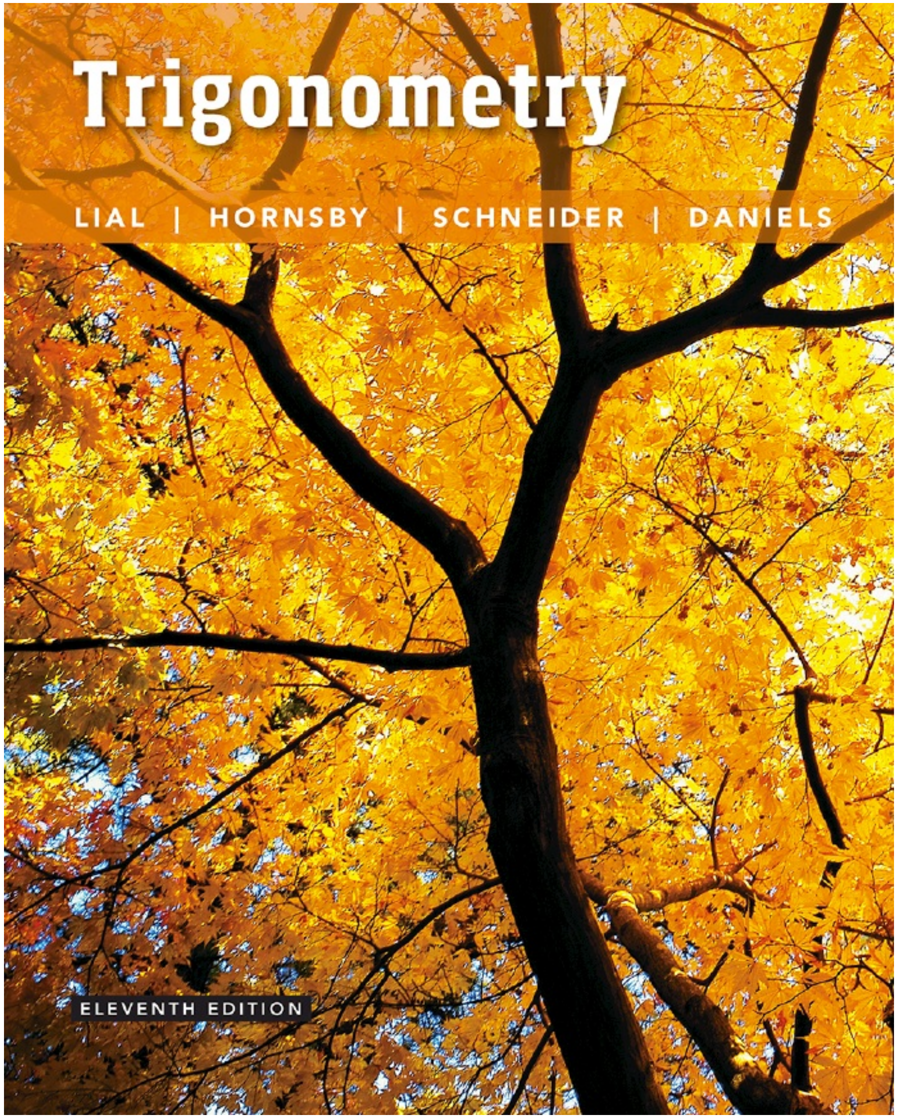
3

Radian Measure and the Unit Circle



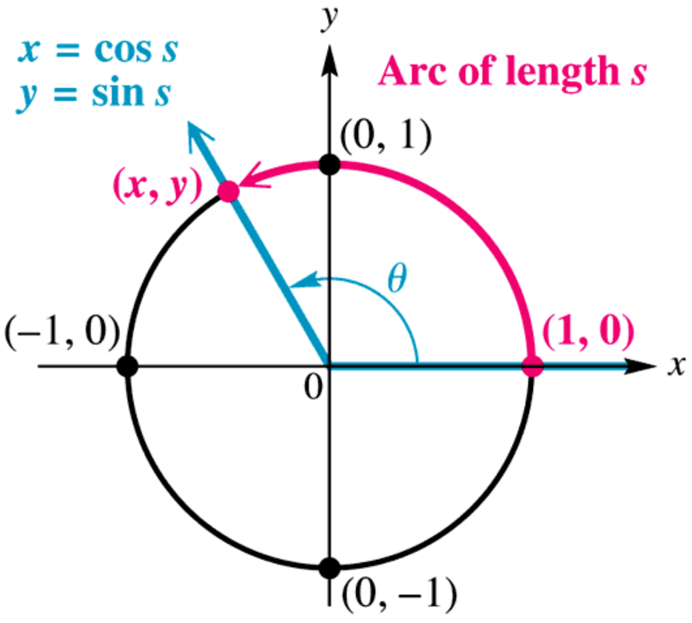
3.3 The Unit Circle and Circular Functions

