The Set of All Natural Numbers in Ernst Zermelo's System of Axioms

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Abstract

Ernst Zermelo's axioms published in 1908 are so far the base of many contemporary set theories. We dare to claim that the set $\mathbb{N}$ – the set of all natural numbers – cannot be properly established on these ideas. Either the *Axiom of Infinity* or the *Axiom of Extensionality* must be broken, so the set $\mathbb{N}$ is impossible. Points below.

1 Preliminaries

The original form of Ernst Zermelo's axioms published in German in 1908 we can find in [1]. German version and its verbatim English translation are presented in [2]. The base for our discussion is the version available in [3].

2 The set $\mathbb{N}$ as the least inductive set

Let $\mathbb{N}$ be the set of all natural numbers – identified as the least inductive set described in the *Axiom of Infinity* (this widely known construction can be found e.g. in [3], chapter 5). $\mathbb{P}(\mathbb{N})$ is the powerset of $\mathbb{N}$ – the set of all subsets of $\mathbb{N}$.

We can define by recursion function $f : \mathbb{N} \longmapsto \mathbb{P}(\mathbb{N})$ as follows:

$$f(n) = \begin{cases} \mathbb{N} \setminus \{n\} & \text{for } n = 0, \\ f(n-1) \setminus \{n\} & \text{for } n \neq 0. \end{cases}$$

Let us sign the domain of f as D_f. Obviously $D_f = \mathbb{N}$.

Another form of f is:

$$f(n) = (\mathbb{N} \setminus \bigcup_{i=0}^{n} \{i\}) \qquad \text{for } n \in \mathbb{N}.$$

The key issue for our discussion is that the image of D_f under the function f must exist. It is an unambiguously determined set that must contain values of f calculated for all(!) elements of D_f as its own elements. Nothing can change these facts. We should not ignore them, but rather make use of them.

No one element of D_f is missing as an argument of f, whose image exists and is an element of $f(D_f)$. This way no one element of D_f is missing as a subtrahend (*lege artis* – as a one-element set) described in the definition of f.

$$\{n \in D_f \mid f(n) \notin f(D_f)\} = \emptyset.$$

Each one element of $f(D_f)$ is a subset of $\mathbb{N}$ (the minuend) without elements that are subtracted according to the definition of f. Since the set $f(D_f)$ contains images of all elements of D_f, all elements of D_f are effectively subtracted, i.e. the result of subtraction always exists and is an element of $f(D_f)$.

Thus $f(D_f)$ must contain a subset of $\mathbb{N}$ (the minuend) with no any element of D_f:

$$\{n \in \mathbb{N} \,|\, n \notin D_f\} \in f(D_f). \tag{1}$$

Now we can ask whether $\emptyset \in f(D_f)$ or not. *(Tertium non datur.)*

• If $\emptyset \in f(D_f)$, then there exists $n \in D_f$ such that $f(n) = \emptyset$. It means that $f(n) = \mathbb{N} \setminus \bigcup\limits_{i=0}^{n} \{i\} = \emptyset$, so $\mathbb{N} \subset \bigcup\limits_{i=0}^{n} \{i\}$ and $\mathbb{N}$ has the last element. In this case the *Axiom of Infinity* is violated.

• If $\emptyset \notin f(D_f)$, then $\{n \in \mathbb{N} \,|\, n \notin D_f\} \neq \emptyset$ – because of (1). It means that $\exists_{n \in \mathbb{N}} \, (n \notin D_f)$. As $D_f = \mathbb{N}$, we have:

$$\exists_n \, (n \in \mathbb{N} \wedge n \notin \mathbb{N}),$$

so we exactly get that $\mathbb{N} \neq \mathbb{N}$ in the sens of the *Axiom of Extensionality.*

3 The set $\mathbb{N}$ as ω

The set of all natural numbers $\mathbb{N}$ can be identified with the ordinal number ω – the first non-zero limit ordinal number. (*Axiom of Replacement* is required.) Just $\mathbb{N} = \omega$. (It can be found e.g. in [3], chapter 12.)
We can repeat our discussion. Either ω is not limit ordinal or $\omega \neq \omega$.

4 Conclusions

In set theories based on Zermelo's ideas the set of all natural numbers $\mathbb{N}$ cannot exist. That is all.

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References

[1] Zermelo Ernst. Untersuchungen über die Grundlagen der Mengenlehre. I. *Mathematische Annalen 65*, 1908. (261-281)

[2] Ebbinghaus Heinz-Dieter, Peckhaus Volker. *Ernst Zermelo – An Approach to His Life and Work.* Springer, 2007.

[3] Moschovakis Yiannis. *Notes on Set Theory.* Springer, 2006.