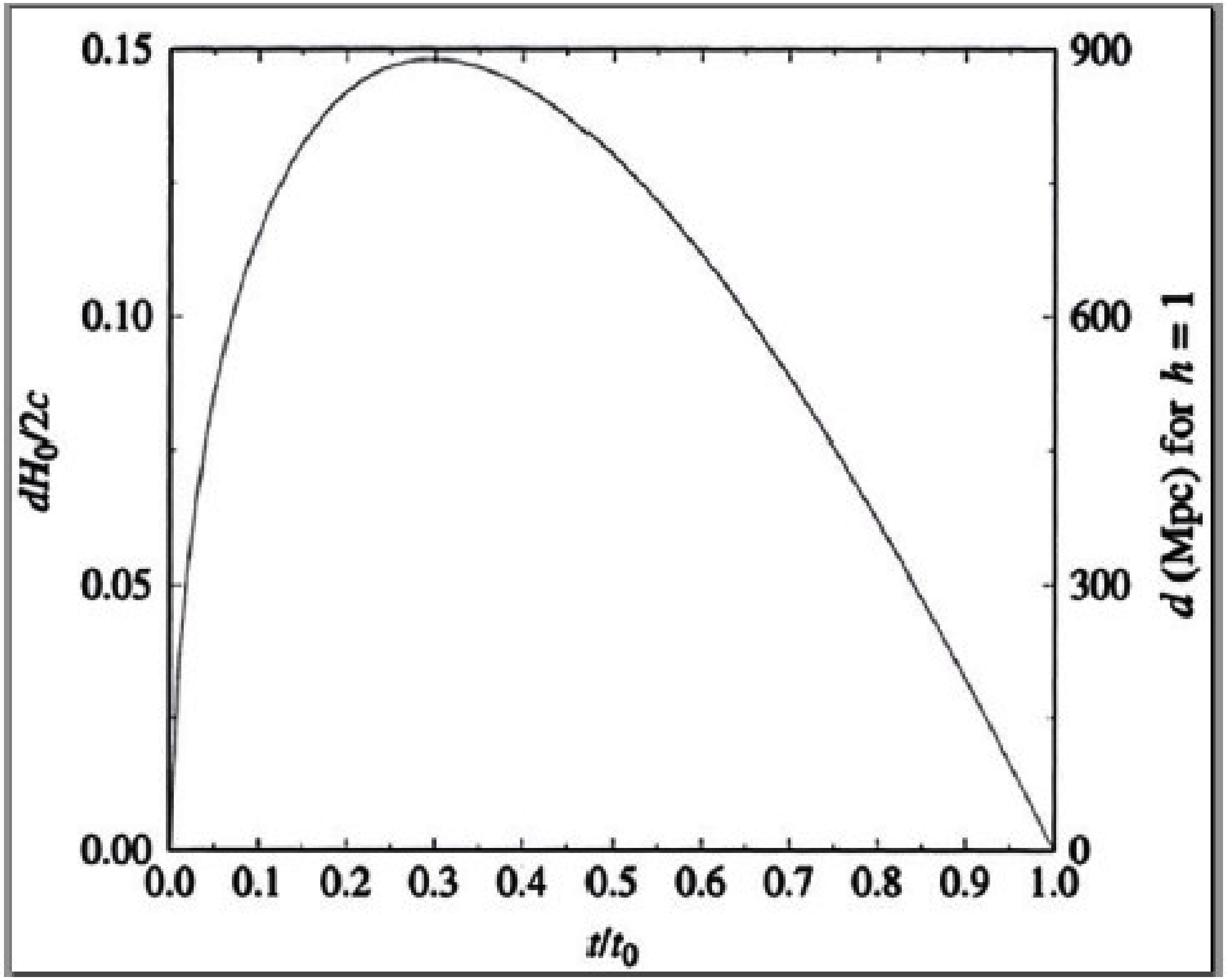


Figure 5.6: Proper distance as a function of time of a photon emitted from the present particle horizon at the time of the Big Bang. The proper distance is expressed as a fraction of $2c/H_0$, the present horizon distance in a flat universe. The right axis shows d in units of h^{-1} Mpc.

We can now see that the initial expansion actually carried the photon away from Earth. Although the photon's co-moving coordinate was always decreasing from an initial value r_{hor} towards Earth's position at r = 0, the scale factor a(t) increased so rapidly that at first the proper distance between the photon and Earth increased with time.

Re-writing in terms of the redshift corresponding to time t (k = 0),

$$s_{\text{hor}}(z) = \frac{2c}{H_0(1+z)^{3/2}} \qquad (\text{for } k=0)$$

we find that at the present time,

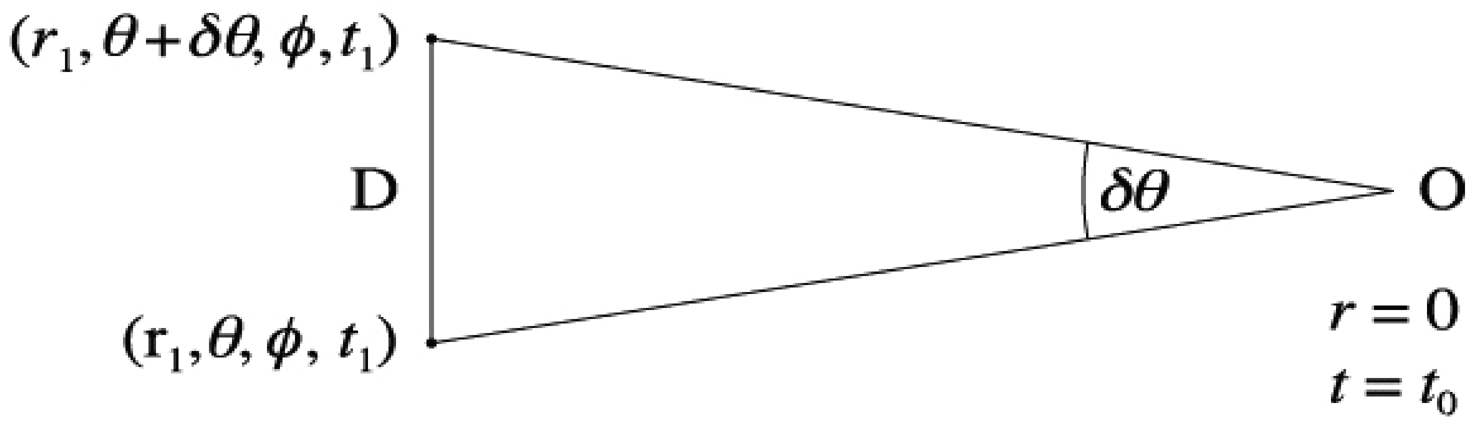
$$s_{\text{hor},0} = \frac{2c}{H_0} = 1.85 \times 10^{28}\, h^{-1}\text{cm} = 6\, h^{-1}\text{Gpc} \qquad (\text{for } k=0)$$

The horizon distance at the epoch of decoupling ($z \simeq 1100$) was

$$s_{\text{hor}}(z=1100) = \frac{2c}{H_0(1+1100)^{3/2}} \qquad (\text{for } k=0)$$

or $s_{\text{hor}}(z=1100) \simeq 164\,\text{kpc}.$

3. **Angular Diameter Distance**



Consider a light source of size D at r = r_1 and t = t_1 subtending an angle $\delta\theta$ at the origin (r = 0, t = t_0). The proper distance D between the two ends of the object is related to $\delta\theta$ by,