Circular Functions ▪ Values of Circular Functions ▪ Determining a Number with a Given Circular Function Value ▪ Function Values as Lengths of Line Segments

Circular Functions

A unit circle has its center at the origin and a radius of 1 unit.

The trigonometric functions of angle *θ* in radians found by choosing a point (*x*, *y*) on the unit circle can be rewritten as functions of the arc length *s*, a real number.



Unit circle $x^2 + y^2 = 1$

When interpreted this way, they are called **circular functions.**

Circular Functions

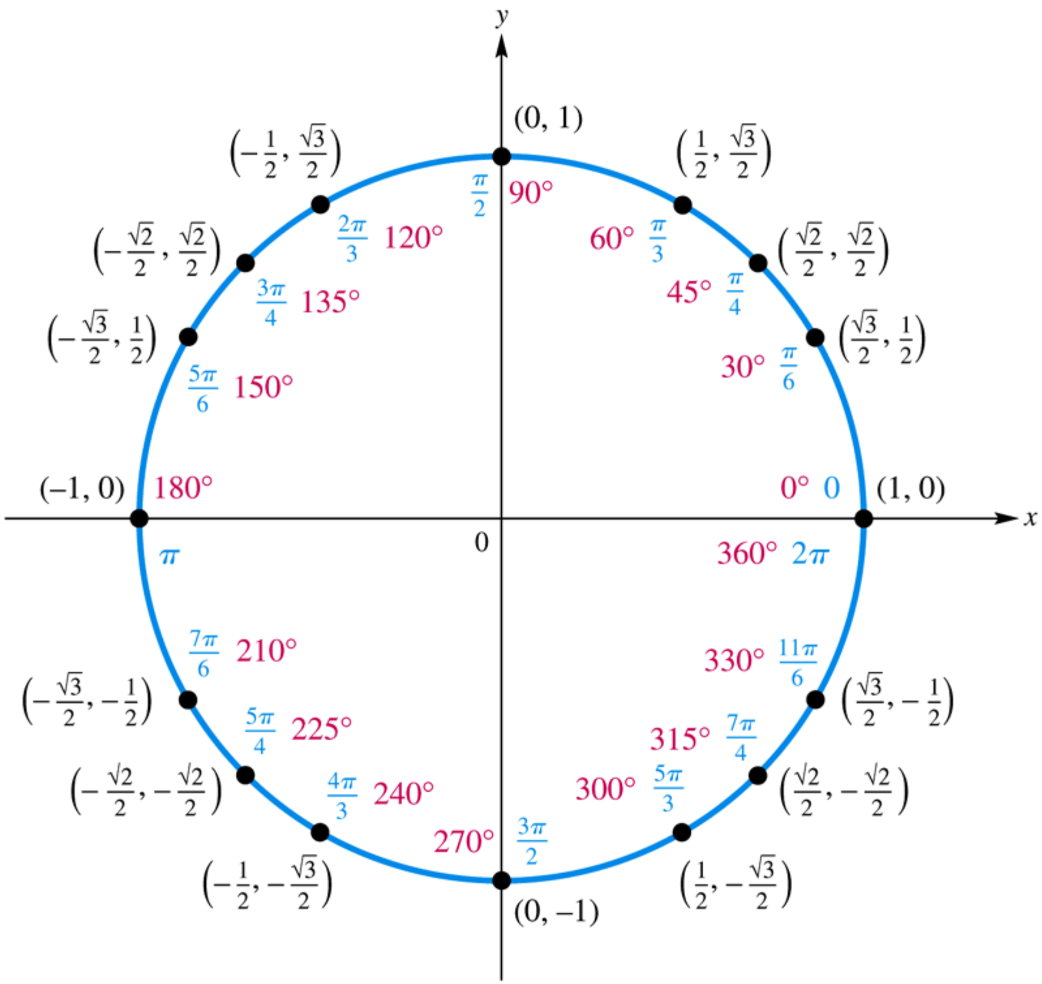
For any real number *s* represented by a directed arc on the unit circle,

