Matrix Analysis with Applications
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Lecture - 39
Non-negativity and Irreducible Matrices

Hello friends. So, welcome to the lecture on Non-negativity and Irreducible Matrices. So, in the last lecture we have discussed about positive matrices where we have seen some properties of positive matrices especially on about the spectral radius of such matrices. We have seen the definition of Perron value and Perron vector also in that lecture.

So, let us continue in the same direction from non positive matrices to non-negative matrices in this lecture.

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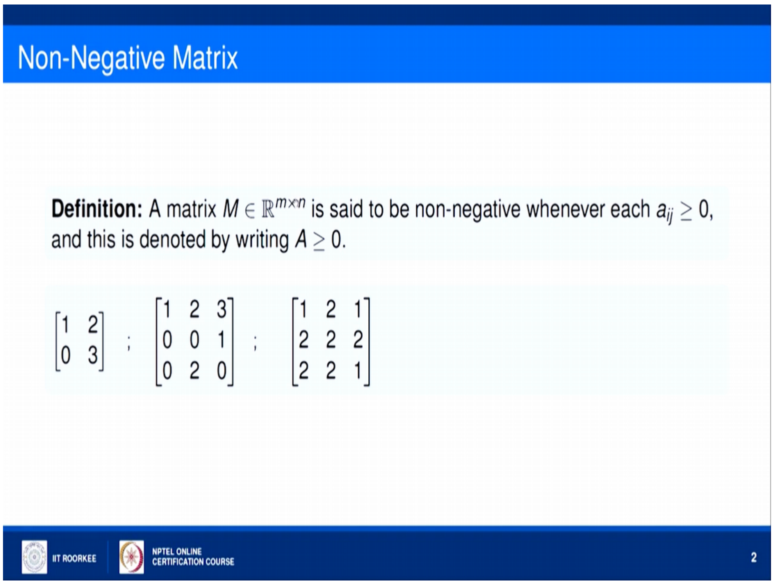

So, a matrix M which is a real matrix of size m by n is said to be a non-negative whenever each entry of this matrix is nonnegative it means 0 is allowed here, unlike the case of positive matrices where we are having each entry is strictly greater than 0 such matrices are denoted by capital A is greater than equals to 0.

So, for example, 1 2 0 3 this 2 by 2 matrix-matrix is a non-negative matrix. Similarly this 3 by 3 matrix is again a non-negative matrix, but these 2 are not positive matrices. This particular matrix is a positive matrix moreover every positive matrix is a non-negative matrix in the sense that it is not having any negative value.

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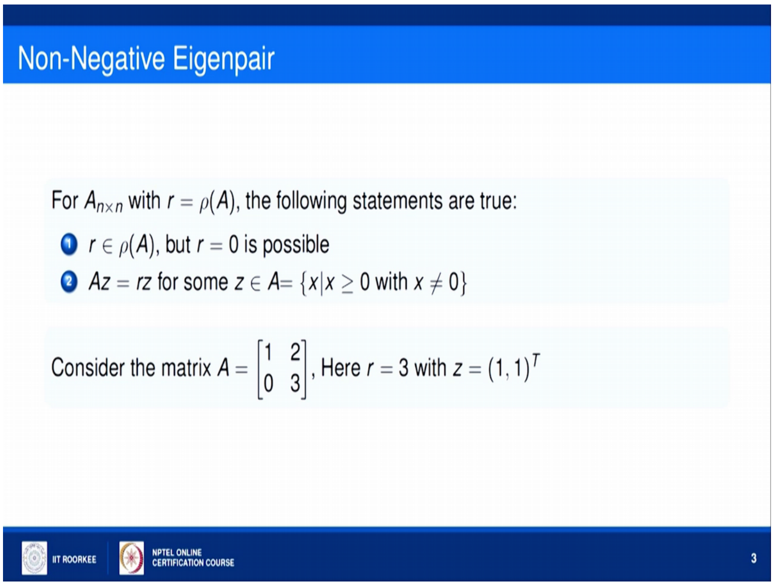

Let us go through the same result which we were having in the case of positive matrices in terms of spectral radius. So, let A be n by n matrix having spectral radius as small r then my following statements are true. The first one is this r which is the spectral radius of A belongs to spectrum of A, but r equals to 0 is possible in this case which was not possible in case of positive matrices, there r will should be strictly greater than 0.

The second is about the eigenvector. So, if Az equals to rz means z be an eigenvector corresponding to eigenvalue r which is the spectral radius for some z belongs to set x and such a vector will be a non-zero vector and a vector which is greater than equals to 0 means each entry is non non-negative. So, for example, if you consider this 2 by 2 matrix then the spectral radius of this matrix is 3 and the eigenvector corresponding to this spectral radius is 1 1 which is not having any negative component.

