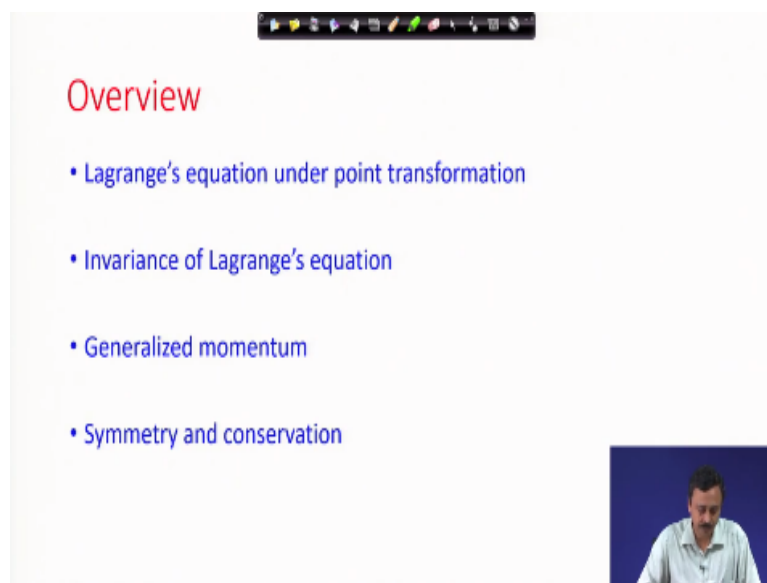


**Advanced Dynamics**  
**Prof. Anirvan Dasgupta**  
**Department of Mechanical Engineering**  
**Indian Institute of Technology, Kharagpur**

**Lecture - 56**  
**Symmetries and Conservation Laws - I**

In this lecture, I am going to discuss some finer issues that are associated with Lagrangian mechanics. We are going to discuss symmetries and their corresponding conservation laws. This is our first lecture on symmetry and conservation laws.

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The image shows a presentation slide with a white background. At the top, there is a small black bar with various icons. Below it, the word "Overview" is written in red. A list of four bullet points follows, each in blue text: "Lagrange's equation under point transformation", "Invariance of Lagrange's equation", "Generalized momentum", and "Symmetry and conservation". In the bottom right corner of the slide, there is a small video inset showing a man in a light blue shirt speaking.

We will start with point transformation and see the invariance of the Lagrange's equation structure under point transformation. Next, we will introduce the concept of generalized momentum, and I am going to show you how the structure of Lagrange's equation of motion compares with Newton's second law of motion.


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## Lagrange's equation under point transformation

Lagrange's equation of motion

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \quad \mathcal{L} = \mathcal{L}(q, \dot{q}, t)$$

Point transformation

$$\left. \begin{aligned} q_i &= q_i(z) \\ \Rightarrow \dot{q}_i &= \frac{\partial q_i}{\partial z} \cdot \frac{dz}{dt} \end{aligned} \right\} \begin{aligned} \tilde{\mathcal{L}} &= \mathcal{L}(q_i(z), \frac{\partial q_i}{\partial z} \cdot \dot{z}, t) \\ &= \mathcal{L}(z, \frac{\partial q_i}{\partial z} \cdot \dot{z}, t) \end{aligned}$$


The above slide introduces the concept of point transformation.

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## Lagrange's equation under point transformation

Lagrange's equation of motion: transformed Lagrangian

$$\tilde{\mathcal{L}} = \mathcal{L}(q_i(z), \frac{\partial q_i}{\partial z} \cdot \dot{z}, t) = \mathcal{L}(z, \frac{\partial q_i}{\partial z} \cdot \dot{z}, t)$$

$$\frac{\partial \tilde{\mathcal{L}}}{\partial \dot{z}} = \frac{\partial q_i}{\partial z} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \quad \frac{d}{dt} \left( \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{z}} \right) = \frac{d}{dt} \left( \frac{\partial q_i}{\partial z} \right) \frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \frac{\partial q_i}{\partial z} \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) = \frac{\partial^2 q_i}{\partial z^2} \dot{z} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \frac{\partial q_i}{\partial z} \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right)$$

$$\frac{\partial \tilde{\mathcal{L}}}{\partial z} = \frac{\partial q_i}{\partial z} \frac{\partial \mathcal{L}}{\partial q_i} + \frac{\partial}{\partial z} \left( \frac{\partial q_i}{\partial z} \right) \cdot \dot{z} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial q_i}{\partial z} \frac{\partial \mathcal{L}}{\partial q_i} + \frac{\partial^2 q_i}{\partial z^2} \dot{z} \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

$$\Rightarrow \frac{d}{dt} \left( \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{z}} \right) - \frac{\partial \tilde{\mathcal{L}}}{\partial z} = \frac{\partial q_i}{\partial z} \left[ \underbrace{\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) - \frac{\partial \mathcal{L}}{\partial q_i}}_{=0} \right]$$

$$\Rightarrow \frac{d}{dt} \left( \frac{\partial \tilde{\mathcal{L}}}{\partial \dot{z}} \right) - \frac{\partial \tilde{\mathcal{L}}}{\partial z} = 0 \Rightarrow \text{Structure of Lagrangian's equations are invariant under point transformation. (But equations can be different)}$$

As shown in the above slide, the structure of Lagrange's equation of motion remains invariant under point transformation.