$$\delta\theta = \frac{D}{a(t_1)\, r_1}$$

We now define the angular diameter distance

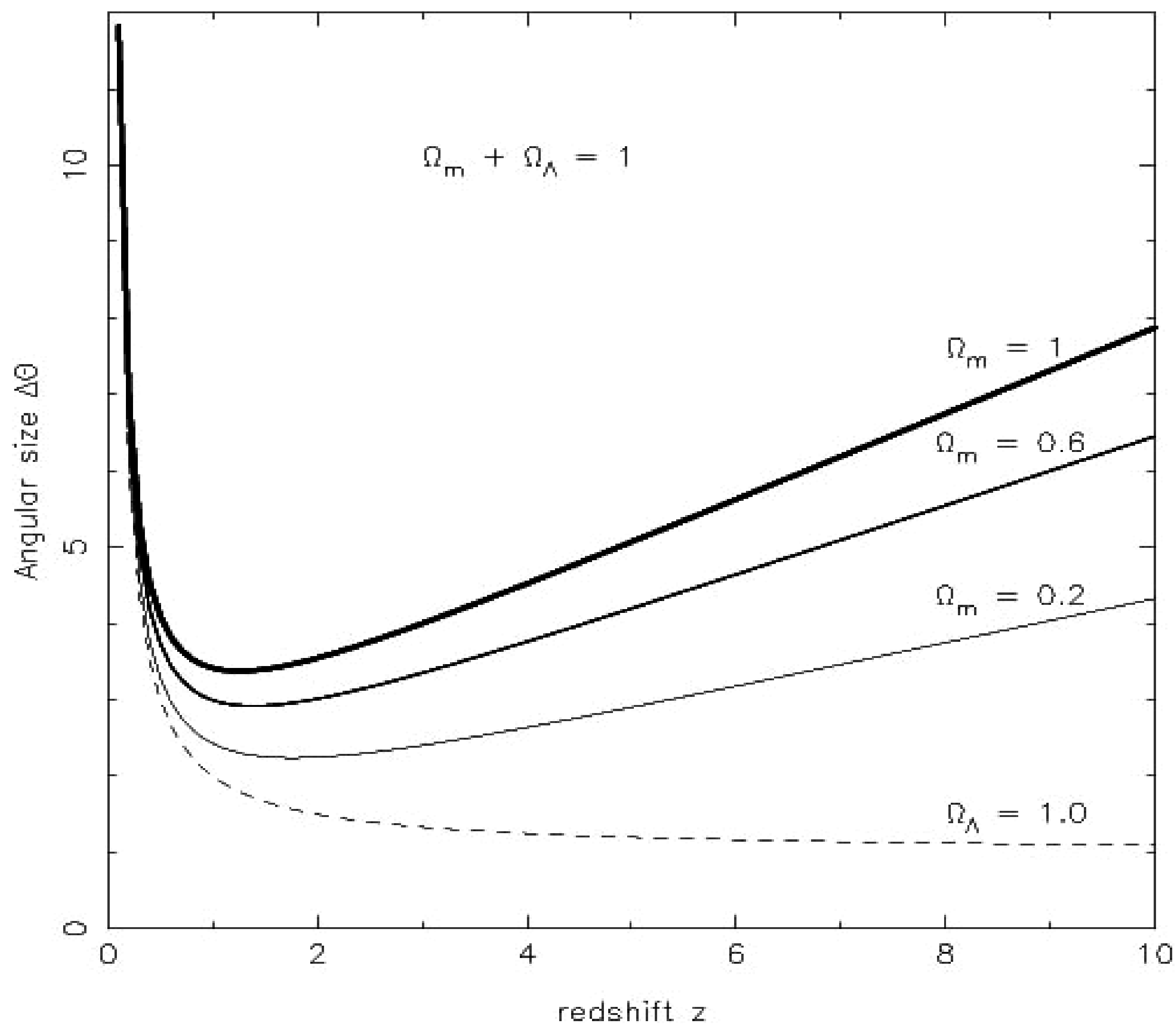
$$d_A \equiv \frac{D}{\delta\theta} \quad \text{so that} \quad d_A = a(t_1)\, r_1 = \frac{r_1}{1+z}$$

We are again studying the propagation of light, so following a similar derivation leads to the expression:

$$d_A(z) = \frac{c}{\sqrt{|\Omega_{\mathrm{k},0}|}H_0(1+z)} \cdot S_k\left(H_0\sqrt{|\Omega_{\mathrm{k},0}|}\int_0^z \frac{dz}{H(z)}\right)$$

$$S_k(x) = \begin{cases} \sin(x) & \text{for } k > 0 \\ x & \text{for } k = 0 \\ \sinh(x) & \text{for } k < 0 \end{cases}$$

Note that $d_A(z)$ has a maximum at z_m, corresponding to the redshift at which objects of a given proper size D will subtend the minimum angle $\delta\theta$ on the sky. At redshifts $z > z_m$ objects of a given proper size will appear bigger on the sky with increasing z.



Things at higher redshift look bigger again because space-time was compressed when the light was emitted, i.e the galaxies were closer to us that they are today!

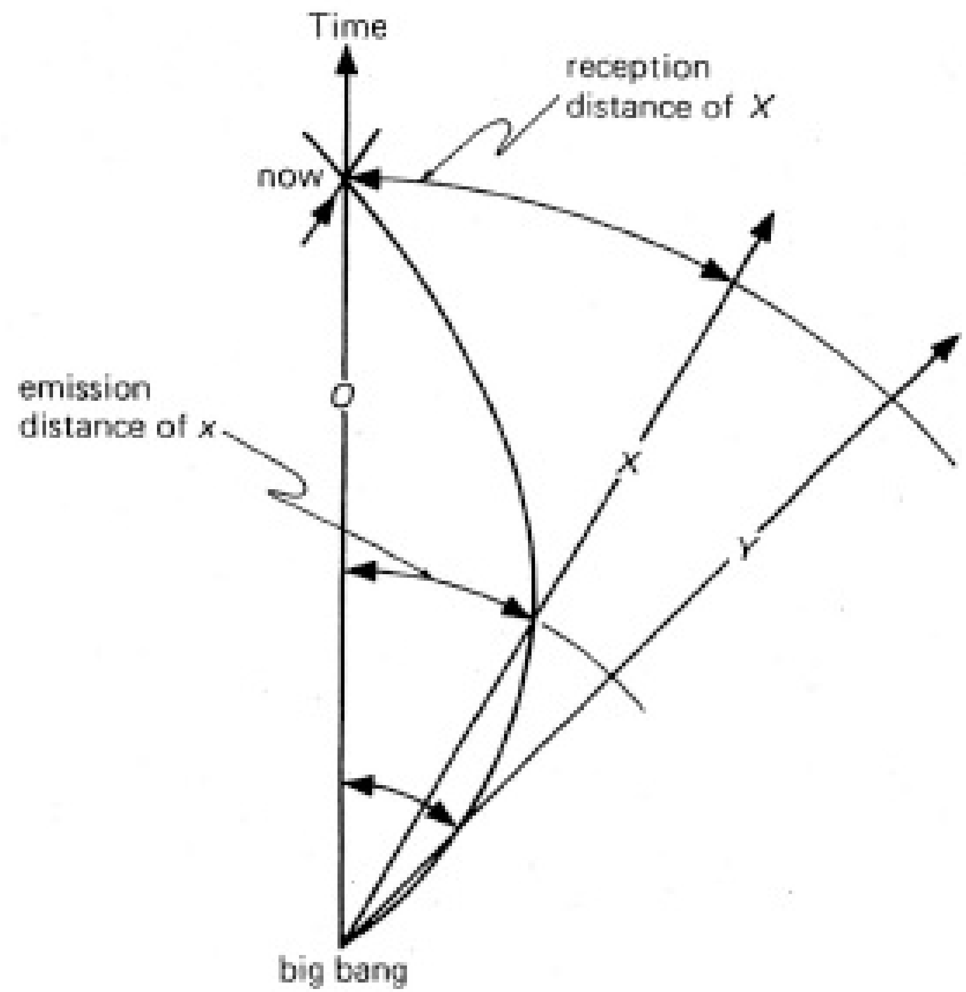
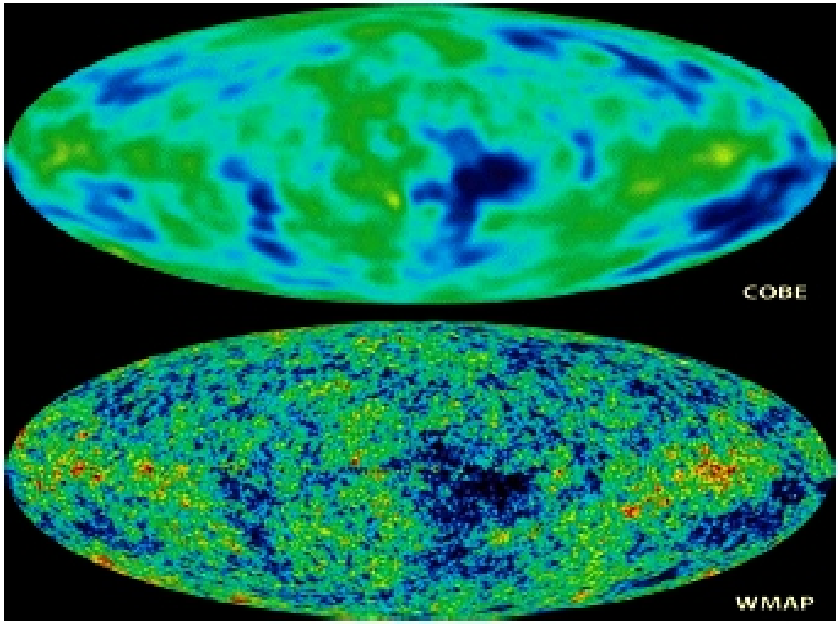
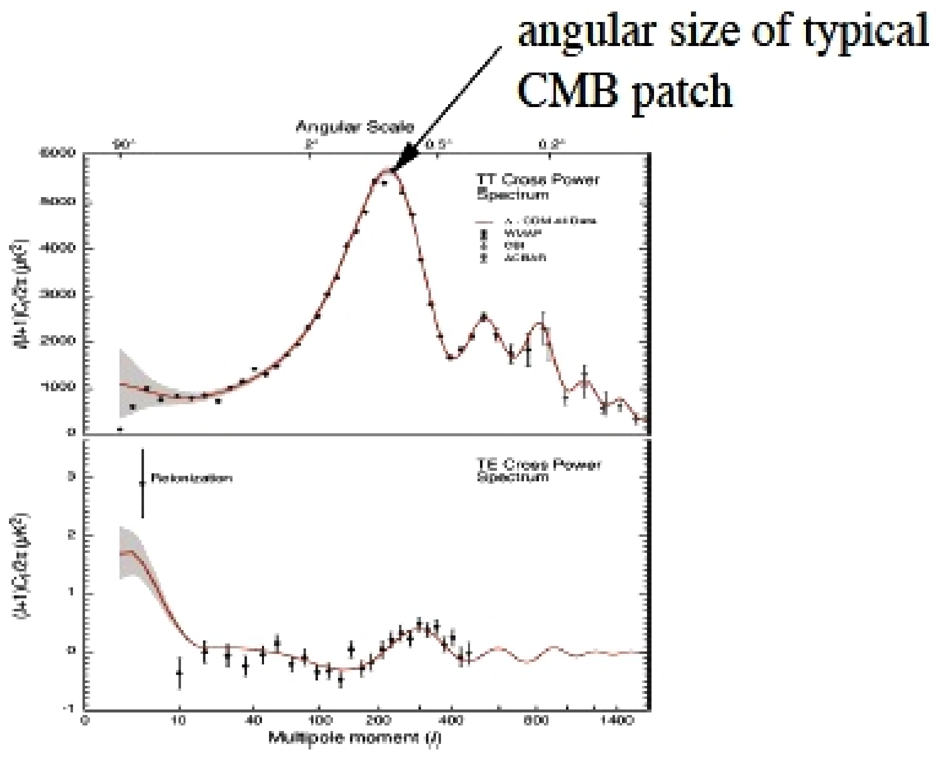


