MA1506 TUTORIAL 11

Question 1

King Xerxes I of Persia has sent a million soldiers to conquer Greece. King Leonidas I of Sparta decides to meet the Persians at Thermopylae. A typical Persian soldier can kill one Spartan per hour, whereas a typical Spartan soldier can kill 11,111,111.1 Persians per hour. Neither side suffers from any disease[1]. How many soldiers does King Leonidas need to take to Thermopylae if he wants to kill all of the Persians but expects no Spartans to go home alive?

Question 2

Show that $u(x, y) = F(y - 3x)$, where F is an arbitrary single variable function, is a solution of the partial differential equation

$$u_x + 3u_y = 0.$$

Find the particular solution which satisfies each of the following conditions *separately*:
(a) $u(0, y) = 4 \sin y$;
(b) $u(x, 0) = e^{x+1}$.

Ans: (a) $u(x, y) = 4 \sin(y - 3x)$; (b) $u(x, y) = e^{\left(-\frac{y}{3}+x+1\right)}$.

Question 3

Solve the following partial differential equations:
(a) $u_{xy} = u_x$ (b) $u_x = 2xyu$.

Ans: (a) $u(x, y) = c(x)e^y + h(y)$; (b) $u(x, y) = c(y)e^{x^2 y}$.

Question 4

Using the method of separation of variables, solve the following partial differential equations:
(a) $yu_x - xu_y = 0$;
(b) $u_x = yu_y, y > 0$;
(c) $u_{xy} = u$;
(d) $xu_{xy} + 2yu = 0, x > 0$.

Ans: (a) $u(x, y) = ke^{c(x^2+y^2)}$; (b) $u(x, y) = ky^c e^{cx}$; (c) $u(x, y) = ke^{cx+y/c}$; (d) $u(x, y) = kx^c e^{-y^2/c}$.

Question 5

Show carefully that the d'Alembert solution of the wave equation, given in lectures, does satisfy the equation and the boundary and initial conditions.

[1] They all die before they have time to get sick.

Question 6

In the lectures we discussed the wave equation on the spatial domain $[0,\pi]$. Show how to solve the wave equation with the same initial conditions and boundary conditions but now with a spatial domain [0,L], where L is any positive number.

Question 7

If a flexible string moves in a fluid that offers a frictional resistance proportional to the speed, then its motion is described by the *damped wave equation*,

$$y_{tt} = c^2 y_{xx} - b y_t,$$

where $b > 0$ is the frictional constant. We assume as usual that the ends of the string are fixed, that is, $y(t,0) = y(t,\pi) = 0$. Show how to separate the variables for this equation [that is, obtain the pair of ordinary differential equations defined by the damped wave equation].

Question 8

Another very important partial differential equation is the *Laplace equation*,

$$u_{xx} + u_{yy} = 0,$$

where x and y are the usual planar Cartesian coordinates. In the simplest case, this equation is studied on a square domain in the plane, say $0 \leq x \leq \pi$, $0 \leq y \leq \pi$. We need four boundary conditions [why?]. Let's take them, for example, to be $u(x,0) = u(0,y) = u(\pi,y) = 0$, and $u(x,\pi) = f(x)$ for some well-behaved function f(x). Separate the variables for the Laplace equation and show that the solution in this example is

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sin(nx) \sinh(ny),$$

where sinh(x) is the hyperbolic sine and where

$$c_n = \frac{\frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx}{\sinh(n\pi)}.$$