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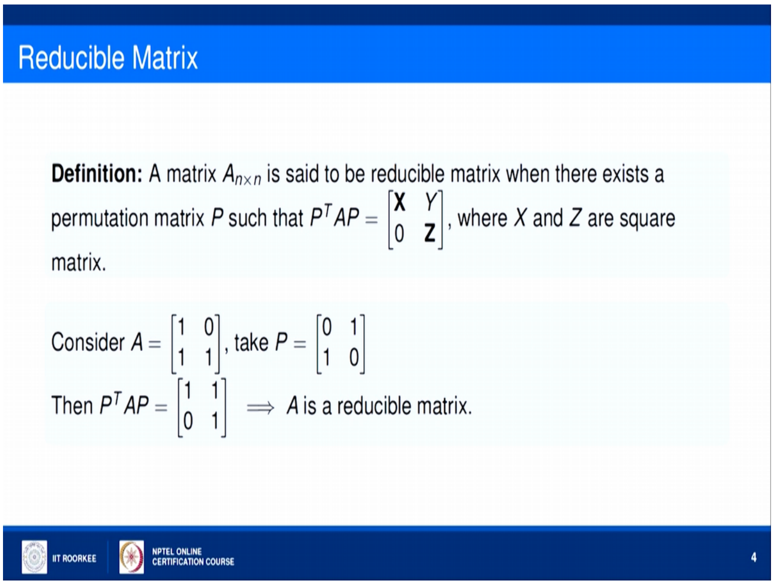

My next definition is reducible matrices. So, A n by n matrix is said to be reducible matrix when there exist a permutation matrix P and you remember that permutation matrix is product of elementary matrices means after applying the elementary row operation on the identity matrix you can get permutation matrix. So, if you are having such a permutation matrix P and you take the product of P transpose AMPLIFIER, if it comes out in this form where x and z are square matrices. It may happen that they are of different order and this matrix form is like this a square sub matrix x y 0 and z then we say that a is a reducible matrix.

So, consider A equals to 1 0 1 1, if I take P as 0 1 1 0 then, so it is a permutation matrix where what I have done just I have interchanged the first and second row of the identity matrix. So, if I calculate P transpose AP we found that the product is an upper triangular matrix. So, here x is 1 by 1, y z is 1 by 1 and it is an upper triangular matrix. So, A can be reduced into upper triangular matrix by this particular transformation. So, hence A is a reducible matrix. So, if A is not reducible then we will say that A is a n irreducible matrix.

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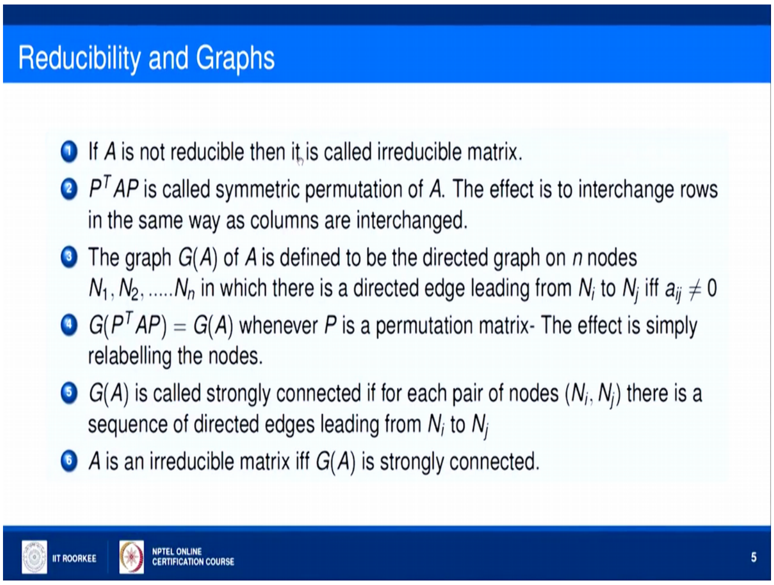

Here the P transpose into A into P is called symmetric permutation of A. It means that we are applying the permutation matrices from both side because if P is permutation matrix. So, P transpose the effect is to interchange rows in the same way as columns are interchanged.

Now, let us see the graph what we can say about the graph of such matrices. So, the graph of matrix A is defined to be the directed graph on n nodes if the size of A is n by n having N 1, N 2, N n as nodes in which there is a directed edge is leading from N i to N j if and only if a ij equals to not equals to 0.

