VAN NOSTRAND MATHEMATICAL STUDIES #10

# lectures on QUASICONFORMAL MAPPINGS

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Lectures on

# QUASICONFORMAL MAPPINGS

*by*

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# CHAPTER I

## *DIFFERENTIABLE QUASICONFORMAL MAPPINGS*

### Introduction

There are several reasons why quasiconformal mappings have recently come to play a very active part in the theory of analytic functions of a single complex variable.

1. The most superficial reason is that q.c. mappings are a natural generalization of conformal mappings. If this were their only claim they would soon have been forgotten.

2. It was noticed at an early stage that many theorems on conformal mappings use only the quasiconformality. It is therefore of some interest to determine when conformality is essential and when it is not.

3. Q.c. mappings are less rigid than conformal mappings, and are therefore much easier to use as a tool. This was typical of the utilitarian phase of the theory. For instance, it was used to prove theorems about the conformal type of simply connected Riemann surfaces (now mostly forgotten).

4. Q.c. mappings play an important role in the study of certain elliptic partial differential equations.

5. Extremal problems in q.c. mappings lead to analytic functions connected with regions or Riemann surfaces. This was a deep and unexpected discovery due to Teichmüller.

6. The problem of moduli was solved with the help of q.c. mappings. They also throw light on Fuchsian and Kleinian groups.

7. Conformal mappings degenerate when generalized to several variables, but q.c. mappings do not. This theory is still in its infancy.

### A. *The Problem and Definition of Grötzsch*

The notion of a quasiconformal mapping, but not the name, was introduced by H. Grötzsch in 1928. If $Q$ is a square and $R$ is a rectangle, not a square, there is no conformal mapping of $Q$ on $R$ which maps vertices on vertices. Instead, Grötzsch asks for the most nearly conformal mapping of this kind. This calls for a measure of approximate conformality, and in supplying such a measure Grötzsch took the first step toward the creation of a theory of q.c. mappings.

All the work of Grötzsch was late to gain recognition, and this particular idea was regarded as a curiosity and allowed to remain dormant for several years. It reappears in 1935 in the work of Lavrentiev, but from the point of view of partial differential equations. In 1936 I included a reference to the q.c. case in my theory of covering surfaces. From then on the notion became generally known, and in 1937 Teichmüller began to prove important theorems by use of q.c. mappings, and later theorems about q.c. mappings.

We return to the definition of Grötzsch. Let $w = f(z)$ $(z = x + iy,\ w = u + iv)$ be a $C^1$ homeomorphism from one region to another. At a point $z_0$ it induces a linear mapping of the differentials

$$\begin{aligned} du &= u_x\, dx + u_y dy \\ dv &= v_x\, dx + v_y dy \end{aligned} \tag{1}$$

which we can also write in the complex form

$$dw = f_z\, dz + f_{\bar z}\, d\bar z \tag{2}$$

with

$$f_z = \tfrac{1}{2}(f_x - if_y), \qquad f_{\bar z} = \tfrac{1}{2}(f_x + if_y) \ . \tag{3}$$

Geometrically, (1) represents an affine transformation from the $(dx, dy)$ to the $(du, dv)$ plane. It maps circles about the origin into similar ellipses. We wish to compute the ratio between the axes as well as their direction.

In classical notation one writes

$$du^2 + dv^2 = E\, dx^2 + 2F\, dxdy + G\, dy^2 \tag{4}$$

with

$$E = u_x^{\,2} + v_x^{\,2}\,, \quad F = u_x u_y + v_x v_y, \quad G = u_y^{\,2} + v_y^{\,2} \ .$$

The eigenvalues are determined from

(5) $$\begin{vmatrix} E-\lambda & F \\ F & G-\lambda \end{vmatrix} = 0$$

and are

(6) $$\lambda_1, \lambda_2 = \frac{E+G \pm[(E-G)^2+4F^2]^{\frac{1}{2}}}{2} .$$

The ratio $a:b$ of the axes is

(7) $$\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{2}} = \frac{E+G+[(E-G)^2+4F^2]^{\frac{1}{2}}}{2(EG-F^2)^{\frac{1}{2}}} .$$

The complex notation is much more convenient. Let us first note that

(8) $$\begin{aligned} f_z &= \tfrac{1}{2}(u_x+v_y) + \tfrac{i}{2}(v_x-u_y) \\ f_{\bar z} &= \tfrac{1}{2}(u_x-v_y) + \tfrac{i}{2}(v_x+u_y) \end{aligned} .$$

This gives

(9) $$|f_z|^2 - |f_{\bar z}|^2 = u_x v_y - u_y v_x = J$$

which is the Jacobian. The Jacobian is positive for sense preserving and negative for sense reversing mappings. For the moment we shall consider only the sense preserving case. Then $|f_{\bar z}| < |f_z|$.

It now follows immediately from (2) that

(10) $$(|f_z| - |f_{\bar z}|)\,|dz| \le |dw| \le (|f_z| + |f_{\bar z}|)\,|dz|$$

where both limits can be attained. We conclude that the ratio of the major to the minor axis is

(11) $$D_f = \frac{|f_z| + |f_{\bar z}|}{|f_z| - |f_{\bar z}|} \ge 1$$

This is called the *dilatation* at the point $z$. It is often more convenient to consider

(12) $$d_f = \frac{|f_{\bar{z}}|}{|f_z|} < 1$$

related to $D_f$ by

(13) $$D_f = \frac{1 + d_f}{1 - d_f}, \qquad d_f = \frac{D_f - 1}{D_f + 1} .$$

The mapping is conformal at $z$ if and only if $D_f = 1$, $d_f = 0$.

The maximum is attained when the ratio

$$\frac{f_{\bar{z}}\, d\bar{z}}{f_z\, dz}$$

is positive, the minimum when it is negative. We introduce now the *complex dilatation*

(14) $$\mu_f = \frac{f_{\bar{z}}}{f_z}$$

with $|\mu_f| = d_f$. The maximum corresponds to the direction

(15) $$\arg dz = \alpha = \tfrac{1}{2} \arg \mu ,$$

the minimum to the direction $\alpha \pm \pi/2$. In the *dw*-plane the direction of the major axis is

(16) $$\arg dw = \beta = \tfrac{1}{2} \arg \nu$$

where we have set

(17) $$\nu_f = \frac{f_{\bar{z}}}{\bar{f}_{\bar{z}}} = \left(\frac{f_z}{|f_z|}\right)^2 \mu_f .$$

The quantity $\nu_f$ may be called the *second complex dilatation.*

We will illustrate by the following self-explanatory figure:


Observe that $\beta - a = \arg f_z$.

*Definition* 1. The mapping $f$ is said to be quasiconformal if $D_f$ is bounded. It is $K$-quasiconformal if $D_f \leq K$.

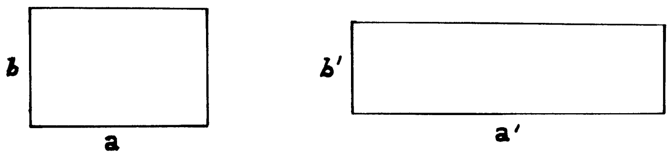
The condition $D_f \leq K$ is equivalent to $d_f \leq k = (K-1)/(K+1)$. A 1-quasiconformal mapping is conformal.

Let it be said at once that the restriction to $C^1$-mappings is most unnatural. One of our immediate aims is to get rid of this restriction. For the moment, however, we prefer to push this difficulty aside.

## B. *Solution of Grötzsch's Problem*

We pass to Grötzsch's problem and give it a precise meaning by saying that $f$ is most nearly conformal if $\sup D_f$ is as small as possible.

Let $R$, $R'$ be two rectangles with sides $a$, $b$ and $a'$, $b'$. We may assume that $a:b \leq a':b'$ (otherwise, interchange $a$ and $b$). The mapping $f$ is supposed to take $a$-sides into $a$-sides and $b$-sides into $b$-sides.




The computation goes

$$a' \le \int_0^a |df(x+iy)| \le \int_0^a (|f_z| + |f_{\bar z}|)\, dx$$

$$a'b \le \int_0^a \int_0^b (|f_z| + |f_{\bar z}|)\, dx\, dy$$

$$a'^2b^2 \le \int_0^a \int_0^b \frac{|f_z| + |f_{\bar z}|}{|f_z| - |f_{\bar z}|}\, dx\, dy \int_0^a \int_0^b (|f_z|^2 - |f_{\bar z}|^2)\, dx\, dy$$

$$= a'b' \int_0^a \int_0^b D_f\, dx\, dy$$

or

(1) $$\frac{a'}{b'} : \frac{a}{b} \le \frac{1}{ab} \iint_R D_f\, dx\, dy$$

and in particular

$$\frac{a'}{b'} : \frac{a}{b} \le \sup D_f \ .$$

The minimum is attained for the affine mapping which is given by

$$f(z) = \tfrac{1}{2}\left(\frac{a'}{a} + \frac{b'}{b}\right) z + \tfrac{1}{2}\left(\frac{a'}{a} - \frac{b'}{b}\right) \bar z$$

THEOREM 1. *The affine mapping has the least maximal and the least average dilatation.*

The ratios $m = a/b$ and $m' = a'/b'$ are called the modules of $R$ and $R'$ (taken with an orientation). We have proved that there exists a $K$-q.c. mapping of $R$ on $R'$ if and only if

(2) $$\frac{1}{K} \le \frac{m'}{m} \le K .$$

C. *Composed Mappings*

We shall determine the complex derivatives and complex dilatations of a composed mapping $g \circ f$. There is the usual trouble with the notation which is most easily resolved by introducing an intermediate variable $\zeta = f(z)$.

The usual rules are applicable and we find

(1) $$\begin{aligned}(g \circ f)_z &= (g_\zeta \circ f)\, f_z + (g_{\bar\zeta} \circ f)\, \bar f_z \\ (g \circ f)_{\bar z} &= (g_\zeta \circ f)\, f_{\bar z} + (g_{\bar\zeta} \circ f)\, \bar f_{\bar z}\end{aligned}$$

When solved they give

(2) $$\begin{aligned}g_\zeta \circ f &= \frac{1}{J}\,[(g \circ f)_z \bar f_{\bar z} - (g \circ f)_{\bar z} \bar f_z] \\ g_{\bar\zeta} \circ f &= \frac{1}{J}\,[(g \circ f)_{\bar z} f_z - (g \circ f)_z f_{\bar z}]\end{aligned}$$

where $J = |f_z|^2 - |f_{\bar z}|^2$ .

For $g = f^{-1}$ the formulas become

(3) $$(f^{-1})_\zeta \circ f = \bar f_{\bar z}/J , \qquad (f^{-1})_{\bar\zeta} \circ f = -f_{\bar z}/J .$$

One derives, for instance,

(4) $$\mu_{f^{-1}} = -\nu_f \circ f^{-1}$$

and, on passing to the absolute values,

(5) $$d_{f^{-1}} = d_f \circ f^{-1}$$

In other words, inverse mappings have the same dilatation at corresponding points.

From (2) we obtain

(6) $$\mu_g \circ f = \frac{f_z}{\bar{f}_{\bar{z}}} \frac{\mu_{g \circ f} - \mu_f}{1 - \bar{\mu}_f \mu_{g \circ f}} .$$

If $g$ is conformal, then $\mu_g = 0$ and we find

(7) $$\mu_{g \circ f} = \mu_f .$$

If $f$ is conformal, $\mu_f = 0$ and

(8) $$\mu_g \circ f = \left(\frac{f'}{|f'|}\right)^2 \mu_{g \circ f} .$$

which can also be written as

(9) $$\nu_g \circ f = \nu_{g \circ f} .$$

In any case, the dilatation is invariant with respect to all conformal transformations.

If we set $g \circ f = h$ we find from (6)

(10) $$\mu_{h \circ f^{-1}} \circ f = \frac{f_z}{\bar{f}_{\bar{z}}} \frac{\mu_h - \mu_f}{1 - \bar{\mu}_f \mu_h} .$$

For the dilatation

(11) $$d_{h \circ f^{-1}} \circ f = \left| \frac{\mu_h - \mu_f}{1 - \mu_f \bar{\mu}_h} \right|$$

and

(12) $$\log D_{h \circ f^{-1}} \circ f = [\mu_h, \mu_f] \ ,$$

the non-euclidean distance (with respect to the metric $ds = \frac{2|dw|}{1-|w|^2}$ in $|w| < 1$).

We can obviously use $\sup [\mu_h, \mu_f]$ as a distance between the mappings $f$ and $h$ (the Teichmüller distance). It is a metric provided one identifies mappings that differ by a conformal transformation.

The composite of a $K_1$-q.c. and a $K_2$-q.c. mapping is $K_1 K_2$- q.c.

D. *Extremal Length*

Let $\Gamma$ be a family of curves in the plane. Each $\gamma \in \Gamma$ shall be a countable union of open arcs, closed arcs or closed curves, and every closed subarc shall be rectifiable. We shall introduce a geometric quantity $\lambda(\Gamma)$, called the *extremal length* of $\Gamma$, which is a sort of average minimal length. Its importance for our topic lies in the fact that it is invariant under conformal mappings and quasi-invariant under q.c. mappings (the latter means that it is multiplied by a bounded factor).

A function $\rho$, defined in the whole plane, will be called *allowable* if it satisfies the following conditions:

1. $\rho \geq 0$ and measurable.
2. $A(\rho) = \iint \rho^2 \, dx \, dy \neq 0, \infty$ (the integral is over the whole plane).

For such a $\rho$, set

$$L_\gamma(\rho) = \int_\gamma \rho \, |dz|$$

if $\rho$ is measurable on $\gamma$*, $L_\gamma(\rho) = \infty$ otherwise. We introduce

$$L(\rho) = \inf_{\gamma \epsilon \Gamma} L_\gamma(\rho)$$

and

*Definition:*

$$\lambda(\Gamma) = \sup_\rho \frac{L(\rho)^2}{A(\rho)}$$

for all allowable $\rho$.

We shall say that $\Gamma_1 < \Gamma_2$ if every $\gamma_2$ contains a $\gamma_1$ (the $\gamma_2$ are fewer and longer).

*Remark.* Observe that $\Gamma_1 \subset \Gamma_2$ implies $\Gamma_2 < \Gamma_1$ !

THEOREM 2. *If* $\Gamma_1 < \Gamma_2$, *then* $\lambda(\Gamma_1) \leq \lambda(\Gamma_2)$.

*Proof.* If $\gamma_1 \subseteq \gamma_2$ then

$$L_{\gamma_1}(\rho) \leq L_{\gamma_2}(\rho)$$

$$\inf L_{\gamma_1}(\rho) \leq \inf L_{\gamma_2}(\rho)$$

and it follows at once that $\lambda(\Gamma_1) \leq \lambda(\Gamma_2)$.

---

* (as a function of arc-length)

*Example 1.* $\Gamma$ is the set of all arcs in a closed rectangle $R$ which joins a pair of opposite sides.

For any $\rho$

$$\int_0^a \rho(x+iy)\,dx \geq L(\rho)$$

$$\iint_R \rho\,dx\,dy \geq b\,L(\rho)$$

$$b^2\,L(\rho)^2 \leq ab \iint_R \rho^2\,dx\,dy \leq ab\,A(\rho)$$

$$\frac{L(\rho)^2}{A(\rho)} \leq \frac{a}{b} .$$

This proves $\lambda(\Gamma) \leq a/b$ .

On the other hand, take $\rho = 1$ in $R$, $\rho = 0$ outside. Then $L(\rho) = a$, $A(\rho) = ab$, hence $\lambda(\Gamma) \geq a/b$. We have proved

$$\lambda(\Gamma) = \frac{a}{b} .$$

*Example 2.* $\Gamma$ is the set of all arcs in an annulus $r_1 \leq |z| \leq r_2$ which join the boundary circles.

Computation:

$$\int_{r_1}^{r_2} \rho\,dr \geq L(\rho) , \qquad \iint \rho\,dr\,d\theta \geq 2\pi L(\rho)$$

$$4\pi^2 L(\rho)^2 \leq 2\pi \log \frac{r_2}{r_1} \iint \rho^2 \, r dr d\theta$$

$$\frac{L(\rho)^2}{A(\rho)} \leq \frac{1}{2\pi} \log \frac{r_2}{r_1} .$$

Equality for $\rho = 1/r$.

*Example 3.* The module of an annulus.

Let $G$ be a doubly connected region in the finite plane with $C_1$ the bounded, $C_2$ the unbounded component of the complement. We say the closed curve $\gamma$ in $G$ *separates* $C_1$ and $C_2$ if $\gamma$ has non-zero winding number about the points of $C_1$. Let $\Gamma$ be the family of closed curves in $G$ which separate $C_1$ and $C_2$. The module $M(G) = \lambda(\Gamma)^{-1}$. Consider, for example, the annulus $G = \{r_1 \leq |z| \leq r_2\}$.

$$L(\rho) \leq \int_0^{2\pi} \rho(re^{i\theta}) \, r d\theta$$

$$\frac{L(\rho)}{r} \leq \int_0^{2\pi} \rho \, d\theta$$

$$L(\rho) \log\left(\frac{r_2}{r_1}\right) \leq \iint \rho \, dr \, d\theta$$

$$L(\rho)^2 \log^2\left(\frac{r_2}{r_1}\right) \leq 2\pi \log\left(\frac{r_2}{r_1}\right) \iint \rho^2 \, r \, dr \, d\theta$$

$$\frac{L(\rho)^2}{A(\rho)} \leq \frac{2\pi}{\log(r_2/r_1)} .$$

Once again $\rho = 1/2\pi r$ gives equality. Indeed, for any $\gamma \in \Gamma$ we have

$$1 \leq |n(\gamma, 0)| = \frac{1}{2\pi} \left| \int_\gamma \frac{dz}{z} \right| \leq \frac{1}{2\pi} \int_\gamma \frac{|dz|}{|z|} = L_\rho(\gamma) ,$$

so $L(\rho) = 1$ and $A(\rho) = \frac{1}{2\pi} \log(r_2/r_1)$. We conclude that $M(G) = \frac{1}{2\pi} \log(r_2/r_1)$.

Suppose that all $\gamma \in \Gamma$ are contained in a region $\Omega$ and let $\phi$ be a $K$-quasiconformal mapping of $\Omega$ on $\Omega'$. Let $\Gamma'$ be the image set of $\Gamma$.

THEOREM 3. $K^{-1} \lambda(\Gamma) \leq \lambda(\Gamma') \leq K \lambda(\Gamma)$.

*Proof.* For a given $\rho(z)$ define $\rho'(\zeta) = 0$ outside $\Omega'$ and

$$\rho'(\zeta) = \frac{\rho}{|\phi_z| - |\phi_{\bar z}|} \circ \phi^{-1}$$

in $\Omega'$. Then

$$\int_{\gamma'} \rho' |d\zeta| \geq \int_\gamma \rho |dz|$$

$$\int\int \rho'^2 \, d\xi \, d\eta = \int\int_\Omega \rho^2 \frac{|\phi_z| + |\phi_{\bar z}|}{|\phi_z| - |\phi_{\bar z}|} \, dx \, dy \leq K A(\rho) .$$

This proves $\lambda' \geq K^{-1} \lambda$, and the other inequality follows by considering the inverse.

COROLLARY. $\lambda(\Gamma)$ is a conformal invariant.

There are two important composition principles.

I. $\Gamma_1 + \Gamma_2 = \{\gamma_1 + \gamma_2 \mid \gamma_1 \epsilon \Gamma_1, \gamma_2 \epsilon \Gamma_2\}$ *

II. $\Gamma_1 \cup \Gamma_2$

THEOREM 4.

a) $\lambda(\Gamma_1 + \Gamma_2) \geq \lambda(\Gamma_1) + \lambda(\Gamma_2)$ ;

b) $\lambda(\Gamma_1 \cup \Gamma_2)^{-1} \geq \lambda(\Gamma_1)^{-1} + \lambda(\Gamma_2)^{-1}$

*if* $\Gamma_1, \Gamma_2$ *lie in disjoint measurable sets.*

*Proof of* a). We may assume that $0 < \lambda(\Gamma_1), \lambda(\Gamma_2) < \infty$ for otherwise the inequality is trivial.

We may normalize so that

$$L_1(\rho_1) = A(\rho_1)$$
$$L_2(\rho_2) = A(\rho_2) \quad .$$

Choose $\rho = \max(\rho_1, \rho_2)$. Then

$$L(\rho) \geq L_1(\rho_1) + L_2(\rho_2) = A(\rho_1) + A(\rho_2)$$
$$A(\rho) \leq A(\rho_1) + A(\rho_2)$$

$$\lambda = \sup \frac{L(\rho)^2}{A(\rho)} \geq A(\rho_1) + A(\rho_2) = \frac{L_1(\rho_1)^2}{A(\rho_1)} + \frac{L_2(\rho_2)^2}{A(\rho_2)}$$

It follows that $\lambda \geq \lambda_1 + \lambda_2$ .

*Proof of* b). If $\lambda = \lambda(\Gamma_1 \cup \Gamma_2) = 0$ there is nothing to prove. Consider an admissible $\rho$ with $L(\rho) > 0$ and set $\rho_1 = \rho$ on $E_1$, $\rho_2 = \rho$ on $E_2$, 0 outside (where $E_1$ and $E_2$ are complementary measurable sets with $\Gamma_1 \subseteq E_1$,

* $\gamma_1 + \gamma_2$ means "$\gamma_1$ followed by $\gamma_2$."

$\Gamma_2 \subseteq E_2$). Then $L_1(\rho_1) \geq L(\rho)$, $L_2(\rho_2) \geq L(\rho)$, and $A(\rho) = A(\rho_1) + A(\rho_2)$. Thus

$$\frac{A(\rho)}{L(\rho)^2} \geq \frac{A(\rho_1)}{L_1(\rho_1)^2} + \frac{A(\rho_2)}{L_2(\rho_2)^2}$$

and hence

$$\lambda^{-1} \geq \lambda_1{}^{-1} + \lambda_2{}^{-1}$$

q.e.d.

E. *A Symmetry Principle.*

For any $\gamma$ let $\bar{\gamma}$ be its reflection in the real axis, and let $\gamma^+$ be obtained by reflecting the part below the real axis and retaining the part above it $(\gamma \cup \bar{\gamma} = \gamma^+ \cup (\gamma^+)^-)$.

The notations $\bar{\Gamma}$ and $\Gamma^+$ are self-explanatory.

THEOREM 5. *If* $\Gamma = \bar{\Gamma}$ *then* $\lambda(\Gamma) = \frac{1}{2}\lambda(\Gamma^+)$.

*Proof.*

1. For a given $\rho$ set $\hat{\rho}(z) = \max(\rho(z), \rho(\bar{z}))$.

Then

$$L_\gamma(\hat{\rho}) = L_{\gamma^+}(\hat{\rho}) \geq L_{\gamma^+}(\rho) \geq L^+(\rho)$$

and

$$A(\hat{\rho}) \leq A(\rho) + A(\bar{\rho}) = 2A(\rho) .$$

This makes

$$\frac{L^+(\rho)^2}{A(\rho)} \leq 2\,\frac{L(\hat{\rho})^2}{A(\hat{\rho})} \leq 2\lambda(\Gamma)$$

and hence $\lambda(\Gamma^+) \leq 2\lambda(\Gamma)$.

2. For given $\rho$ set

$$\rho^+(z) = \begin{cases} \rho(z) + \rho(\overline{z}) & \text{in upper halfplane} \\ 0 & \text{in lower halfplane} \end{cases}$$

Then

$$L_{\gamma^+}(\rho^+) = L_{\gamma^+ + (\gamma^+)^-}(\rho) = L_{\gamma+\gamma^-}(\rho)$$
$$= L_\gamma(\rho) + L_{\overline{\gamma}}(\rho) \geq 2L(\rho) \quad .$$

On the other hand

$$A(\rho^+) \leq 2\int \rho^2 + \overline{\rho}^2 = 2A(\rho)$$

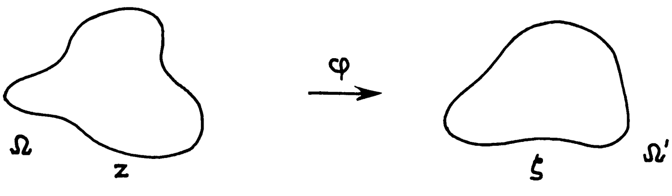
and hence

$$\frac{L(\rho)^2}{A(\rho)} \leq \tfrac{1}{2}\,\frac{L_{\gamma^+}(\rho^+)^2}{A(\rho^+)} \leq \tfrac{1}{2}\lambda(\Gamma^+)$$

$$\lambda(\Gamma) \leq \tfrac{1}{2}\lambda(\Gamma^+) \quad .$$

q.e.d.

## F. *Dirichlet Integrals*




Let $\phi$ be a $K$-q.c. mapping from $\Omega$ to $\Omega'$. The Dirichlet integral of a $C^1$ function $u(\zeta)$ is

$$D(u) = \iint_{\Omega'} (u_\xi^{\,2} + u_\eta^{\,2})\, d\xi\, d\eta = 4\iint |u_\zeta|^2\, d\xi\, d\eta \quad .$$

For the composite $u \circ \phi$ we have

$$(u \circ \phi)_z = (u_\zeta \circ \phi)\phi_z + (u_{\bar\zeta} \circ \phi)\bar\phi_z$$
$$|(u \circ \phi)_z| \le (|u_\zeta| \circ \phi)(|\phi_z| + |\phi_{\bar z}|)$$

$$D(u \circ \phi) \le 4 \iint_\Omega (|u_\zeta| \circ \phi)^2 (|\phi_z| + |\phi_{\bar z}|)^2 \, dx\, dy$$
$$= 4 \iint_{\Omega'} |u_\zeta|^2 \left(\frac{|\phi_z| + |\phi_{\bar z}|}{|\phi_z| - |\phi_{\bar z}|}\right) \circ \phi^{-1} \, d\xi\, d\eta$$

and thus

(1) $$D(u \circ \phi) \le K D(u) .$$

*Dirichlet integrals are quasi-invariant.*

There is another formulation of this. We may consider merely corresponding Jordan regions with boundaries $\gamma, \gamma'$. Let $v$ on $\gamma'$ and $v \circ \phi$ on $\gamma$ be corresponding boundary values. There is a minimum Dirichlet integral $D_0(v)$ for functions with boundary values $v$, attained for the harmonic function with these boundary values. Clearly,

(2) $$D_0(v \circ \phi) \le K D_0(v) .$$

One may go a step further and assume that $v$ is given only on part of the boundary. For instance, if $v = 0$ and $v = 1$ on disjoint boundary arcs we get a new proof of the quasi-invariance of the module.

In order to define the Dirichlet integral it is not necessary to assume that $u$ is of class $C^1$. Suppose that $u(z)$ is continuous with compact support. Thus we can form the Fourier transform

$$\hat{u}(\xi, \eta) = \frac{1}{2\pi} \iint_{\Omega} e^{i(x\xi + y\eta)} u(x, y)\, dx\, dy$$

and we know that

$$(u_x)\hat{} = -i\xi\hat{u}$$
$$(u_y)\hat{} = -i\eta\hat{u} \quad .$$

It follows by the Plancherel formula that

$$D(u) = \iint (\xi^2 + \eta^2)\,|\hat{u}|^2\, d\xi\, d\eta$$

and this can be taken as *definition* of $D(u)$.

# CHAPTER II

## *THE GENERAL DEFINITION*

### A. *The Geometric Approach*

All mappings $\phi$ will be topological and sense preserving from a region $\Omega$ to a region $\Omega'$.

*Definition A.* $\phi$ is $K$-q.c. if the modules of quadrilaterals are $K$-quasi-invariant.

A quadrilateral is a Jordan region $Q$, $\overline{Q} \subset \Omega$, together with a pair of disjoint closed arcs on the boundary (the $b$-arcs). Its module $m(Q) = a/b$ is determined by conformal mapping on a rectangle


The conjugate $Q^*$ is the same $Q$ with the complementary arcs (the $a$-arcs). Clearly, $m(Q^*) = m(Q)^{-1}$.

The condition is $m(Q') \leq Km(Q)$. It clearly implies a double inequality,

$$K^{-1}m(Q) \leq m(Q') \leq Km(Q) .$$

Trivial properties:

1. If $\phi$ is of class $C^1$, the definition agrees with the earlier.

2. $\phi$ and $\phi^{-1}$ are simultaneously $K$-q.c.

3. The class of $K$-q.c. mappings is invariant under conformal mappings.

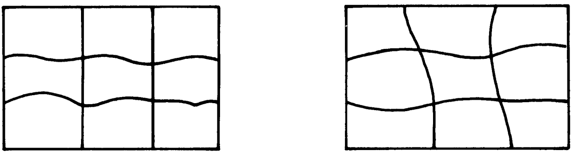
4. The composite of a $K_1$-q.c. and a $K_2$-q.c. mapping is $K_1 K_2$-q.c.