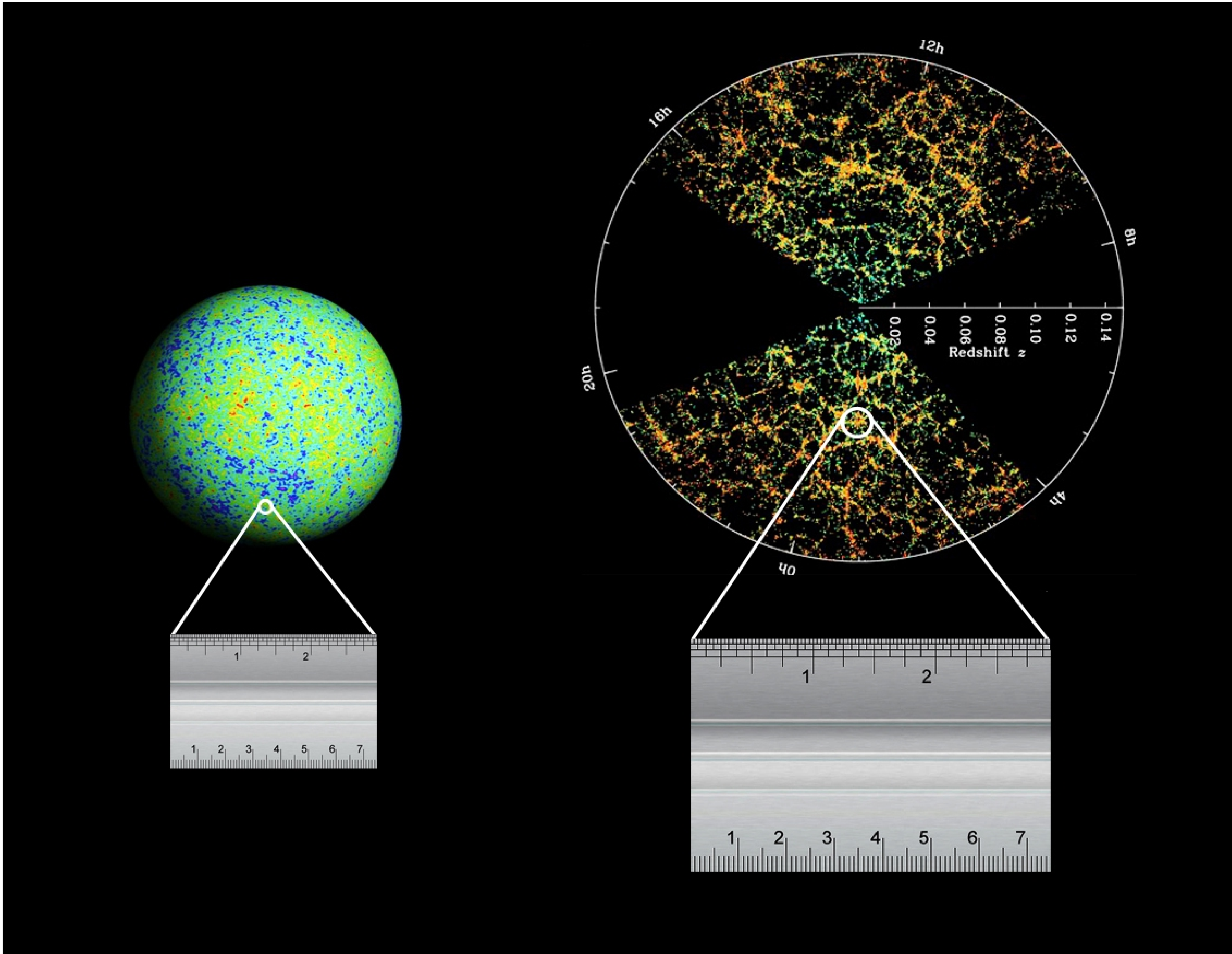
Figure 5.9: O's lightcone curves back into the Big Bang. The diagram shows the reception and emission distances of galaxies X and Y. Although galaxy Y has a greater reception distance, its emission distance is smaller than that of X. Thus Y, which is now further away than X, was closer to us than X at the time of the emission of the light which we now see (reproduced from E. R. Harrison's *Cosmology*).

The dependence of the angular diameter distance on cosmological prompted a number of tests of the geometry of the universe based on measuring the angular size of different sources: **STANDARD RULERS**

One excellent standard ruler is the first peak in the angular power spectrum of the temperature fluctuations of the CMB. One can calculate the typical size of an overdense region at the time the microwave photons started to stream free. As we also know the redshift of this last scattering surface, we can compare their ratio to the observed angular size and hence obtain a very accurate measurement of the curvature of the universe. The favoured solution is that we live in a flat universe, with k=0.






12h
16h
8h
20h
4h
0h
0.02
0.04
0.06
0.08
0.10
0.12
0.14
Redshift z
1 2
1 2 3 4 5 6 7
1 2
1 2 3 4 5 6 7

Horizon "problem"

s_{hor}(z = 1100) = 164 kpc. To find the angle subtended on the sky by this diameter we divide by the angular distance which, is given by:

$$d_A(z) = \frac{2c}{H_0}\left(1 - \frac{1}{(1+z)^{1/2}}\right) \cdot \frac{1}{1+z} \qquad (\text{for } k = 0)$$