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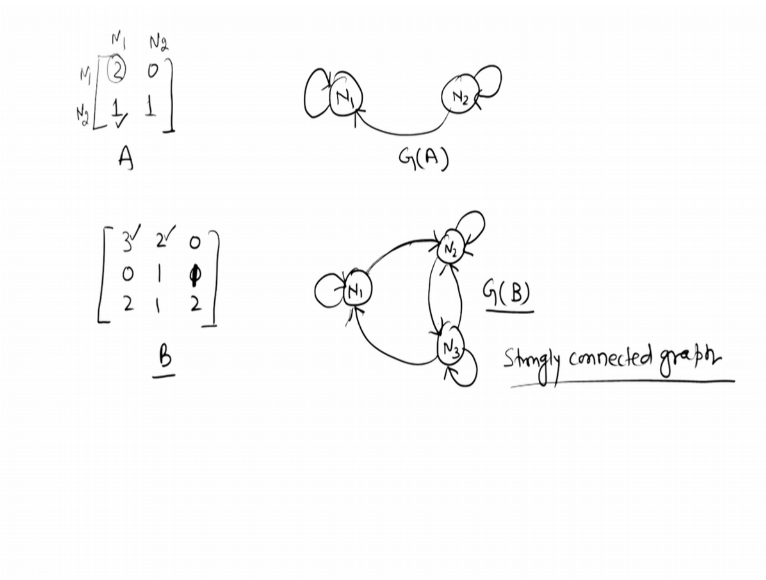

So, if I am having a matrix let us say 2 by 2 matrix 2 0, 1 1. Then here I will be having 2 nodes in the graph of this matrix and I will be having an edge if there is a non-zero entry at this position. So, if you see a 1 1 position here I am having a non-zero entry. So, I am having an edge from 1 to 1 and edge will be an a direct edge if you see from you can write like this N 1, N 2, N 1, N 2. So, if you see from N 1 to N 2 this particular entry is 0. So, there is no connection from N 1 to N 2 a direct connection.

If I go from N 2 to N 1, yes I am having an edge due to this non-zero entry and then I am having N 2 to N 2. So, this if this matrix is A this is graph of A. If I take a 3 by 3 matrix 3 2 0 0 1 0 2 1 2 then here it is a 3 by 3 matrix, so graph of this matrix will be having 3 nodes let us say N 1, N 2 and N 3. So, I am having an edge N 1 to 1 to 1 due to this non-zero entry then I am having an edge from N 1 to N 2 due to this entry 1 to 3 I am having 0. So, I do not have any edge from 1 to 3. Then 2 to 1 0 2 to 2 each yes I am having an edge 2 to 3 I do not have 3 to 1 I am having an edge 3 to 1 3 to 2 yes and then 3 to 3. So, this is the graph of if this is my matrix B then this is graph of B. So, in this way we can define the graph of a given square matrix.

Now, so if G A is the graph of A then G of P transpose Ap equals to G A, whenever P is a permutation matrix the effect is simply relabelling the nodes. Here this graph is called strongly connected if for each pair of nodes N i N j there is a connection of direct edges leading from N i to N j means you are you can reach from N i to N j by a sequence of edges. So, if you go here like if I talk do I have a connection from N 1 to N 3 I do not have, because I cannot move from N 1 to N 3 by using any sequence of edges of this graph.

If am having an edge like this means 2 to 3 and 3 is 1. So, if this entry is one. So, I will be having this edge. Now, I am having a connection from N 1 to N 3 because I will go like this from N 1 to N 2 and then N 2 to N 3. I can move from N 1 to N 2 I can move N 2 to N 3 in the same way I can move N 2 to N 1, because I will reach from N 2 to N 3 and then N 2 to N 1. And then I can also move N 2 to N 3, I can move from N 3 to N 1 as well as from N 3 to n 2. So hence, I am having connection between all the edges. So, this graph is a strongly connected graph in this case.

If I remove this particular edge, if I do not take this edge then this particular graph is not a strongly connected graph, but if I take this edge here then it becomes a a strongly connected graph. So, if I see the relation between a strongly connected graph and irreducible matrices then I will be having a very elegant relationship between these 2 and that is A is an irreducible matrix if and only if G A is a strongly connected. So, let us take example.