$K$-quasiconformality is a local property:

THEOREM 1. *If $\phi$ is $K$-q.c. in a neighborhood of every point, then it is $K$-q.c. in $\Omega$.*

*Proof.*

Q



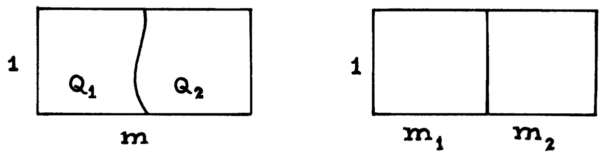
Q

We subdivide $Q$ into vertical strips $Q_i$ and then each image, $Q_i'$ into horizontal strips $Q_{ij}'$ with respect to the rectangular structure of $Q_i'$. Set $m_{ij} = m(Q_{ij})$ etc. Then

$$m = \Sigma\, m_i\ , \qquad \frac{1}{m_i} \geq \Sigma_j \frac{1}{m_{ij}}$$

$$m' \geq \Sigma\, m_i'\ , \qquad \frac{1}{m_i'} = \Sigma_j \frac{1}{m_{ij}'} \ .$$

If the subdivision is sufficiently fine we shall have $m_{ij} \leq Km'_{ij}$. This gives $m \leq Km'$.




LEMMA. *$m = m_1 + m_2$ only if the dividing line is $x = m_1$.*

*Proof.* Let the conformal mapping functions be $f_1, f_2$. Set $\rho = |f_1'|$ in $Q_1$, $\rho = |f_2'|$ in $Q_2$, $\rho = 0$ everywhere. else. Then, integrating over $Q$

$$\iint (\rho^2 - 1)\, dx\, dy = 0$$

$$\iint (\rho - 1)\, dx\, dy \geq 0 .$$

But $\iint (\rho-1)^2\, dx\, dy = \iint [(\rho^2-1) - 2(\rho-1)]\, dx\, dy \leq 0$. Hence $\rho = 1$ a.e. and this is possible only if $f_1 = f_2 = z$.

THEOREM 2. *A 1-q.c. map is conformal.*

*Proof.* We must have equality everywhere in the proof of Theorem 1. This shows that the rectangular map is the identity.

B. *The Analytic Definition*

We shall say that a function $u(x, y)$ is ACL (absolutely continuous on lines) in the region $\Omega$ if for every closed rectangle $R \subset \Omega$ with sides parallel to the $x$ and $y$-axes, $u(x, y)$ is absolutely continuous on a.e. horizontal and a.e. vertical line in $R$. Such a function has of course, partial derivatives $u_x, u_y$ a.e. in $\Omega$.

The definition carries over to complex valued functions.

*Definition B.* A topological mapping $\phi$ of $\Omega$ ia $K$-q.c. if

1) $\phi$ is ACL in $\Omega$;

2) $|\phi_{\bar{z}}| \leq k|\phi_z|$ a.e. $(k = \frac{K-1}{K+1})$ .

We shall prove that this definition is equivalent to the geometric definition. It follows from $B$ that $\phi$ is sense preserving.

We shall say that $\phi$ is *differentiable* at $z_0$ (in the sense of Darboux) if

$$\phi(z) - \phi(z_0) = \phi_z(z_0)(z-z_0) + \phi_{\bar{z}}(z_0)(\bar{z} - \bar{z}_0) + o(|z-z_0|) .$$

The first lemma we shall prove is an amazing result due to Gehring and Lehto.

LEMMA 1. *If $\phi$ is topological* and has partial derivatives a.e., then it is differentiable a.e.*

By Egoroff's theorem the limits

$$\phi_x(z) = \lim_{h \to 0} \frac{\phi(z+h) - \phi(z)}{h}$$

$$\phi_y(z) = \lim_{k \to 0} \frac{\phi(z+ik) - \phi(z)}{k}$$

---

* The result holds as soon as $\phi$ is an open mapping.

are taken on *uniformly* except on a set $\Omega - E$ of arbitrarily small measure. It will be sufficient to prove that $\phi$ is differentiable a.e. on $E$.

*Remark.* Usually Egoroff's theorem is formulated for sequences. We obtain (1) if we apply it to

$$\sup_{0<|h|<1/n} \left| \frac{\phi(z+h)-\phi(z)}{h} - \phi_x(z) \right| .$$

The set $E$ is measurable, and therefore it intersects a.e. horizontal line in a measurable set. On such a line a.e. point in $E$ is a point of linear density 1. The same holds for vertical lines. Therefore a.e. $x_0, y_0 \in E$ is a point of linear density 1 for the intersections of $E$ with $x = x_0$ and $y = y_0$. It will be sufficient to prove that $\phi$ is differentiable at such a point $z_0 = x_0 + iy_0$, and for simplicity we assume $z_0 = 0$.

Because of the uniformity $\phi_x$ and $\phi_y$ are continuous on $E$. Given $\epsilon > 0$ we can therefore find a $\delta > 0$ such that

$$|\phi_x(z) - \phi_x(0)| < \epsilon$$
$$|\phi_y(z) - \phi_y(0)| < \epsilon$$

(2)
$$\left|\frac{\phi(z+h)-\phi(z)}{h} - \phi_x(z)\right| < \epsilon$$

$$\left|\frac{\phi(z+ik)-\phi(z)}{k} - \phi_y(z)\right| < \epsilon$$

as soon as $|x| < \delta$, $|y| < \delta$, $|h| < \delta$, $|k| < \delta$ and $z \in E$.

We follow the argument which is used when one proves that a function with continuous partial derivatives is differentiable. It is based on the identity

$$\phi(x+iy) - \phi(0) - x\phi_x(0) - y\phi_y(0) =$$
$$[\phi(x+iy) - \phi(x) - y\phi_y(x)] + [\phi(x) - \phi(0) - x\phi_x(0)]$$
$$+ [y(\phi_y(x) - \phi_y(0))] \ .$$

If $x \in E$ we are able to derive by (2) that

(3) $$|\phi(x+iy) - \phi(0) - x\phi_x(0) - y\phi_y(0)| \leq 3\epsilon|z| \ .$$

The same reasoning can be repeated if $y \in E$. Thus we have proved what we wish to prove if either $x \in E$ or $y \in E$.

We now want to make use of the fact that $0$ is a point of linear density 1. Let $m_1(x)$ be the measure of that part of $E$ which lies on the segment $(-x, x)$. Then $(m_1(x))/2|x| \to 1$ for $x \to 0$, and we can choose $\delta$ so small that

$$m_1(x) > \frac{2+\epsilon}{1+\epsilon}|x|$$

for $|x| < \delta$. For such an $x$ the interval $(\frac{x}{1+\epsilon}, x)$ cannot be free from points of $E$, for if it were we should have

$$m_1(x) \leq |x| + \frac{|x|}{1+\epsilon} = \frac{2+\epsilon}{1+\epsilon}|x| \ .$$

The same reasoning applies on the $y$-axis. If $|z| < \frac{\delta}{1+\epsilon}$ we conclude that there are points $x_1, x_2, y_1, y_2 \in E$ with

$$\frac{x}{1+\epsilon} < x_1 < x < x_2 < (1+\epsilon)x, \ \frac{y}{1+\epsilon} < y_1 < y < y_2 < (1+\epsilon)y_2{}^*$$

We may then conclude that (3) holds on the perimeter of the

---

* We are assuming for convenience that $z$ lies in the first quadrant.

rectangle $(x_1, x_2) \times (y_1, y_2)$.

To complete the reasoning we use the fact that $\phi$ satisfies the maximum principle. There is a point $z^* = x^* + iy^*$ on the perimeter such that

$$\begin{aligned}
&|\phi(x+iy) - \phi(0) - x\phi_x(0) - y\phi_y(0)| \\
&\quad \le |\phi(x^*+iy^*) - \phi(0) - x\phi_x(0) - y\phi_y(0)| \\
&\quad \le 3\epsilon|z^*| + |x-x^*|\,|\phi_x(0)| + |y-y^*|\,|\phi_y(0)| \\
&\quad \le 3\epsilon(1+\epsilon)|z| + \epsilon|\phi_x(0)|\,|z| + \epsilon|\phi_y(0)|\,|z| \quad .
\end{aligned}$$

This is an estimate of the desired form, and the lemma is proved.

We can say a little more. If $E$ is a Borel set in $\Omega$ we define $A(E)$ as the area of its image. This defines a locally finite additive measure, and according to a theorem of Lebesgue such a measure has a symmetric derivative a.e., that is,

$$J(z) = \lim \frac{A(Q)}{m(Q)}$$

when $Q$ is a square of center $z$ whose side tends to zero. Moreover,

$$\int_E J(z)\,dx\,dy \le A(E)$$

(we cannot yet guarantee equality). But if $\phi$ is differentiable at $z$ it is immediate that $J(z)$ is the Jacobian, and we have proved that the Jacobian is locally integrable.

But $J = |\phi_z|^2 - |\phi_{\bar z}|^2$ and if 2) is fulfilled we obtain

$$|\phi_{\bar z}|^2 \le |\phi_z|^2 \le \frac{J}{1-k^2} \quad .$$

We conclude that the partial derivatives are locally square integrable.

Moreover, if $h$ is a test function ($h \in C^1$ with compact support) we find at once by integration over horizontal or vertical lines and subsequent use of Fubini's theorem

$$\begin{aligned} \int\int \phi_x\, h\, dx\, dy &= -\int\int \phi\, h_x\, dx\, dy \\ \int\int \phi_y\, h\, dx\, dy &= -\int\int \phi\, h_y\, dx\, dy \end{aligned} \tag{4}$$

In other words, $\phi_x$ and $\phi_y$ are distributional derivatives of $\phi$.

More important still is the converse:

LEMMA 2. *If $\phi$ has locally integrable distributional derivatives, then $\phi$ is* ACL.

*Proof.* We assume the existence of integrable functions $\phi_1, \phi_2$ such that

$$\begin{aligned} \int\int \phi_1\, h\, dx\, dy &= -\int\int \phi\, h_x\, dx\, dy \\ \int\int \phi_2\, h\, dx\, dy &= -\int\int \phi\, h_y\, dx\, dy \end{aligned} \tag{5}$$

for all test functions.

Consider a rectangle $R_\eta = \{0 \le x \le a,\ 0 \le y \le \eta\}$ and choose $h = h(x)\,k(y)$ with support in $R_\eta$. In

$$\int\int_{R_\eta} \phi_1\, h(x)\,k(y)\, dx\, dy = -\int\int_{R_\eta} \phi\, h'(x)\,k(y)\, dx\, dy$$

we can first let $k$ tend boundedly to 1. This gives

$$\iint_{R_\eta} \phi_1 h(x)dx\,dy = -\iint_{R_\eta} \phi h'(x)dx\,dy$$

and hence

$$\int_0^a \phi_1(x, \eta)h(x)dx = -\int_0^a \phi(x, \eta)h'(x)dx$$

for almost all $\eta$. Now let $h = h_n$ run through a sequence of test-functions such that $0 \le h_n \le 1$ and $h_n = 1$ on $(\frac{1}{n}, a-\frac{1}{n})$. We obtain

$$\phi(a, \eta) - \phi(0, \eta) = \int_0^a \phi_1(x, \eta)\,dx \tag{6}$$

for almost all $\eta$. The exceptional set does depend on $a$, but we may conclude that (6) holds a.e. for all rational $a$. By continuity it is then true for all $a$, and we have proved that $\phi(x, \eta)$ is absolutely continuous for almost all $\eta$. Moreover, we have $\phi_x = \phi_1$, $\phi_y = \phi_2$ a.e.

q.e.d.

In other words, $B$ is equivalent to $B'$, $\phi$ has locally integrable distributional derivatives which satisfy

$$|\phi_{\bar{z}}| \le k|\phi_z| .$$

It is now easy to show that $B$ is invariant under conformal mapping. We prove, a little more generally:

LEMMA 3. *If $\omega$ is a $C^2$ topological mapping and if $\phi$ has locally integrable distributional derivatives, so does $\phi \circ \omega$, and they are given by*

(6)
$$
\begin{aligned}
(\phi \circ \omega)_x &= (\phi_\xi \circ \omega)\,\frac{\partial \xi}{\partial x} + (\phi_\eta \circ \omega)\,\frac{\partial \eta}{\partial x} \\
(\phi \circ \omega)_y &= (\phi_\xi \circ \omega)\,\frac{\partial \xi}{\partial y} + (\phi \circ \omega)\,\frac{\partial \eta}{\partial y}
\end{aligned}
$$

*Proof.* We note first that

$$\begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{pmatrix}$$

are reciprocal matrices (at correspon ding points). This makes

$$\begin{pmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{pmatrix} = \frac{1}{J}\begin{pmatrix} \eta_y & -\xi_y \\ -\eta_x & \xi_x \end{pmatrix}$$

with $J = \xi_x \eta_y - \xi_y \eta_x$, the Jacobian of $\omega$.

For any test function $h \circ \omega$ we can put

$$
\begin{aligned}
&\int\!\!\int [(\phi_\xi \circ \omega)\,\xi_x + (\phi_\eta \circ \omega)\,\eta_x](h \circ \omega)\,dx\,dy \\
&\quad = \int\!\!\int [(\phi_\xi \circ \omega)\,\frac{\xi_x}{J} + (\phi_\eta \circ \omega)\,\frac{\eta_x}{J}](h \circ \omega)\,J\,dx\,dy \\
&\quad = \int\!\!\int (\phi_\xi y_\eta - \phi_\eta y_\xi)\,h\;d\xi\,d\eta \\
&\quad = \int\!\!\int \phi(-(hy_\eta)_\xi + (hy_\xi)_\eta)\,d\xi\,d\eta \\
&\quad = \int\!\!\int \phi(-h_\xi y_\eta + h_\eta y_\xi)\,d\xi\,d\eta \\
&\quad = \int\!\!\int (\phi \circ \omega)(-(h_\xi \circ \omega)\,\frac{\xi_x}{J} - (h_\eta \circ \omega)\,\frac{\eta_x}{J})\,J\,dx\,dy \\
&\quad = \int\!\!\int (\phi \circ \omega)(-(h_\xi \circ \omega)\,\xi_x - (h_\eta \circ \omega)\,\eta_x)\,dx\,dy \\
&\quad = -\int\!\!\int (\phi \circ \omega)(h \circ \omega)_x\,dx\,dy
\end{aligned}
$$

q.e.d.

The last lemma enables us to prove:

$$B \Longrightarrow A\ .$$

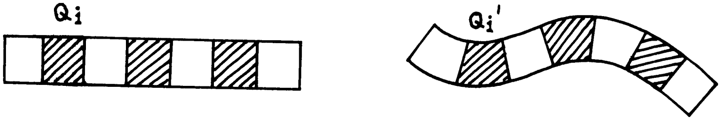
Indeed, in view of the lemma we need only prove that a rectangle with modulus $m$ is mapped on a quadrilateral with $m' \leq Km$, and this proof is exactly the same as in the differentiable case.

We shall now concentrate on proving the converse:

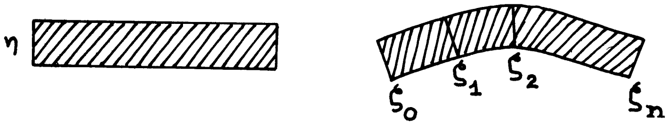
$$A \Longrightarrow B .$$

First we prove: if $\phi$ is q.c. in the geometric sense, then it is ACL.

Let $A(\eta)$ be the image area under the mapping $\phi$ of the rectangle $a \leq x \leq \beta$, $y_0 \leq y \leq \eta$. Because $A(\eta)$ is increasing the derivative $A'(\eta)$ exists a.e., and we shall assume that $A'(0)$ exists.




In the figure, let $Q_i$ be rectangles with height $\eta$ and base $b_i$. Let $b_i'$ be the length of the image of $b_i$. We wish to show, first of all: If $\eta$ is sufficiently small, then the length of any curve in $Q_i'$ which joins the "vertical" sides is nearly $b_i'$.




To do so we first determine a polygon so that

$$\sum_{1}^{n} |\zeta_k - \zeta_{k-1}| \geq b_i' - \frac{\epsilon}{2} .$$

Next, take $\eta$ so small that the variation of $\phi$ on vertical segments is $< \epsilon/4n$. Draw the lines through $\zeta_k$ that correspond to verticals. Any transversal must intersect all these lines. This gives a length

$$\geq \sum_{1}^{n} |\zeta_k - \zeta_{k-1}| - \frac{\epsilon}{2} \geq b_i' - \epsilon .$$

On using the euclidean metric we have now, if $\epsilon <$ min $\frac{1}{2} b_i'$,

$$m_i(Q_i') \geq \frac{b_i'^2}{4A_i} , \qquad \frac{b_i'^2}{4A_i} \leq K \frac{b_i}{\eta}$$

$$(\Sigma\, b_i')^2 \leq \Sigma \frac{b_i'^2}{b_i} \cdot \Sigma\, b_i \leq 4K \frac{A(\eta)}{\eta} \cdot (\Sigma\, b_i) \; . \qquad *$$

But $A(\eta)/\eta \to A'(0) < \infty$. This shows that $\Sigma\, b_i' \to 0$ with $\Sigma\, b_i$, and hence $\phi$ is absolutely continuous.

q.e.d.

Finally, if $\phi$ is $K$-q.c. in the geometric sense, it is easy to prove that the inequality

$$|\phi_{\bar{z}}| \leq k\,|\phi_z| \qquad \left(k = \frac{K-1}{K+1}\right)$$

holds at all points where $\phi$ is differentiable. This proves that $A \Longrightarrow B$.

We have now proved the equivalence of the geometric and analytic definitions, and we have two choices for the analytic definition.

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* The proof needs an obvious modification if $b_i' = \infty$..

COROLLARY 1. *If the topological mapping* $\phi$ *satisfies* $\phi_{\bar{z}} = 0$ *a.e., and if* $\phi$ *is either* ACL *or has integrable distributional derivatives, then* $\phi$ *is conformal.*

Observe the close relationship with Weyl's lemma.

COROLLARY 2. *If* $\phi$ *is q.c. and* $\phi_{\bar{z}} = 0$ *a.e., then* $\phi$ *is conformal.*

Finally, we shall prove:

THEOREM 3. *Under a q.c. mapping the image area is an absolutely continuous set function. This means that null sets are mapped on null sets, and that the image area can always be represented by*

$$A(E) = \iint\limits_E J \, dx\, dy \, .$$

*Proof.* $\phi = u + iv$ can be approximated by $C^2$ functions $u_n + iv_n$ in the sense thet $u_n \to u$, $v_n \to v$ and

$$\iint |u_x - (u_n)_x|^2 \, dx\, dy \to 0$$

$$\iint |v_x - (v_n)_x|^2 \, dx\, dy \to 0 \qquad \text{etc.}$$

Consider rectangles $R$ such that $u$ and $v$ are absolutely continuous on all sides.

$$\iint\limits_R [(u_m)_x (v_n)_y - (u_m)_y (v_n)_x] \, dx\, dy = \int\limits_{\partial R} u_m \, dv_n \, .$$

As $m, n \to \infty$ the double integral tends to $\iint_R J\, dx\, dy$. For $m \to \infty$ the line integral tends to

$$\int_{\partial R} u\, dv_n = -\int_{\partial R} v_n\, du$$

and for $n \to \infty$ this tends to

$$-\int_{\partial R} v\, du = \int_{\partial R} u\, dv$$

(all because $u$ and $v$, and therefore $uv$, are absolutely continuous on $\partial R$).

We have proved

$$\iint_R J\, dx\, dy = \int_{\partial R} u\, dv$$

for these $R$. It is not precisely trivial, but in any case fairly easy to prove that the line integral does represent the image area, and that proves the theorem.

COROLLARY. $\phi_z \neq 0$ *a.e.*

Otherwise there would exist a set of positive measure which is mapped on a nullset, and consideration of the inverse mapping would lead to a contradiction.

*Remark.* It can now be concluded that Dirichlet integrals are $K$-quasi-invariant under arbitrary $K$-q.c. mappings. It is possible to prove the same for extremal lengths.

## CHAPTER III

## *EXTREMAL GEOMETRIC PROPERTIES*

### A. *Three Extremal Problems*

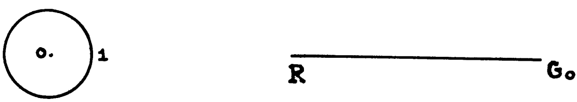
Let $G$ be a doubly connected region in the finite plane, $C_1$ the bounded and $C_2$ the unbounded component of its complement. We want to find the largest value of the module $M(G)$ under one of the following conditions:

I. (Grötzsch) $C_1$ is the unit disk $(|z| \leq 1)$ and $C_2$ contains the point $R > 1$.

II. (Teichmüller) $C_1$ contains $0$ and $-1$; $C_2$ contains a point at distance $P$ from the origin.

III. (Mori) $\operatorname{diam}(C_1 \cap \{|z| \leq 1\}) \geq \lambda$; $C_2$ contains the origin.

We claim that the maximum of $M(G)$ is obtained in the following symmetric cases.

I. 

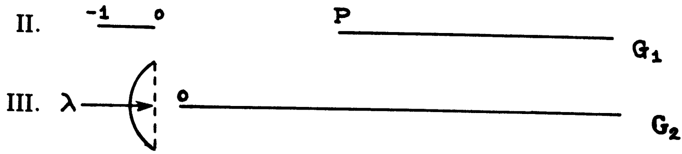



Fig. 1.

*Case I.* Let $\Gamma$ be the family of closed curves that separate $C_1$ and $C_2$. We know that $\lambda(\Gamma) = M(G)^{-1}$.

Compare $\Gamma$ with the family $\tilde{\Gamma}$ of closed curves that lie in the complement of $C_1 \cup \{R\}$, have zero winding number about $R$ and nonzero winding number about the origin. Evidently, $\Gamma \subset \tilde{\Gamma}$, and hence $\lambda(\Gamma) \geq \lambda(\tilde{\Gamma})$. But $\tilde{\Gamma}$ is a symmetric family, and hence $\lambda(\tilde{\Gamma}) = \frac{1}{2}\lambda(\tilde{\Gamma}^+)$ by our symmetry principle. Similarly, if $\Gamma_0$ is the family $\Gamma$ in the alleged extremal case*, then $\lambda(\Gamma_0) = \frac{1}{2}\lambda(\Gamma_0{}^+)$.

We show that $\tilde{\Gamma}^+ = \Gamma_0{}^+$. Each curve $\tilde{\gamma}$ in $\tilde{\Gamma}$ has points $P_1, P_2$ on $(-\infty, -1)$ and $(1, R)$ respectively. Say they divide $\tilde{\gamma}$ into arcs $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ so that $\tilde{\gamma} = \tilde{\gamma}_1 + \tilde{\gamma}_2$. Then $\tilde{\gamma}^+ = \tilde{\gamma}_1{}^+ + \tilde{\gamma}_2{}^+ = (\tilde{\gamma}_1' + \tilde{\gamma}_2{}^{+-})^+$. Here $\tilde{\gamma}_1{}^+ + \tilde{\gamma}_2{}^{+-}$ belongs to $\Gamma_0$, and we conclude that $\tilde{\gamma}^+ \in \Gamma_0{}^+$. Thus $\tilde{\Gamma}^+ \subset \Gamma_0{}^+$, and the opposite inclusion is trivial.

We have proved $\lambda(\Gamma) \geq \lambda(\tilde{\Gamma}) = \lambda(\Gamma_0)$, hence $M(G) \leq M(G_0)$.

q.e.d.

---

* We enlarge $\Gamma_0$ slightly by allowing it to contain curves which contain segments on the edges of the cut $(R, \infty)$. This is, of course, harmless.

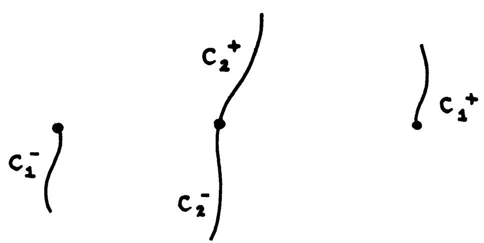
*Case II.* Let $z = f(\zeta)$ map $|\zeta| < 1$ conformally onto $C_1 \cup G$ with $f(0) = 0$. Koebe's one-quarter theorem gives $|f'(0)| \leq 4P$ with equality for $G = G_1$. Suppose $f(a) = -1$. The distortion theorem gives

$$1 = |f(a)| \leq \frac{|a|\ |f'(0)|}{(1-|a|)^2} \leq \frac{4P\,|a|}{(1-|a|)^2}$$

and equality is again attained when $G = G_1$. In other words $|a| \geq |a_1|$ where $a_1$ belongs to the symmetric case. The module $M(G)$ equals the module between the image of $C_1$ and the unit circle.

Finally, by inversion and application of Case I, if $|a|$ is given the module is largest for a line segment, and it increases when $|a|$ decreases. Hence $G_1$ is extremal.

*Case III.* Open up the plane by $\zeta = \sqrt{z}$. We get a figure which is symmetric with respect to the origin




with two component images of $C_1$ and two of $C_2$. The composition laws tells us that $M(G) \leq \frac{1}{2} M(\hat{G})$ where $\hat{G}$ is the region between $C_1^-$ and $C_1^+$. It is clear that equality holds in the symmetric situation.

By assumption, $C_1$ contains points $z_1, z_2$, with $|z_1| \leq 1$, $|z_2| \leq 1$, $|z_1 - z_2| \geq \lambda$. Let $\zeta_1, \zeta_2 \in C_1^+$, $-\zeta_1, -\zeta_2 \in C_1^-$ be the corresponding points in the $\zeta$-plane. The linear transformation

$$w = \frac{\zeta + \zeta_1}{\zeta - \zeta_1} \cdot \frac{\zeta_1 + \zeta_2}{\zeta_1 - \zeta_2}$$

carries $(-\zeta_1, -\zeta_2)$ into $(0, 1)$ and $\zeta_1 \to \infty$, $\zeta_2 \to w_0$ where

$$w_0 = -\left(\frac{\zeta_2 + \zeta_1}{\zeta_2 - \zeta_1}\right)^2 .$$

On setting $u = (\zeta_2 + \zeta_1)/(\zeta_2 - \zeta_1)$ one has

$$u + \frac{1}{u} = \frac{2(\zeta_2^{\,2} + \zeta_1^{\,2})}{\zeta_2^{\,2} - \zeta_1^{\,2}} = \frac{2(z_1 + z_2)}{z_2 - z_1} .$$

Since

$$|z_2 + z_1|^2 = 2(|z_1|^2 + |z_2|^2) - |z_2 - z_1|^2 \leq 4 - \lambda^2 ,$$

we get

$$|u| - \frac{1}{|u|} \leq \frac{2}{\lambda}\sqrt{4-\lambda^2}$$

$$|u| \leq \frac{2 + \sqrt{4 - \lambda^2}}{\lambda}$$

$$|w_0| \leq \left(\frac{2 + \sqrt{4-\lambda^2}}{\lambda}\right)^2 .$$

One verifies that equality holds for the symmetric case, and by use of Case II it follows again that $M(G)$ is a maximum for the case in Figure 1.

In order to conform with the notations in Künzi's book *
the extremal modules will be denoted

I. $$\frac{1}{2\pi} \log \Phi(R)$$

II. $$\frac{1}{2\pi} \log \Psi(P)$$

III. $$\frac{1}{2\pi} \log X(\lambda) \quad .$$

There are simple relations between these functions. Obviously, reflection of $G_0$ gives a twice as wide ring of type $G_1$, and one finds

(1) $$\Phi(R)^2 = \Psi(R^2-1) \quad .$$

Another relation is obtained by mapping the outside of the unit circle on the outside of the segment $(-1, 0)$. This gives

(2) $$\Phi(R) = \Psi(\tfrac{1}{4}(\sqrt{R} - \frac{1}{\sqrt{R}})^2)$$

and together with (1), we find the identity

(3) $$\Phi(R) = \Phi[\tfrac{1}{2}(\sqrt{R} + \frac{1}{\sqrt{R}})]^2 \quad .$$

The previous computation in Case III gives, for instance,

(4) $$X(\lambda) = \Phi\left(\frac{\sqrt{4+2\lambda} + \sqrt{4-2\lambda}}{\lambda}\right) \quad .$$

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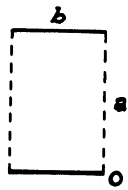
* Hans P. Künzi, *Quasikonforme Abbildungen, in Ergebnisse der Mathematik*, Springer Verlag, Berlin, 1960.