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### Invariance of Lagrange's equation

Consider  $L = L(\bar{q}, \dot{\bar{q}}, t)$       Lagrange's equations:  $\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\bar{q}}} \right) - \frac{\partial L}{\partial \bar{q}} = 0$


Let  $\bar{L} = L(\bar{q}, \dot{\bar{q}}, t) + \frac{d\Lambda}{dt}$       where  $\Lambda = \Lambda(\bar{q}, t)$

$\Rightarrow \bar{L} = L(\bar{q}, \dot{\bar{q}}, t) + \frac{\partial \Lambda}{\partial t} + \frac{\partial \Lambda}{\partial \bar{q}} \cdot \dot{\bar{q}}$

Lagrange's equation:  $\frac{\partial \bar{L}}{\partial \dot{\bar{q}}} = \frac{\partial L}{\partial \dot{\bar{q}}} + \frac{\partial \Lambda}{\partial \dot{\bar{q}}}$        $\frac{d}{dt} \left( \frac{\partial \bar{L}}{\partial \dot{\bar{q}}} \right) = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\bar{q}}} \right) + \frac{\partial^2 \Lambda}{\partial t \partial \dot{\bar{q}}} + \frac{\partial}{\partial \dot{\bar{q}}} \left( \frac{\partial \Lambda}{\partial \bar{q}} \right) \cdot \dot{\bar{q}}$

$\frac{\partial \bar{L}}{\partial \bar{q}} = \frac{\partial L}{\partial \bar{q}} + \frac{\partial^2 \Lambda}{\partial \bar{q} \partial t} + \frac{\partial}{\partial \bar{q}} \left( \frac{\partial \Lambda}{\partial \bar{q}} \right) \cdot \dot{\bar{q}}$

$\Rightarrow \frac{d}{dt} \left( \frac{\partial \bar{L}}{\partial \dot{\bar{q}}} \right) - \frac{\partial \bar{L}}{\partial \bar{q}} = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\bar{q}}} \right) - \frac{\partial L}{\partial \bar{q}} = 0$



Next, we look at the effect of adding a total time derivative to the Lagrangian. It is to be noted that the function  $\Lambda$  is a function of only the generalized coordinates and time.

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### Invariance of Lagrange's equation

Consider  $L = L(\bar{q}, \dot{\bar{q}}, t)$       Lagrange's equations:  $\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\bar{q}}} \right) - \frac{\partial L}{\partial \bar{q}} = 0$

Let  $\bar{L} = L(\bar{q}, \dot{\bar{q}}, t) + \frac{d\Lambda}{dt}$       where  $\Lambda = \Lambda(\bar{q}, t)$


$\Rightarrow \bar{L} = L(\bar{q}, \dot{\bar{q}}, t) + \frac{\partial \Lambda}{\partial t} + \frac{\partial \Lambda}{\partial \bar{q}} \cdot \dot{\bar{q}}$

Lagrange's equation:  $\frac{\partial \bar{L}}{\partial \dot{\bar{q}}} = \frac{\partial L}{\partial \dot{\bar{q}}} + \frac{\partial \Lambda}{\partial \dot{\bar{q}}}$        $\frac{d}{dt} \left( \frac{\partial \bar{L}}{\partial \dot{\bar{q}}} \right) = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\bar{q}}} \right) + \frac{\partial^2 \Lambda}{\partial t \partial \dot{\bar{q}}} + \frac{\partial}{\partial \dot{\bar{q}}} \left( \frac{\partial \Lambda}{\partial \bar{q}} \right) \cdot \dot{\bar{q}}$

$\frac{\partial \bar{L}}{\partial \bar{q}} = \frac{\partial L}{\partial \bar{q}} + \frac{\partial^2 \Lambda}{\partial \bar{q} \partial t} + \frac{\partial}{\partial \bar{q}} \left( \frac{\partial \Lambda}{\partial \bar{q}} \right) \cdot \dot{\bar{q}}$