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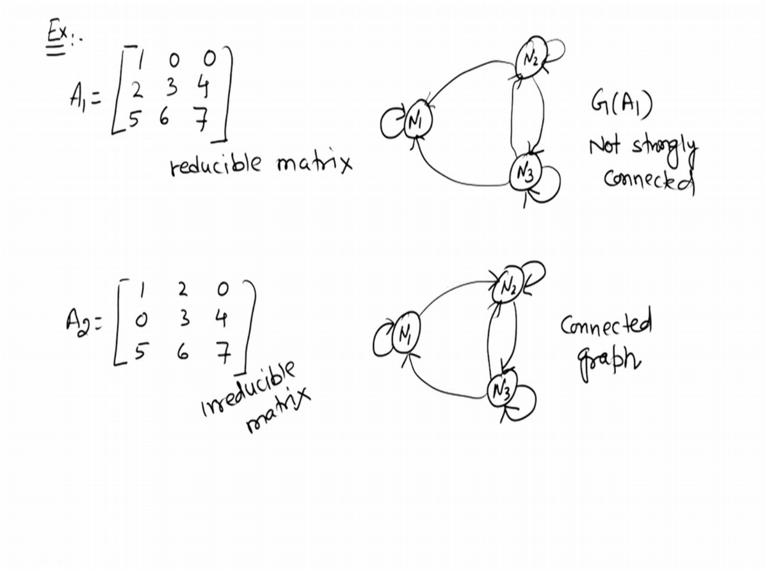

So, if someone ask you given this matrix 1 0 0. So, it is a matrix let us say a 1 2 3 4, 5 6 7 check whether it is an irreducible matrix or not. So, just make the graph of this matrix, N 1 N 2 and N 3. So, we are making the graph associated with this matrix. So, I am having n as 1 to N 1 I do not have N 1 to N 2, N 1 to N 3 this is 0 then 2 to 1 2 to 2 2 to 3, 3 to 1 3 to 2 and then 3 to 3. So, what I am having this is the graph associated with this particular matrix. Now, is it is strongly connected if I want to go from N 1 to N 2 or N 1 2 to N 3 it is not possible here in this case because there is no sequence of edges by following that I can move from N 1 to N 2 or N 1 to N 3. So, it is not a strongly connected. It means this matrix is not irreducible. So, it is a reducible matrix.

On the other hand if I take another example let us say A 2 let us say 1 2 0, 0 3 4, 5 6 7. So, please note that I am having the same number of 0 in this matrix also whatever I was having in the case of matrix A 1. So, again if a make a graph here N 1, N 2, N 3. So, 1 to 1, 1 to 2, 1 to 3 there is no direct edge 0 2 to 1 no, 2 to 2, 2 to 3 yes 3 to 1 3 to 2 and 3 to 3. So, now, if you check this I can move from N 1 to N 2 using this edge. N 1 to N 3 by moving first N 2 and then following this edge I can move from N to 2 and N 2 to N 3 as well as N 2 to N 1 I can move N 3 to N 1 as well as N 3 to N 2. So, it is a connected graph.

There are another method if you are having more number of nodes like twenty notes. So, there are several algorithm for checking this particular thing whether the graph is a strongly connected or not. So, there are different algorithm, but I am not discussing those in the this particular lecture, go that is the part of a separate course just I am making an understanding of matrix and how can we use graphs for checking whether the given matrix is reducible or not. So, here it is a strongly connected graph. So, it is a irreducible matrix.

So, this is the relation between these two matrices my next result is the Perron-Frobenius theorem in case of non-negative and irreducible matrices in case of non-negative and irreducible matrices.

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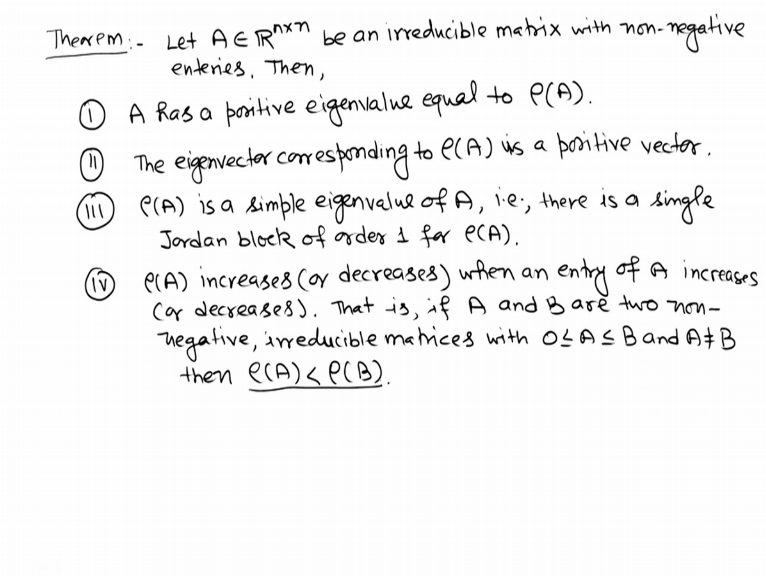

So, the statement of this theorem is something like that let a belongs to n by n matrix having real entries and it is an irreducible matrix with non-negative entries. It means it is irreducible as well as non-negative matrix. Then we are having few statements we can make few statements for this particular matrix that, A has a positive eigenvalue equal to its spectral radius.