B. *Elliptic and Modular Functions*

The elliptic integral

(1) $$w = \int_0^z \frac{dz}{\sqrt{(z+1)\,z\,(z-P)}}$$

maps the upper half of the normal region $G_1$ on a rectangle




with sides

(2) $$a = \int_0^P \frac{dz}{\sqrt{(z+1)\,z\,(P-z)}}$$

$$b = \int_P^\infty \frac{dz}{\sqrt{(z+1)\,z\,(z-P)}} \quad .$$

Evidently,

(3) $$\frac{1}{2\pi} \log \Psi(P) = \frac{a}{2b} \quad .$$

This is an explicit expression, but it is not very convenient for a study of asymptotic behavior. Anyway, we want to study the connection with elliptic functions in much greater detail.

We recall that Weierstrass' $\wp$-function is defined by

(4) $$\wp(z) = \frac{1}{z^2} + \Sigma' \left[ \frac{1}{(z - m\omega_1 - n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2} \right]$$

and satisfies the differential equation

(5) $$\wp'(z)^2 = 4(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3)$$

where

(6) $$e_1 = \wp\left(\frac{\omega_1}{2}\right), \quad e_2 = \wp\left(\frac{\omega_2}{2}\right), \quad e_3 = \wp\left(\frac{\omega_1+\omega_2}{2}\right) .$$

It follows that the $e_k$ are distinct.

We set $\tau = \omega_2/\omega_1$ and consider only the halfplane $\operatorname{Im} \tau > 0$. In that halfplane

(7) $$\rho(\tau) = \frac{e_3 - e_1}{e_2 - e_1}$$

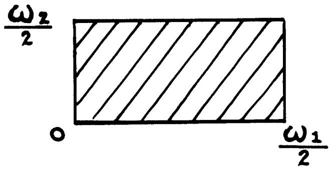
is an analytic function $\neq 0, 1$. It is this function we wish to study.

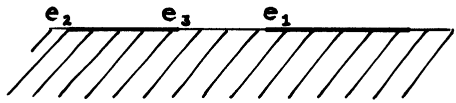
$\rho$ is invariant under modular transformations $\dfrac{a\tau + b}{c\tau + d}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ mod 2, for $\wp$ does not change and the $\omega_k/2$ change by full periods. The transformation $\tau' = \tau + 1$ replaces $\rho$ by $1/\rho$, and $\tau' = -1/\tau$ changes $\rho$ into $1 - \rho$. In other words

(8) $$\begin{aligned} \rho(\tau + 1) &= \rho(\tau)^{-1} \\ \rho\left(-\frac{1}{\tau}\right) &= 1 - \rho(\tau) \end{aligned}$$

and these relations determine the behavior of $\rho(\tau)$ under the whole modular group.

For purely imaginary $\tau$ the mapping by $\wp$ is as indicated:



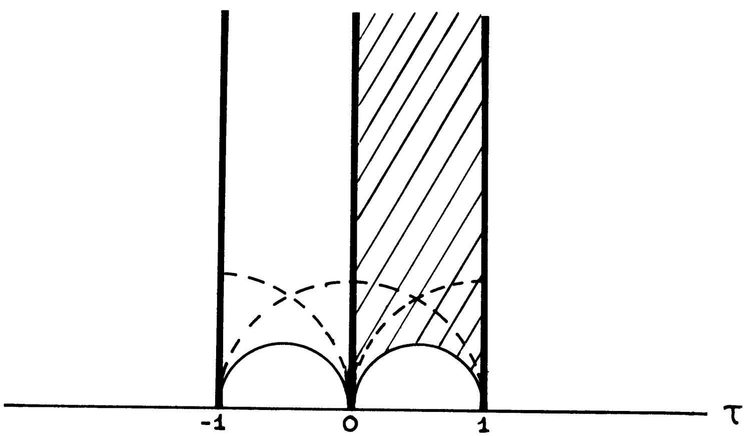




The image is similar to the normal region with $P = \rho/1-\rho$, and we find

$$(9) \qquad \tau(\rho) = \frac{i}{\pi} \log \Psi\left(\frac{\rho}{1-\rho}\right) \quad ,$$

$0 < \rho < 1$. It is evident from this formula that $\rho$ varies monotonically from 0 to 1 when $\tau$ goes from 0 to $\infty$ along the imaginary axis.

It can be seen by direct computation that $\rho \to 1$ as soon as $\operatorname{Im} \tau \to \infty$, uniformly in the whole halfplane. If we make use of the relations (8) it is readily seen that the correspondence between $\tau$ and $\rho$ is as follows:



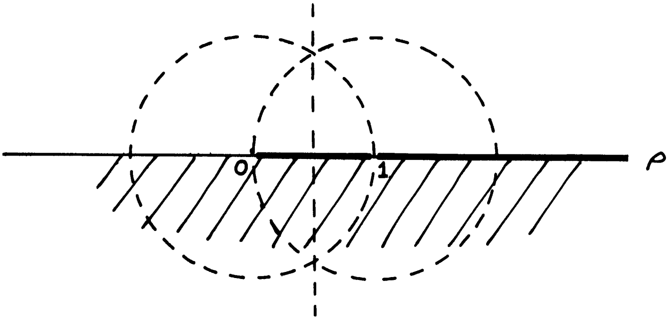


Fig. 2.

We shall let $\tau(\rho)$ denote the branch of the inverse function determined by this choice of the fundamental region. Observe the symmetries. They yield

$$(10) \qquad \begin{aligned} \tau(\tfrac{1}{2}) &= i & \tau(-1) &= \frac{1+i}{2} \\ \tau(2) &= \pm 1 + i & \tau\left(\frac{1-i\sqrt{3}}{2}\right) &= \frac{1+i\sqrt{3}}{2} \end{aligned}$$

Relation (8) gives

$$(11) \qquad \begin{aligned} \tau\left(\frac{1}{\rho}\right) &= \tau(\rho) \pm 1 \\ \tau(1-\rho) &= -\frac{1}{\tau(\rho)} \quad . \end{aligned}$$

Also $$\tau(\bar{\rho}) = -\bar{\tau}(\rho) \quad .$$

It is clear that $e^{\pi i\tau}$ is analytic at $\rho = 1$ with a simple zero. Therefore one can write

$$(12) \qquad \rho - 1 \sim a\, e^{i\pi\tau} \quad , \qquad (a > 0)$$

but the determination of the constant requires better knowledge of the function..

It will be of some importance to know how $|\rho - 1|$ varies when $\operatorname{Im} \tau$ is kept fixed. In other words, if we write $\tau = s + it$

we would like to know the sign of $\partial/\partial s \log|\rho-1|$. This harmonic function is evidently zero on the lines $s = 0$, $s = 1$, and a look at Figure 2 shows that it is positive on the half-circle from 0 to 1. This proves

$$\frac{\partial}{\partial s} \log|\rho-1| > 0$$

in the right half of the fundamental region, provided that the maximum principle is applicable. However, equation (12) shows that $\partial/\partial s \log|\rho-1| \to 0$ as $\tau \to \infty$. Use of the relation (8) shows that the same is true in the other corners, and therefore the use of the maximum principle offers no difficulty.

We conclude that $|\rho-1|$ is smallest on the imaginary axis and biggest on $\operatorname{Re}\tau = \pm 1$.

The estimates of the function $\rho(\tau)$ obtained by geometric comparison are not strong enough, but classical explicit developments are known. For the sake of completeness we shall derive the most important formula.

The function

$$\frac{\wp(z) - \wp(u)}{\wp(z) - \wp(v)}$$

has zeros at $z = \pm u + m\omega_1 + n\omega_2$

and poles at $z = \pm v + m\omega_1 + n\omega_2$.

A function with the same periods, zeros and poles is

$$F(z) = \prod_{n=-\infty}^{\infty} \frac{1 - e^{2\pi i \frac{(n\omega_2 \pm u - z)}{\omega_1}}}{1 - e^{2\pi i \frac{(n\omega_2 \pm v - z)}{\omega_1}}}$$

(the product has one factor for each $n$ and each $\pm$). It is easy to see that the product converges at both ends..

It will be convenient to use the notation $q = e^{\pi i \tau} = e^{\pi i \omega_2/\omega_1}$. We separate the factor $n = 0$ and combine the factors $\pm n$. This gives

$$F(z) = \frac{1 - e^{2\pi i \frac{u-z}{\omega_1}}}{1 - e^{2\pi i \frac{v-z}{\omega_1}}} \cdot \frac{1 - e^{2\pi i \frac{u+z}{\omega_1}}}{1 - e^{2\pi i \frac{v+z}{\omega_1}}}$$

$$\cdot \prod_{n=1}^{\infty} \frac{1 - q^{2n} e^{2\pi i \frac{\pm u \pm z}{\omega_1}}}{1 - q^{2n} e^{2\pi i \frac{\pm v \pm z}{\omega_2}}}$$

with one factor for each combination of signs.

It is clear that

$$\frac{\wp(z) - \wp(u)}{\wp(z) - \wp(v)} = \frac{F(z)}{F(0)} .$$

In order to compute

$$1 - \rho = \frac{e_2 - e_3}{e_3 - e_1}$$

we have to choose $z = \omega_2/2$, $u = (\omega_1 + \omega_2)/2$, $v = \omega_1/2$ which gives

$$e^{\frac{2\pi i z}{\omega_1}} = q, \quad e^{\frac{2\pi i u}{\omega_1}} = -q, \quad e^{\frac{2\pi i v}{\omega_1}} = -1 .$$

Substitution gives

$$F(z) = \frac{2}{1+q^{-1}} \cdot \frac{1+q^{-2}}{1+q^{-1}} \cdot \prod_{1}^{\infty} \frac{(1+q^{2n+2})(1+q^{2n})^2(1+q^{2n-2})}{(1+q^{2n+1})^2 (1+q^{2n-1})^2}$$

$$= 4 \prod_{1}^{\infty} \left( \frac{1+q^{2n}}{1+q^{2n-1}} \right)^4$$

and

$$F(0) = \frac{1+q}{2} \cdot \frac{1+q^{-1}}{2} \prod \frac{(1+q^{2n+1})^2(1+q^{2n-1})^2}{(1+q^{2n})^4}$$

$$= \frac{1}{4q} \prod_1^\infty \left( \frac{1+q^{2n-1}}{1+q^{2n}} \right)^4 .$$

Finally

$$1-\rho = 16q \prod_1^\infty \left( \frac{1+q^{2n}}{1+q^{2n-1}} \right)^8 \tag{13}$$

A similar computation gives

$$\rho = \prod_1^\infty \left( \frac{1-q^{2n-1}}{1+q^{2n-1}} \right)^8 . \tag{14}$$

((14) is obtained by the relation $\tau(1/\rho) = \tau(\rho) \pm 1$ .)

We return to formula (9). It gives

$$\log \Psi(P) = \pi \operatorname{Im} \tau\left(\frac{P}{1+P}\right) = \pi \operatorname{Im} \tau\left(1+\frac{1}{P}\right) \tag{15}$$

or

$$\Psi(P) = q(\rho)^{-1} \quad \text{for} \quad \rho = \frac{P}{P+1}$$

and, by (13),

$$1-\rho = \frac{1}{P+1} = 16q \prod_1^\infty \left( \frac{1+q^{2n}}{1+q^{2n-1}} \right)^8$$

or

$$\frac{\Psi(P)}{P+1} = 16 \prod_1^\infty \left( \frac{1+q^{2n}}{1+q^{2n-1}} \right)^8 . \tag{16}$$

This gives the basic inequality

(17) $$\Psi(P) \leq 16(P+1) \quad .$$

Because $\Phi(R) = \Psi(R^2-1)^{1/2}$ it follows further that

(18) $$\frac{\Phi(R)}{R} = 4 \prod_1^{\infty} \left( \frac{1+q^{2n}}{1+q^{2n-1}} \right)^4$$

and

(19) $$\Phi(R) \leq 4R$$

For $X(\lambda)$ we obtain

(20) $$\lambda X(\lambda) \leq 4(\sqrt{4+2\lambda} + \sqrt{4-2\lambda})$$

$$\lambda X(\lambda) \leq 16 \quad .$$

C. *Mori's Theorem*

Let $\zeta = \phi(z)$ be a $K$-q.c. mapping of $|z| < 1$ onto $|\zeta| < 1$, normalized by $\phi(0) = 0$.

*Mori's theorem:*

(1) $$|\phi(z_1) - \phi(z_2)| < 16|z_1 - z_2|^{1/K} \qquad (z_1 \neq z_2)$$

and 16 cannot be replaced by a smaller constant.

*Remark.* The theorem implies that $\phi$ satisfies a Hölder condition, and this was known earlier. It follows that $\phi$ has a continuous extension to the closed disk $|z| \leq 1$. If we apply the theorem to the inverse mapping we see that the extension is a homeomorphism.

COROLLARY. *Every q.c. mapping of a disk on a disk can be extended to a homeomorphism of the closed disks.*

In proving Mori's theorem we shall first *assume* that $\phi$ has an extension to the closed disk.. It will be quite easy to remove this restriction.

For the proof we shall also need this lemma:

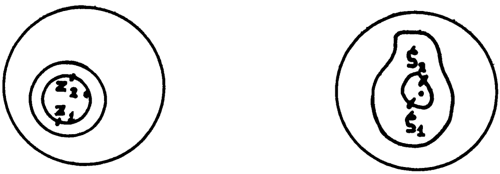
LEMMA. *If $\phi(z)$ with $\phi(0) = 0$ is K-q.c. and homeomorphic on the boundary, then the extension obtained by setting $\phi(1/\bar{z}) = 1/(\bar{\phi}(z))$ is a K-q.c. mapping.*

*Proof.* It is clear that the extended mapping is $K$-q.c. inside and outside the unit circle. It follows very easily that it is ACL even on rectangles that overlap the unit circle. The $K$-q.c. follows.

*Proof of* (1). There is nothing to prove if $|z_1 - z_2| \geq 1$, say. We assume that $|z_1 - z_2| < 1$.

Construct an annulus $A$ whose inner circle has the segment $z_1, z_2$ for diameter and whose outer circle is a concentric circle of radius ½.

Case (i). $A$ lies inside the unit circle..




Consider the mapping

$$w = \frac{\zeta - \zeta_1}{1 - \bar{\zeta}_1 \zeta} .$$

We obtain

$$\frac{1}{2\pi}\log\frac{1}{|z_2-z_1|} \leq \frac{K}{2\pi}\log\Phi\left(\left|\frac{1-\bar{\zeta}_1\zeta_2}{\zeta_2-\zeta_1}\right|\right)$$

$$\leq \frac{K}{2\pi}\log\frac{8}{|\zeta_2-\zeta_1|} .$$

This gives

$$|\zeta_2-\zeta_1| \leq 8|z_2-z_1|^{1/K} .$$

*Case* (ii). $A$ does not contain the origin. Now the image of the inner continuum intersects $|\zeta| < 1$ in a set with diameter $\geq |\zeta_1-\zeta_2|$, and the outer continuum contains the origin. Hence

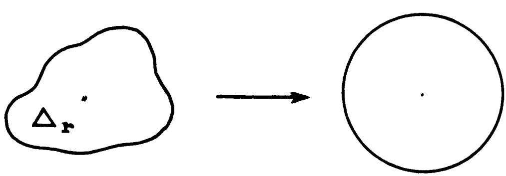
$$\frac{1}{2\pi}\log\frac{1}{|z_2-z_1|} \leq \frac{K}{2\pi}\log X(|\zeta_1-\zeta_2|)$$

$$< \frac{K}{2\pi}\log(16/|\zeta_1-\zeta_2|)$$

and we obtain

$$|\zeta_2-\zeta_1| < 16|z_2-z_1|^{1/K} .$$

To rid ourselves of the hypothesis that $\phi$ is continuous on $|z| = 1$, consider the image region $\Delta_r$ of $|z| < r$



and map it conformally on $|w| < 1$ by $\psi_r$ with $\psi_r(0) = 0$, $\psi_r'(0) > 0$. The function $\psi_r(\phi(rz))$ is $K$-q.c. and continuous on $|z| = 1$. Therefore,

$$|\psi_r(\phi(rz_2)) - \psi_r(\phi(rz_1))| < 16|z_2 - z_1|^{1/K} .$$

Now we can let $r \to 1$. It is a simple matter to show that $\psi_r$ tends to the identity, and we obtain

$$|\phi(z_2) - \phi(z_1)| \leq 16|z_2 - z_1|^{1/K}$$

But this implies continuity on the boundary, and hence the strict inequality is valid.

To show that 16 is the best constant we consider two Mori regions (Case III) $G(\lambda_1)$ and $G(\lambda_2)$. If they are mapped on annuli we can map them on each other by stretching the radii in constant ratio

$$K = \frac{\log X(\lambda_1)}{\log X(\lambda_2)} > 1 \qquad (\lambda_1 < \lambda_2) .$$

Because the unit circle is a line of symmetry it is clear that we obtain a $K$-q.c. mapping which makes $\lambda_1$ correspond to $\lambda_2$. We have now

$$\frac{16-\epsilon}{\lambda_2} \leq X(\lambda_2) = X(\lambda_1)^{1/K} \leq \left(\frac{16}{\lambda_1}\right)^{1/K}$$

for small enough $\lambda_2$. Thus

$$\lambda_2 \geq (\lambda_1)^{1/K} \frac{16-\epsilon}{16^{1/K}}$$

and for large $K$ the constant factor is arbitrarily close to 16.

There are of course strong consequences with regard to normal and compact families of quasiconformal mappings.

THEOREM 1. *The $K$-quasiconformal mappings of the unit disk onto itself, normalized by $\phi(0) = 0$, form a sequentially compact family with respect to uniform convergence.*

*Proof.* By Mori's theorem the functions $\phi$ are equicontinuous. It follows by Ascoli's theorem that every infinite sequence contains a uniformly convergent subsequence: $\phi_n \to \phi$. Because Mori's theorem can be applied to the inverse $\phi_n^{-1}$ it follows that $\phi$ is schlicht. It is rather obviously $K$-q.c.

For conformal mappings it is customary to normalize by $\phi(0) = 0$, $|\phi'(0)| = 1$. For q.c. mappings this does not make sense: we must normalize at two points.

For an arbitrary region $\Omega$ we normalize by $\phi(a_1) = b_1$, $\phi(a_2) = b_2$ where of course $a_1 \neq a_2$, $b_1 \neq b_2$.

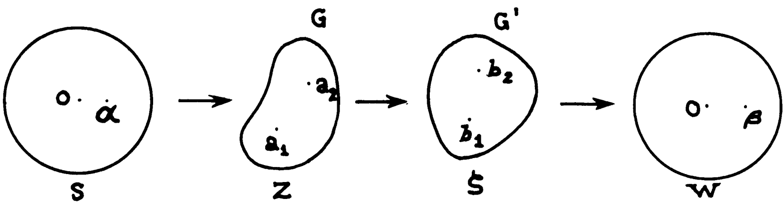
THEOREM 2. *With this fixed normalization the $K$-q.c. mappings in $\Omega$ satisfy a uniform Hölder condition*

$$|\phi(z_1) - \phi(z_2)| \leq M|z_1 - z_2|^{1/K}$$

*on every compact set. The family of such mappings is sequentially compact with respect to uniform convergence on compact sets.*

*Proof.* For any $z_1, z_2 \epsilon \Omega$ there exists a simply connected region $G \subset \Omega$, not the whole plane, which contains $z_1$, $z_2$, $a_1$, $a_2$. If $A$ is a compact set in $\Omega$, then $A \times A$ can be covered by a finite number of $G \times G$. Hence it is sufficient to prove the existence of $M$ for $G$.

We map $G$ and $G' = \phi(G)$ conformally on unit disks as follows:




The compact set may be represented by $|s| \leq r_0 < 1$. Mori's theorem gives

$$(1-|s|) \leq 16(1-|w|)^{1/K} .$$

Hence $|s| \leq r_0$ implies $|w| \leq \rho_0$ (depending only on $r_0$) and a similar reasoning gives $\beta \leq \beta_0$.

We know that

$$|w_1 - w_2| \leq 16|s_1 - s_2|^{1/K} .$$

If we can show that $\zeta(w)$ and $s(z)$ satisfy uniform Lipschitz conditions

$$|\zeta_1 - \zeta_2| \leq C_1|w_1 - w_2|, \quad |s_1 - s_2| \leq C_2|z_1 - z_2|$$

we are through. These are familiar consequences of the distortion theorem.

For instance

$$|z_2 - z_1| \geq \frac{|z(s_1)|}{4} \quad \left| \frac{s_1 - s_2}{1 - s_1\bar{s}_2} \right| \geq C\,|z'(s_1)|\;|s_1 - s_2|$$

$$|z'(s_1)| \geq C_0|z'(0)|$$

$$|a_2 - a_1| \leq C'|z'(0)|$$

and hence

$$|s_1 - s_2| \leq \frac{C'}{CC_0|a_2 - a_1|} \; |z_1 - z_2| \; .$$

The other inequality is easier.

It now follows that every sequence of mappings has a convergent subsequence. To prove that the limit mapping is schlicht we must reverse the inequalities. Although $G'$ is now a variable region this is possible because $\beta < \beta_0$, a number that depends only on $a$ (that is, on $G$, $a_1$, $a_2$).

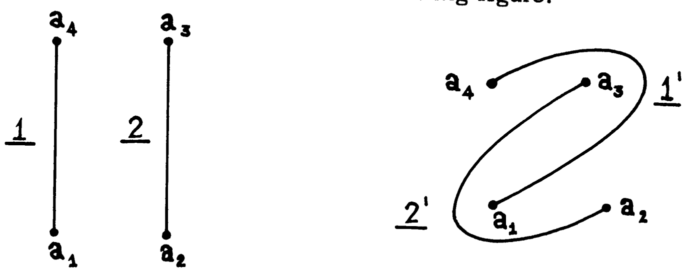
D. *Quadruplets*

Let $(a_1, a_2, a_3, a_4)$ and $(b_1, b_2, b_3, b_4)$ be two ordered quadruples of distinct complex numbers. There exists a conformal mapping of the whole extended plane which takes $a_k$ into $b_k$ if and only if the cross-ratios are equal. If they are not equal it is natural to consider the following problem.

*Problem* 1. For what $K$ does there exist a $K$-q.c. mapping which transforms one quadruple into another.

It was Teichmüller who first pointed out that the problem becomes more natural if one puts topological side conditions on the mapping.

To illustrate, consider the following figure:




There exists a topological mapping of the extended plane which takes the line 1 into 1′ and 2 into 2′. Obviously, as a self-mapping of the sphere punctured at $a_1, a_2, a_3, a_4$ this mapping is quite different from the identity mapping (it is not homotopic). Therefore, although the cross-ratios are equal it makes sense to ask whether there exists a K-q.c. mapping of this topological figure.

We shall have to make this much more precise. First of all, there exist periods $\omega_1, \omega_2$ such that, for $\tau = \omega_2/\omega_1$,

$$\rho(\tau) = \frac{e_3 - e_1}{e_2 - e_1} = \frac{a_3 - a_1}{a_2 - a_1} : \frac{a_3 - a_4}{a_2 - a_4}$$

Evidently, a linear transformation throws $(a_1, a_2, a_3, a_4)$ into $(e_1, e_2, e_3, \infty)$, and we may as well assume that this is the original quadruple.

Let $\Omega$ be the finite $\zeta$-plane punctured at $e_1, e_2, e_3$, and let $P$ be the $z$-plane with punctures at all points

$m(\omega_1/2) + n(\omega_2/2)$. The projection $\wp$ defines $P$ as a covering surface of $\Omega$. In this situation we know that the fundamental group $F$ of $\Omega$ has a normal subgroup $G$ which is isomorphic to the fundamental group of $P$, and the group $\Gamma$ of cover transformations is isomorphic to $F/G$. We know $F$, $G$, and $\Gamma$ explicitly. $F$ is a free group with three generators $\sigma_1, \sigma_2, \sigma_3$, representing loops around $e_1, e_2, e_3$. $G$ is the least normal subgroup that contains $\sigma_1^2, \sigma_2^2, \sigma_3^2$, $(\sigma_1\sigma_2\sigma_3)^2$, and $\Gamma$ is the group of all transformations $z \to \pm z + m\omega_1 + n\omega_2$. The translation subgroup $\Gamma_0$ consists of all transformations $z \to z + m\omega_1 + n\omega_2$. It is generated by $A_1 z = z + \omega_1$ and $A_2 z = z + \omega_2$.

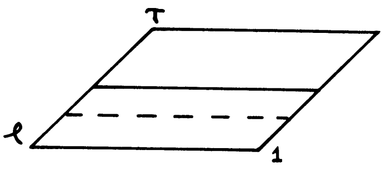
Consider now a second quadruple $(e_1^*, e_2^*, e_3^*, \infty)$ and use corresponding notations $\Omega^*$, $F^*$, etc. A topological mapping $\phi: \Omega \to \Omega^*$ induces an isomorphism of $F$ on $F^*$ which maps $G$ on $G^*$. This means that the covering $\phi \circ \wp$: $P \to \Omega^*$ and $\wp^*: P^* \to \Omega^*$ correspond to the same subgroup $G^*$ of $F^*$, so there is a topological mapping $\psi: P \to P^*$ such that $\phi \circ \wp = \wp^* \circ \psi$. Clearly, $A_1^* = \psi \circ A_1 \circ \psi^{-1}$ and $A_2^* = \psi \circ A_2 \circ \psi^{-1}$ are generators of $\Gamma_0^*$. We write $A_1^* z = z + \omega_1^*$, $A_2^* z = z + \omega_2^*$ and call $(\omega_1^*, \omega_2^*)$ the base determined by $\psi$. Since $\phi$ does not uniquely determine $\psi$, we must check the effect of a change of $\psi$ on the base. We may replace $\psi$ by $T^* \circ \psi$, $T^* \in \Gamma^*$, and we find that the base $(\omega_1^*, \omega_2^*)$ either stays fixed or goes into $(-\omega_1^*, -\omega_2^*)$. In any case, the ratio $\tau^* = \omega_2^*/\omega_1^*$ is determined by $\phi$, and we say that two homeomorphisms of $\Omega$ on $\Omega^*$ are equivalent if they determine the same $\tau^*$. By methods which are beyond our scope here, one can show that two maps of $\Omega$ on $\Omega^*$ are equivalent if and only if they are homotopic.

*Problem* 2. For what $K$ does there exist a $K$-q.c. mapping of $\Omega$ on $\Omega^*$ which is equivalent to a given $\phi_0$.

We need a preliminary calculation of extremal length. Denote by $\{\gamma_1\}$ the family of closed curves in $\Omega$ which lift to arcs in $P$ with endpoints $z$ and $z+\omega_1$. In general, $\{m\gamma_1 + n\gamma_2\}$ denotes the family of curves which lift to arcs with endpoints $z$ and $z + m\omega_1 + n\omega_2$.

LEMMA. *The extremal length of* $\{\gamma_1\}$ *is* $2/\text{Im}\,\tau$, $(\tau = \omega_2/\omega_1)$.

*Proof.* We may assume that $\omega_1 = 1$, $\omega_2 = \tau$. Consider the segment $\ell$ in the figure:




Its projection is a curve $\gamma \in \{\gamma_1\}$. For a given $\rho$ in $\Omega$ set $\tilde{\rho} = \rho(\wp(z))|\wp'(z)|$. We obtain

$$\int_\ell \tilde{\rho}\, dx \geq L(\rho), \qquad L(\rho)^2 \leq \int_\ell \tilde{\rho}^2\, dx$$

$$L(\rho)^2\, \frac{\text{Im}\,\tau}{2} \leq \iint \tilde{\rho}^2\, dx\, dy = A(\rho)$$

and we conclude that

$$\lambda \leq \frac{2}{\operatorname{Im} \tau} .$$

In the opposite direction, choose $\rho$ so that $\tilde{\rho} = 1$. If a curve $\gamma \in \{\gamma_1\}$ is lifted to $P$ its image $\tilde{\gamma}$ leads from a point $z$ to $z+1$. Hence the length of $\tilde{\gamma}$ is $\geq 1$, and we have

$$L(\rho) = 1 , \quad A(\rho) = \tfrac{1}{2} \operatorname{Im} \tau .$$

This completes the proof.

Of course, $\omega_1$ may be replaced by any $c\omega_2 + d\omega_1$ where $(c, d) = 1$. The result is that

$$\text{(1)} \qquad \lambda\{c\gamma_2 + d\gamma_1\} = \frac{2}{\operatorname{Im}\left(\frac{a\tau+b}{c\tau+d}\right)} = \frac{2\,|c\tau+d|^2}{\operatorname{Im} \tau}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a unimodular matrix of integers.