For large values of z,

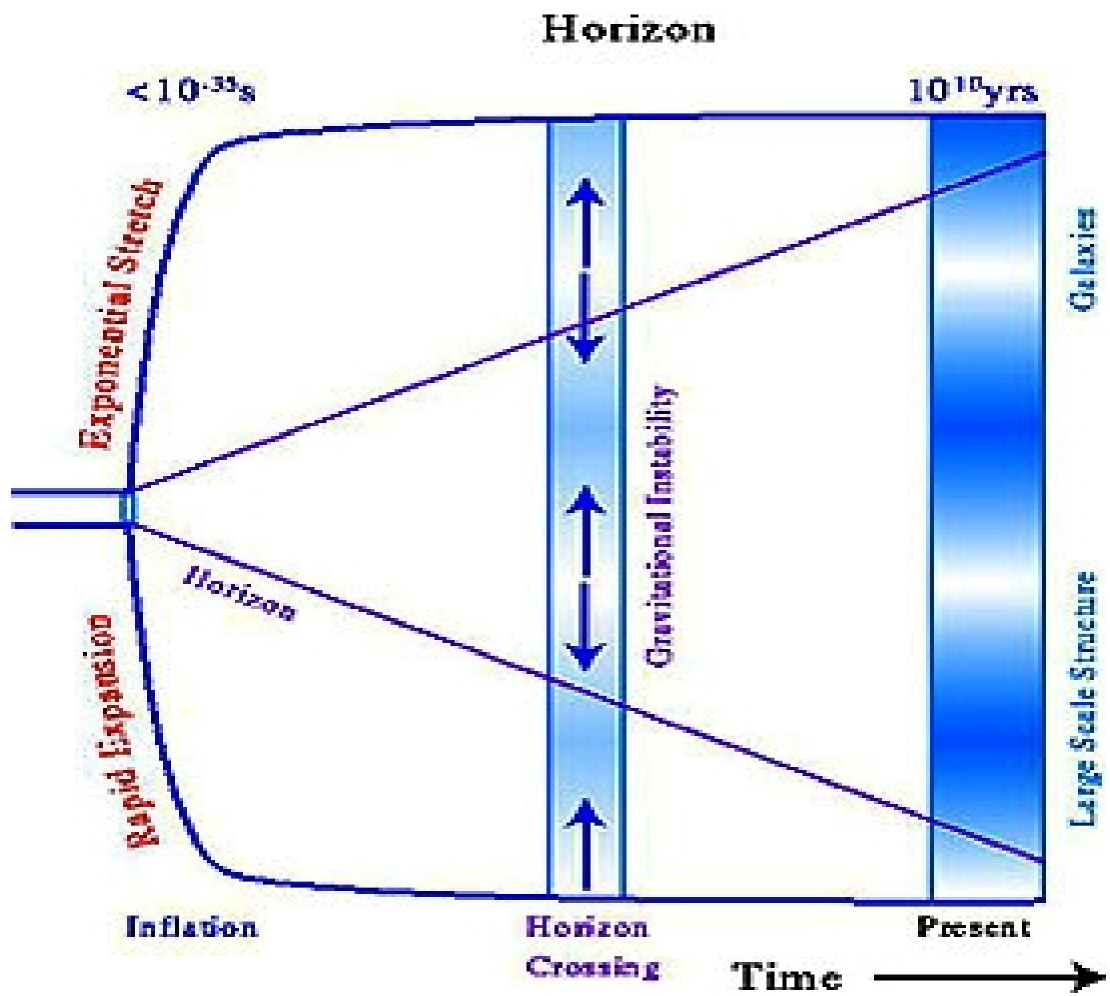
$$\Theta_{\text{hor}} = \frac{s_{\text{hor}}(z = 1100)}{d_A(z = 1100)} \approx \left(\frac{1}{1+z}\right)^{1/2} \approx \left(\frac{1}{1+1100}\right)^{1/2}$$

$$\Theta_{\text{hor}} \approx \frac{1}{33}\,\text{radians} = 1.7\,\text{degrees}$$

Why is the CMB radiation so isotropic over angular scales much larger than the horizon scale at the time of decoupling?

The solution to the horizon problem provided by inflationary theories is that there must have been a very early period of rapid expansion, when the scale factor of the universe increased exponentially:

$a(t) \propto \exp(Ht)$.

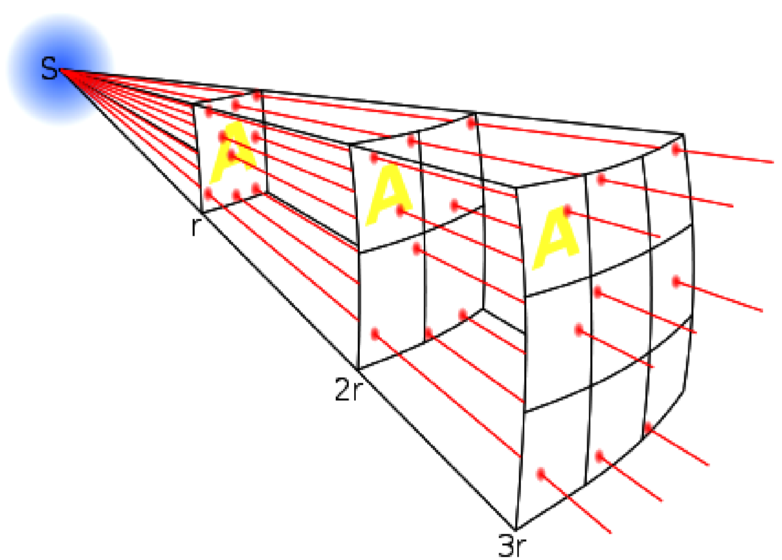


3. **Luminosity Distance (standard candles)**

The luminosity distance d_L is defined to satisfy the relation:

$$F_{\text{obs}} = \frac{L}{4\pi d_L^2}$$

where F_{obs} is the observed flux from an astronomical source and L is its absolute luminosity. We define flux as the energy that passes per unit time through a unit area (so that the energy per unit time, or the power, collected by a telescope of area A is F A); and luminosity as the total power (energy per unit time) emitted by the source at all wavelengths.



At distance r_1, photons are spread over a sphere of area

$$A = r_1^2 \int\int \sin\theta \, d\theta \, d\phi = 4\pi r_1^2$$

Recall that photons emitted with wavelength λ_1 at time intervals δt_1 are received (by an observer on the surface of the sphere) at time intervals δt_0 and with wavelength λ_0. Both wavelengths and time intervals are related by

$$\frac{\lambda_1}{\lambda_0} = \frac{\delta t_1}{\delta t_0} = \frac{a_1}{a_0}$$

Now consider a single photon: $E=h\nu = hc/\lambda$

Emitted power: $P_{\mathrm{em}} = \dfrac{h\nu_1}{\delta t_1}$

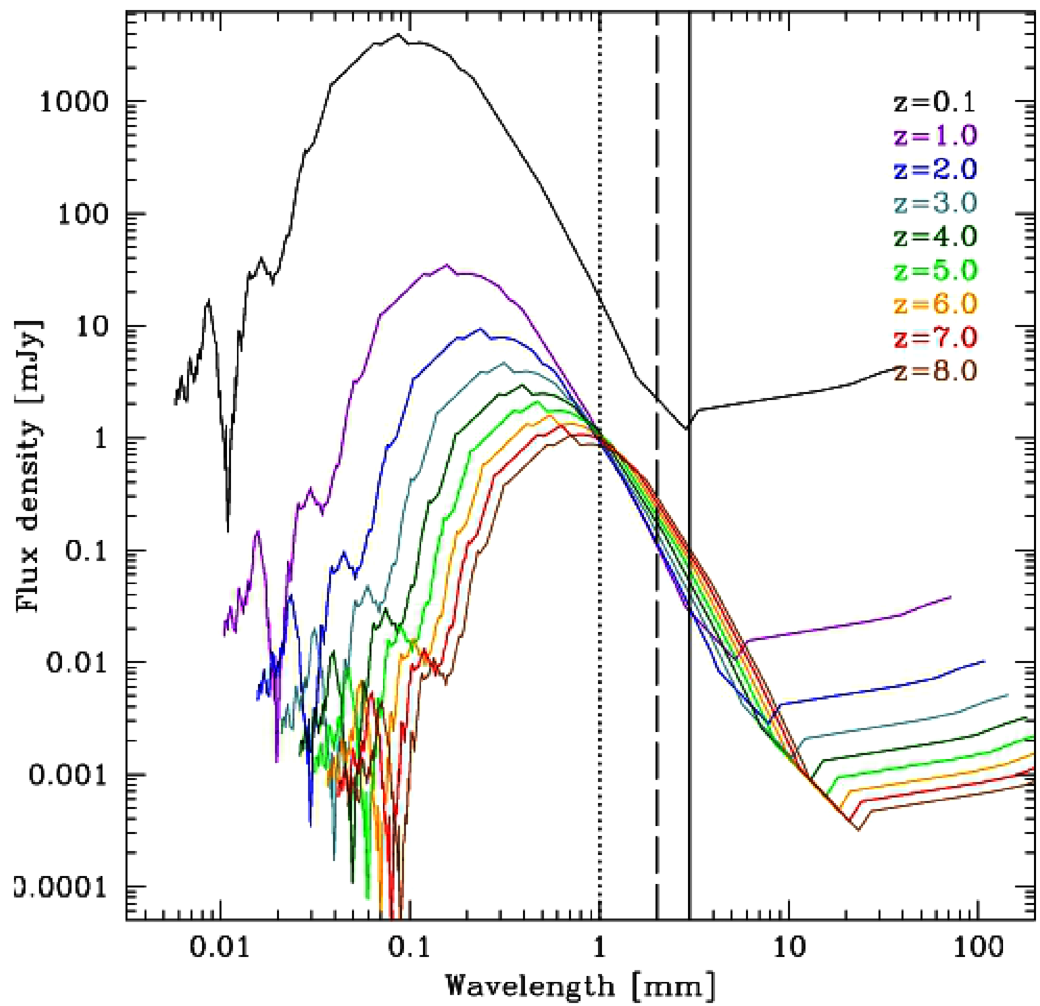
Received power: $P_{\mathrm{obs}} = \dfrac{h\nu_0}{\delta t_0} = \dfrac{h\nu_1}{\delta t_1} \cdot \dfrac{a_1^2}{a_0^2}$