The second is the eigenvector corresponding to rho a is A positive vector. The third is this particular eigenvalue rho A which is the spectral radius also is a simple eigenvalue that is at in other words I can write it there is a single Jordan block of order 1 for this particular eigenvalue or index of this eigenvalue is 1. The fourth result is this eigenvalue rho A increases or decreases when an entry of A increases or decreases. So, in other way the spectral radius of A, if a irreducible and non-negative defense on entries of A. If you increase any of the entry of A spectral radius will increase if you decrease the entry of A spectral radius will decrease. That is if A and B are 2 non-negative irreducible matrices with each entry of A is greater than equals to 0 and each entry of B is greater than equals to corresponding entries of A and A not equals to B means some of the entries are equal to A and some of the entries are greater than the corresponding entries of A.

Then according to this I can say that spectral radius of A will be strictly less than the spectral radius of B. So, this theorem is called Perron-Frobenius theorem in case of irreducible and non-negative matrices. And these for beautiful properties of such matrices can be used in many applications in various engineering discipline. So, here I am not taking the proof of this matrix theorem because it is quite big. So, one can follow the book if someone is interested in the proof we will see the applications of all these.

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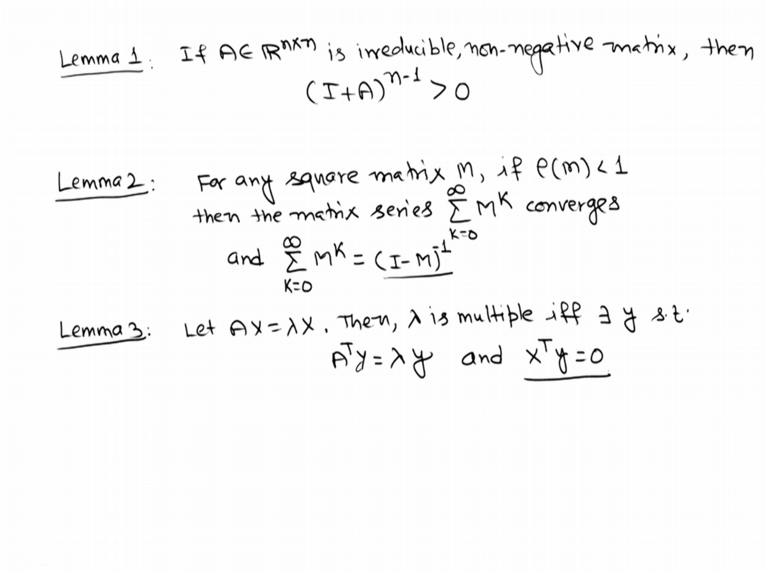

Let us see few more consequences of the Perron-Frobenius theorem. So, let me write as lemma 1. So, if A is a n by n matrix having real entries and it is irreducible and non-negative matrix then I plus A raised to power n minus 1 will be a positive matrix. And the proof of this can be done like this the matrix I plus A raised to n power n minus 1 will be the linear combination of the matrices I A, A square, A cube upto A raised to power n minus 1 with positive coefficients. Why positive coefficients? Because A is a non-negative matrix, so there you cannot take mean there will not be negative coefficients comes come came into picture. So, using that fact and using the result from the Perron-Frobenius theorem we can prove this particular result.

The next is another lemma on the Perron-Frobenius theorem. It say me that for any square matrix M, if the spectral radius of this matrix is less than 1 then the matrix series let us say summation K equals to 0 to infinity M raised to power K. Means it is I plus M plus M square plus M cube up to infinity this infinite series converges and in particular K equals to 0 to infinity M raised to power K will become I minus M raised to power minus 1.

Means, if you open the binomial expansion of this you will get this infinite series the another beautiful properties of such matrices we see that let AX equals to lambda X means A is a square matrix and X is an eigenvector of these corresponding to eigenvalue lambda. Then lambda is multiple means it is not simple means the index of this is greater than 1, if and only if there exist y another vector y such that A transpose y equals to lambda y. And x and y are orthogonal means the dot product of these 2 vectors are 0.

So, these 3 properties we will use in the proof of Perron-Frobenius theorem and then 1 can prove it.