We are now in a position to solve Problem 2. We lift the mapping $\phi\colon \Omega \to \Omega^*$ to $\psi\colon P \to P^*$ and choose the base $(\omega_1^*, \omega_2^*)$ determined by $\psi$. It is clear that the image under $\phi$ of $\{\gamma_1\}$ is the class $\{\gamma_1^*\}$ corresponding to $\omega_1^*$. If $\phi$ is $K$-q.c. it follows that

$$K^{-1} \operatorname{Im} \tau \leq \operatorname{Im} \tau^* \leq K \operatorname{Im} \tau .$$

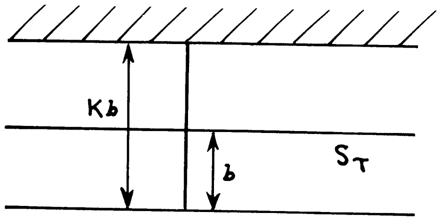
But $\{c\gamma_2 + d\gamma_1\}$ is also mapped on $\{c\gamma_2^* + d\gamma_1^*\}$ and we have also

$$\operatorname{Im} \frac{a\tau^* + b}{c\tau^* + d} \leq K \operatorname{Im} \frac{a\tau + b}{c\tau + d}$$

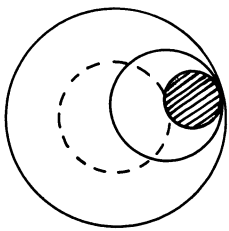
for all unimodular transformations.

To interpret this result geometrically, let $S$ be any unimodular transformation and let $U$ be an auxiliary linear transformation which maps the unit disk $|w| < 1$ on the upper half-plane with $U(0) = \tau$.

Mark the halfplanes bounded by horizontal lines through $S\tau$ and $K S\tau$.




We know that $S\tau^*$ does not lie in the shaded part. Mapping back by $U^{-1}\ S^{-1}$ we see that $U^{-1}\ \tau^*$ does not lie in a shaded circle



which is tangent to the unit circle at $U^{-1}\ S^{-1}(\infty)$ and whose radius depends only on $K$. But the points $S^{-1}(\infty)$ are dense on the real axis, and hence the $U^{-1}\ S^{-1}(\infty)$ are dense on the circle. This shows that $\tau^*$ is restricted to a smaller circle. An invariant way of expressing this result is to say that the non-euclidean distance between $\tau$ and $\tau^*$ is at most equal to that between $ib$ and $iKb$, or

$$d[\tau, \tau^*] \le \log K \ .$$

THEOREM 3. *There exists a K-q.c. mapping equivalent to* $\phi_0$ *if and only if* $d[\tau, \tau^*] \leq \log K$.

We have not yet proved the existence. But this is immediate, for we need only consider the affine mapping

$$\psi(z) = \frac{(\tau^* - \bar{\tau})z + (\tau - \tau^*)\bar{z}}{\tau - \bar{\tau}}$$

which is such that $\psi(-z) = -\psi(z)$, $\psi(z+1) = \psi(z) + 1$, and $\psi(z+\tau) = \psi(z) + \tau^*$. As such, $\psi$ covers a mapping $\phi$ of $\Omega$ on $\Omega^*$ which is equivalent to $\phi_0$, and the dilatation is precisely $e^{d[\tau, \tau^*]}$.

What about Problem 1? Here it is assumed that $\phi_0$ maps $e_1, e_2, e_3$ on $e_1^*, e_2^*, e_3^*$ in this order. When does $\phi$ have the same property? Assume that $\phi_0$ is covered by $\psi_0$: $P \to P^*$ which determines the base $(\omega_1^*, \omega_2^*)$ and that $\phi$ lifts to $\psi$ determining $(c\omega_2^* + d\omega_1^*,\ a\omega_2^* + b\omega_1^*)$. It is found that $\phi$ will map $e_1, e_2, e_3$ on $e_1^*, e_2^*, e_3^*$ if and only if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 2$.

The set of linear transformations $\frac{a\tau+b}{c\tau+d}$ such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is unimodular and congruent to the identity mod 2 is the congruence subgroup (of level 2) of the modular group. The solution to Problem 1 is therefore as follows:

THEOREM 4. *There exists a K-q.c. of* $\Omega$ *on* $\Omega^*$ *which preserves the order of the points* $e_i$ *if and only if the non-euclidean distance from* $\tau$ *to the nearest equivalent point of* $\tau^*$ *under the congruence subgroup is* $\leq \log K$.

In connection with the lemma it is of some interest to determine the linear transformation $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 2$ for which the extremal length (1) is a minimum. We claim that this is the case for the point $\tau$ that lies in the fundamental region

$$|\tau \pm \tfrac{1}{2}| \leq \tfrac{1}{2}, \quad |\operatorname{Re} \tau| \leq 1 .$$

Then

$$|c\tau + d| \geq |c \operatorname{Re} \tau + d| \geq |d| - |c|$$

and

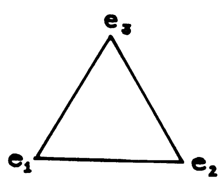
$$|c\tau + d| = |c(\tau \pm \tfrac{1}{2}) \mp \frac{c}{2} + d|$$

$$\geq \tfrac{1}{2}|c| - ||d| - \tfrac{1}{2}|c|| = |d| \text{ or } |c| - |d| .$$

Under the parity conditions $|d| \geq 1$ and either $|d| - |c| \geq 1$ or $|c| - |d| \geq 1$. It follows that $|c\tau + d|^2 \geq 1$ and thus $\lambda$ is smallest for the identity transformation.

COROLLARY *Let $\phi$ be any K-q.c. mapping of the finite plane onto itself with $K < \sqrt{3}$. Then the vertices of any equilateral triangle are mapped on the vertices of a triangle with the same orientation.*

Such a mapping can be approximated by piecewise affine mappings. The triangle corresponds to




$$\rho = \frac{e_3 - e_1}{e_2 - e_1} = \frac{1 + i\sqrt{3}}{2}$$

and we know that the corresponding $\tau = \dfrac{-1 + i\sqrt{3}}{2}$ . The nearest point with a real $\rho$ is $\dfrac{-1 + i}{2}$ , and the non-euclidean distance is $\log\sqrt{3}$. Hence $K < \sqrt{3}$ guarantees that $\operatorname{Im}\rho^* > 0$, and this means that the orientation is preserved.

The remarkable feature of this corollary is that it requires no normalization. The result is at the same time global and local.

# CHAPTER IV

## *BOUNDARY CORRESPONDENCE*

### A. *The M-condition*

We have shown that a q.c. mapping of a disk on itself induces a topological mapping of the circumference. How regular is this mapping? Can it be characterized by some simple condition? The surprising thing is that it can.

Things become slightly simpler if we map the upper halfplane on itself and assume that $\infty$ corresponds to $\infty$. The boundary correspondence is then given by a continuous increasing real-valued function $h(x)$ such that $h(-\infty) = -\infty$ and $h(+\infty) = +\infty$. What conditions must it satisfy?

We suppose first that there exists a $K$-q.c. mapping $\phi$ of the upper halfplane on itself with boundary values $h(x)$. It can immediately be extended by reflection to a $K$-q.c. mapping of the whole plane, and we are therefore in a position to apply the results of the preceding chapter. Namely, let $e_1 < e_3 < e_2$

be real points which are mapped on $e_1', e_3', e_2'$. If

$$\rho = \frac{e_3 - e_1}{e_2 - e_1}, \qquad \rho' = \frac{e_3' - e_1'}{e_2' - e_1'}$$

and $\tau, \tau'$ are the corresponding values on the imaginary axis we have

$$K^{-1} \operatorname{Im} \tau \leq \operatorname{Im} \tau' \leq K \operatorname{Im} \tau \quad .$$

We shall use only the very simplest case where $e_1, e_3, e_2$, are equidistant points $x-t, x, x+t$ and consequently $\rho = \frac{1}{2}$, which corresponds to $\tau = i$. In this case we have that

$$K^{-1} \leq \operatorname{Im} \tau(\rho') \leq K \quad .$$

Equivalently, this means that

$$\rho(iK^{-1}) \leq \rho' \leq \rho(iK)$$

or

(1) $$1 - \rho(iK) \leq \rho' \leq \rho(iK) \quad .$$

Actually, what we prefer are bounds for

$$\frac{e_2' - e_3'}{e_3' - e_1'} = \frac{1-\rho'}{\rho'}$$

and from (1) we get

$$\frac{1-\rho(iK)}{\rho(iK)} \leq \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \leq \frac{\rho(iK)}{1-\rho(iK)} \quad .$$

Even better, let us recall that $\rho(\tau+1) = 1/\rho(\tau)$. For this reason the lower bound can be written as $\rho(1+iK) - 1$ and by our product formula (Chapter III, B, (13)) we find

$$\rho(1+iK) - 1 = 16\, e^{-\pi K} \prod^{\infty} \left( \frac{1+e^{-2n\pi K}}{1-e^{-(2n-1)\pi K}} \right)^8 .$$

We have thus proved

$$M(K)^{-1} \leq \frac{h(x+t)-h(x)}{h(x)-h(x-t)} \leq M(K) \tag{2}$$

where

$$M(K) = \frac{1}{16} e^{\pi K} \prod_{1}^{\infty} \left( \frac{1-e^{-(2n-1)\pi K}}{1+e^{-2n\pi K}} \right)^{8} . \tag{3}$$

We call (2) an *M*-condition. Obviously (3) gives the best possible value, the upper bound

$$M(K) < \frac{1}{16} e^{\pi K} .$$

THEOREM 1. *The boundary values of a K-q.c. mapping satisfy the* $M(K)$ *-condition* (2).

It is important to study the consequences of an *M*-condition

$$M^{-1} \leq \frac{h(x+t)-h(x)}{h(x)-h(x+t)} \leq M \tag{4}$$

even quite apart from its importance for q.c. mappings. Let $H(M)$ denote the family of all such $h$. Observe that is is invariant under linear transformations $S$: $x \to ax+b$ both of the dependent and the independent variables. In other words, if $h \in H(M)$ then $S_1 \circ h \circ S_2 \in H(M)$. We shall let $H_0(M)$ denote the subset of functions $h$ that are normalized by $h(0) = 0$, $h(1) = 1$.

For $h \in H_0(M)$ we have immediately

$$\frac{1}{M+1} \leq h(½) \leq \frac{M}{M+1} \tag{5}$$

and by induction this generalizes to

$$(6) \qquad \frac{1}{(M+1)^n} \leq h\left(\frac{1}{2^n}\right) \leq \left(\frac{M}{M+1}\right)^n .$$

This is true, with the inequality reversed, for negative $n$ as well. In other words we have for instance

$$h(2^n) \leq (M+1)^n$$

which shows that the $h \in H_0(M)$ are uniformly bounded on any compact intervals (apply to $h(x)$ and $1 - h(1-x)$).

They are also equicontinuous. Indeed, for any fixed $a$ the function

$$\frac{h(a+x) - h(a)}{h(a+1) - h(a)}$$

is normalized. Hence $0 \leq x \leq 1/2^n$ gives

$$h(a+x) - h(a) \leq (h(a+1) - h(a))\left(\frac{M}{M+1}\right)^n$$

which proves the equicontinuity on compact sets. As a consequence:

LEMMA 1. *The space $H_0(M)$ is compact (under uniform convergence on compact sets).*

Indeed, the limit function of a sequence must satisfy (4), and this immediately makes it strictly increasing.

Actually, the compactness characterizes the $H_0(M)$.

LEMMA 2. *Let $H_0$ be a set of normalized homeomorphisms $h$ which is compact and stable under composition with linear mappings. Then $H_0 \subset H_0(M)$ for some $M$.*

*Proof.* Set $\alpha = \inf h(-1)$, $\beta = \sup h(-1)$ for $h \in H_0$. There exists a sequence such that $h_n(-1) \to \alpha$, and a subsequence which converges to a homeomorphism. Therefore $\alpha > -\infty$, and the same reasoning gives $\beta < 0$.

For any $h \in H_0$ the mapping

$$k(x) = \frac{h(y+tx) - h(y)}{h(y+t) - h(y)}, \qquad t > 0$$

is in $H_0$. Hence

$$\alpha \leq \frac{h(y-t) - h(y)}{h(y+t) - h(y)} \leq \beta$$

or

$$-\frac{1}{\alpha} \leq \frac{h(y+t) - h(y)}{h(y) - h(y-t)} \leq -\frac{1}{\beta}$$

which is an $M$-condition.

We shall also need the following more specific information:

LEMMA 3. *If* $h \in H_0(M)$ *then*

$$\frac{1}{M+1} \leq \int_0^1 h(x)\,dx \leq \frac{M}{M+1} .$$

*Proof.* Let us set $F(x) = \sup h(x)$, $h \in H_0(M)$. This is a curious function that seems very difficult to determine explicitly. However, some estimates are easy to come by.

We have already proved that $F(\tfrac{1}{2}) \leq M/(M+1)$. Because

$$\frac{h(tx)}{h(t)} \in H_0(M)$$

we obtain, for $x = ½$,

$$\frac{h(\frac{t}{2})}{h(t)} \leq F(½)$$

and hence

(7) $$F(\tfrac{t}{2}) \leq F(\tfrac{1}{2})\, F(t) \qquad \text{for} \qquad t > 0 \; .$$

Similarly

$$\frac{h((1-t)x + t) - h(t)}{1 - h(t)} \in H_0(M)$$

gives

$$\frac{h(\frac{1+t}{2}) - h(t)}{1 - h(t)} \leq F(½) \; .$$

For $t < 1$ this gives

$$h(\tfrac{1+t}{2}) \leq F(½) + (1 - F(½))h(t)$$

and

(8) $$F(\tfrac{1+t}{2}) \leq F(½) + (1 - F(½))F(t) \; .$$

Adding (7) and (8)

(9) $$F(\tfrac{t}{2}) + F(\tfrac{1+t}{2}) \leq F(½) + F(t) \; .$$

Now

$$\int_0^1 F(t)dt = ½ \int_0^2 F(\tfrac{t}{2})\, dt =$$

$$= ½ \int_0^1 (F(\tfrac{t}{2}) + F(\tfrac{1}{2} + \tfrac{t}{2}))\, dt \leq ½ F(½) + ½ \int_0^1 F(t)dt \; .$$

Hence

$$\int_0^1 F(t)dt \leq F(½)$$

and the assertion follows.

The opposite inequality follows on applying the result to $1 - h(1-t)$.

*Remark.* From

$$\frac{1}{M+1} \leq h(½) \leq \frac{M}{M+1}$$

the weaker inequalities

$$\frac{1}{2(M+1)} \leq \int_0^1 h\,dt \leq \frac{2M+1}{2(M+1)}$$

are immediate, and since they serve the same purpose, Lemma 3 is a luxury.

B. *The Sufficiency of the M-condition.*

We shall prove the converse:

THEOREM 2. *Every mapping* $h$ *which satisfies an* $M$-*condition is extendable to a* $K$-*q.c. mapping for a* $K$ *that depends only on* $M$.

The proof is by an explicit construction. We shall indeed set $\phi(x, y) = u(x, y) + iv(x, y)$ where

$$(1) \qquad \begin{aligned} u(x, y) &= \frac{1}{2y} \int_{-y}^{y} h(x+t)dt \\ v(x, y) &= \frac{1}{2y} \int_{0}^{y} (h(x+t) - h(x-t))dt \quad . \end{aligned}$$

It is clear that $v(x, y) \geq 0$ and tends to 0 for $y \to 0$. Moreover, $u(x, 0) = h(x)$, as desired.

The formulas may be rewritten

$$(1)' \quad u = \frac{1}{2y} \int_{x-y}^{x+y} h(t)dt$$

$$v = \frac{1}{2y} \left( \int_{x}^{x+y} h(t)dt - \int_{x-y}^{x} h(t)dt \right)$$

and in this form it is evident that the partial derivatives exist and are

$$u_x = \frac{1}{2y} (h(x+y) - h(x-y))$$

$$u_y = -\frac{1}{2y^2} \int_{x-y}^{x+y} h\,dt + \frac{1}{2y} (h(x+y) + h(x-y))$$

$$v_x = \frac{1}{2y} (h(x+y) - 2h(x) + h(x-y))$$

$$v_y = -\frac{1}{2y^2} \int_{x}^{x+y} h\,dt - \int_{x-y}^{x} h\,dt) + \frac{1}{2y} (h(x+y) - h(x-y)) .$$

The following simplification is possible: if we replace $h(t)$ by $h_1(t) = h(at + b)$, $a > 0$, the $M$-condition remains in force and $\phi(z)$ is replaced by $\phi_1(z) = \phi(az + b)$. Thus $\phi_1(i) = \phi(ai + b)$, and since $ai + b$ is arbitrary we need only study the dilatation at the point $i$. Also, we are still free to choose $h(0) = 0$, $h(1) = 1$.

The derivatives now become

$$u_x = \tfrac{1}{2}(1-h(-1)),$$

$$u_y = -\tfrac{1}{2}\int_{-1}^{1} h\,dt + \tfrac{1}{2}(1+h(-1)),$$

$$v_x = \tfrac{1}{2}(1+h(-1))$$

$$v_y = -\tfrac{1}{2}\left(\int_0^1 h\,dt - \int_{-1}^{0} h\,dt\right) + \tfrac{1}{2}(1-h(-1)) .$$

The (small) dilatation is given by

$$d = \left| \frac{(u_x - v_y) + i(v_x + u_y)}{(u_x + v_y) + i(v_x - u_y)} \right| .$$

To simplify we are going to set

$$\xi = 1 - \int_0^1 h\,dt, \qquad \beta = -h(-1),$$

$$\eta\beta = -h(-1) + \int_{-1}^{0} h\,dt \qquad (>0).$$

Thus

$$u_x = \tfrac{1}{2}(1+\beta) \qquad v_x = \tfrac{1}{2}(1-\beta)$$

$$u_y = \tfrac{1}{2}(\xi - \eta\beta) \qquad v_y = \tfrac{1}{2}(\xi + \eta\beta)$$

giving

$$d = \left| \frac{((1-\xi) + \beta(1-\eta)) + i((1+\xi) - \beta(1+\eta))}{((1+\xi) + \beta(1+\eta)) + i((1-\xi) - \beta(1-\eta))} \right| ,$$

$$d^2 = \frac{1 + \xi^2 + \beta^2(1+\eta^2) - 2\beta(\xi+\eta)}{1 + \xi^2 + \beta^2(1+\eta^2) + 2\beta(\xi+\eta)} ,$$

$$\frac{1+d^2}{1-d^2} = \tfrac{1}{2}\left[\frac{1}{\beta}\,\frac{1+\xi^2}{\xi+\eta} + \beta\,\frac{1+\eta^2}{\xi+\eta}\right] .$$

We have proved the estimates

$$M^{-1} \leq \beta \leq M, \qquad \frac{1}{M+1} \leq \xi \leq \frac{M}{M+1},$$

$$\frac{1}{M+1} \leq \eta \leq \frac{M}{M+1}$$

(the last follows by symmetry).

This gives, for instance

$$\frac{1+d^2}{1-d^2} < M(M+1)$$

$$D < 2M(M+1) .$$

As soon as we have this estimate we know that the Jacobian is positive, from which it follows that the mapping $\phi$ is locally one to one.

It must further be shown that $\phi(z) \to \infty$ for $z \to \infty$. By (1)′ we have

$$u = \frac{1}{2y}\left(\int_{x-y}^{x} h\,dt + \int_{x}^{x+y} h\,dt\right)$$

$$v = \frac{1}{2y}\left(\int_{x}^{x+y} h\,dt - \int_{x-y}^{x} h\,dt\right)$$

$$u^2 + v^2 = \frac{1}{2y^2}\left[\left(\int_{x}^{x+y} h\,dt\right)^2 + \left(\int_{x-y}^{x} h\,dt\right)^2\right] .$$

If $x \geq 0$, $$u^2 + v^2 > \frac{1}{2y^2}\left(\int_{0}^{y} h\,dt\right)^2 .$$

If $x \leq 0$, $$u^2 + v^2 > \frac{1}{2y^2}\left(\int_{-y}^{0} h\,dt\right)^2$$

and both tend to $\infty$ for $y \to \infty$. When $y$ is bounded it is equally clear that $u^2 + v^2 \to \infty$ with $z$.

Now $\zeta = \phi(z)$ defines the upper halfplane as a smooth unlimited covering of itself. By the monodromy theorem it is a homeomorphism.

1. *Remark:* Beurling and Ahlfors proved $D < M^2$. To do so they had to introduce an extra parameter in the definition of $\phi$.

2. *Remark:* It may be asked how regular a function $h(t)$ is that satisfies an $M$-condition. For a long time it was believed that the boundary correspondence would always be absolutely continuous. But this is not so, for it is possible to construct functions $h$ that satisfy the $M$-condition without being absolutely continuous.

C. *Quasi-isometry.*

For conformal mappings of a halfplane onto itself noneuclidean distances are invariant. For q.c.-mappings, are they quasi-invariant? Of course not. If noneuclidean distances are multiplied by a bounded factor (in both directions) we shall say that the mapping is quasi-isometric.

THEOREM 3. *The mapping $\phi$ constructed in $B$ is quasi-isometric. Indeed, it satisfies*

(2) $$A^{-1} d[z_1, z_2] \leq d[\phi(z_1), \phi(z_2)] \leq A\, d[z_1, z_2]$$

*with a constant $A$ that depends only on $M$.*

It is sufficient to prove (2) infinitesimally; that is,

$$A^{-1}\frac{|dz|}{y} \le \frac{|d\phi|}{v} \le A\frac{|dz|}{y} \quad . \tag{3}$$

Again, affine mappings of the $z$-plane are of no importance, and it is therefore sufficient to consider the point (0, 1). We have

$$v(i) = \tfrac{1}{2}\left(\int_0^1 h\,dt - \int_{-1}^0 h\,dt\right)$$

and the estimates

$$\frac{1}{2M} \le v(i) \le \frac{M}{2}$$

are immediate (using Lemma 3).

This is to be combined with

$$\frac{1}{D}|\phi_z|\,|dz| \le |d\phi| \le 2|\phi_z|\,|dz|$$

and

$$|\phi_z|^2 = \tfrac{1}{8}\,[(1+\xi^2) + \beta^2(1+\eta^2) + 2\beta(\xi+\eta)] \quad .$$

One finds that (2) holds with

$$A = 4M^2(M+1) \quad .$$

We omit the details.

## D. *Quasiconformal Reflection.*

Consider now a $K$-q.c. mapping $\phi$ of the whole plane onto itself. The real axis is mapped on a simple curve $L$ that goes to $\infty$ in both directions. Is it possible to characterize $L$ through geometric properties?*

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* For instance, we know already that $L$ has zero area.

First some general remarks. Suppose that $L$ divides the plane into $\Omega$ and $\Omega^*$ corresponding to the upper halfplane $H$ and the lower halfplane $H^*$. Let $j$ denote the reflection $z \to \bar{z}$ that interchanges $H$ and $H^*$. Then $\phi \circ j \circ \phi^{-1}$ is a sense-reversing $K^2$-q.c. mapping which interchanges $\Omega$, $\Omega^*$ and keeps $L$ pointwise fixed. We say that $L$ admits a $K^2$-q-c- reflection.

Conversely, suppose that $L$ admits a $K$-q.c. reflection $\omega$. Let $f$ be a conformal mapping from $H$ to $\Omega$. Define


$$(1) \qquad \begin{cases} F = f & \text{in } H, \\ F = \omega \circ f \circ j & \text{in } H^*. \end{cases}$$

It is clear that $F$ is $K^2$-q.c. So we see that $L$ admits a reflection if and only if it is the image of a line under a q.c. mapping of the whole plane. Moreover, we are free to choose this mapping so that it is conformal in one of the half-planes. We say that the conformal mapping $f$ admits a $K^2$-q.c. extension to the whole plane.

We can also consider the conformal mapping $f^*$ from $H^*$ to $\Omega^*$. The mapping $j \circ f^{*-1} \circ \omega \circ f$ is a quasiconformal mapping of $H$ on itself. Its restriction to the $x$-axis is $h(x) = f^{*-1} \circ f$, and we know that it must satisfy an $M$-condition. Observe that $L$ determines $h$ uniquely except for linear transformation ($h(x)$ can be replaced by $A h(ax+b) + B$).

On the other hand, suppose that $h$ is given and satisfies an $M$-condition. We know that there exists a q.c. mapping with these boundary values: we make it a sense-reversing mapping $\iota$ from $H$ to $H^*$. We cannot yet prove it, but there exists a mapping $\phi$ of the whole plane upon itself which is conformal in $H$ and such that $\phi \circ \iota \circ j$ is conformal in $H^*$. (This condition determines $\mu$ in the whole plane, and Theorem 3 of Chapter V will assert that we can determine $\phi$ when $\mu_\phi$ is given.) Thus $\phi$ maps the real axis on a line $L$ which in turn determines $h$.

How unique is $L$? Suppose $L_1$ and $L_2$ admit q.c. reflections $\omega_1$ and $\omega_2$, and denote the corresponding conformal mappings by $f_1, f_1^*, f_2, f_2^*$. Assume that they determine the same $h = f_1^{*-1} \circ f_1 = f_2^{*-1} \circ f_2$. The mapping

$$g = \begin{cases} f_2 \circ f_1^{-1} & \text{in } \Omega_1 , \\ f_2^* \circ f_1^{*-1} & \text{in } \Omega_1^* , \end{cases}$$

is conformal in $\Omega_1 \cup \Omega_1^*$ and continuous on $L_1$. Is it conformal in the whole plane? To prove that this is so we show that $g$ is quasiconformal, for we know that a q.c. mapping which is conformal a.e. is conformal.

Introduce $F_1$ and $F_2$ as in (1). Form

$$G = F_2^{-1} \circ f_2^* \circ f_1^{*-1} \circ F_1 \quad *$$

in $H^*$. It reduces to identity on the real axis, and we set $G = z$ in $H$. Then $G$ is q.c. Hence $F_2 \circ G \circ F_1^{-1}$ is quasiconformal. It reduces to $f_2^* \circ f_1^{*-1}$ in $\Omega_1^*$ and to $f_2 \circ f_1^{-1}$ in $\Omega_1$; that is, to $g$.

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* $G = j \circ f_2^{-1} \circ \omega_2 \circ f_2^* \circ f_1^{*-1} \circ \omega_1 \circ f_1 \circ j$ .

We conclude that $g$ is conformal. Hence $f_2$ differs from $f_1$ only by a linear transformation, and $L$ is essentially unique.

There are two main problems:

*Problem 1.* To characterize $L$ by geometric properties.

*Problem 2.* To characterize $f$ (and $f^*$).

We shall solve Problem 1. I don't know how to solve Problem 2. The characterization should be in analytic properties of the invariant $f''/f'$.

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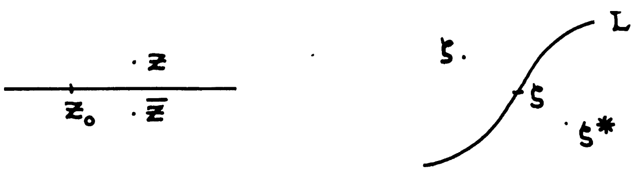
We begin by showing that a $K$-q.c. reflection retains many characteristics of an ordinary reflection.




We shall set $\zeta^* = \omega(\zeta)$ and suppose that $\zeta = \phi(z)$, $\zeta^* = \phi(\bar{z})$, $\zeta_0 = \phi(z_0)$ when $z_0$ is real.

All numerical functions of $K$ alone, not always the same, will be denoted by $C(K)$.

LEMMA 1.

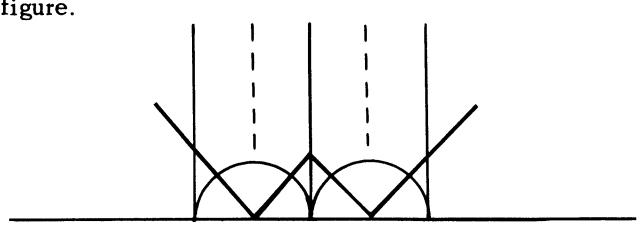
$$C(K)^{-1} \leq \left| \frac{\zeta^* - \zeta_0}{\zeta - \zeta_0} \right| \leq C(K)$$

*Proof.* We note that

$$\rho = \frac{z_0 - z}{z_0 - \overline{z}}$$

satisfies $|\rho| = 1$. For any such $\rho$ there is a corresponding $\tau$ situated on the lines $\operatorname{Re} \tau = \pm \frac{1}{2}$, $\operatorname{Im} \tau \geq \frac{1}{2}$.