Flux measured on a sphere at distance r_1:

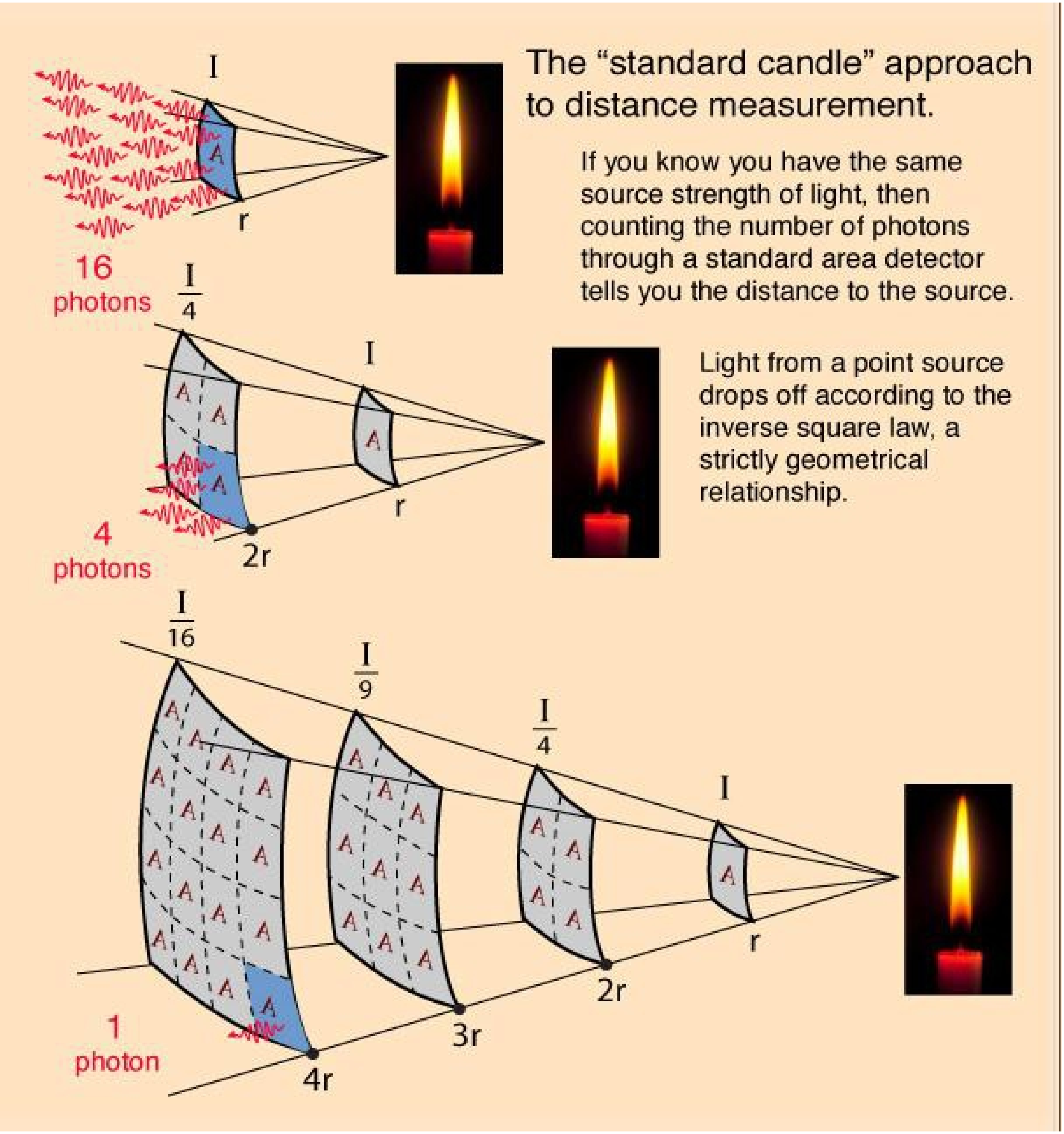
$$F_{\mathrm{obs}} = L \cdot \frac{1}{4\pi a_0^2 r_1^2} \cdot \frac{a_1^2}{a_0^2}$$

This implies $d_{\mathrm{L}} = \dfrac{r_1}{a} = (1+z)\, r_1 = (1+z)^2 \cdot d_{\mathrm{A}}$

In practice, we do not record the light emitted at all wavelengths from an astronomical source, but rather only a part of its electromagnetic spectrum, between $\lambda - \Delta\lambda$ and $\lambda + \Delta\lambda$. This introduces an additional term into the expression for the luminosity distance, which accounts for the fact that astronomical sources do not emit the same power at all wavelengths. This factor is termed the K-correction.



Cosmological Tests using Supernovae as Standard Candles



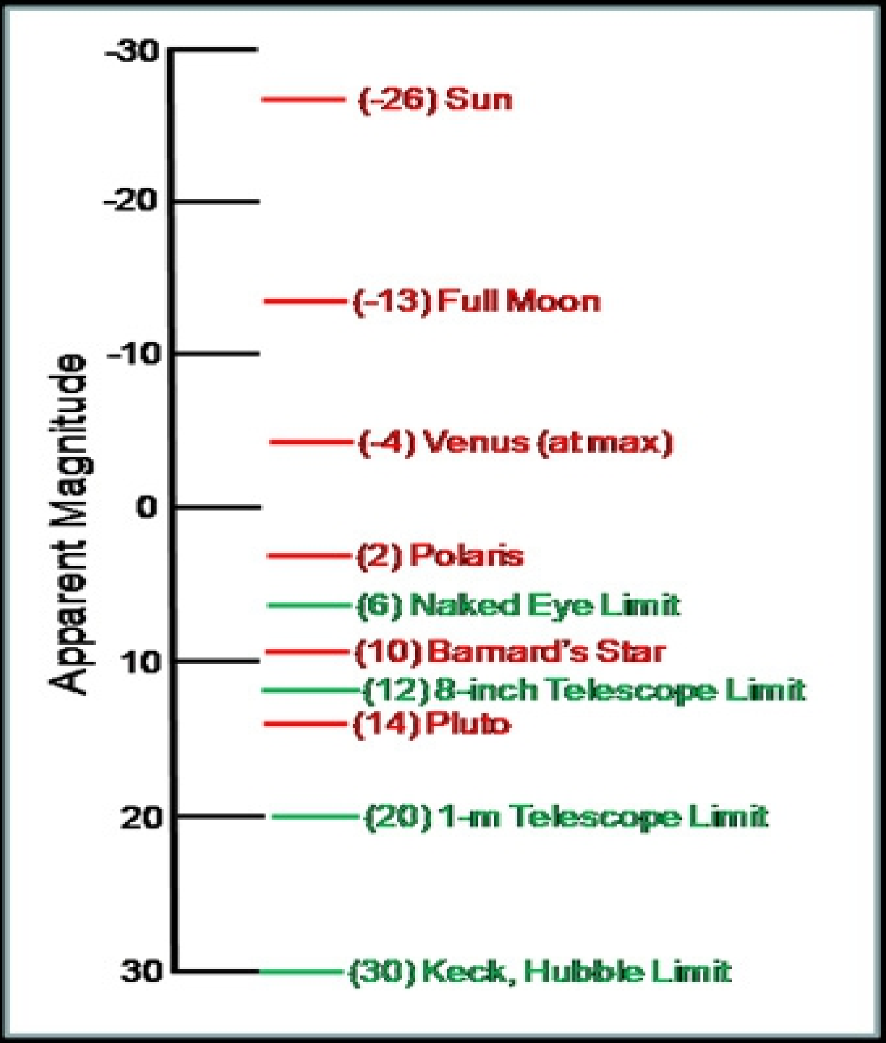
Apparent Magnitude

(useful for describing how bright objects appear from the Earth)

The original magnitude system of Hipparchus had:

magnitude 1 – the brightest stars
magnitude 2 …
magnitude 3 …
magnitude 4 …
magnitude 5 …
magnitude 6 – the faintest stars

Today the magnitude system has been extended to include much fainter and brighter objects.