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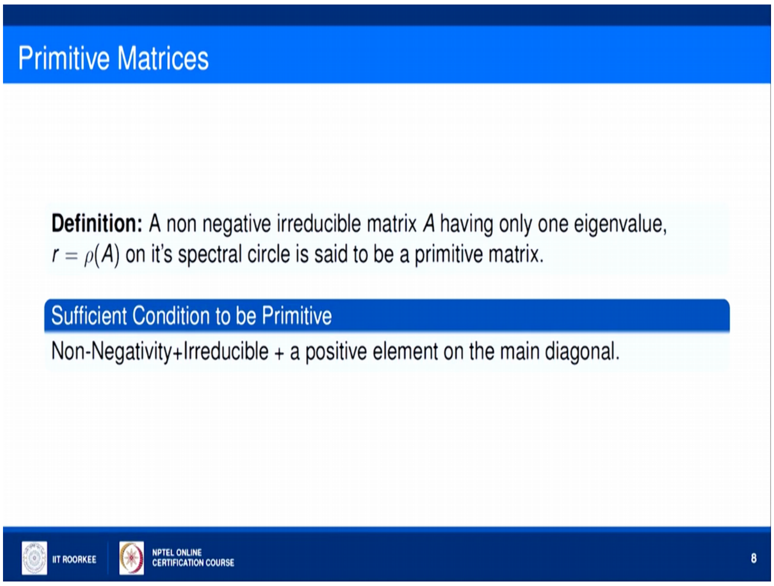

Now, after irreducible and irreducible matrices, let us come to next definition that is about the primitive matrices. So, a non-negative irreducible matrix a having only 1 eigenvalue let us say that is spectral radius on its spectral circle is said to be a primitive matrix.

So, if in the case of non-negative and irreducible matrix there exists only 1 eigenvalue which is on the spectral circles then the matrix is called primitive. If there are more than 1 eigenvalue those are lie on the spectral circle then the matrix is called imprimitive matrix. And that the number of those eigenvalues those are spectral circle is called the index of imprimitivity. So, that number is called index of imprimitivity.

So, sufficient condition to be primitive for a given matrices it should be non-negative, it should be irreducible, and there should be a positive element on the main diagonal of that matrix. If these 3 conditions hold for a given matrix then the matrix is a primitive matrix.

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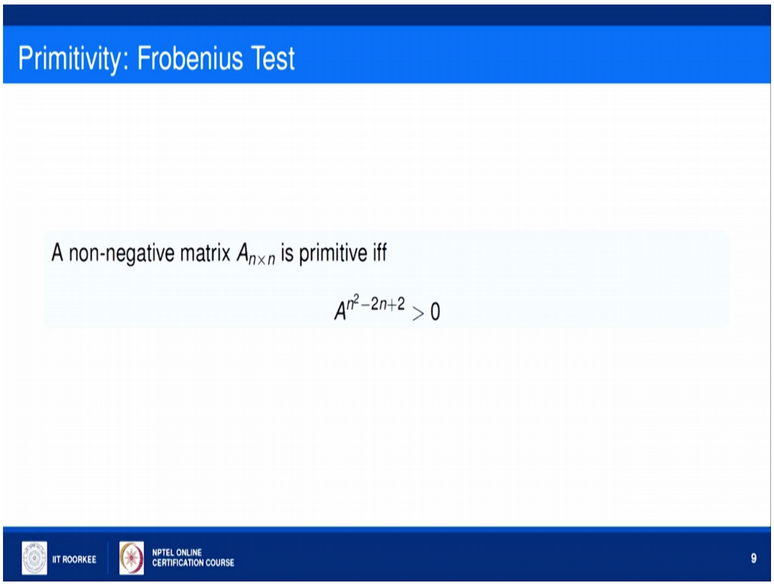

Another test for checking the primitivity which is very useful when you are solving the examples is the Frobenius test and Frobenius test tells us that a non-negative matrix A is primitive if and only if n size of is n by n, A raised to power n square minus 2 n plus 2 is a positive matrix. So, if this happens then A is primitive.

Let us see an example of this.