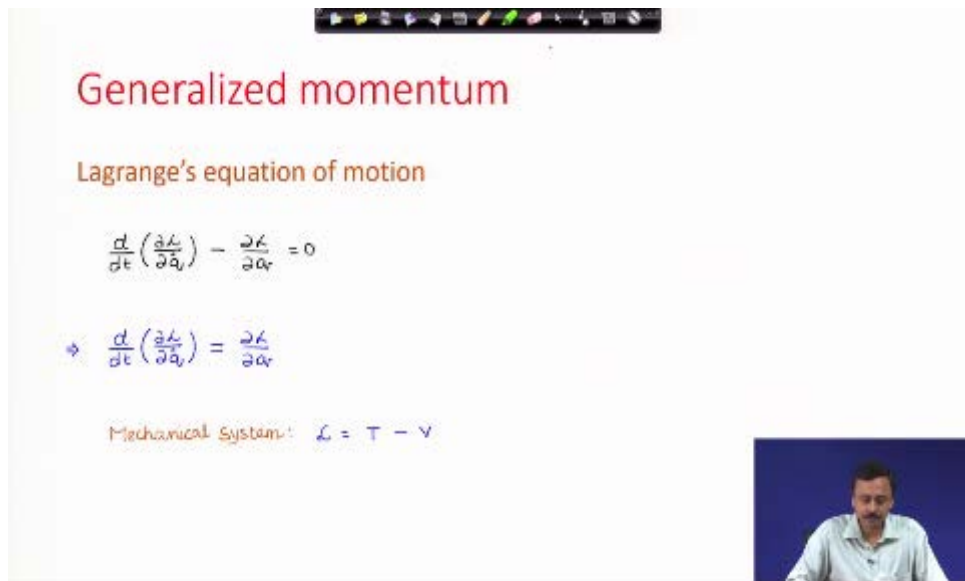
$\Rightarrow \frac{d}{dt} \left( \frac{\partial \bar{L}}{\partial \dot{\bar{q}}} \right) - \frac{\partial \bar{L}}{\partial \bar{q}} = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\bar{q}}} \right) - \frac{\partial L}{\partial \bar{q}} = 0$

$\Rightarrow$  Lagrange's equations are invariant under addition of a total time derivative term to  $L$ .



It is observed that the Lagrange's equations are invariant under the addition of a total time derivative term to  $L$ . This idea can be used to simplify the Lagrangian by removing any part of the Lagrangian that is a total time derivative of a function involving the generalized coordinates and time.

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**Generalized momentum**

Lagrange's equation of motion

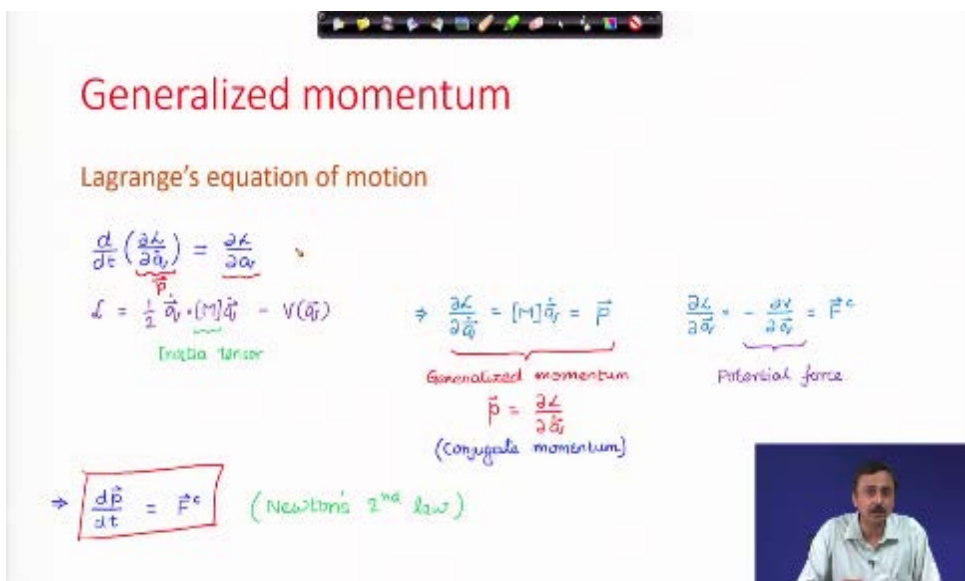
$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\alpha}_r} \right) - \frac{\partial \mathcal{L}}{\partial \alpha_r} = 0$$
$$\Rightarrow \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\alpha}_r} \right) = \frac{\partial \mathcal{L}}{\partial \alpha_r}$$

Mechanical system:  $\mathcal{L} = T - V$



Now, we will introduce this very important concept of generalized momentum.