Note: Logarithmic scale. A a first magnitude star is about 2.512 times as bright as a second magnitude star.

Define a distance modulus:

Absolute mag

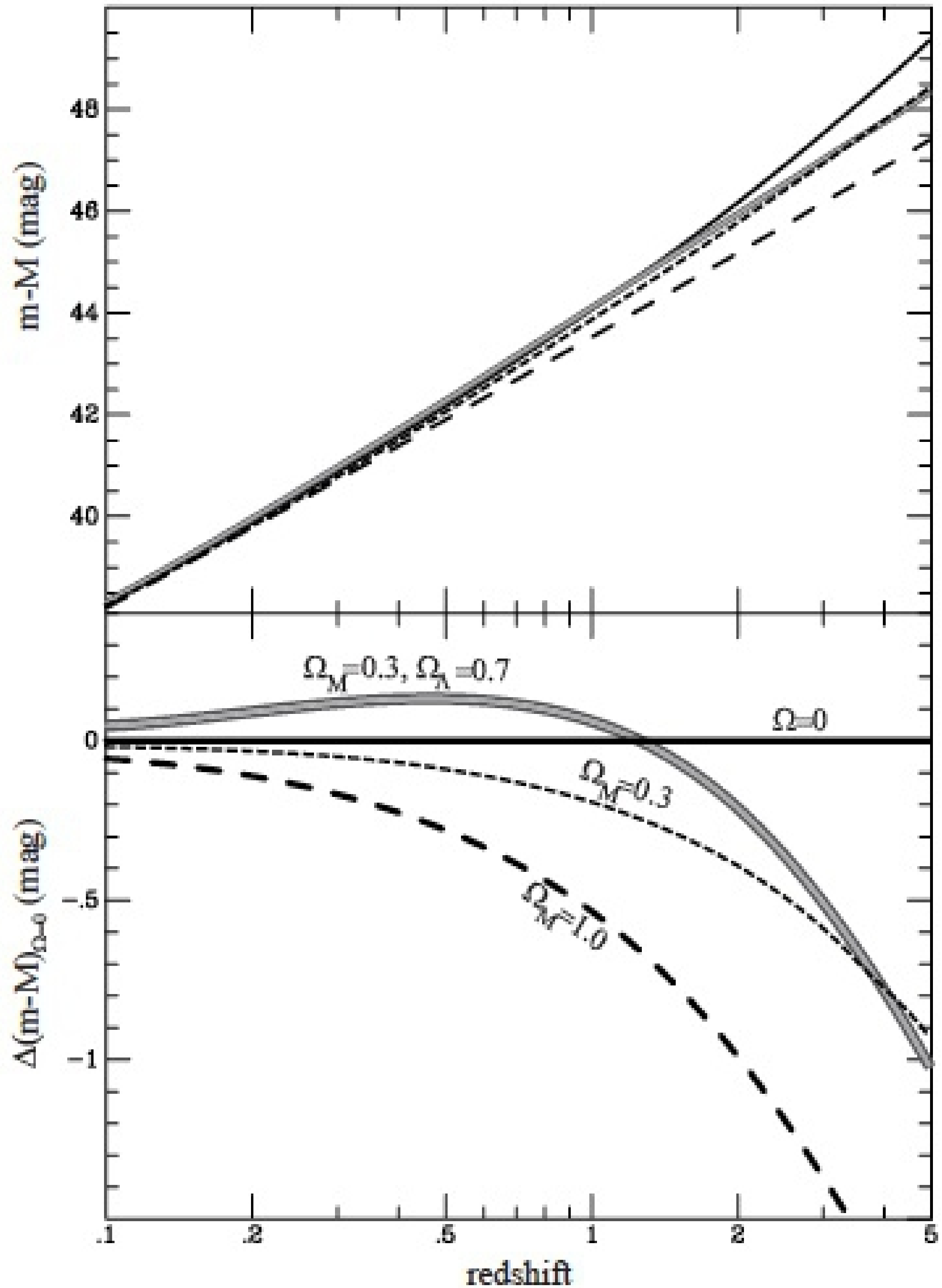
Apparent mag

$$M - m = 2.5 \log \left(\frac{d_{\mathrm{L},0}}{d_{\mathrm{L}}}\right)^2 = 5 \log \left(\frac{d_{\mathrm{L},0}}{d_{\mathrm{L}}}\right)$$

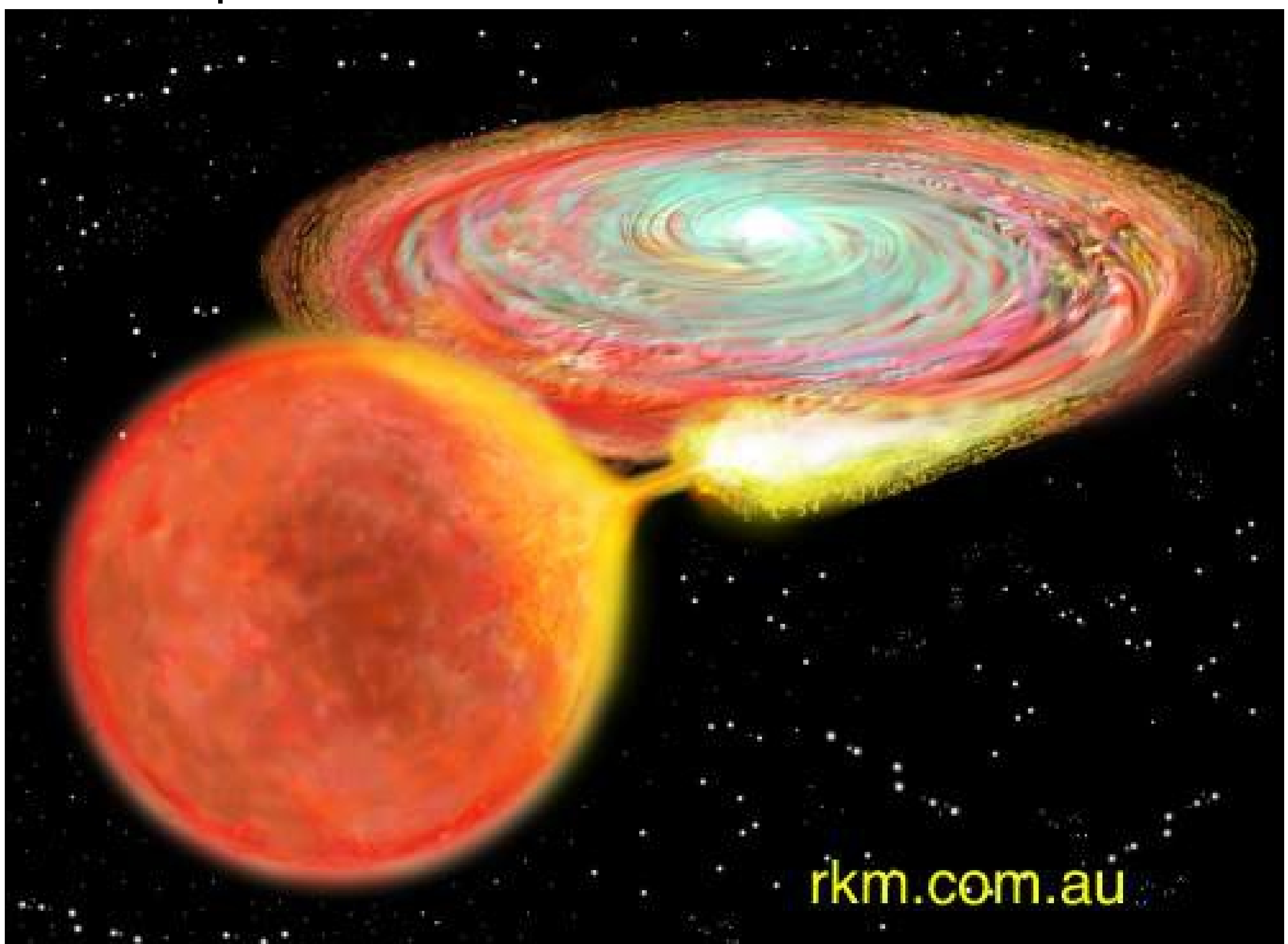
If we set $d_{L,0}$ at 1 Mpc:

$$m = M + 5 \log d_{\mathrm{L}} + 25$$

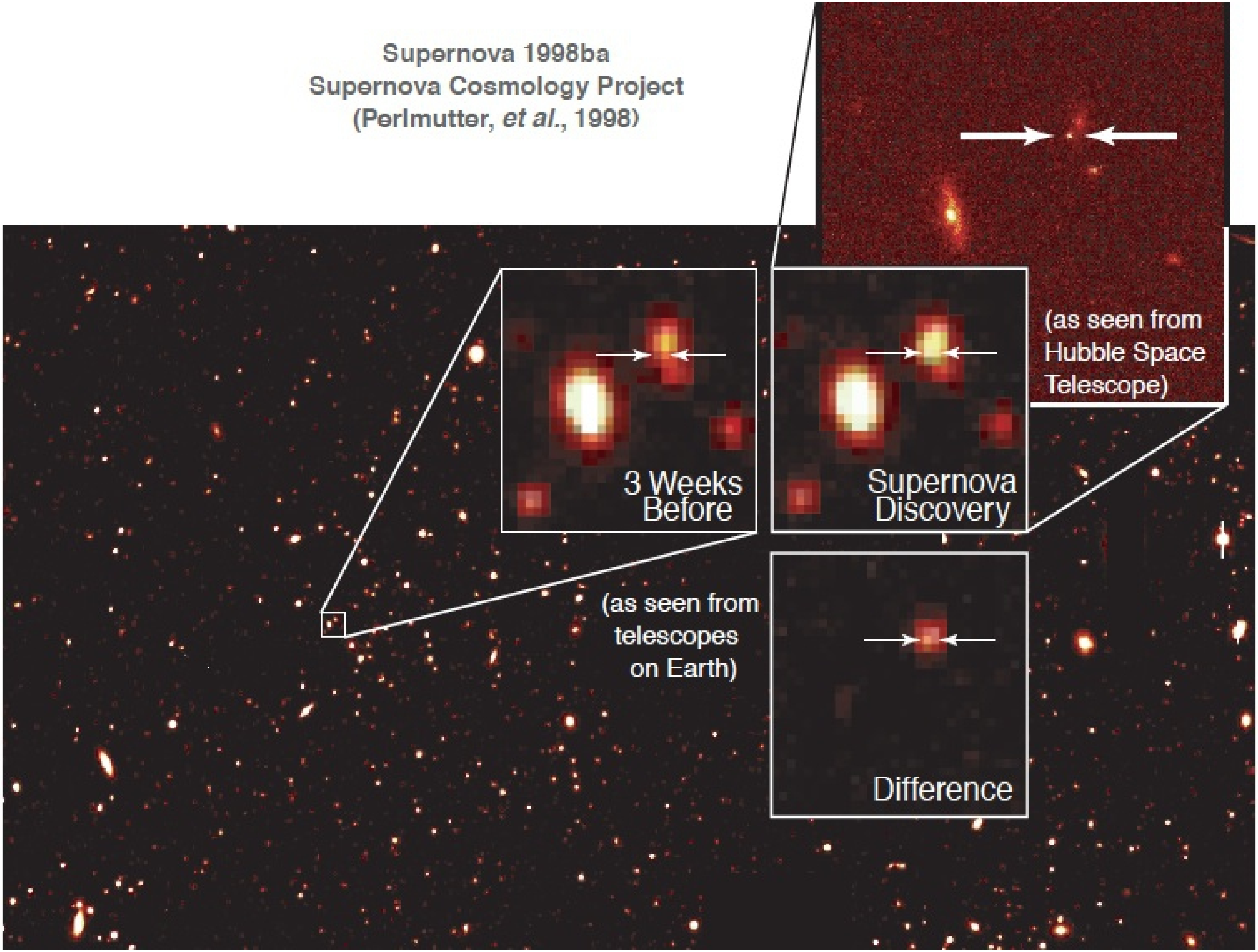
Sensitivity of distance modulus to cosmology

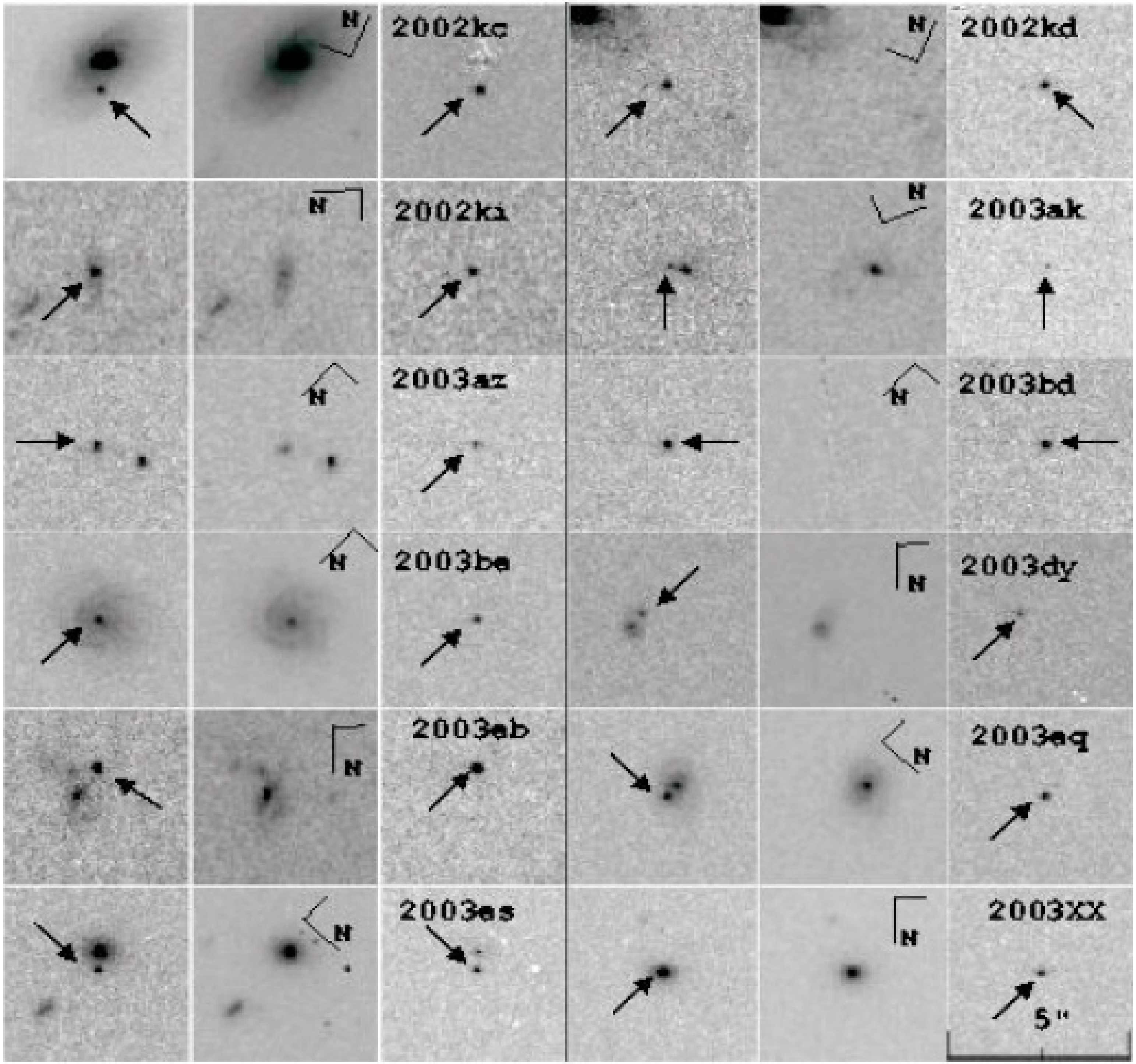


SN of type Ia are thought to be nuclear explosions of carbon/oxygen white dwarfs in binary systems. The white dwarf (a stellar remnant supported by the degenerate pressure of electrons)accretes matter from an evolving companion and its mass increases toward the Chandrasekhar limit of 1.44 solar masses (this is the mass above which the degenerate electrons become relativistic and the white dwarf unstable). Near this limit there is a nuclear detonation in the core in which carbon (or oxygen) is converted to iron. A nuclear flame propagates tot he exterior and blows the white dwarf apart.