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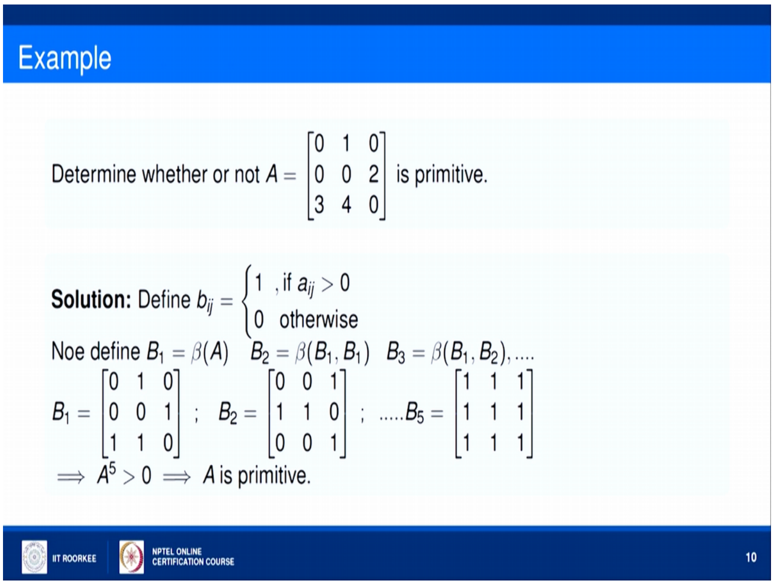

So, determine whether or not this particular non-negative matrix is primitive or not.

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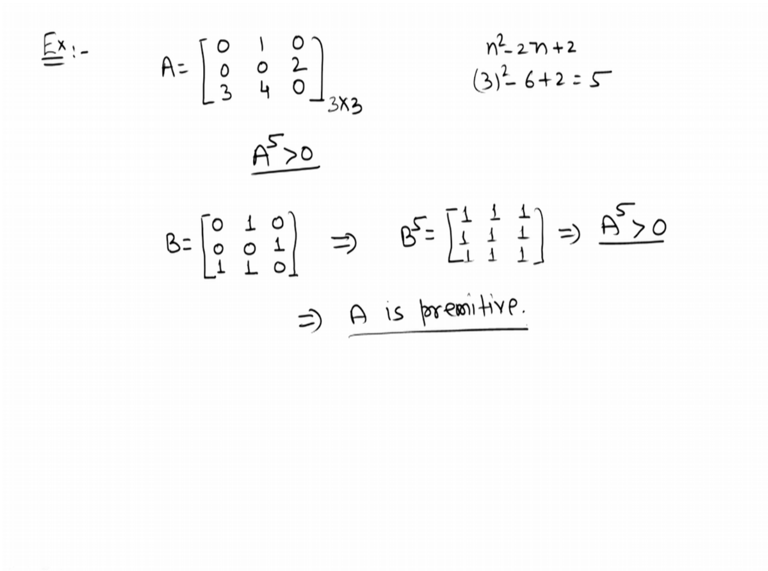

So, here this example is, so what I need to check I am having this matrix A and I have to check whether this matrix is a primitive matrix or not. One of the way is use just find out the spectral circles of this and eigenvalues of this matrix if only 1 eigenvalue is lie on the spectral circle then it is primitive otherwise it is not.

Another is Frobenius test, so how to use Frobenius test? It is a 3 by 3 matrix. So, here n square minus 2 n plus 2 becomes 3 square minus 6 plus 2, so it is 5. So, we have need to check A raised to power 5 if it is a positive matrix then A is a primitive matrix. So, here if I need to calculate a raised to power 5 what I need to do I have to multiply A 5 times. Instead of A let us write a Boolean matrix corresponding to A, which is a matrix of 0 and 1 only. So, the Boolean matrix corresponding to A is a matrix of 0 and 1 of the same size if there is a 0 entry it will be having 0 if there is a non-zero entry there we will be having 1. So, like first row of B will become 0 1 0, second row will become 0 0 because these 2 are 0 and entry and here instead of 2 I will write 1, then 1 1 0.

So, A raised to power 5 will be a positive matrix means all the values all the entries in these matrix will be positive when B raised to power 5 will be having of all the entries as 1. So, if I calculate I what I found B raised to power equals to 1 1 1, 1 1 1, 1 1 1 which implies that A raised to power 5 is a positive matrix which implies A is primitive. So, this is the way of applying Frobenius test for checking the primitivity of a given matrix.