We conclude that there is a $\tau'$ corresponding to $\rho' = (\zeta_0 - \zeta)/(\zeta_0 - \zeta^*)$ that is within n.e. distance $\log K$ of these lines. This means that $\tau'$ lies in a $W$-shaped region indicated in our figure.



We note that the points $\pm 1$ where $\rho = \infty$ are shielded from the $W$-region. Also, $\rho \to 1$ as $\operatorname{Im} \tau \to \infty$ and $\rho \to -1$ at the points $\pm \frac{1}{2}$, provided that they are approached within an angle. Therefore $\rho'$ is bounded by a constant $C(K)$ and this proves the lemma.

*Remark.* It is not necessary to investigate the behavior of $\rho(\tau)$ at $\tau = \pm \frac{1}{2}$, for the true region to which $\tau'$ is restricted does not reach these points.

Let $\delta(\zeta)$ be the shortest euclidean distance from $\zeta$ to $L$.

LEMMA 2.

$$C(K)^{-1} \leq \frac{\delta(\zeta^*)}{\delta(\zeta)} \leq C(K) \quad .$$

The proof is trivial.

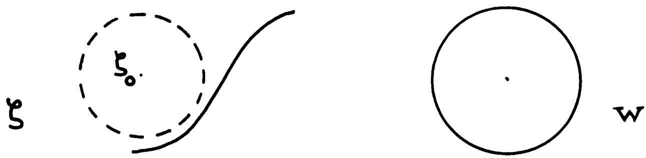
The regions $\Omega$ and $\Omega^*$ carry their own noneuclidean metrics defined by

$$\lambda\,|d\zeta| = \frac{|dz|}{y}$$

for $\zeta = f(z)$. The reflection $\omega$ induces a $K$-q.c. mapping of $H$ on $H^*$, and we know that it can be changed to a $C(K)$ quasi-isometric mapping. It follows that we can replace $\omega$ by a reflection $\omega'$ such that (at points $\zeta^* = \omega'(\zeta)$)

$$C(K)^{-1}\lambda|d\zeta| \leq \lambda^*|d\zeta^*| \leq C(K)\,\lambda|d\zeta| \quad .$$

But it is elementary to estimate $\lambda(\zeta)$ in terms of $\delta(\zeta)$.




For this purpose map $\Omega$ conformally on $|w| < 1$ with $w(\zeta_0) = 0$. Schwarz's lemma gives

$$|w'(\zeta_0)| \leq \frac{1}{\delta(\zeta_0)} \quad .$$

But the noneuclidean line element at the origin is $2|dw|$. So

$$\lambda(\zeta_0) = 2|w'(\zeta_0)| \leq \frac{2}{\delta(\zeta_0)} \quad .$$

In the other direction Koebe's distortion theorem gives

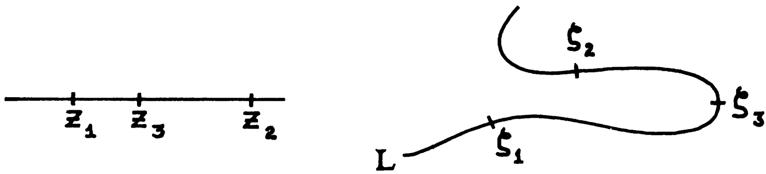
$$\delta(\zeta_0) \geq \tfrac{1}{4}\,\frac{1}{|w'(\zeta_0)|} \,,$$

$$\lambda(\zeta_0) \geq \frac{1}{2\delta(\zeta_0)} \quad .$$

Combining the results with Lemma 2 we conclude

LEMMA 3. *If there exists a K-q.c. reflection across L, then there is also a C(K)-q.c. reflection which is differentiable and changes euclidean lengths at most by a factor C(K).*

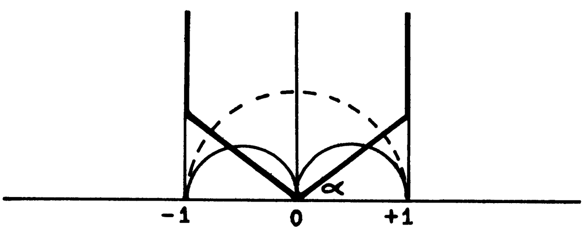
This is a surprising result, for a priori one would expect the stretching to satisfy only a Hölder condition.




Now consider three points on $L$ such that $\zeta_3$ lies between $\zeta_1, \zeta_2$. Now $\rho = (z_1 - z_3)/(z_1 - z_2)$ is between 0 and 1 which means that $\tau$ lies on the imaginary axis. Therefore $\tau^*$ lies in an angle

$$2 \text{ arc tan } K^{-1} \leq \arg \tau^* \leq \pi - 2 \text{ arc tan } K^{-1} \quad .$$

It can also be chosen so that $|\text{Re } \tau^*| \leq 1$, so $\tau^*$ is restricted to the following region




Again it is obvious that $|\rho|$ has a maximum $C(K)$ and we have proved:

THEOREM 4. *If* $\zeta_1, \zeta_3, \zeta_2$ *are any three points on* L *such that* $\zeta_3$ *separates* $\zeta_1, \zeta_2$ *then*

$$\left| \frac{\zeta_3 - \zeta_1}{\zeta_1 - \zeta_2} \right| \leq C(K) .$$

It is more symmetric to write

$$\left| \zeta_3 - \frac{\zeta_1 + \zeta_2}{2} \right| \leq C(K) \, |\zeta_1 - \zeta_2|$$

and in this form the best value of $C(K)$ can be computed. It corresponds to the point $e^{i\alpha}$ and can be computed explicitly.

E. *The Reverse Inequality.*

We shall prove that the condition in the last theorem is not only necessary but also sufficient. In other words:

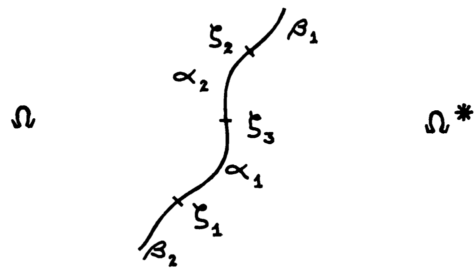
THEOREM 5. *A necessary and sufficient condition for* L *to admit a q.c. reflection is the existence of a constant* C *such that*

$$(1) \qquad \left| \frac{\zeta_3 - \zeta_1}{\zeta_2 - \zeta_1} \right| \leq C$$

*for any three points on* L *such that* $\zeta_3$ *is between* $\zeta_1$ *and* $\zeta_2$.

More precisely, if $K$ is given $C$ depends only on $K$, and if $C$ is given there is a $K$-q.c. reflection where $K$ depends only on $C$.

Introduce notations by this figure:




Let $\lambda_k$ be the extremal distance* from $a_k$ to $\beta_k$ in $\Omega$, and $\lambda_k^*$ the corresponding distance in $\Omega^*$. Thus $\lambda_1\lambda_2 = 1$, $\lambda_1^*\lambda_2^* = 1$. Through the conformal mapping of $\Omega$, let $\zeta_1, \zeta_3, \zeta_2$ correspond to $x-t$, $x$, $x+t$. This means that $\lambda_1 = \lambda_2 = 1$. Through the conformal mapping of $\Omega^*$, $\zeta_1, \zeta_3, \zeta_2$ correspond to $h(x-t)$, $h(x)$, $h(x+t)$. If we can show that $\lambda_1^*$ is bounded it follows at once that $h$ satisfies an $M$-condition, and hence that a q.c. reflection exists.

We show first that $\lambda_1 = 1$ implies

$$(2) \qquad C^{-2}e^{-2\pi} \leq \frac{|\zeta_2 - \zeta_1|}{|\zeta_3 - \zeta_2|} \leq C^2 e^{2\pi} \quad .$$

Indeed, it follows from (1) that the points of $\beta_2$ are at distance $\geq C^{-1}|\zeta_2 - \zeta_1|$ from $\zeta_2$, while the points of $a_2$ have distance $\leq C|\zeta_3 - \zeta_2|$ from $\zeta_2$. If the upper bound in (2) did not hold, $a_2$ and $\beta_2$ would be separated by a circular annulus whose radii have the ratio $e^{2\pi}$. In such an annulus the ex-

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* The extremal distance from $a$ to $\beta$ in $E$ is the extremal length of the family of arcs in $E$ joining $a$ to $\beta$.

tremal distance between the circles is $1$, and the comparison principle for extremal lengths would yield $\lambda_2 > 1$, contrary to hypothesis. This proves the upper bound, and we get the lower bound by interchanging $\zeta_1$ and $\zeta_3$.

Consider points $\zeta \in \alpha_2$, $\zeta' \in \beta_2$. By repeated application of (1)

$$|\zeta - \zeta'| \geq C^{-1}|\zeta - \zeta_1| \geq C^{-2}|\zeta_1 - \zeta_2| ,$$

and with the help of (2) we conclude that the shortest distance between $\alpha_2$ and $\beta_2$ is $\geq C^{-4} e^{-2\pi}|\zeta_2 - \zeta_3|$. To simplify notations write

$$M_1 = C|\zeta_2 - \zeta_3|, \qquad M_2 = C^{-4} e^{-2\pi}|\zeta_2 - \zeta_3| .$$

Because of (1), all points on $\alpha_2$ are within distance $M_1$ from $\zeta_2$.

Let $\Gamma^*$ be the family of all arcs in $\Omega^*$ joining $\alpha_2$ and $\beta_2$. Then

$$\lambda_2^* = \lambda(\Gamma^*) \geq \frac{L(\rho)^2}{A(\rho)}$$

where $\rho$ is any allowable function (recall Chapter I, D). We choose $\rho = 1$ in the disk $\{\,|\zeta - \zeta_2| \leq M_1 + M_2\}$, $\rho = 0$ outside that disk. Then $L_\gamma(\rho) \geq M_2$ for all curves $\gamma \in \Gamma^*$, whether $\gamma$ stays within the disk or not. We conclude that

$$\lambda_2^* \geq \frac{1}{\pi}\left(\frac{M_2}{M_1 + M_2}\right)^2$$

Since $\lambda_1^* \lambda_2^* = 1$, $\lambda_1^*$ has a finite upper bound, and the theorem is proved.

## CHAPTER V

## *THE MAPPING THEOREM*

### A. *Two Integral Operators.*

Our aim is to prove the existence of q.c. mappings $f$ with a given complex dilatation $\mu_f$. In other words, we are looking for solutions of the Beltrami equation

$$f_{\bar{z}} = \mu f_z \quad , \tag{1}$$

where $\mu$ is a measurable function and $|\mu| \leq k < 1$ a.e. The solution $f$ is to be topological, and $f_{\bar{z}}$, $f_z$ shall be locally integrable distributional derivatives. We recall from Chapter II that they are then also locally square integrable. It will turn out that they are in fact locally $L^p$ for a $p > 2$.

The operator $P$ acts on functions $h \in L^p$, $p > 2$, (with respect to the whole plane) and it is defined by

$$Ph(\zeta) = -\frac{1}{\pi} \int\int h(z)\left(\frac{1}{z-\zeta} - \frac{1}{z}\right) dx\,dy \quad . \tag{2}$$

(All integrals are over the whole plane.)

LEMMA 1. *Ph is continuous and satisfies a uniform Hölder condition with exponent* $1 - 2/p$.

The integral (2) is convergent because $h \in L^p$ and $\zeta/(z(z-\zeta)) \in L^p$, $1/p + 1/q = 1$. Indeed, $1 < q < 2$, and for such an exponent $|z(z-\zeta)|^{-q}$ converges at $0$, $\zeta(\neq 0)$ and $\infty$. The Hölder inequality yields, for $\zeta \neq 0$,

$$|Ph(\zeta)| \leq \frac{|\zeta|}{\pi} \|h\|_p \left\| \frac{1}{z(z-\zeta)} \right\|_q .$$

A change of variable shows that

$$\iint |z(z-\zeta)|^{-q}\, dx\, dy = |\zeta|^{2-2q} \iint |z(z-1)|^{-q}\, dx\, dy$$

and we find

$$|Ph(\zeta)| \leq K_p \|h\|_p |\zeta|^{1-2/p} \tag{3}$$

with a constant $K_p$ that depends only on $p$. ((3) is trivially fulfilled if $\zeta = 0$.)

We apply this result to $h_1(z) = h(z+\zeta_1)$. Since

$$Ph_1(\zeta_2-\zeta_1) = -\frac{1}{\pi} \int h(z+\zeta_1)\left(\frac{1}{z+\zeta_1-\zeta_2} - \frac{1}{z}\right) dx\, dy$$

$$= -\frac{1}{\pi} \int h(z)\left(\frac{1}{z-\zeta_2} - \frac{1}{z-\zeta_1}\right) dx\, dy$$

$$= Ph(\zeta_2) - Ph(\zeta_1)$$

we obtain

$$|Ph(\zeta_1) - Ph(\zeta_2)| \leq K_p \|h\|_p |\zeta_1 - \zeta_2|^{1-2/p} \tag{4}$$

which is the assertion of the lemma.

The second operator, $T$, is initially defined only for functions $h \in C_0^2$ ($C^2$ with compact support), namely as the Cauchy principal value

$$(5) \qquad Th(\zeta) = \lim_{\epsilon \to 0} -\frac{1}{\pi} \iint\limits_{|z-\zeta|>\epsilon} \frac{h(z)}{(z-\zeta)^2}\, dx\, dy \quad .$$

We shall prove:

LEMMA 2. *For $h \in C_0^2$, $Th$ exists and is of class $C^1$. Furthermore,*

$$(6) \qquad \begin{aligned} (Ph)_{\bar z} &= h \\ (Ph)_z &= Th \end{aligned}$$

*and*

$$(7) \qquad \iint |Th|^2\, dx\, dy = \iint |h|^2\, dx\, dy \quad .$$

*Proof.* We begin by verifying (6) under the weaker assumption $h \in C_0^1$. This is clearly enough to guarantee that

$$(8) \qquad \begin{aligned} (Ph)_{\bar\zeta} &= -\frac{1}{\pi} \iint \frac{h_{\bar z}}{z-\zeta}\, dx\, dy \\ (Ph)_{\zeta} &= -\frac{1}{\pi} \iint \frac{h_z}{z-\zeta}\, dx\, dy \quad . \end{aligned}$$

Let $\gamma_\epsilon$ be a circle with center $\zeta$ and radius $\epsilon$. By use of Stokes' formula we find

$$-\frac{1}{\pi}\iint \frac{h_{\bar z}}{z-\zeta}\,dx\,dy = \frac{1}{2\pi i}\iint \frac{h_{\bar z}}{z-\zeta}\,dz\,d\bar z$$

$$= -\frac{1}{2\pi i}\iint \frac{dh\,dz}{z-\zeta}$$

$$= \lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{\gamma_\epsilon}\frac{h\,dz}{z-\zeta} = h(\zeta)$$

and

$$-\frac{1}{\pi}\iint \frac{h_z}{z-\zeta}\,dx\,dy = \frac{1}{2\pi i}\iint \frac{dh\,d\bar z}{z-\zeta}$$

$$= \lim_{\epsilon\to 0}\left[-\frac{1}{2\pi i}\int_{\gamma_\epsilon}\frac{h\,d\bar z}{z-\zeta} + \frac{1}{2\pi i}\iint_{|z-\zeta|>\epsilon}\frac{h\,dz\,d\bar z}{(z-\zeta)^2}\right]$$

$$= Th(\zeta) \quad .$$

We have proved (6).

Observe that (8) can be written in the form

$$\text{(9)}\qquad \begin{aligned} P(h_{\bar z}) &= h - h(0) \\ P(h_z) &= Th - Th(0) \quad . \end{aligned}$$

Under the assumption $h \epsilon C_0^2$ we may apply (6) to $h_z$, and by use of the second equation (9) we find

$$\text{(10)}\qquad \begin{aligned} (Th)_{\bar z} &= P(h_z)_{\bar z} = h_z \\ (Th)_z &= P(h_z)_z = T(h_z) = P(h_{zz}) + Th_z(0) \quad . \end{aligned}$$

These relations show that $Th \epsilon C^1$, $Ph \epsilon C^2$.

Because $h$ has compact support it is immediate from the definitions that $Ph = O(1)$ and $Th = O(|z|^{-2})$ as $z \to \infty$. We have now sufficient information to justify all steps in the calculation

$$\int\int |Th|^2 \, dx\, dy = -\frac{1}{2i}\int\int (Ph)_z (\overline{Ph})_{\bar{z}} \, dz\, d\bar{z}$$

$$= \frac{1}{2i}\int\int Ph(\overline{Ph})_{\bar{z}z} \, dz\, d\bar{z} = \frac{1}{2i}\int\int (Ph)\bar{h}_{\bar{z}} \, dz\, d\bar{z}$$

$$= -\frac{1}{2i}\int\int \bar{h}(Ph)_{\bar{z}} \, dz\, d\bar{z} = \int\int |h|^2 \, dx\, dy$$

which proves the isometry.

q.e.d.

The functions of class $C_0^{\,2}$ are dense in $L^2$. For this reason the isometry permits us to extend the operator $T$ to all of $L^2$, by continuity. Unfortunately, we cannot extend $P$ in the same way, for the integral becomes meaningless when $h$ is only in $L^2$, and even if we use principal values the difficulties are discouraging.

The key for solving the difficulty lies in a lemma of Zygmund and Calderon, to the effect that the isometry relation (7) can be replaced by

$$\|Th\|_p \leq C_p \|h\|_p \tag{11}$$

for any $p > 1$, with the additional information that $C_p \to 1$ for $p \to 2$. Naturally, this enables us to extend $T$ to $L^p$, and for $p > 2$ the $P$-transform is well defined.

The proof of the Zygmund-Calderon inequality is given in section D of this chapter. At present we prove:

LEMMA 3. *For* $h \in L^p$, $p > 2$, *the relations*

$$(Ph)_{\bar{z}} = h \qquad (12)$$
$$(Ph)_z = Th$$

*hold in the distributional sense.*

We have to show that

$$\iint (Ph)\phi_{\bar{z}} = -\iint \phi h \qquad (13)$$
$$\iint (Ph)\phi_z = -\iint \phi Th$$

for all test functions $\phi \in C_0^1$. We know that the equations hold when $h \in C_0^2$. Suppose that we approximate $h$ by $h_n \in C_0^2$ in the $L^p$-sense. The right hand members have the right limits since $\|Th - Th_n\|_p \leq C_p \|h - h_n\|_p$. On the left hand side we know by Lemma 1 that

$$|P(h-h_n)| \leq K_p \|h-h_n\|_p |z|^{1-2/p}$$

and since $\phi$ has compact support the result holds.

## B. *Solution of the Mapping Problem.*

We are interested in solving the equation

$$f_{\bar{z}} = \mu f_z \qquad (1)$$

where $\|\mu\|_\infty \leq k < 1$. To begin with we treat the case where $\mu$ has compact support so that $f$ will be analytic at $\infty$.

We shall use a fixed exponent $p > 2$ such that $kC_p < 1$.

THEOREM 1. *If* $\mu$ *has compact support there exists a unique solution* $f$ *of* (1) *such that* $f(0) = 0$ *and* $f_z - 1 \in L^p$.

*Proof.* We begin by proving the uniqueness. This proof will also suggest the existence proof.

Let $f$ be a solution. Then $f_{\bar{z}} = \mu f_z$ is of class $L^p$, and we can form $P(f_{\bar{z}})$. The function

$$F = f - P(f_{\bar{z}})$$

satisfies $F_{\bar{z}} = 0$ in the distributional sense. Therefore $F$ is analytic (Weyl's lemma). The condition $f_z - 1 \in L^p$ implies $F' - 1 \in L^p$, and this is possible only for $F' = 1$, $F = z + a$. The normalization at the origin gives $a = 0$, and we have

(2) $$f = P(f_{\bar{z}}) + z \quad .$$

It follows that

(3) $$f_z = T(\mu f_z) + 1 \ .$$

If $g$ is another solution we get

$$f_z - g_z = T[\mu(f_z - g_z)]$$

and by Zygmund-Calderón we would get

$$\|f_z - g_z\|_p \leq k\, C_p \|f_z - g_z\|_p$$

and since $kC_p < 1$ we must have $f_z = g_z$ a.e. Because of the Beltrami equation we have also $f_{\bar{z}} = g_{\bar{z}}$. Hence $f - g$ and $\bar{f} - \bar{g}$ are both analytic. The difference must be a constant, and the normalization shows that $f = g$.

To prove the existence we study the equation

$$h = T(\mu h) + T\mu \quad . \tag{4}$$

The linear operator $h \to T(\mu h)$ on $L^p$ has norm $\leq k\,C_p < 1$. Therefore the series

$$h = T\mu + T\mu T\mu + T\mu T\mu T\mu + \cdots \tag{5}$$

converges in $L^p$. It is obviously a solution of (4).

If $h$ is given by (5) we find that

$$f = P[\mu(h+1)] + z \tag{6}$$

is the desired solution of the Beltrami equation. In the first place $\mu(h+1) \in L^p$ (this is where we use the fact that $\mu$ has compact support) so that $P[\mu(h+1)]$ is well defined and continuous. Secondly, we obtain

$$\begin{aligned} f_{\bar z} &= \mu(h+1) \\ f_z &= T[\mu(h+1)] + 1 = h + 1 \end{aligned} \tag{7}$$

and $f_z - 1 = h \in L^p$.

The function $f$ will be called the *normal solution* of (1).

We collect some estimates. From (4)

$$\|h\|_p \leq kC_p \|h\|_p + C_p \|\mu\|_p$$

so

$$\|h\|_p \leq \frac{C_p}{1 - kC_p} \|\mu\|_p \tag{8}$$

and by (7)

(9) $$\|f_{\bar{z}}\|_p \leq \frac{1}{1-kC_p} \|\mu\|_p .$$

The Hölder condition gives, from (3),

(10) $$|f(\zeta_1) - f(\zeta_2)| \leq \frac{K_p}{1-kC_p} \|\mu\|_p |\zeta_1-\zeta_2|^{1-2/p} + |\zeta_1-\zeta_2| .$$

Let $\nu$ be another Beltrami coefficient, still bounded by $k$, and denote the corresponding normal solution by $g$. We obtain

$$f_z - g_z = T(\mu f_z - \nu g_z)$$

and hence

$$\|f_z-g_z\|_p \leq \|T[\nu(f_z-g_z)]\|_p + \|T[(\mu-\nu)f_z]\|_p$$
$$\leq kC_p\|f_z-g_z\|_p + C_p\|(\mu-\nu)f_z\|_p .$$

We suppose that $\nu \to \mu$ a.e. (for instance through a sequence) and that the supports are uniformly bounded. The following conclusions can be made:

LEMMA 1. $\|g_z-f_z\|_p \to 0$ *and* $g \to f$, *uniformly on compact sets.**

More important, we want now to show that $f$ has derivatives if $\mu$ does. For this purpose we need first a slight generalization of Weyl's lemma:

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* Since $f-g$ is analytic at $\infty$, the convergence is in fact uniform in the entire plane.

LEMMA 2. *If* $p$ *and* $q$ *are continuous and have locally integrable distributional derivatives that satisfy* $p_{\bar{z}} = q_z$, *then there exists a function* $f \in C^1$ *with* $f_z = p$, $f_{\bar{z}} = q$.

It is sufficient to show that

$$\int_\gamma p\,dz + q\,d\bar{z} = 0$$

for any rectangle $\gamma$. We use a smoothing operator. For $\epsilon > 0$ define $\delta_\epsilon(z) = 1/\pi\epsilon^2$ for $|z| \le \epsilon$, $\delta_\epsilon(z) = 0$ for $|z| > \epsilon$. The convolutions $p * \delta_\epsilon * \delta_{\epsilon'}$ and $g * \delta_\epsilon * \delta_{\epsilon'}$ are of class $C^2$ and

$$(p * \delta_\epsilon * \delta_{\epsilon'})_{\bar{z}} = (q * \delta_\epsilon * \delta_{\epsilon'})_z \quad .$$

Therefore

$$\int_\gamma (p * \delta_\epsilon * \delta_{\epsilon'})\,dz + (q * \delta_\epsilon * \delta_{\epsilon'})\,dz = 0$$

and the result follows on letting $\epsilon$ and $\epsilon'$ tend to $0$.

We apply the lemma to prove:

LEMMA 3. *If* $\mu$ *has a distributional derivative*

$$\mu_z \in L^p, \qquad p > 2,$$

*then* $f \in C^1$, *and it is a topological mapping.*

*Proof.* We try to determine $\lambda$ so that the system

$$\begin{aligned} f_z &= \lambda \\ f_{\bar{z}} &= \mu\lambda \end{aligned} \tag{11}$$

has a solution. The preceding lemma tells us that this will be so if

$$(12) \qquad \lambda_{\bar{z}} = (\mu\lambda)_z = \lambda_z \mu + \lambda \mu_z$$

or

$$(\log \lambda)_{\bar{z}} = \mu (\log \lambda)_z + \mu_z \quad .$$

The equation

$$q = T(\mu q) + T\mu_z$$

can be solved for $q$ in $L^p$, and we set

$$\sigma = P(\mu q + \mu_z) + \text{constant}$$

in such a way that $\sigma \to 0$ for $z \to \infty$. Thus $\sigma$ is continuous and

$$\sigma_{\bar{z}} = \mu q + \mu_z$$

$$\sigma_z = T(\mu q + \mu_z) = q \quad .$$

Hence $\lambda = e^{\sigma}$ satisfies (12), and (11) can be solved with $f \in C^1$. We may of course normalize by $f(0) = 0$, and then $f$ is the normal solution since $\sigma \to 0$, $\lambda \to 1$, $f_z \to 1$ at $\infty$.

The Jacobian $|f_z|^2 - |f_{\bar{z}}|^2 = (1 - |\mu|^2) e^{2\sigma}$ is strictly positive. Hence the mapping is locally one-one, and since $f(z) \to \infty$ for $z \to \infty$ it is a homeomorphism.

*Remark.* Doubts may be raised whether $(e^{\sigma})_z = e^{\sigma}\sigma_z$ in the distributional sense. They are dissolved by remarking that we can approximate $\sigma$ by smooth functions $\sigma_n$ in the sense that $\sigma_n \to \sigma$ a.e. and $(\sigma_n)_z \to \sigma_z$ in $L^p$ (locally). Then for any test function $\phi$,

$$\int e^{\sigma}\phi_z = \lim \int e^{\sigma_n}\phi_z = \lim\left(-\int \phi e^{\sigma_n}(\sigma_n)_z\right)$$

$$= -\int \phi e^{\sigma}\sigma_z$$

q.e.d.

Under the hypothesis of Lemma 3 the inverse function $f^{-1}$ is again $K$-q.c. with a complex dilatation $\bar{\mu}^1 = \mu_{f^{-1}}$ which satisfies $|\bar{\mu}^1 \circ f| = |\mu|$.

Let us estimate $\|\bar{\mu}^1\|_p$. We have

$$\int\int |\bar{\mu}^1|^p \, d\xi\, d\eta = \int\int |\mu|^p (|f_z|^2 - |f_{\bar{z}}|^2)\, dx\, dy$$

$$\leq \int |\mu|^p |f_z|^2\, dx\, dy = \int |\mu|^{p-2} |f_{\bar{z}}|^2\, dx\, dy$$

$$\leq \left(\int |\mu|^p\, dx\, dy\right)^{\frac{p-2}{p}} \left(\int |f_{\bar{z}}|^p\, dx\, dy\right)^{\frac{2}{p}}$$

and thus

$$\|\bar{\mu}^1\|_p \leq (1-kC_p)^{-2/p}\, \|\mu\|_p \; .$$

If we apply the estimate (10) to the inverse function we find

$$|z_1 - z_2| \leq K_p(1-kC_p)^{-1-\frac{2}{p}}\, \|\mu\|_p\, |f(z_1)-f(z_2)|^{1-\frac{2}{p}} \tag{13}$$

$$+\; |f(z_1)-f(z_2)| \, .$$

It is now obvious how to prove

THEOREM 2 *For any* $\mu$ *with compact support and* $\|\mu\|_\infty \leq k < 1$ *the normal solution of the Beltrami equation is a q.c. homeomorphism with* $\mu_f = \mu$.

We can find a sequence of functions $\mu_n \in C^1$ with $\mu_n \to \mu$ a.e., $|\mu_n| \leq k$ and $\mu_n = 0$ outside a fixed circle. The normal solutions $f_n$ satisfy (13) with $f_n$ in the place of $f$ and $\mu_n$ in the place of $\mu$. Because $f_n \to f$ and $\|\mu_n\|_p \to \|\mu\|_p$, $f$ satisfies (13), and hence $f$ is one-one.