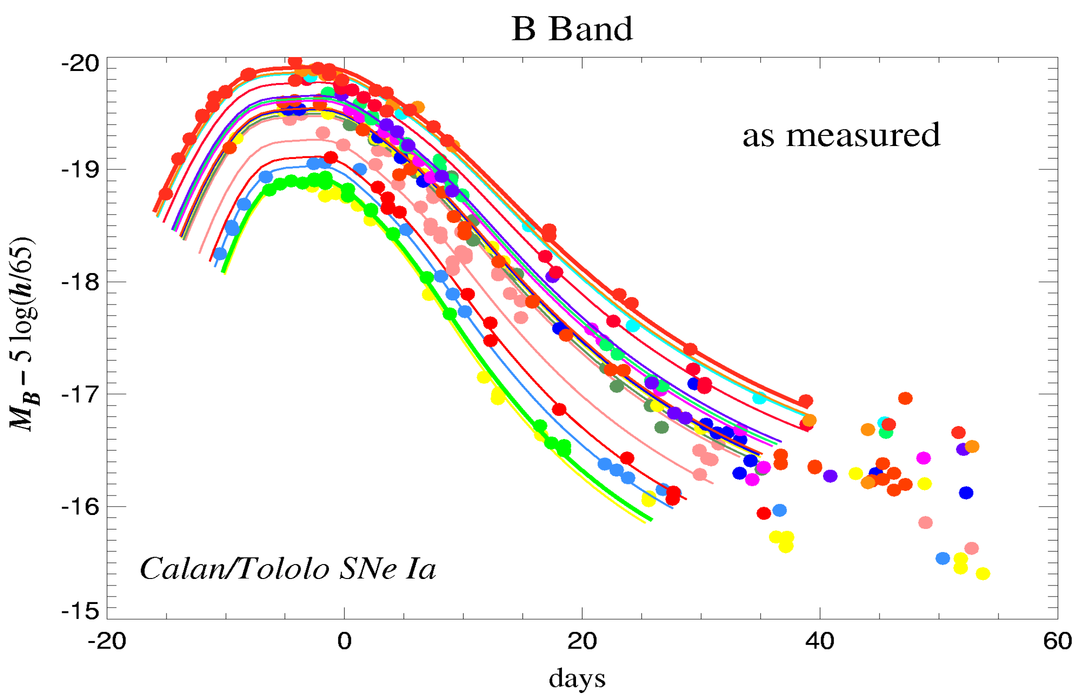


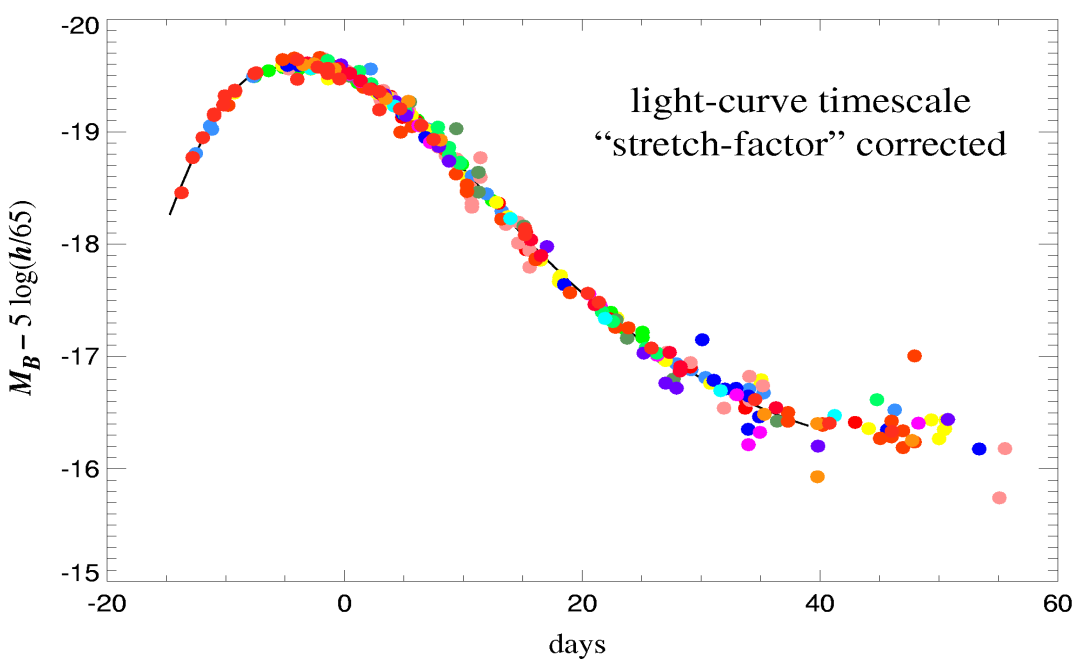
Supernova 1998ba
Supernova Cosmology Project
(Perlmutter, *et al.*, 1998)





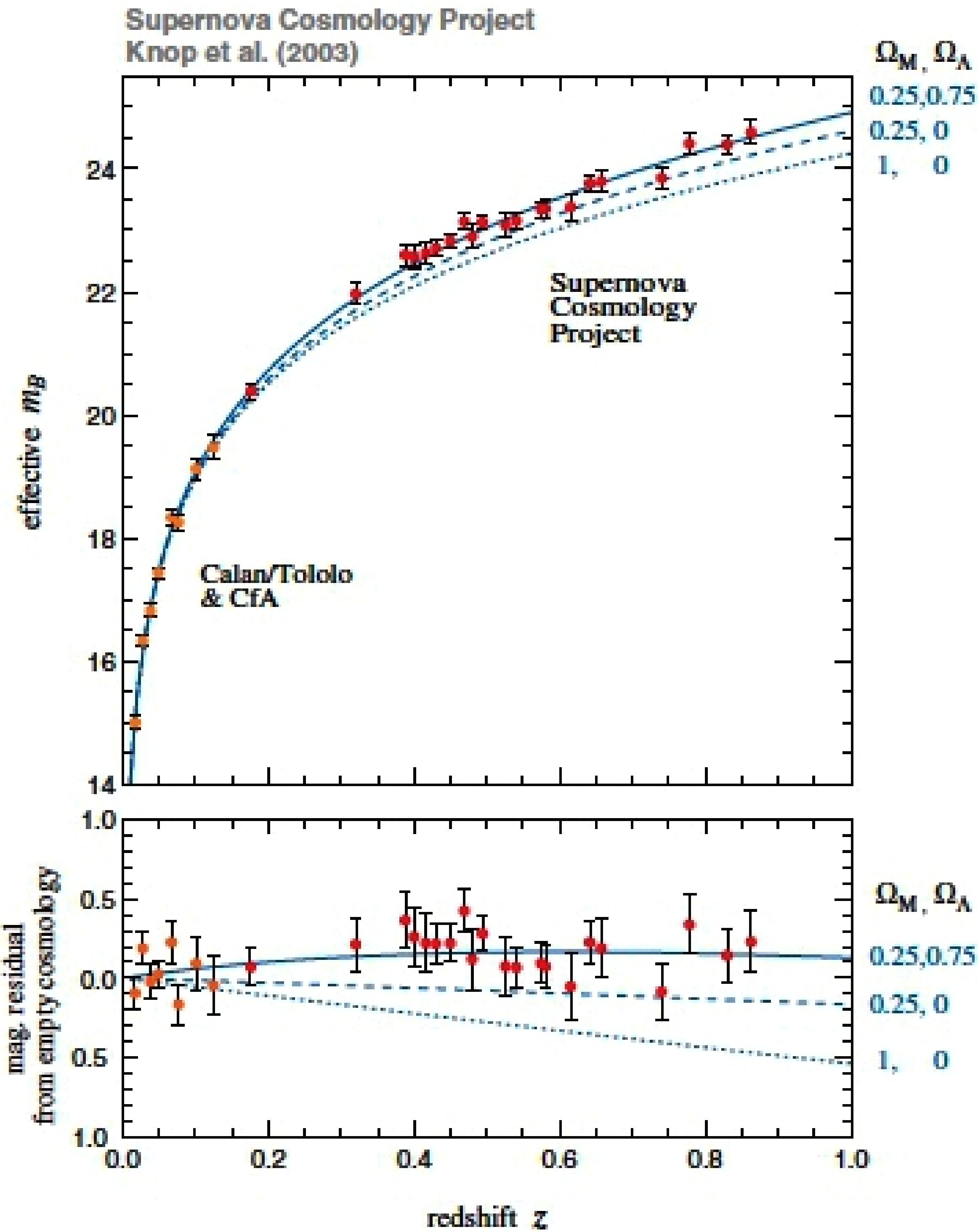
Supernovae in distant galaxies found by HST





Kim, *et al.* (1997)

Light curves can be scaled to yield a "universal" shape --
So the peak brightness can serve as a standard candle, provided astronomers can track the supernovae as it fades



Supernova Cosmology Project
Knop et al. (2003)
Ω_M, Ω_Λ
0.25,0.75
0.25, 0
1, 0
24
22
20
18
16
14
effective m_B
Supernova Cosmology Project
Calan/Tololo & CfA
1.0
0.5
0.0
0.5
1.0
mag. residual from empty cosmology
Ω_M, Ω_Λ
0.25,0.75
0.25, 0
1, 0
0.0
0.2
0.4
0.6
0.8
1.0
redshift z