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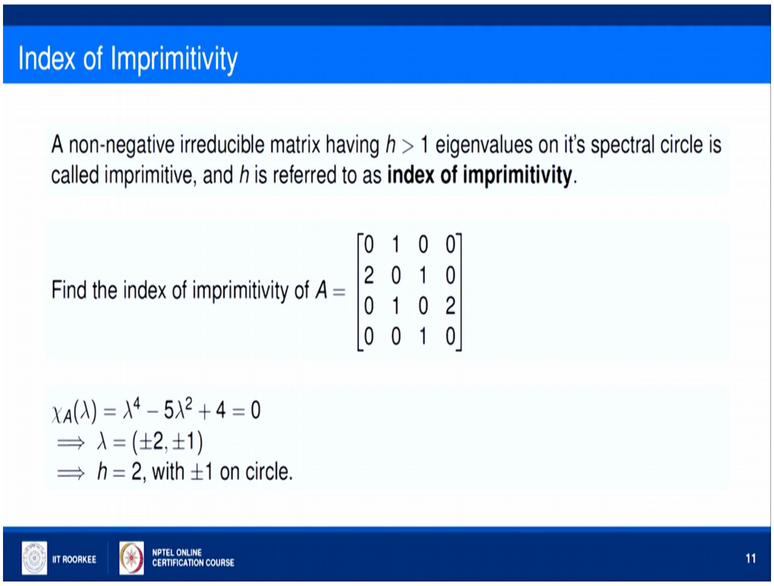

Now, if matrix is not primitive it is called imprimitive. And the number of eigenvalues those lie on the spectral circle is called the index of imprimitivity. If someone ask you find the index of imprimitivity of A, where A is this 4 by 4 matrix then what I will be having the characteristic polynomial of this matrix A is lambda raised to power 4 minus 5 lambda square plus 4 equals to 0. So, here eigenvalues are plus minus 2 and plus minus 1.

If I make the spectral circle of this matrix, so the first row will give me the spectral circle as the center at 0 0 and radius is 1, second row will give me lambda or absolute value of lambda less than equals to 2 plus 1 3. So, it will become center at origin 0 0 and radius is 3.

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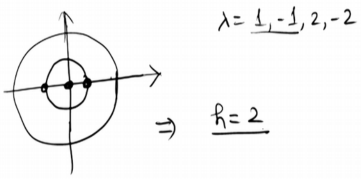

And then 4th row will give the same fifth row will give as the first row. So, these 2 are the spectral circle for the given matrix and eigenvalues are 1, minus 1, 2, minus 2. So, if you check these 2 eigenvalues lie on the spectral circle. So, hence here h equals to 2 means index of imprimitivity is 2 here. If there is only 1 eigenvalue which is on the spectral circle then the matrix will become a primitive matrix, please note that.

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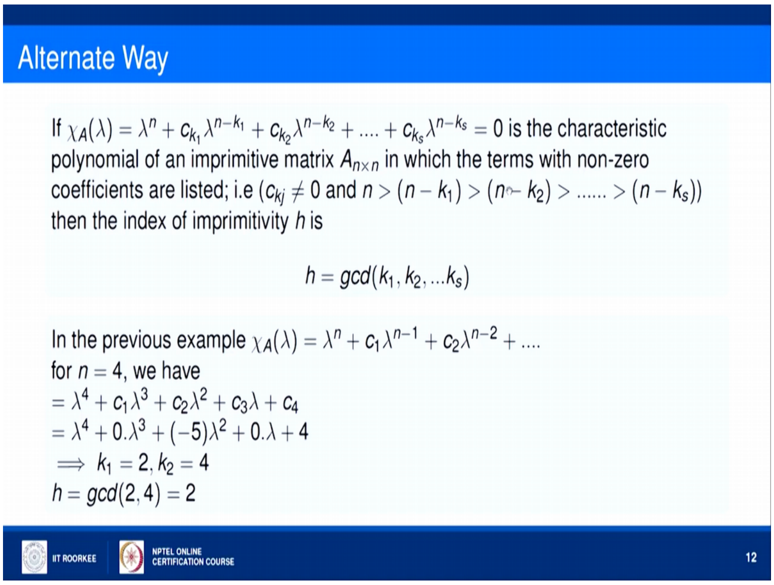

Another way of checking index of imprimitivity you just write the characteristic polynomial and now, see this polynomial like this, this is my just what you have to do you have to count the non-zero coefficients.

So, it is let us say K 0, it is K 1 then so K 0, K 1, K 2, K 3, K 4. Now, see the subscript where the coefficients are non-zero. So, if you see the coefficient of which is corresponding to K 2 is minus 5 which is non-zero. So, here take K 1 as 2. Again the coefficient which is corresponding to fifth location is non-zero, so that is my it is I am taking K 5. So, it means there are 2 and 4 position second and 4th position where the coefficients are non-zero. So, GCD of these two will give you the index of imprimitivity. So, GCD of 2 and 4 is 2 and this process is mentioned here.

So, in this lecture we have learned about reducible, irreducible, their respective graph, how to check whether the given matrix is reducible or not, and then about the primitive matrices. In the next lecture we will learn the polar decomposition of a given matrix. These are the references for this lecture.

Thank you very much.