We recall the results from Chapter II. Because $f$ is a uniform limit of $K$-q.c. mappings $f_n$, it is itself $K$-q.c. As such it has locally integrable partial derivatives, and these partial derivatives are also distributional derivatives. We know further that $f_z \neq 0$ a.e. so that $\mu_f = f_{\bar{z}}/f_z$ is defined a.e. and coincides with $\mu$.

(Incidentally, the fact that $f$ maps null sets on null sets can be proven more easily than before. Let e be an open set of finite measure. Then

$$\begin{aligned}
\text{mes } f_n(e) &= \int_e (|f_{n,z}|^2 - |f_{n,\bar{z}}|^2)\, dx\, dy \\
&\leq \iint_e |f_{n,z}|^2\, dx\, dy \\
&\leq \left( \int |f_{n,z}|^p \right)^{2/p} \quad (\text{mes } e)^{1-2/p}
\end{aligned}$$

and since $\|f_{n,z}\|_p$ is bounded we conclude that $f$ is indeed absolutely continuous in the sense of area.)

We shall now get rid of the assumption that $\mu$ has compact support.

THEOREM 3. *For any measurable* $\mu$ *with* $\|\mu\|_\infty < 1$ *there exists a unique normalized q.c. mapping* $f^\mu$ *with complex dilatation* $\mu$ *that leaves* $0, 1, \infty$ *fixed.*

*Proof.* 1) If $\mu$ has compact support we need only normalize $f$.

2) Suppose $\mu = 0$ in a neighborhood of $0$. Set

$$\tilde{\mu}(z) = \mu\left(\frac{1}{z}\right) \frac{z^2}{\bar{z}^2} .$$

Then $\tilde{\mu}$ has compact support. We claim that

$$f^\mu(z) = \frac{1}{f^{\tilde{\mu}}(1/z)} .$$

Indeed,

$$f^\mu_z(z) = \frac{1}{f^{\tilde{\mu}}(1/z)^2} \frac{1}{z^2} f^{\tilde{\mu}}_z(1/z)$$

$$f^\mu_{\bar{z}}(z) = \frac{1}{f^{\tilde{\mu}}(1/z)^2} \frac{1}{\bar{z}^2} f^{\tilde{\mu}}_{\bar{z}}(1/z) .$$

*Remark:* The computation is legitimate because $f^{\tilde{\mu}}$ is differentiable a.e.

3) In the general case, set $\mu = \mu_1 + \mu_2$, $\mu_1 = 0$ near $\infty$, $\mu_2 = 0$ near $0$. We want to find $\lambda$ so that

$$f^\lambda \circ f^{\mu_2} = f^\mu, \qquad f^\lambda = f^\mu \circ (f^{\mu_2})^{-1} .$$

In Chapter I we showed that this is so for

$$\lambda = \left[ \left( \frac{\mu - \mu_2}{1 - \mu\bar{\mu}_2} \right) \left( \frac{f_z^{\mu_2}}{\bar{f}_{\bar{z}}^{\mu_2}} \right) \right] \circ (f^{\mu_2})^{-1}$$

and this $\lambda$ has compact support. The problem is solved.

THEOREM 4. *There exists a $\mu$-conformal mapping of the upper halfplane on itself with* 0, 1, $\infty$ *as fixed points.*

Extend the definition of $\mu$ by

$$\hat{\mu}(z) = \overline{\mu(\bar{z})} \quad .$$

One verifies by the uniqueness that $f^{\hat{\mu}}(\bar{z}) = \bar{f}^{\hat{\mu}}(z)$. Therefore the real axis is mapped on itself, and so is the upper halfplane.

COROLLARY. *Every q.c. mapping can be written as a finite composite of q.c. mappings with dilatation arbitrarily close to* 1.

We may as well assume that $f = f^{\mu}$ is given as a q.c. mapping of the whole plane. For any $z$, divide the noneuclidean geodesic between 0 and $\mu(z)$ in $n$ equal parts, the successive end points being $\mu_k(z)$. Set

$$f_k = f^{\mu_k} \quad .$$

Then

$$\mu_{(f_{k+1} \circ f_k^{-1})} = \left( \frac{\mu_{k+1} - \mu_k}{1 - \mu_{k+1}\,\bar{\mu}_k} \; \frac{(f_k)_z}{(\bar{f}_k)_{\bar{z}}} \right) \circ f_k^{-1} \quad .$$

If $f^\mu$ is $K$-q.c. it is clear that $g_k = f_{k+1} \circ f_k^{-1}$ is $K^{1/n}$-q.c., and

$$f = g_n \circ \cdots \circ g_2 \circ g_1 .$$

C. *Dependence on Parameters.*

In what follows $f^\mu$ will always denote the solution of the Beltrami equation with fixpoints at $0, 1, \infty$. We shall need the following lemma:

LEMMA. *If* $k = \|\mu\|_\infty \to 0$ *then* $\|f_z^\mu - 1\|_{1,p} \to 0$ *for all* $p$.

Note: $\| \quad \|_{R,p}$ means the $p$-norm over $|z| \leq R$.

Suppose first that $\mu$ has compact support, and let $F^\mu$ be the normal solution obtained in Theorem 1. We know that $h = F_z^\mu - 1$ is obtained from

$$h = T(\mu h) + T\mu$$

and this implies $\|h\|_p \leq C\|\mu\|_p \to 0$. Here $p$ is arbitrary, for the condition $kC_p < 1$ is fulfilled as soon as $k$ is sufficiently small.

Since $f^\mu = F^\mu/F^\mu(1)$ and $F^\mu(1) \to 1$, the assertion follows for all $\mu$ with compact support.

Let us write $\check{f}(z) = 1/f(\frac{1}{z})$. Then it is also true that $\|\check{f}^\mu{}_z - 1\|_{1,p} \to 0$ when $\mu$ has compact support. It is again sufficient to prove the corresponding statement for the normal solution $F^\mu$. One has

$$\iint_{|z|<1} |\check{F}_z - 1|^p = \iint_{|z|>1} \left| \frac{z^2 F_z(z)}{F(z)^2} - 1 \right|^p \frac{dx\,dy}{|z|^4} .$$

The part $\iint_{1<|z|<R}$ can be written

$$\iint_{1<|z|<R} \left| \frac{z^2(F_z(z) - 1)}{F(z)^2} + \frac{z^2}{F(z)^2} - 1 \right|^p \frac{dx\,dy}{|z|^4}$$

and tends to zero because $\|F_z - 1\|_{R,p} \to 0$. For $|z| > R$, $F$ may be assumed analytic, and $F(z) \to z$ uniformly.

In the general situation we can write $f = \check{g} \circ h$ where $\mu_h = \mu_f$ inside the unit disk and $h$ is analytic outside. Then $\mu_h$ and $\mu_g$ are bounded by $k$ and have compact support. Since

$$f_z = (\check{g}_z \circ h)h_z + (\check{g}_{\bar{z}} \circ h)\bar{h}_z = (\check{g}_z \circ h)h_z ,$$

we get

$$\|f_z - 1\|_{1,p} \leq \|[(\check{g}_z - 1) \circ h]h_z\|_{1,p} + \|h_z - 1\|_{1,p} .$$

For the first integral we obtain

$$\iint_{|z|<1} |(\check{g}_z - 1) \circ h|^p |h_z|^p \, dx\,dy =$$

$$\leqq \frac{1}{1-k^2} \iint_{h\{|z|<1\}} |\check{g}_z - 1|^p |h_z \circ h^{-1}|^{p-2} \, dx\,dy$$

$$\leqq \frac{1}{1-k^2} \left( \iint_{h\{|z|<1\}} |\check{g}_z - 1|^{2p} \cdot \iint_{|z|<1} |h_z|^{2p-4} \right)^2 \longrightarrow 0 .$$

One of these integrals is taken over a region slightly bigger than the unit disk, but this is of no importance. The lemma is proved.

It is now assumed that $\mu$ depends on a real or complex parameter $t$ and that

$$\mu(z, t) = t\nu(z) + t\varepsilon(z, t)$$

where $\nu$ and $\varepsilon \in L^\infty$ and $\|\varepsilon(z, t)\|_\infty \to 0$ for $t \to 0$. We shall show that $f^\mu = f(z, t)$ has a $t$-derivative at $t = 0$.

For any $|\zeta| < 1$ we can write

$$f(\zeta) = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)dz}{z - \zeta} - \frac{1}{\pi} \iint_{|z|<1} \frac{f_{\bar{z}}(z)}{z - \zeta}\, dx\, dy$$

(the Pompeiu formula). Replace $z$ by $1/z$ in the line integral. It becomes

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{f(1/z)dz}{z(1-z\zeta)} = A + B\zeta + \frac{\zeta^2}{2\pi i} \int_{|z|=1} \frac{\check{f}(z)^{-1}}{1 - z\zeta}\, dz$$

$$= A + B\zeta - \frac{\zeta^2}{\pi} \iint_{|z|<1} \frac{\check{f}_{\bar{z}}(z) z\, dx\, dy}{\check{f}(z)^2(1-z\zeta)} .$$

The convergence is guaranteed as soon as $t$ is sufficiently small to make $K < 2$. Indeed, the inverse of $\check{f}$ satisfies a Hölder condition with exponent $1/K$. For small $|z|$ we have thus, $|\check{f}(z)| > m|z|^K$ and hence

$$\int_{|z|=\delta} \frac{|\check{f}(z)|^{-1}|z|\,|dz|}{|1 - z\zeta|} = O(\delta^{2-K}) \to 0 .$$

The constants can be recovered from the normalization $f(0) = 0$, $f(1) = 1$. All told we get

$$f(\zeta) = \zeta - \frac{1}{\pi} \iint\limits_{|z|<1} f_{\bar z}(z)\left(\frac{1}{z-\zeta} - \frac{\zeta}{z-1} + \frac{\zeta-1}{z}\right) dx\, dy$$

$$- \frac{1}{\pi} \iint\limits_{|z|<1} \frac{\check{f}_{\bar z}(z)}{\check{f}(z)^2} \left(\frac{\zeta^2 z}{1-z\zeta} - \frac{\zeta z}{1-z}\right) dx\, dy \quad .$$

In the first integral, set

$$f_{\bar z} = \mu f_z = \mu(f_z - 1) + \mu$$

and use a corresponding expression for $\check{f}_{\bar z}$ with $\check{\mu}(z) = (z/\bar z)^2 \mu(1/z)$. Because

$$\|f_z - 1\|_{1,p} \to 0 \quad \text{and} \quad \frac{\mu}{t} \to \nu$$

we find

$$\dot{f}(\zeta) = \lim_{t\to 0} \frac{f(\zeta)-\zeta}{t}$$

$$= -\frac{1}{\pi} \iint\limits_{|z|<1} \nu(z)\left(\frac{1}{z-\zeta} - \frac{\zeta}{z-1} + \frac{\zeta-1}{z}\right) dx\, dy$$

$$- \frac{1}{\pi} \iint\limits_{|z|<1} \nu(1/z) \cdot \frac{1}{\bar z^2}\left(\frac{\zeta^2 z}{1-z\zeta} - \frac{\zeta z}{1-z}\right) dx\, dy \quad .$$

The convergence is clearly uniform when $\zeta$ ranges over a compact subset of $|\zeta| < 1$.

Finally, if $1/z$ is taken as integration variable in the second integral, it transforms to the same integrand as the first, and we have

$$\dot{f}(\zeta) = -\frac{1}{\pi}\iint \nu(z) R(z, \zeta)\, dx\, dy$$

where

$$R(z, \zeta) = \frac{1}{z-\zeta} - \frac{\zeta}{z-1} + \frac{\zeta-1}{z} = \frac{\zeta(\zeta-1)}{z(z-1)(z-\zeta)} .$$

By considering $f(rz)$, one verifies easily that this formula is valid for all $\zeta$, and that the convergence is uniform on compact sets.

For arbitrary $t_0$ we assume

$$\mu(t) = \mu(t_0) + \nu(t_0)(t-t_0) + o(t-t_0)$$

in the same sense as above, and we consider

$$f^{\mu(t)} = f^{\lambda} \circ f^{\mu(t_0)}$$

where

$$\lambda = \lambda(t) = \left(\frac{\mu(t)-\mu(t_0)}{1-\mu(t)\bar{\mu}(t_0)} \cdot \frac{f_z^{\mu_0}}{\bar{f}_{\bar{z}}^{\mu_0}}\right) \circ (f^{\mu_0})^{-1} .$$

Clearly, $\lambda(t) = (t-t_0)\dot{\lambda}(t_0) + o(t-t_0)$ where

$$\dot{\lambda}(t_0) = \left(\frac{\nu(t_0)}{1-|\mu_0|^2} \cdot \frac{f_z^{\mu_0}}{\bar{f}_{\bar{z}}^{\mu_0}}\right) \circ (f^{\mu_0})^{-1}$$

and

$$\frac{\partial}{\partial t} f(z, t) = \dot{f} \circ f^{\mu_0}$$

$$= -\frac{1}{\pi}\iint \left(\frac{\nu(t_0)}{1-|\mu_0|^2} \; \frac{f_z^{\mu_0}}{\bar{f}_{\bar{z}}^{\mu_0}}\right) \circ (f^{\mu_0})^{-1} R(z, f^{\mu_0}(\zeta))dx\, dy$$

$$= -\frac{1}{\pi}\iint \nu(t_0, z)(f_z^{\mu_0})^2 R(f^{\mu_0}(z), f^{\mu_0}(\zeta))\, dx\, dy$$

which is the general perturbation formula.

We may sum up our results as follows:

THEOREM 5. *Suppose*

$$\mu(t+s)(z) = \mu(t)(z) + s\nu(t)(z) + s\,\varepsilon(s,t)(z)$$

*where*

$$\nu(t),\ \mu(t),\ \varepsilon(s,t) \in L^\infty, \quad \|\mu(t)\|_\infty < 1,$$

*and* $\|\varepsilon(s,t)\|_\infty \to 0$ as $s \to 0$.

*Then*

$$f^{\mu(t+s)}(\zeta) = f^{\mu(t)}(\zeta) + t\dot{f}(\zeta,t) + o(s)$$

*uniformly on compact sets, where*

$$\dot{f}(\zeta,t) = -\frac{1}{\pi}\iint \nu(t)(z)\,R\big(f^{\mu(t)}(z),\ f^{\mu(t)}(\zeta)\big)\,(f_z^{\mu(t)}(z))^2\,dx\,dy\ .$$

If $\nu(t)$ depends continuously on $t$ (in the $L^\infty$ sense) then we can prove, moreover, that $(\partial/\partial t)\,f(z,t)$ is a continuous function of $t$. It is enough to prove this for $t = 0$, so we have to show that

$$\iint \nu(t,z)(f_z^\mu(z))^2 R(f^\mu(z), f^\mu(\zeta)) \to \iint \nu(z)\,R(z,\zeta)\,dx\,dy\ .$$

By inversion, the integral over the plane can again be written as the sum of two integrals over $|z| \le 1$, and they can be treated in a similar manner. We consider only the first part.

The important thing is that the improper integral converges uniformly; that is,

$$\iint\limits_{\substack{|z-\zeta|<\delta \\ |z|<1}} |f_z^\mu(z)|^2\,|R(f^\mu(z),\ f^\mu(\zeta)|\,dx\,dy < \varepsilon$$

in a uniform manner. Indeed, this integral is comparable to

$$\iint_{f^{\mu}[z:\ |z-\zeta|<\delta]} |R(z,\ f^{\mu}(\zeta))|\, dx\, dy \quad .$$

The region of integration can be replaced by a disk with center $f^{\mu}(\zeta)$ and radius $< 2\delta$, say. The value of the integral is then uniformly of order $O(\delta)$.

For the remaining part of the plane (with a fixed $\delta$) the difference

$$\iint |f_z^{\mu}(z)|^2\ |R(f^{\mu}(z), f^{\mu}(\zeta))\nu(t,\ z) - R(z,\ \zeta)\nu(z)|$$

tends rather trivially to 0, and it is also clear that

$$\iint \Big( |f_z^{\mu}(z)^2 - 1|\ \ |R(z,\ \zeta)\nu(z)| \Big) \to 0$$

since $\|f^{\mu} - 1\|_p \to 0$ and the other factor is bounded.

D. *The Calderón-Zygmund Inequality.*

We are going to prove that the operator

$$Th(\zeta) = \lim_{\epsilon \to 0} -\frac{1}{\pi} \iint_{|z-\zeta|>\epsilon} \frac{h(z)}{(z-\zeta)^2}\, dx\, dy \quad ,$$

already defined for $h \in C_0^2$, can be extended to $L^p$, $p \geq 2$, so that

(1) $$\|Th\|_p \leq C_p \|h\|_p \quad .$$

We prove first a one-dimensional analogue, due to Riesz. The proof is taken from Zygmund, *Trigonometric Series*, first edition (Warsaw, 1935).

LEMMA. *For* $f \in C_0^1$ *on the real line, set*

$$Hf(\xi) = \text{pr.v.} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x)}{x-\xi} \, dx \quad .$$

*Then* $\|Hf\|_p \leq A_p \|f\|_p \quad (p \geq 2)$ *with* $A_2 = 1$.

*Proof.* Set

$$F(\zeta) = u + iv = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x)}{x-\zeta} \, dx \quad ,$$

$$\zeta = \xi + i\eta, \qquad \eta > 0 \ .$$

The imaginary part $v$ is the Poisson integral, and it is immediate that

$$v(\xi) = f(\xi) \ .$$

The real part is

$$u(\xi, \eta) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x-\xi}{(x-\xi)^2 + \eta^2} f(x) \, dx$$

$$= \frac{1}{\pi} \int_{0}^{\infty} \frac{f(\xi + x) - f(\xi - x)}{x} \frac{x^2}{x^2 + \eta^2} \, dx$$

$$\to Hf(\xi) \qquad \text{as} \qquad \eta \to 0 .$$

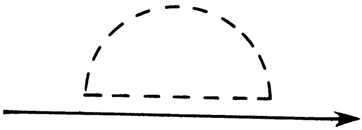
We observe now that

$$\Delta |u|^p = p(p-1) |u|^{p-2} (u_x^2 + u_y^2)$$
$$\Delta |v|^p = p(p-1) |v|^{p-2} (u_x^2 + u_y^2)$$
$$\Delta |F|^p = p^2 |F|^{p-2} (u_x^2 + u_y^2) \quad .$$

It follows that

$$\Delta (|F|^p - \frac{p}{p-1} |u|^p) = p^2 (|F|^{p-2} - |u|^{p-2})(u_x^2 + u_y^2)$$
$$\geq 0 \ .$$

We apply Stokes' formula to a large semicircle



and find easily

$$\frac{\partial}{\partial \eta} \int_{-\infty}^{\infty} \left( |F(\zeta)|^p - \frac{p}{p-1} \, |u(\zeta)|^p \right) d\xi \leq 0 \; .$$

It is easy to see that the integral goes to zero for $\eta \to \infty$. Hence

$$\int_{-\infty}^{\infty} |F(x+i\eta)|^p \, dx \geq \frac{p}{p-1} \int_{-\infty}^{\infty} |u(x+i\eta)|^p \, dx$$

for fixed $\eta > 0$. We observe that

$$\left( \int |F|^p \, d\xi \right)^{2/p} = \|u^2 + v^2\|_{p/2} \leq \|u^2\|_{p/2} + \|v^2\|_{p/2} \; .$$

Thus

$$\left( \frac{p}{p-1} \right)^{2/p} \|u^2\|_{p/2} \leq \|u\|^2_{p/2} + \|v^2\|_{p/2} \; ,$$

$$\|u^2\|_{p/2} \leq \frac{1}{\left( \left( \frac{p}{p-1} \right)^{2/p} - 1 \right)} \|v^2\|_{p/2} \; ,$$

$$\int |u|^p \, d\xi \leq \frac{1}{\left( \left( \frac{p}{p-1} \right)^{2/p} - 1 \right)^{p/2}} \int |v|^p \, d\xi \; .$$

Letting $\eta \to 0$ we get the desired inequality.

We continue the proof of the Calderón-Zygmund inequality (1), following the method in Vekua, *Generalized Analytic Functions,* Pergamon Press, 1962.

We define the operator

$$T^*f(\zeta) = \frac{1}{2\pi} \iint f(z+\zeta) \, \frac{dx\,dy}{z|z|} , \qquad f \in C_0^{\,2} ,$$

again as a principal value. On setting $z = re^{i\theta}$ we see that it can be written

$$T^*f(\zeta) = \frac{1}{2} \int_0^{\pi} \left( \frac{1}{\pi} \int_0^{\infty} \frac{f(\zeta+re^{i\theta}) - f(\zeta-re^{i\theta})}{r} \, dr \right) e^{-i\theta} \, d\theta .$$

This implies

$$\|T^*f\|_p \le \frac{\pi}{2} \max_{\theta} \left\| \frac{1}{\pi} \int_0^{\infty} \frac{f(\zeta+re^{i\theta}) - f(\zeta-re^{i\theta})}{r} \, dr \right\|_p$$

The norm on the right does not change if we replace $\zeta$ by $\zeta e^{i\theta}$, and then the integral becomes $Hf_\theta(\zeta)$ where $f_\theta(z) = f(ze^{i\theta})$. The norm is of course a 2-dimensional norm. But it is clear that our estimate of the 1-dimensional norm can be used, for we obtain

$$\|Hf_\theta\|_p^p = \iint |Hf_\theta(u+iv)|^p \, du\,dv$$

$$\le A_p^p \int dv \int |f_\theta(u+iv)|^p \, du = A_p^p \, \|f_\theta\|_p^{\,p}$$

so that, finally, $\|T^* f\|_p \le \frac{\pi}{2} A_p \|f\|_p$ .

We extend $T^*$ to $L^p$ by continuity. The proof of (1) will now be completed by showing that $Tf = -T^* T^* f$ for $f \in C_0^2$. This, of course, allows us to extend $T$ to $L^p$ also.

We have $\frac{\partial}{\partial z} \frac{1}{|z|} = -\frac{1}{2z|z|}$ . If $f \in C_0^1$ we obtain

$$T^* f(\zeta) = -\frac{1}{\pi} \iint f(z+\zeta) \frac{\partial}{\partial z} \frac{1}{|z|} \, dx\, dy$$

$$= \frac{1}{\pi} \iint f_z(z+\zeta) \frac{1}{|z|} \, dx\, dy$$

$$(2) \qquad = \frac{1}{\pi} \frac{\partial}{\partial \zeta} \iint f(z) \frac{dx\, dy}{|z-\zeta|}$$

$$= \frac{1}{\pi} \frac{\partial}{\partial \zeta} \iint f(z) \left( \frac{1}{|z-\zeta|} - \frac{1}{|z|} \right) dx\, dy \; .$$

For any test function $\phi$ it follows that

$$\iint T^* f(\zeta) \phi_\zeta(\zeta) \, d\xi\, d\eta$$

$$= -\frac{1}{\pi} \iiiint f(z) \left( \frac{1}{|z-\zeta|} - \frac{1}{|z|} \right) \phi(\zeta) \, dx\, dy\, d\xi\, d\eta \; .$$

This remains true for $f \in L^p$, because the integral on the right is absolutely convergent (compare the proof for $Pf$). This means that (2) is valid in the distributional sense.

We can now write

$$T^*T^*f(w)$$

$$= \frac{1}{\pi} \frac{\partial}{\partial w} \int\int T^*f(\zeta) \left( \frac{1}{|\zeta - w|} - \frac{1}{|\zeta|} \right) d\xi\, d\eta$$

$$= \frac{1}{\pi^2} \frac{\partial}{\partial w} \left[ \int\int \left( \frac{1}{|\zeta - w|} - \frac{1}{|\zeta|} \right) d\xi\, d\eta \int\int \frac{f_z\, dx\, dy}{|z - \zeta|} \right]$$

$$= \frac{1}{\pi^2} \frac{\partial}{\partial w} \left[ \int\int f_z\, dx\, dy \int\int \frac{1}{|z - \zeta|} \left( \frac{1}{|\zeta - w|} - \frac{1}{|\zeta|} \right) d\xi\, d\eta \right]$$

$$= - \frac{1}{\pi^2} \frac{\partial}{\partial w} \int\int f \frac{\partial}{\partial z} \left( \int\int \frac{1}{|z - \zeta|} \left( \frac{1}{|\zeta - w|} \quad \frac{1}{|\zeta|} \right) d\xi\, d\eta \right) dx\, dy \quad .$$

(One should check the behavior for $z = 0$ and $z = w$. The integral blows up logarithmically, and the boundary integral over a small circle tends to zero.) We have to compute

$$\frac{\partial}{\partial z} \lim_{R \to \infty} \int\int_{|\zeta - w| < R} \frac{1}{|z - \zeta|} \left( \frac{1}{|\zeta - w|} - \frac{1}{|\zeta|} \right) d\xi\, d\eta \quad .$$

The differentiation and limit can be interchanged, for it is quite evident that

$$\frac{\partial}{\partial z} \int\int_{|\zeta - w| > R} \frac{1}{|z - \zeta|} \left( \frac{1}{|\zeta - w|} - \frac{1}{|\zeta|} \right) d\xi\, d\eta$$

tends to zero, uniformly on compact sets. Moreover, we can replace our expression by

$$\lim \frac{\partial}{\partial z} \iint\limits_{|\zeta - w| < R} \frac{1}{|z-w|} \frac{1}{|\zeta - w|} d\xi\, d\eta - \frac{\partial}{\partial z} \iint\limits_{|\zeta| < R} \frac{d\xi\, d\eta}{|\zeta||z-\zeta|}$$

for the difference, which is an integral over two slivers, tends uniformly to zero.

By an obvious change of variable the first integral becomes

$$\frac{\partial}{\partial z} \iint\limits_{|\zeta| < R/|z-w|} \frac{d\xi\, d\eta}{|\zeta||1-\zeta|}$$

$$= \frac{\partial}{\partial z} \int_0^{R/|z-w|} \int_0^{2\pi} \frac{dr\, d\theta}{|1-re^{i\theta}|}$$

$$= -\tfrac{1}{2} \frac{R}{(z-w)|z-w|} \int_0^{2\pi} \frac{d\theta}{\left|1-\frac{Re^{i\theta}}{|z-w|}\right|}$$

and it is obvious that the limit is $-\dfrac{\pi}{z-w}$. Similarly, the second limit is $-\pi/z$.

We now obtain

$$T^*T^*f(w) = \frac{\partial}{\partial w}\left[\frac{1}{\pi}\iint f(z)\left(\frac{1}{z-w} - \frac{1}{z}\right) dx\, dy\right]$$

$$= -\frac{\partial}{\partial w} Pf(w) = -Tf(w) .$$

This completes the proof of (1).

The fact that $C_p \to 1$ as $p \to 2$ is a consequence of the

*Riesz-Thorin convexity theorem:*

*The best constant* $C_p$ → *is such that* $\log C_p$ *is a convex function of* $1/p$.

*Proof.* We consider $p_1 = \frac{1}{a_1}$, $p_2 = \frac{1}{a_2}$ both $\geq 2$. Assume

$$\|Tf\|_{1/a_1} \leq C_1 \|f\|_{1/a_1}$$

$$\|Tf\|_{1/a_2} \leq C_2 \|f\|_{1/a_2} \quad .$$

If $a = (1-t)a_1 + t a_2$ the contention is that

$$\|Tf\|_{1/a} \leq C_1^{\,1-t}\, C_2^{\,t}\, \|f\|_{1/a} \qquad (0 \leq t \leq 1)\,.$$

Conjugate exponents will correspond to $a$ and $a'$ with $a + a' = 1$. Similar meanings for $a_1'$, $a_2'$. We note first that

$$\|Tf\|_{1/a} = \sup_g \int Tf \cdot g \quad dx\, dy$$

where $g$ ranges over all functions of norm one in $L^{1/a'}$. Because simple functions (measurable functions with a finite number of values) with compact support are dense in every $L^p$ it is no restriction to assume that $f$ and $g$ are such functions.

For such fixed $f, g$ set

$$I = \int Tf \cdot g \;\; dx\, dy \quad .$$

The idea of the proof is to make $f$ and $g$ depend analytically on a complex variable $\zeta$. $I$ will be a particular value of an analytic function $\Phi(\zeta)$, and we shall be able to estimate its modulus by use of the maximum principle.