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**Generalized momentum**

Lagrange's equation of motion

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_r} \right) = \frac{\partial \mathcal{L}}{\partial q_r}$$
$$\mathcal{L} = \frac{1}{2} \dot{q}_r \cdot [m] \dot{q}_r - V(q_r)$$


Inertia tensor

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{q}_r} = [m] \dot{q}_r = \vec{p}$$

Generalized momentum  
 $\vec{p} = \frac{\partial \mathcal{L}}{\partial \dot{q}_r}$   
(conjugate momentum)

$$\frac{\partial \mathcal{L}}{\partial q_r} = - \frac{dV}{dq_r} = \vec{F}^c$$

Potential force

$$\Rightarrow \frac{d\vec{p}}{dt} = \vec{F}^c \quad (\text{Newton's 2nd law})$$


Using a conventional structure of the Lagrangian, we show the correspondence of the Lagrange's equation of motion with the Newton's second law in the above slide.

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## Generalized momentum

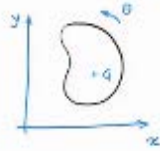

**Rigid body in a plane**

Generalized coordinates:  $\vec{q}_i = (x_c, y_c, \theta)$

Equations of motion:  $\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) = \frac{\partial \mathcal{L}}{\partial q_i}$

Lagrangian:  $\mathcal{L} = \frac{1}{2} m (\dot{x}_c^2 + \dot{y}_c^2) + \frac{1}{2} I_c \dot{\theta}^2 - V(x_c, y_c, \theta)$

$\left. \begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{x}_c} &= m \dot{x}_c = p_x \\ \frac{\partial \mathcal{L}}{\partial \dot{y}_c} &= m \dot{y}_c = p_y \\ \frac{\partial \mathcal{L}}{\partial \dot{\theta}} &= I_c \dot{\theta} = p_\theta \end{aligned} \right\} \text{Generalized momentum}$	$p_{q_i} = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$	$\begin{aligned} \frac{\partial V}{\partial x_c} = -\frac{\partial V}{\partial x_c} = f_x &\Rightarrow \frac{d p_x}{dt} = f_x \\ \frac{\partial V}{\partial y_c} = -\frac{\partial V}{\partial y_c} = f_y &\Rightarrow \frac{d p_y}{dt} = f_y \\ \frac{\partial V}{\partial \theta} = -\frac{\partial V}{\partial \theta} = M_\theta &\Rightarrow \frac{d p_\theta}{dt} = M_\theta \end{aligned}$
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The above slide considers a generic example of a rigid body in a plane. The equations of motion obtained from the Lagrange's equation correspond to the structure of Newton-Euler equations of motion through the concept of generalized momentum.

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## Generalized momentum: examples

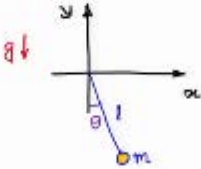
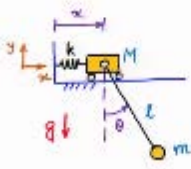
$\mathcal{L} = \frac{1}{2} m l^2 \dot{\theta}^2 + m g l c \theta$

$\rightarrow p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m l^2 \dot{\theta}$

$\mathcal{L} = \frac{1}{2} (m+M) \dot{x}^2 + \frac{1}{2} m (\dot{x}^2 \theta^2 + 2 \dot{x} \dot{\theta} c \theta) - \frac{1}{2} k x^2 + m g l c \theta$

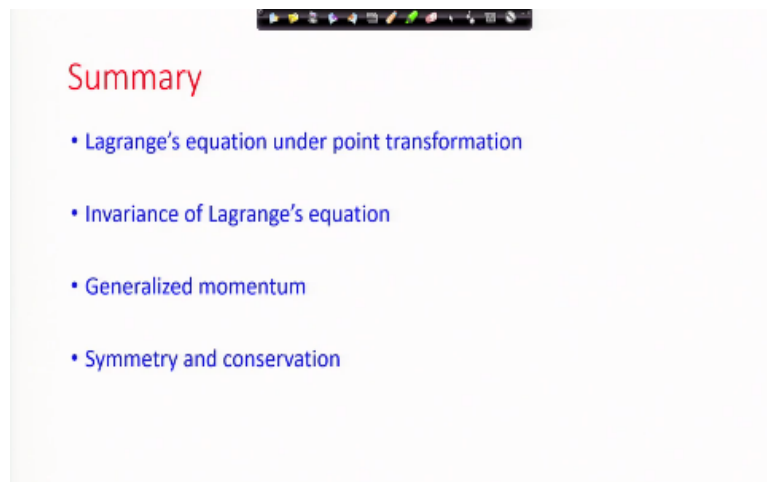
$\rightarrow p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = (m+M) \dot{x} + m l \dot{\theta} c \theta$

$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m l^2 \dot{\theta} + m l \dot{x} c \theta$

Two examples are used to explain the concept of generalized momentum.

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Summary

- Lagrange's equation under point transformation
- Invariance of Lagrange's equation
- Generalized momentum
- Symmetry and conservation