For any complex $\zeta$, define

$$F(\zeta) = |f|^{\frac{a(\zeta)}{a}} \frac{f}{|f|}$$

$$G(\zeta) = |g|^{\frac{a(\zeta)'}{a'}} \frac{g}{|g|}$$

where $a(\zeta) = (1-\zeta)a_1 + \zeta a_2$ and $a(\zeta)' = 1 - a(\zeta)$. Observe that $\zeta$ enters as a parameter: $F(\zeta)$ and $G(\zeta)$ are functions of $z$. We agree that $F(\zeta) = 0$, $G(\zeta) = 0$ whenever $f = 0$, $g = 0$. Note that $F(t) = f$, $G(t) = g$. We set

$$\phi(\zeta) = TF(\zeta) \cdot G(\zeta)\, dx\, dy \quad .$$

$F(\zeta)$ is itself a simple function $\Sigma F_i \chi_i$ which makes $TF(\zeta) = \Sigma F_i T\chi_i$. Similarly $G(\zeta) = \Sigma G_j \chi_j^*$, say, and we find

$$\Phi(\zeta) = \sum F_i G_j \iint T \chi_i \cdot \chi_j^* \, dx\, dy \quad .$$

We see at once that it is an exponential polynomial: $\Phi(\zeta) = \Sigma a_i e^{\lambda_i \zeta}$ with *real* $\lambda_i$. Therefore $\Phi(\zeta)$ is bounded if $\xi = \operatorname{Re} \zeta$ stays bounded.

Consider now the special cases $\xi = 0$ and $\xi = 1$. For $\xi = 0$ we have $\operatorname{Re} a(\xi) = a_1$ and hence

$$|F(\zeta)| = |f|^{a_1/a}$$

$$|G(\zeta)| = |g|^{a_1'/a'} \quad .$$

It follows that

$$\|F(\zeta)\|_{1/a_1} = (\|f\|_{1/a})^{a_1/a}$$

$$\|G(\zeta)\|_{1/a_1'} = (\|g\|_{1/a'})^{a_1'/a'} = 1 \quad .$$

For simplicity we may assume that $\|f\|_{1/a} = 1$ (this is merely a normalization).

We now get

$$|\phi(\zeta)| \leq \|TF(\zeta)\|_{1/a_1} \|G(\zeta)\|_{1/a_1'} \leq C_1 .$$

A symmetric reasoning shows that

$$|\phi(\zeta)| \leq C_2$$

on $\xi = 1$. We now conclude that

$$\log|\phi(\zeta)| - (1-\xi)\log C_1 - \xi \log C_2 \leq 0$$

on the boundary of the strip $0 \leq \xi \leq 1$. Since the lefthand number is subharmonic the inequality holds in the whole strip, and for $\zeta = t$ we obtain the desired result.

# CHAPTER VI

## *TEICHMÜLLER SPACES*

### A. *Preliminaries*

Let $S$ be a Riemann surface whose universal covering $\tilde{S}$ is conformally isomorphic to the upper halfplane $H$. The cover transformations of $\tilde{S}$ over $S$ are represented by linear transformations of $H$ upon itself which form a discontinuous subgroup $\Gamma$ of the group $\Omega$ of all such transformations. We can write

$$S = \Gamma \backslash H \quad \text{(orbits)}$$

and the canonical mapping

$$\pi : H \to \Gamma \backslash H$$

is a complex analytic projection of $H$ on $S$.

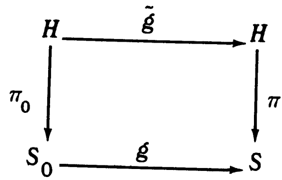
Conjugate subgroups represent conformally equivalent Riemann surfaces. Indeed, if $\Gamma_0 = B_0 \Gamma B_0^{-1}$, $B_0 \in \Omega$ then $z \to B_0 z$ maps orbits of $\Gamma$ on orbits of $\Gamma_0$ (for $B_0 Az = (B_0 AB_0^{-1}) B_0 z$ ). Therefore $B_0$ determines a one-one conformal mapping of $S_0 = \Gamma_0 \backslash H$ on $S$.

Conversely, let there be given a topological mapping

$$g : S_0 \to S \ .$$

It can be lifted to a topological mapping $\tilde{g} : \tilde{S}_0 \to \tilde{S}$ which obviously satisfies

$$\pi \circ \tilde{g} = g \circ \pi_0 \quad .$$




If $g$ is conformal, so is $\tilde{g}$, and we have $\tilde{g} = B_0 \epsilon \Omega$ and $\Gamma_0 = B_0 \Gamma B_0^{-1}$.

The classes of conformally equivalent Riemann surfaces correspond to classes of conjugate discontinuous subgroups of $\Omega$ (without elliptic fixpoints).

But even if $g$ is not conformal, it is still true that

$$A = \tilde{g} \circ A_0 \circ \tilde{g}^{-1} \epsilon \Gamma$$

whenever $A_0 \epsilon \Gamma_0$, for

$$\begin{aligned} \pi \circ A &= \pi \circ \tilde{g} \circ A_0 \circ \tilde{g}^{-1} \\ &= g \circ \pi_0 \circ A_0 \circ \tilde{g}^{-1} \\ &= g \circ \pi_0 \circ \tilde{g}^{-1} \\ &= \pi \ . \end{aligned}$$

In other words, $\tilde{g}$ defines an isomorphism $\theta$ such that

$$A_0^{\theta} = \tilde{g} \circ A_0 \circ \tilde{g}^{-1} \quad .$$

It is not quite unique, for we may replace $\tilde{g}$ by $B \circ \tilde{g} \circ B_0$ where $B \epsilon \Gamma$, $B_0 \epsilon \Gamma_0$. This changes $\theta$ into $\theta'$ with

$$A_0^{\theta'} = B \circ \tilde{g} \circ (B_0 A_0 B_0^{-1}) \circ \tilde{g} \circ B^{-1}$$

which means that we compose $\theta$ with inner automorphisms of $\Gamma_0$ and $\Gamma$. We say that $\theta$ and $\theta'$ are equivalent isomorphisms.

LEMMA. *$g_1$ and $g_2$ determine equivalent isomorphisms $\theta_1$ and $\theta_2$ if and only if they are homotopic.*

*Proof.* If $g_1$, $g_2$ are homotopic they can be deformed into each other via $g(t)$, say, which depends continuously on $t$. We can then find $\tilde{g}(t)$ so that it varies continuously with $t$, and since

$$A_0^{\theta(t)}(z) = \tilde{g}(t) \circ A_0 \circ \tilde{g}(t)^{-1}$$

has values in a discrete set it must actually be constant.

Conversely, suppose $g_1$, $g_2$ determine equivalent $\theta_1$, $\theta_2$. By changing $\tilde{g}_1$, $\tilde{g}_2$ we may suppose that $\theta_1 = \theta_2$, and hence

$$\tilde{g}_2^{-1} \tilde{g}_1 A_0 = A_0 \tilde{g}_2^{-1} \tilde{g}_1 .$$

Define $\tilde{g}(t, z)$ as the point which divides the noneuclidean line segment between $\tilde{g}_1(t, z)$ and $\tilde{g}_2(t, z)$ in the ratio $t: (1-t)$.

Because

$$\begin{aligned} \tilde{g}_1(Az) &= A\tilde{g}_1(z) \\ \tilde{g}_2(Az) &= A\tilde{g}_2(z) \end{aligned} \qquad (A = A_0^{\theta})$$

it follows that

$$\tilde{g}(t, Az) = A\tilde{g}(t, z) .$$

Hence $g(t) = \pi \circ \tilde{g}(t) \circ \pi_0^{-1}$ is a mapping from $S_0$ to $S$, and we have shown that $g_1$ and $g_2$ are homotopic.

*Definition of* $T(S_0)$ *(the Teichmüller space):*

Consider all pairs $(S, f)$ where $S$ is a Riemann surface and $f$ is a sense-preserving q.c. mapping of $S_0$ onto $S$. We say that $(S_1, f_1) \sim (S_2, f_2)$ if $f_2 \circ f_1^{-1}$ is homotopic to a conformal mapping of $S_1$ on $S_2$. The equivalence classes are the *points* of $T(S_0)$, and $(S_0, I)$ is called the initial point of $T(S_0)$.

Every $f$ determines a q.c. mapping $\tilde{f}$ of $H$ on itself, and thereby an isomorphism $\theta$ of $\Gamma_0$. Two isomorphisms correspond to the same Teichmüller point if and only if they differ by an inner automorphism of $\Omega$.

The space $T(S_0)$ has a natural Teichmüller metric: the distance of $(S_1, f_1)$, $(S_2, f_2)$ is $\log K$ where $K$ is the smallest maximal dilatation of a mapping homotopic to $f_2 \circ f_1^{-1}$.

Let us compare $T(S_0)$ and $T(S_1)$. Let $g$ be a q.c. mapping of $S_0$ on $S_1$. The mapping

$$(S, f) \to (S, f \circ g)$$

induces a mapping of $T(S_1)$ onto $T(S_0)$. Indeed, if $(S, f) \sim (S', f')$, then $(S, f \circ g) \sim (S', f' \circ g)$. This mapping is clearly isometric.

## B. *Beltrami Differentials.*

A q.c. mapping $f: S_0 \to S$ induces a mapping $\tilde{f}$ of $H$ on itself which satisfies

$$\tilde{f} \circ A_0 = A \circ \tilde{f} \tag{1}$$

for $A = A_0^{\theta}$. Conversely, if $\tilde{f}$ satisfies (1) it induces a mapping $f$.*

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* For typographical simplicity both mappings will henceforth be denoted by $f$.

From (1) we obtain

$$(A' \circ f) f_z = (f_z \circ A_0) A_0'$$
$$(A' \circ f) f_{\bar{z}} = (f_{\bar{z}} \circ A) \overline{A_0'}$$

and thus the complex dilatation $\mu_f$ satisfies

$$\mu_f = (\mu_f \circ A_0) \overline{A_0'} / A_0'$$

or

(2) $$\mu(A_0 z) = \mu(z) A_0'(z) / \overline{A_0'(z)} \quad .$$

A measurable and essentially bounded function $\mu$ which satisfies (2) for all $A_0 \in \Gamma_0$ is called a *Beltrami differential* with respect to $\Gamma_0$. Another way to express the condition is to say that

$$\mu(z) \frac{d\bar{z}}{dz}$$

is invariant under $\Gamma_0$.

Conversely, if $\mu_f$ does satisfy (2), then

$$\mu_{f \circ A_0} = \mu_f$$

and it follows that $f \circ A_0$ is an analytic function of $f$, or that

$$A = f \circ A_0 \circ f^{-1}$$

is analytic, and hence a linear transformation.

The linear space of Beltrami differentials will be denoted by $B(\Gamma_0)$ and its open unit ball with respect to the $L^\infty$ norm is denoted by $B_1(\Gamma_0)$ .

For every $\mu \in B_1(\Gamma_0)$ we know that there exists a corresponding $f^\mu$ which maps $H$ on itself. We normalize it so that it leaves $0, 1, \infty$ fixed. It is then unique.

We set

$$A^\mu = f^\mu \circ A_0 \circ (f^\mu)^{-1}$$

and write $\Gamma^{\mu}$ for the corresponding group, $\theta^{\mu}$ for the isomorphism. Since $\theta^{\mu}$ represents a point in Teichmüller space we have actually defined a mapping

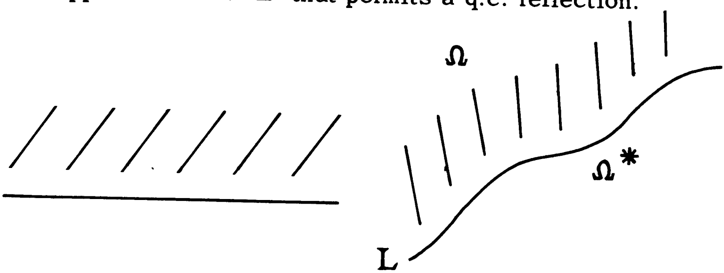
$$B_1(\Gamma_0) \rightarrow T(S_0) \quad .$$

It is clearly continuous from $L^{\infty}$ to the Teichmüller metric.

There is an obvious equivalence relation: $\mu_1 \sim \mu_2$ if $\theta^{\mu_1}$ and $\theta^{\mu_2}$ are equivalent isomorphisms. It is very difficult to recognize this equivalence by direct comparison of $\mu_1$ and $\mu_2$. Because we cannot solve the global problem we shall be content to solve the local problem for infinitesimal deformations.

Before embarking on this road we shall discuss a different approach which has several advantages. The mapping $f^{\mu}$ was obtained by extending $\mu$ to the lower halfplane by symmetry. If instead we extend $\mu$ to be identically zero in the lower halfplane we get a new mapping that we shall call $f_{\mu}$ (again normalized by fixpoints at 0, 1, $\infty$).

Clearly, $f_{\mu}$ gives a q.c. mapping of the upper halfplane and a conformal mapping of the lower halfplane. The real axis is mapped on a line $L$ that permits a q.c. reflection.



It is again true that

$$A_\mu = f_\mu \circ A_0 \circ f_\mu^{-1}$$

is conformal, and hence a linear transformation (it is conformal in $\Omega$ and $\Omega^*$, and it is q.c., hence conformal). We get a new group $\Gamma_\mu$ which is discontinuous on $\Omega \cup \Omega^*$. We call it a Fuchsoid group. We also get a surface $\Gamma_\mu \backslash \Omega = S$ and a q.c. mapping $S_0 \to S$ as well as a conformal mapping $\overline{S_0} \to \Gamma_\mu \backslash \Omega^*$ where $\overline{S_0}$ has the conjugate complex structure of $S_0$.

Observe that $f^\mu$ and $f_\mu$ are defined for *all* $\mu \in L^\infty$ with $\|\mu\|_\infty < 1$ $(\mu \in B_1)$ even when the group $\Gamma_0$ reduces to the trivial group. Our first result is

LEMMA 1. $f^\mu = f^\nu$ on the real axis if and only if $f_\mu = f_\nu$ on the real axis, and hence in $H^*$.

*Proof* 1). If $f_\mu = f_\nu$ on the real axis if and only if $f_\mu(H)$ and $f_\nu(H)$ are the same and therefore

$$f_\mu \circ (f^\mu)^{-1} = f_\nu \circ (f^\nu)^{-1} ,$$

for both are normalized conformal mappings of $H$ on the same region.

2) Suppose $f^\mu = f^\nu$ on the real axis. The mapping $h = (f^\nu)^{-1} \circ f^\mu$ reduces to the identity on the real axis, so $h$ can be extended to a q.c. mapping of the whole plane by putting $h(z) = z$ in $H^*$. Consider the q.c. mapping $A = f_\nu \circ h \circ (f_\mu)^{-1}$. In $f_\mu(H^*)$, $A = f_\nu \circ (f_\mu)^{-1}$ is conformal. In $f_\mu(H)$,

$$A = f_\nu \circ (f^\nu)^{-1} \circ f^\mu \circ (f_\mu)^{-1}$$

is conformal. Thus $A$ is a linear transformation, and the normalization makes it the identity. This means $f_\nu = f_\mu$ in $H^*$.

q.e.d.

We shall now make the assumption that $\Gamma_0$ is *of the first kind,* which means that it is not discontinuous at any point on the real axis. It is then known that the orbits on the real axis are dense. In particular the fixpoints are dense. Under these conditions we can prove

LEMMA 2. *$\mu_1$ and $\mu_2$ in $B(\Gamma_0)$ determine the same Teichmüller point if and only if $f^{\mu_1} = f^{\mu_2}$ on the real axis.*

*Proof.* If $f^{\mu_1} = f^{\mu_2}$ on the real axis, then $A^{\mu_1} = A^{\mu_2}$ on the real axis and hence identically. Therefore $\theta^{\mu_1} = \theta^{\mu_2}$, and $\mu_1$ and $\mu_2$ determine the same Teichmüller point.

Conversely, suppose $\theta^{\mu_1}$ and $\theta^{\mu_2}$ are equivalent isomorphisms. This means there is a linear transformation $S \in \Omega$ such that

$$A^{\mu_2} \circ S = S \circ A^{\mu_1} \quad \text{for all } A \text{ in } \Gamma_0 .$$

We conclude that $S$ maps the fixpoints of $A^{\mu_1}$ on the fixpoints of $A^{\mu_2}$ (attractive on attractive). Since they correspond to each other we see that

$$S \circ f^{\mu_1} = f^{\mu_2}$$

on the real axis. By the normalization, $S$ must be the identity.

q.e.d.

COROLLARY. *If $\Gamma_0$ is of the first kind, $\mu_1$ and $\mu_2$ determine the same Teichmüller point if and only if*

$$f_{\mu_1} = f_{\mu_2} \quad \textit{in} \quad H^*.$$

This means we may *identify* the Teichmüller space $T(S_0)$ with the space of conformal mappings of $H^*$ of the form $f_\mu$, $\mu \in B_1(\Gamma_0)$.

An even better characterization is by consideration of the Schwarzian derivative

$$\{f_\mu, z\} = \frac{f_\mu'''}{f_\mu'} - \frac{3}{2}\left(\frac{f_\mu''}{f_\mu'}\right)^2 .$$

Let us recall its properties with respect to composition. Consider

$$F(z) = f(\zeta(z))$$

and let primes mean differentiation. We get

$$F'(z) = f'(\zeta)\zeta'(z) ,$$

$$\frac{F''}{F'} = \frac{f''(\zeta)}{f'(\zeta)}\zeta' + \frac{\zeta''}{\zeta'} ,$$

$$\frac{F'''}{F'} - \left(\frac{F''}{F'}\right)^2 = \left(\frac{f'''(\zeta)}{f'(\zeta)} - \left(\frac{f''(\zeta)}{f'(\zeta)}\right)^2\right)\zeta'^2 + \frac{f''(\zeta)}{f'(\zeta)}\zeta'' + \frac{\zeta'''}{\zeta'} - \left(\frac{\zeta''}{\zeta'}\right)^2 ,$$

$$\{F, z\} = \{f, \zeta\}\zeta'(z)^2 + \{\zeta, z\} .$$

For a better formulation, let us denote the Schwarzian by $[f]$. The formula reads

$$[f \circ g] = ([f] \circ g)(g')^2 + [g] .$$

There are two special cases. For $f = A$, a linear transformation,

$$[A \circ g] = [g]$$

and for $g = A$

$$[f \circ A] = ([f] \circ A)(A')^2 .$$

On setting $\phi_\mu = [f_\mu]$,

$$(\phi_\mu \circ A)A'^2 = [f_\mu \circ A] = [A_\mu \circ f_\mu]$$
$$= [f_\mu] = \phi_\mu .$$

We see that $\phi_\mu$ satisfies

$$(\phi_\mu \circ A)(A')^2 = \phi_\mu$$

which makes it a *quadratic differential.* ($\phi\, dz^2$ is invariant).

The following theorem is due to Nehari:

LEMMA 3. *If $f$ is schlicht in the halfplane $H^*$, then* $|[f]| \leq \frac{3}{2} y^{-2}$ .

*Proof.* Suppose that $F(\zeta) = \zeta + \frac{b_1}{\zeta} + \frac{b_2}{\zeta^2} + \cdots$ is schlicht for $|\zeta| > 1$. The integral $\frac{1}{2i} \int_{|\zeta|=r} \bar{F}\, dF$ measures the area enclosed by the image of $|\zeta| = r$, and is therefore positive. One computes

$$\frac{1}{2i} \int_{|\zeta|=r} \bar{F}\, dF = \frac{1}{2i} \int (\bar{\zeta} + \frac{\bar{b}_1}{\bar{\zeta}} + \cdots)(1 - \frac{b_1}{\zeta^2} - \cdots)\, d\zeta$$
$$= \pi ( r^2 - \frac{|b_1|}{r^2} - \cdots ) .$$

It follows that $|b_1| \leq 1$. (More economically, $|b_1|^2 + 2|b_2|^2 + \cdots + n|b_n|^2 + \cdots \leq 1$, which is Bieberbach's Flächensatz.)

Note that

$$F' = 1 - \frac{b_1}{\zeta^2} + \cdots$$
$$F'' = \frac{2b_1}{\zeta^3} + \cdots$$
$$F''' = - \frac{6b_1}{\zeta^4} + \cdots$$

gives $[F] = -\dfrac{6b_1}{\zeta^4} + \cdots$ and hence

$$\lim_{\zeta \to \infty} |\zeta^4 [F]| \leq 6 \ .$$

Set $\zeta = Uz = (z-\bar{z}_0)/(z-z_0)$, $z_0 = x_0 + iy_0$, $y_0 < 0$. Consider $F(\zeta) = f(U^{-1}\zeta)$. Then

$$[f] = ([F] \circ U) U'^2 \ .$$

Here

$$U' = \frac{-2iy_0}{(z-z_0)^2} \ ,$$

$$U \sim \frac{2iy_0}{z-z_0} \qquad (z \to z_0)$$

$$U'^2 \sim -\frac{1}{4y_0^2} U^4 \qquad (z \to z_0) \ .$$

On going to the limit we find

$$[f](z_0) = -\frac{1}{4y_0^2} \lim [F] \cdot \zeta^4$$

and then

$$|[f]| \leq 3/2 \ \frac{1}{y^2} \ .$$

q.e.d.

In view of the lemma it is natural to define a norm on the quadratic differentials by

$$\|\phi\| = \sup |\phi(z)| \, y^2 \ .$$

C. $\Delta$ *is Open.*

We have defined a mapping $\mu \to \phi_\mu$ from the unit ball $B_1(\Gamma)$ to the space $Q(\Gamma)$ of quadratic differentials with finite norm. The image of $B_1(\Gamma)$ under this mapping will be denoted by $\Delta(\Gamma)$. It is our aim to show that $\Delta(\Gamma)$ is an open subset of $Q(\Gamma)$.

The question makes sense even in the case where $\Gamma$ is the trivial group consisting only of the identity. In this case the spaces will be denoted by $B_1$, $\Delta$, $Q$.

THEOREM. $\Delta$ *is an open subset of* $Q$.

Clearly, $\Delta$ consists of the Schwarzians $[f]$ of functions $f$ which are schlicht holomorphic in the lower halfplane and have a q.c. extension to the upper halfplane. We know already that all $\phi$ in $\Delta$ satisfy $\|\phi\| \leq 3/2$.

LEMMA 1. *Every holomorphic* $\phi$ *with* $\|\phi\| < 1/2$ *is in* $\Delta$.

*Proof.* We choose two linearly independent solutions of the differential equation

$$\eta'' = -\tfrac{1}{2}\phi\eta \tag{1}$$

which we may normalize by $\eta_1'\eta_2 - \eta_2'\eta_1 = 1$. It is easy to check that $f = \eta_1/\eta_2$ satisfies $[f] = \phi$. Observe that the solutions of (1) have at most simple zeros. Hence $f$ has at most simple poles, and at other points $f' \neq 0$.

We want to show that $f$ is schlicht and has a q.c. extension to the upper halfplane. To construct the extension we consider

$$F(z) = \frac{\eta_1(z) + (\bar{z} - z)\eta_1'(z)}{\eta_2(z) + (\bar{z} - z)\eta_2'(z)}, \qquad (z \in H^*) .$$

We remark first that the numerator and denominator do not vanish simultaneously (because $\eta_1' \eta_2 - \eta_2' \eta_1 = 1$). Therefore $F$ is defined everywhere, but could be $\infty$.

Simple computations yield

$$(2) \qquad F_{\bar{z}} = \frac{1}{(\eta_2 + (\bar{z} - z)\eta_2')^2}$$

$$F_z = \frac{\frac{1}{2}\phi(\bar{z}-z)^2}{(\eta_2 + (\bar{z} - z)\eta_2')^2}$$

and thus

$$\frac{F_z}{F_{\bar{z}}} = \tfrac{1}{2}\phi(\bar{z}-z)^2$$

The assumption implies $|F_z| \leq k|F_{\bar{z}}|$ for some $k < 1$. Hence $F$ is q.c. but sense-reversing.

The extension will be defined by

$$(3) \qquad \hat{f}(z) = \begin{cases} f(z) & z \in H^* \\ F(\bar{z}) & z \in H \end{cases}$$

It must be shown that $\hat{f}$ is a q.c. 1-1 mapping.

This is easy if $\phi$ is very regular. Let us assume, specifically, that $\phi$ remains analytic on the real axis, and that it has a zero of at least order 4 at $\infty$. It is immediate that $f$ and $F$ agree on the real axis, and it is also evident that $\hat{f}$ is locally schlicht. The assumption at $\infty$ means that there are solutions of (1) whose power series expansions at $\infty$ begin with 1 and $z$ respectively. We have thus

$$\eta_1 = a_1 z + b_1 + O\left(\frac{1}{|z|}\right)$$

$$\eta_2 = a_2 z + b_2 + O\left(\frac{1}{|z|}\right)$$

with $a_1 b_2 - a_2 b_1 = 1$. This gives

$$F(z) = \frac{a_1\bar{z} + b_1 + O(|z|^{-1})}{a_2\bar{z} + b_2 + O(|z|^{-1})} \longrightarrow \frac{a_1}{a_2}$$

which is also the limit of $f$.

Now the schlichtness of $\hat{f}$ follows by the monodromy theorem. We may of course, compose $\hat{f}$ with a linear transformation to make it normalized.

To prove the general case we use approximation. Put $S_n z = (2nz-i)/(iz+2n)$. Then $S_n H^* \subset\subset H^*$ and $S_n z \to z$ for $n \to \infty$. Set $\phi_n(z) = \phi(S_n z) S_n'(z)^2$. We have

$$y^2|\phi_n(z)| = |\phi(S_n z)|\,|S_n'(z)|^2 y^2 < |\phi(S_n z)|(\operatorname{Im} S_n(z))^2$$

and hence $\|\phi_n\| \leq \|\phi\|$. Now $\phi_n$ has all the regularity properties. We can find mappings $\hat{f}_n$ with $[\hat{f}_n] = \phi_n$ in $H^*$ and uniformly bounded dilatations. By compactness there exists a subsequence which converges to a solution $\hat{f}_n$ of the original problem.

If $\phi_n \to \phi$ it is not hard to see that a normalized solution of $\eta'' = -\frac{1}{2}\phi_n\eta$ converges to a normalized solution of $\eta'' = -\frac{1}{2}\phi\eta$. Therefore, if we choose the same normalizations we may conclude that $\hat{f}_0 = \hat{f}$ in $H$ and in $H^*$. Hence $\hat{f}$ can be extended by continuity to the real axis and is a solution of the problem with

$$\mu = -2\bar{\phi}y^2 .$$

But if $\phi \in Q(\Gamma)$ one verifies that $\mu \in B(\Gamma)$, and we conclude

LEMMA 2. *The origin of* $Q(\Gamma)$ *is an interior point of* $\Delta(\Gamma)$.

Suppose now that $\phi_0 \in \Delta$ and $[f_0] = \phi_0$ where $f_0 = f_{\mu_0}$. We assume that $f_0$ maps $H$ on $\Omega$, $H^*$ on $\Omega^*$. Then the boundary curve $L$ of $\Omega$ admits a q.c. reflection $\lambda$. According to Lemma 3 of Chapter IV D, we can choose $\lambda$ so that corresponding euclidean lengths have bounded ratio. This means that $\lambda$ is $C(K)$-q.c. and

$$C(K)^{-1} \leq |\lambda_{\bar{z}}| \leq C(K)$$

provided that $f_0$ is $K$-q.c.

If $[f] = \phi$ the composition rule for Schwarzians gives

$$\phi - \phi_0 = \{f, f_0\} f_0'^2 \quad .$$

The noneuclidean metric in $\Omega^*$ is such that

$$\rho(\zeta)|d\zeta| = \frac{|dz|}{-y} \quad .$$

Therefore, $\|\phi - \phi_0\| \leq \epsilon$ shows that $g = f \circ f_0^{-1}$ satisfies

$$|[g](\zeta)| \leq \epsilon \rho(\zeta)^2 \quad .$$

For sufficiently small $\epsilon$ we have to show that $g$ has a q.c. extension.

We set $\psi = [g]$ and determine normalized solutions $\eta_1$, $\eta_2$ of

$$\eta'' = -\tfrac{1}{2}\psi\eta \quad .$$

This time we construct

$$g(\zeta) = \eta_1(\zeta)/\eta_2(\zeta), \qquad \zeta \in \Omega^*$$

$$\hat{g}(\zeta) = \frac{\eta_1(\zeta^*) + (\zeta-\zeta^*)\eta_1'(\zeta^*)}{\eta_2(\zeta^*) + (\zeta-\zeta^*)\eta_2'(\zeta^*)}, \qquad \zeta \in \Omega .$$

(Recall that $\zeta^* = \lambda(\zeta)$.) Computation gives

$$\mu_{\hat{g}}(\zeta) = \frac{\frac{1}{2}(\zeta-\zeta^*)^2 \psi(\zeta^*)\lambda_{\bar{\zeta}}(\zeta)}{1+\frac{1}{2}(\zeta-\zeta^*)^2 \psi(\zeta^*)\lambda_{\zeta}(\zeta)}, \qquad \zeta \in \Omega .$$

But $|\lambda_\zeta| < |\lambda_{\bar{\zeta}}| \le C(K)$ and $|\zeta-\zeta^*| < C\rho(\zeta^*)^{-1}$. Hence

(4) $$|\mu_{\hat{g}}| \le \frac{\epsilon \cdot C(K)}{1-\epsilon \cdot C(K)} < 1$$

as soon as $\epsilon$ is small enough.

It must again be shown that $\hat{g}$ is continuous and schlicht. There is no difficulty if $L$ is analytic and $\psi$ is analytic on $L$ with a zero of order four at $\infty$.

The general case again requires an approximation argument. Let $f_n = f_0 \circ S_n$, where $S_n$ is as in the proof of Lemma 1, and let $L_n$ be the image of the real axis under $f_n$. $L_n$ is an analytic curve that admits a $K$-q.c. reflection, and $\psi$ is analytic on $L_n$.

The Poincaré density $\rho_n$ of $\Omega_n^* = f_n(H^*)$ is $\ge \rho$, so that $|\psi| \le \epsilon\rho^2$ implies $|\psi| \le \epsilon\rho_n^2$. Therefore we can construct a sequence of normalized q.c. mappings $\hat{g}_n$ such that $[\hat{g}_n] = \psi$ in $\Omega_n^*$ and $\mu_{\hat{g}_n}$ satisfies the inequality (4). A subsequence of the $\hat{g}_n$ tends to a q.c. limit $\hat{g}$ which is equal to $g$ in $\Omega^*$. This proves Theorem 1. Since $\mu_{\hat{g}}$ satisfies (4) we conclude:

COROLLARY. *For every sequence of* $\phi_n \in \Delta$ *converging to* $\phi_0 = [f_{\mu_0}] \in \Delta$, *there exist* $\mu_n \to \mu_0$ *such that* $[f_{\mu_n}] = \phi_n$.

The proof is by writing $\phi = [\hat{g} \circ f_0]$.

We come now to the most delicate part:

THEOREM 2. $\Delta(\Gamma)$ *is an open subset of* $Q(\Gamma)$.

*Remark:* This was first proved by Bers. The idea of the proof that follows is due to Clifford Earle.

Given any $\mu_0 \in B$, we construct a mapping

$$\beta_0 : \Delta \to \Delta$$

as follows: Given $\phi \in \Delta$ there exists a $\mu \in B_1$ such that $\phi = \phi_\mu$. With this $\mu$ we determine $\lambda$ by

$$f^\lambda = f^\mu \circ (f^{\mu_0})^{-1} \tag{5}$$

and set $\beta_0(\phi) = \phi_\lambda$. It is unique, for if $\phi_\mu = \phi_{\mu_1}$, then $f^\mu$ has the same boundary values as $f^{\mu_1}$. Hence $f^\lambda$ has the same boundary values as $f^{\lambda_1}$, and hence $\phi_\lambda = \phi_{\lambda_1}$.

It is evident that $\beta_0$ is 1-1, and it carries $\phi_{\mu_0}$ into zero. Moreover, $\beta_0$ is continuous. For if $\phi_n \to \phi = [f_\mu]$ we have just proved the existence of $\mu_n \to \mu$ such that $\phi_n = [f_{\mu_n}]$. The corresponding $\phi_{\lambda_n}$ converge to $\phi_\lambda$.

LEMMA. (Earle). $\phi_\mu \in Q(\Gamma)$ *if and only if for every* $A \in \Gamma$ *there exists a linear transformation* $B$ *such that*

$$f^\mu \circ A \circ (f^\mu)^{-1} = B \quad \text{on} \quad \mathsf{R}.$$

*Proof.* $\phi_\mu \in Q(\Gamma)$ is equivalent to $[f_\mu \circ A] = [f_\mu]$, and this is true if and only if $f_\mu \circ A \circ f_\mu^{-1} = C$, a linear transformation, in $\Omega_\mu^*$.

1) If $B$ exists, then

$$C = f_\mu \circ A \circ f_\mu^{-1} = f_\mu \circ (f^\mu)^{-1} \circ B \circ f^\mu \circ (f_\mu)^{-1}$$

on $f^\mu(\mathsf{R})$. The first expression is holomorphic in $\Omega_\mu^*$, the second in $\Omega_\mu$. Hence $C$ is a linear transformation.*

2) If $C$ exists, then

$$B = f^\mu \circ A \circ (f^\mu)^{-1} = f^\mu \circ f_\mu^{-1} \circ C \circ f_\mu \circ (f^\mu)^{-1}$$

on $\mathsf{R}$. But the last expression is a conformal mapping of $H$ on itself, hence a linear transformation. The lemma is proved.

Assume now that $\mu_0 \in B_1(\Gamma)$. We find that $\beta_0$ maps $\Delta(\Gamma)$ on $\Delta(\Gamma^{\mu_0})$, for

$$f^\lambda \circ A^{\mu_0} \circ (f^\lambda)^{-1} = f^\mu \circ A \circ (f^\mu)^{-1} = A^\mu .$$

From the lemma we infer that $Q(\Gamma) \cap \Delta$ is mapped on $Q(\Gamma^{\mu_0}) \cap \Delta$. The origin has a neighborhood $N$ in $Q(\Gamma^{\mu_0})$ which is contained in $\Delta(\Gamma^{\mu_0})$. Write $N = Q(\Gamma^{\mu_0}) \cap N_0 = Q(\Gamma^{\mu_0}) \cap \Delta \cap N_0$ where $N_0$ is a neighborhood in $\Delta$. Then

$$\beta^{-1}(N) = Q(\Gamma) \cap \Delta \cap \beta^{-1}(N_0) = Q(\Gamma) \cap \beta^{-1}(N_0)$$

which is a neighborhood of $\phi_{\mu_0}$ in $Q(\Gamma)$. From $N \subset \Delta(\Gamma^{\mu_0})$

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* Because it is quasiconformal and conformal a.e.

we get $\beta^{-1}(N) \subset \Delta(\Gamma)$ and this proves that $\phi_{\mu_0}$ has a neighborhood in $Q(\Gamma)$ which is contained in $\Delta(\Gamma)$.

q.e.d.

*Conclusion:*

If the group $\Gamma$ of cover transformations of $H$ over $S_0$ is of the first kind, the Teichmüller space $T(S_0)$ is identified with $\Delta(\Gamma)$, an open subset of $Q(\Gamma)$. The Teichmüller metric defines the same topology as the norm in $Q(\Gamma)$.

Consider the case where $S_0$ is a compact Riemann surface of genus $g > 1$. Then $Q(\Gamma)$ has complex dimension $3g-3$. Choose a basis $\phi_1, \ldots, \phi_{3g-3}$. Every $\phi \in Q(\Gamma)$ is a linear combination

$$\phi = \tau_1 \phi_1 + \cdots + \tau_{3--3} \phi_{3g-3}$$

with complex coefficients. We find that $\Delta(\Gamma)$ can be identified with a bounded open set in $C^{3g-3}$.

We can show, moreover, that the parametrization by $\tau = (\tau_1, \ldots, \tau_{3g-3})$ defines the Riemann surfaces of genus $g$ as a *holomorphic family* of Riemann surfaces.

According to Kodaira and Spencer a holomorphic family may be described as follows:

There is given an $(n+1)$-dimensional complex manifold $V$ and a holomorphic mapping $\pi: V \to M$ on an $n$-dimensional complex manifold $M$. Each fiber $\pi^{-1}(\tau)$, $\tau \in M$ is a Riemann surface.

The complex structures of $V$ and $M$ are related in the following way: There is an open covering $\{U_a\}$ of $V$ so that for each $\alpha$ there is given a holomorphic homeomorphism $h_\alpha$: $U_\alpha \to C \times M$ connected by $\phi_{\alpha\beta} = h_\alpha \circ h_\beta^{-1}$ (for intersectint $U_\alpha$, $U_\beta$). For any $\tau \in M$ the restriction of $\phi_{\alpha\beta}$ to $h_\beta(U_\alpha \cap U_\beta \cap \pi^{-1}(\tau))$ shall be complex analytic (in fact, these functions determine the complex structure of $\pi^{-1}(\tau)$.)

In our case, $M$ will be the set $\Delta(\Gamma)$. Every $\tau \in \Delta(\Gamma)$ determines a $\phi = [f_\mu]$, and $f_\mu$ is uniquely defined in $H^*$. Hence $\Omega_\mu^*$, $\Omega_\mu$, and the group $\Gamma_\mu$ are determined by $\tau$. To emphasize the dependence on $\tau$ we change the notations to $\phi_\tau$, $\Omega_\tau$, $\Gamma_\tau$, etc. Since $f_\tau$ is determined in $H^*$ by the differential equation $[f_\tau] = \phi_\tau$, we know that $f_\tau$ depends holomorphically on the parameter $\tau$. For $A \in \Gamma$, the corresponding $A_\tau \in \Gamma_\tau$ is determined by the condition $f_\tau \circ A = A_\tau \circ f_\tau$ in $H^*$. This means that $A_\tau$ depends holomorphically on $\tau$, a fact that will be important.

The Riemann surfaces in our family will be $S(\tau) = \Omega_\tau/\Gamma_\tau$ and $V$ will be their union. Thus the points of $V$ are orbits $\Gamma_\tau\zeta$ with $\zeta \in \Omega_\tau$ and $\tau \in \Delta(\Gamma)$. The projection $\pi$: $V \to M$ is defined so that $\pi^{-1}(\tau) = S(\tau)$.

Consider a point $\Gamma_{\tau_0}\zeta_0$ in $V$. We pick a fixed $\zeta_0$ from the orbit and determine an open neighborhood $N(\zeta_0)$ such that $\bar{N}$ is compact, contained in $\Omega_{\tau_0}$, and does not meet its images under $\Gamma_{\tau_0}$. The neighborhood $N(\varepsilon\ \zeta_0, \tau_0)$ of $\Gamma_{\tau_0}\zeta_0$ will consist of all $\Gamma_\tau\zeta$ such that $\|\phi_\tau - \phi_{\tau_0}\| < \varepsilon$ and $\zeta \in N(\zeta_0)$. Here $\varepsilon$ shall be so small that $\bar{N}$ does not meet its images under $\Gamma_\tau$. This is possible because $A_\tau$ is near $A_{\tau_0}$ when $\phi_\tau$ is near $\phi_{\tau_0}$. As a consequence, there is only one $\zeta \in N(\zeta_0) \cap \Gamma_\tau\zeta$ and the mapping $h: \Gamma_\tau\zeta \to (\zeta, \tau)$ is well-defined in $N(\varepsilon, \zeta_0, \tau_0)$.

The neighborhoods $N(\varepsilon, \zeta_0, \tau_0)$ shall be a base for the topology of $V$. We assert, in addition, that the parameter mappings $h: \Gamma_\tau\zeta \to (\zeta, \tau)$ defined on the basic neighborhoods make $\pi: V \to M$ into a holomorphic family. Indeed, if two basic neighborhoods $U_0$ and $U_1$ intersect, then in $U_0 \cap U_1$, $h_0(\Gamma_\tau\zeta) = (\zeta, \tau)$ and $h_1(\Gamma_\tau\zeta) = (A_\tau\zeta, \tau)$ for some $A_\tau \in \Gamma_\tau$. Hence the mapping $h_1 \circ h_0^{-1}$ is given by $(\zeta, \tau) \to (A_\tau\zeta, \tau)$, which we know is holomorphic in both $\tau$ and $\zeta$. It is evident that the parameter mappings define a complex structure on $V$ which agrees with the conformal structure of the surfaces $S(\tau)$ and makes the mapping $\pi: V \to M$ holomorphic.

D. *The Infinitesimal Approach.*

We continue with a direct investigation of $f^\mu$, $A^\mu$ that does not make use of the mappings $f_\mu$.

For any function $F(\mu)$ and any $\gamma \in L^\infty$ we set

$$\lim_{t\to 0} \frac{F(\mu + t\gamma) - F(\mu)}{t} = \dot{F}(\mu)[\nu]$$

when the limit exists, and we omit the argument $\mu$ when the derivative is taken for $\mu = 0$. We are assuming that $t$ is real.

We have already derived the representation

$$\dot{f}[\nu](\zeta) = -\frac{1}{\pi}\iint \nu(z) R(z, \zeta)\, dx\, dy$$

where

$$R(z, \zeta) = \frac{1}{z-\zeta} - \frac{1-\zeta}{z} - \frac{\zeta}{z-1} .$$

We apply the formula to the symmetric case: $\nu(\overline{z}) = \overline{\nu}(z)$, and we write more explicitly

(1) $$\dot{f}[\nu](\zeta) = -\frac{1}{\pi}\iint_H \nu(z) R(z, \zeta)\, dx\, dy - \frac{1}{\pi}\iint_H \overline{\nu}(z) R(\overline{z}, \zeta)\, dx\, dy .$$

It is evident that $\dot{f}[\nu]$ is linear in the real sense, but not in the complex sense. To obtain a complex linear functional we form

(2) $$\Phi[\nu] = \dot{f}[\nu] + i\dot{f}[i\nu]$$

and we find

(3) $$\Phi[\nu](\zeta) = -\frac{2}{\pi}\iint_H \overline{\nu}(z) R(\overline{z}, \zeta)\, dx\, dy .$$

It is holomorphic for $\zeta \in H$. Its third derivative is

(4) $$\Phi'''(\zeta) = \phi[\nu](\zeta) = -\frac{12}{\pi}\iint_H \frac{\overline{\nu}(z)}{(\overline{z}-\zeta)^4}\, dx\, dy .$$

If $\nu \in B(\Gamma)$ one verifies that $\phi$ is a quadratic differential $(\phi \in Q(\Gamma))$.

From

$$f^{\mu}(Az) = A^{\mu} f^{\mu}(z)$$

with $\mu = t\nu$ we obtain after differentiation

(5) $$\dot{f}[\nu] \circ A = \dot{A}[\nu] + A' \dot{f}[\nu] .$$

(the existence of $\dot{A}$ requires some, but not much, proof). Since $\dot{f}[\nu]_{\bar{z}} = \nu$ differentiation of (5) gives

$$(\nu \circ A)\bar{A}' = \dot{A}_{\bar{z}} + A'\nu$$

and hence $\dot{A}_{\bar{z}} = 0$ because $\nu \in B(\Gamma)$. We conclude that the $\dot{A}$ are analytic functions. Moreover, $\dot{A}/A'$ is real on the real axis and can hence be extended by symmetry to the whole plane. The explicit formula for $\dot{f}[\nu]$ shows that it is $o(|z|^2)$ at $\infty$. The apparent singularity of $\dot{A}/A'$ at $A^{-1}\infty$ is therefore removable, and there is at most a double pole at $\infty$. We conclude that

$$\frac{\dot{A}}{A'} = P_A \tag{6}$$

where $P_A$ is a second degree polynomial. From

$$(A_1A_2)^{\bullet} = (\dot{A}_1 \circ A_2) + (A_1{}' \circ A_2)\dot{A}_2$$
$$(A_1A_2)' = (A_1{}' \circ A_2)A_2{}'$$

we deduce that

$$P_{A_1A_2} = \frac{P_{A_1} \circ A_2}{A_2{}'} + P_{A_2} \ . \tag{7}$$

We shall say that $\nu$ is *trivial*, and we write $\nu \in N(\Gamma)$, if all $\dot{A}[\nu] = 0$. There is a whole slew of equivalent conditions:

LEMMA 1. *The following conditions are all equivalent.*

a) $\dot{A}[\nu] = 0$ *for all* $A \in \Gamma$,
b) $P_A = 0$ *for all* $A \in \Gamma$,
c) $\dot{f}[\nu] = 0$ *on* $\mathsf{R}$,
d) $\Phi[\nu] \equiv 0$,
e) $\phi[\nu] \equiv 0$,
f) $\displaystyle\iint_{\Gamma\backslash H} \nu\phi \, dx\,dy = 0$ *for all* $\phi \in Q(\Gamma)$.*

*Proof.* a) $\Longleftrightarrow$ b) by (6). c) $\Longrightarrow$ a) by (5). Conversely, if $\dot{A} = 0$ then since $\dot{f}(0) = 0$ it follows that $\dot{f}(A0) = 0$ for all $A$. These points are dense on R (we are assuming that the group is of the first kind) and hence, by continuity, $\dot{f} = 0$ on R.

The definition of $\Phi$, together with (5) and (6), gives

$$\frac{\Phi \circ A}{A'} - \Phi = P_A[\nu] + i\,P_A[i\nu] \quad .$$

If $\Phi \equiv 0$, it follows that $P_A[\nu] + i\,P_A[i\nu] = 0$ on R. But both polynomials are real on R, hence identically zero, so d) $\Longrightarrow$ b). Conversely, if $\dot{f}[\nu] = 0$ on R, then $\Phi$ is purely imaginary on R and can be extended to be analytic in the whole plane. Since $\Phi = o(|z|^2)$ it is a first degree polynomial. But it vanishes at 0 and 1. Hence $\Phi \equiv 0$ and c) $\Longrightarrow$ d).

Since $\Phi''' = \phi$, d) $\Longrightarrow$ e). Conversely, if $\phi = 0$, $\Phi$ is a polynomial and we reason as above to conclude $\Phi \equiv 0$.

Condition f) is the most important one. We prove it only for the case that $\Gamma\backslash H$ is compact, and we represent it as a compact fundamental polygon $S$ with matched sides. If $\dot{A} = 0$, then $\dot{f} \circ A = A'\dot{f}$. We know further that $\dot{f}_{\bar{z}} = \nu$. By Stokes' formula we find

$$\iint_S \nu\phi\, dx\, dy = -\frac{1}{2i} \iint_S \dot{f}_{\bar{z}}\, \phi\, dz\, d\bar{z} = \frac{1}{2i} \int_{\partial S} \dot{f}\, \phi\, dz \quad .$$

* For $\Gamma\backslash H$ non-compact, the condition shall hold for all $\phi \in Q(\Gamma)$ such that $\iint_{\Gamma\backslash H} |\phi|\, dx\, dy < \infty$.

But the condition means that $\dot{f}/dz$ is invariant. Hence

$$\dot{f}\,\phi\,dz = \frac{\dot{f}}{dz}\cdot\phi\,dz^2$$

is invariant, and the boundary integral vanishes.

To prove the opposite, consider a real $\zeta$ so that (3) takes the form

$$\overline{\Phi(\zeta)} = -\frac{2}{\pi}\iint_H \nu(z)R(z,\zeta)\,dx\,dy \quad .$$

We introduce a Poincaré $\theta$-series

$$\psi(z) = \sum R(Az,\zeta)A'(z)^2 \quad .$$

From the fact that $\int_H |R(z,\zeta)|dx\,dy < \infty$ it is quite easy to deduce that the series converges. We obtain now

$$\overline{\Phi(\zeta)} = -\frac{2}{\pi}\sum\iint_{A(S)} \nu(z)R(z,\zeta)\,dx\,dy$$

$$= -\frac{2}{\pi}\sum\iint_S \nu(Az)R(Az,\zeta)|A'(z)|^2\,dx\,dy$$

$$= -\frac{2}{\pi}\sum\iint \nu(z)R(Az,\zeta)A'(z)^2\,dx\,dy$$

$$= -\frac{2}{\pi}\iint \nu\,\psi\,dx\,dy$$

which is zero if f) holds. Hence $\Phi$ is identically zero. The lemma is proved.

We need the following lemma.

LEMMA 2. *Suppose* $\phi$ *is analytic in* $H$ *and*

$$\sup|\phi|y^2 < \infty .$$

*Then*

$$\phi(\zeta) = \frac{12}{\pi}\iint_H \frac{\phi(z)\,y^2}{(\bar{z}-\zeta)^4}\,dx\,dy \quad . \tag{8}$$

For the proof we note that

$$\frac{y^2}{(\bar{z}-\zeta)^4} = -\tfrac{1}{4}\frac{(\bar{z}-z)^2}{(\bar{z}-\zeta)^4} = -\tfrac{1}{4}\left[\frac{1}{(\bar{z}-\zeta)^2} - \frac{2(z-\zeta)}{(\bar{z}-\zeta)^3} + \frac{(z-\zeta)^2}{(\bar{z}-\zeta)^4}\right]$$

$$= -\tfrac{1}{4}\frac{\partial}{\partial\bar{z}}\left[-\frac{1}{(\bar{z}-\zeta)} + \frac{z-\zeta}{(\bar{z}-\zeta)^2} - \tfrac{1}{3}\frac{(z-\zeta)^2}{(\bar{z}-\zeta)^3}\right] \quad .$$

Assume first that $\phi$ is still analytic on $\mathsf{R}$. Then integration by parts gives

$$\frac{12}{\pi}\iint_H \frac{\phi\,y^2\,dx\,dy}{(\bar{z}-\zeta)^4} = -\frac{3}{2\pi i}\int_{\mathsf{R}} (-\tfrac{1}{3})\,\frac{\phi(z)}{z-\zeta}\,dz = \phi(\zeta) \quad .$$

It is easy to complete the proof by applying this formula to $\phi(z+i\varepsilon)$, $\varepsilon > 0$, for the hypothesis guarantees absolute convergence.

Compare formula (8) with (4). We have defined an anti-linear mapping

$$\Lambda:\ \nu \to \phi[\nu]$$

from $B(\Gamma)$ to $Q(\Gamma)$. On the other hand we may define

$$\Lambda^*:\ \phi \to -\bar{\phi}\,y^2$$

which is a mapping from $Q(\Gamma)$ to $B(\Gamma)$. Lemma 2 tells us that $\Lambda\Lambda^*$ is the identity.

By Lemma 1, e), $\nu \in N(\Gamma)$ if and only if $\Lambda\nu = 0$. From $\Lambda\Lambda^* = I$ we conclude that

$$\nu - \Lambda^* \Lambda \nu \in N(\Gamma).$$

In other words, $\nu$ is equivalent $\mod N(\Gamma)$ to $-\bar{\phi}[\nu] y^2$. Of course this is the only $\Lambda^* \phi$ which is equivalent to $\nu$, for if $-\phi y^2 \in N(\Gamma)$ then $\iint_{\Gamma \backslash H} |\phi(z)|^2 y^2 \, dx\, dy = 0$, hence $\phi = 0$.

We conclude:

$\Lambda$ establishes an isomorphism of $B(\Gamma)/N(\Gamma)$ on $Q(\Gamma)$. The inverse isomorphism of $Q(\Gamma)$ on $B(\Gamma)/N(\Gamma)$ is given by $\Lambda^*$.

A compact surface $S$, of genus $g > Q$, determines a group $\Gamma$ generated by linear transformations $A_1, \ldots, A_{2g}$ which satisfy

(9) $$A_1 A_2 A_1^{-1} A_2^{-1} \cdots A_{2g-1} A_{2g} A_{2g-2}^{-1} A_{2g}^{-1} = I.$$

We say that $\{A_1, \ldots, A_{2g}\}$ is a canonical set. If it belongs to a surface, $A_1$ and $A_2$ have four distinct fixpoints. By passing to a conjugate subgroup we can make $A_1$ have fixpoints at $0$, $\infty$ and $A_2$ to have a fixpoint at 1. When this is so we say that the generating system is normalized.

Set

$V$ = set of normalized canonical systems,

$T$ = set of normalized canonical systems that come from a surface of genus $g$.

It can be shown that $V$ is a real analytic manifold of dimension $6g-6$. We are going to prove that $T$ is an open subset of $V$, and that it carries a natural complex structure.

Let $S$ determine $\Gamma$ with normalized generators $(A) = (A_1, \ldots, A_{2g})$. We choose a basis $\nu_1, \ldots, \nu_{3g-3}$ of $B(\Gamma)/N(\Gamma)$ and set

$$\nu(\tau) = \tau_1\nu_1 + \cdots + \tau_{3g-3}\,\nu_3$$

and $\tau_k = t_k + it_k'$. For small $\tau$ we obtain a system $(A)^{\nu(\tau)} \in T$. The points of the manifold $V$ near $(A)$ can be expressed by local parameters $u_1, \ldots, u_{6g-6}$, and the mapping $\tau \to (A)^{\nu(\tau)}$ takes the form

$$u_j = h_j(t_1, t_1', \ldots, t_{3g-3}, t_{3g-3}') \; .$$

We have to show

1) The $h_j$ are continuously differentiable,

2) The Jacobian is $\neq 0$ at $\tau = 0$.

1) has already been proved. The coefficients of the $A_k$ are differentiable functions of the $u_k$. Therefore, if all $\dot{u}_k[\nu]$ are $0$, so are all $\dot{A}_k[\nu]$ and hence all $\dot{A}[\nu]$. Observe that

$$\frac{\partial h_j}{\partial t_k} = \dot{u}_j[\nu_k] \;, \qquad \frac{\partial h_j}{\partial t_k'} = \dot{u}_j[i\nu_k]$$

If the Jacobian at the origin were zero there would exist real numbers $\xi_k, \eta_k$ such that

$$\Sigma\; \xi_k \frac{\partial h_j}{\partial t_k} + \eta_k \frac{\partial h_j}{\partial t_k'} = 0 \quad \text{all } j \; .$$

This would mean

$$\dot{u}_j\,[\Sigma\,(\xi_k + i\eta_k)\,\nu_k] = 0$$

and hence

$$\dot{A}[\,\Sigma\,(\xi_k + i\eta_k)\,\nu_k] = 0 \; .$$

Hence $\Sigma\,(\xi_k + i\eta_k)\nu_k \in N(\Gamma)$, and this is possible only if all $\xi_k$, $\eta_k = 0$. We have proved that the Jacobian does not vanish.

The proof shows that $T$ is an open subset of $V$. It also follows that if $\|\mu\|$ is small enough, then there exist unique complex numbers $\tau_1(\mu), \ldots, \tau_{3g-3}(\mu)$ such that

$$A^{\tau_1(\mu)\nu_1 + \cdots + \tau_{3g-3}(\mu)\nu_{3g-3}} = A^{\mu} .$$

Set $\mu = t_\rho$ and differentiate with respect to $t$ for $t = 0$. We obtain

$$\dot{A}[\dot{\tau}_1[\rho]\nu_1 + \cdots + \dot{\tau}_{3g-3}[\rho]\nu_{3g-3}] = \dot{A}[\rho] .$$

This implies

$$\dot{\tau}_1[\rho]\nu_1 + \cdots + \dot{\tau}_{3g-3}[\rho]\nu_{3g-3} - \rho \in N(\Gamma) .$$

Replace $\rho$ by $i\rho$. We can then eliminate the term $\rho$ to obtain

$$\sum_1^{3g-3} (\dot{\tau}_k[\rho] + i\,\dot{\tau}_k[i\rho])\nu_k \in N(\Gamma)$$

from which it follows that

$$\dot{\tau}_k[i\rho] = i\,\dot{\tau}_k[\rho] .$$

In other words, the $\dot{\tau}_k$ are complex linear functionals, which means that the $\tau_k(\mu)$ are differentiable in the complex sense at $\mu = 0$.

From this we can prove that the coordinate mappings $(A^{\mu}) \to (\tau_1(\mu), \ldots, \tau_{3g-3}(\mu))$ define a complex structure on $T$. Indeed, we must show that on overlapping neighborhoods the coordinates are analytic functions, and it suffices to show this at the origin of a given coordinate system. Take $\mu(\tau) = \Sigma\, \tau_i\nu_i$ and $\mu_0 = \mu(\tau_0)$ near zero in $B(\Gamma)$. Define $\lambda(\tau)$ in

$B(\Gamma^{\mu_0})$ by $f^{\mu(\tau)} = f^{\lambda(\tau)} \circ f^{\mu_0}$. By the formulas of section C in Chapter I, we have

$$\lambda(\tau) \circ f^{\mu_0} = \frac{\mu(\tau) - \mu_0}{1 - \bar{\mu}_0 \mu(\tau)} \left( f_z^{\mu_0} / |f_z^{\mu_0}| \right)^2 ,$$

so that $\lambda$ depends analytically on $\tau$.

Now choose a basis $\lambda_1, \ldots, \lambda_{3g-3}$ for $B(\Gamma^{\mu_0})/N(\Gamma^{\mu_0})$. Near $(A^{\mu_0})$ we have coordinate functions $\sigma_1(\lambda), \ldots, \sigma_{3g-3}(\lambda)$. This means that for $\tau$ near $\tau_0$ we can write uniquely

$$(A^{\mu(\tau)}) = \left( (A^{\mu_0})^{\lambda(\tau)} \right) = \left( (A^{\mu_0})^{\Sigma \sigma_i(\lambda(\tau)) \lambda_i} \right) .$$

Because $\sigma_i(\lambda)$ is complex analytic at $\lambda = 0$, we conclude that $\sigma_i$ is a complex analytic function of $\tau$ at $\tau_0$. This is exactly what we required.