$$\sin s = y \qquad \csc s = \frac{1}{y} \ (y \neq 0)$$

$$\cos s = x \qquad \sec s = \frac{1}{x} \ (x \neq 0)$$

$$\tan s = \frac{y}{x} \ (x \neq 0) \qquad \cot s = \frac{x}{y} \ (y \neq 0)$$

The Unit Circle



The unit circle $x^2 + y^2 = 1$

The Unit Circle

- The unit circle is symmetric with respect to the *x*-axis, the *y*-axis, and the origin.

  If a point (a, b) lies on the unit circle, so do $(a, -b)$, $(-a, b)$ and $(-a, -b)$.

The Unit Circle

- For a point on the unit circle, its **reference arc** is the shortest arc from the point itself to the nearest point on the *x*-axis.

For example, the quadrant I real number $\frac{\pi}{3}$ is associated with the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ on the unit circle.

s	Quadrant of s	Symmetry Type and Corresponding Point	$\cos s$	$\sin s$
$\dfrac{\pi}{3}$	I	not applicable; $\left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$	$\dfrac{1}{2}$	$\dfrac{\sqrt{3}}{2}$
$\pi - \dfrac{\pi}{3} = \dfrac{2\pi}{3}$	II	y-axis; $\left(-\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$	$-\dfrac{1}{2}$	$\dfrac{\sqrt{3}}{2}$
$\pi + \dfrac{\pi}{3} = \dfrac{4\pi}{3}$	III	origin; $\left(-\dfrac{1}{2}, -\dfrac{\sqrt{3}}{2}\right)$	$-\dfrac{1}{2}$	$-\dfrac{\sqrt{3}}{2}$
$2\pi - \dfrac{\pi}{3} = \dfrac{5\pi}{3}$	IV	x-axis; $\left(\dfrac{1}{2}, -\dfrac{\sqrt{3}}{2}\right)$	$\dfrac{1}{2}$	$-\dfrac{\sqrt{3}}{2}$

The Unit Circle

Because cos $s = x$ and sin $s = y$, we can replace x and y in the equation of the unit circle

$$x^2 + y^2 = 1$$

and obtain the Pythagorean identity

$$\cos^2 s + \sin^2 s = 1.$$

Domains of the Circular Functions

Sine and Cosine Functions: $(-\infty, \infty)$

Tangent and Secant Functions:

$$\left\{s \mid s \neq (2n+1)\frac{\pi}{2}, \text{ where } n \text{ is any integer}\right\}$$

Cotangent and Cosecant Functions:

$$\{s \mid s \neq n\pi, \text{ where } n \text{ is any integer}\}$$

Evaluating A Circular Function

Circular function values of real numbers are obtained in the same manner as trigonometric function values of angles measured in radians.

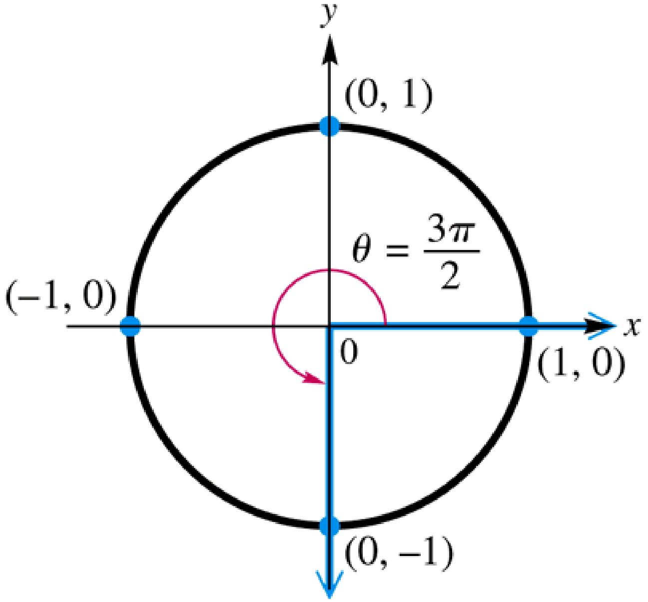
This applies both to methods of finding exact values (such as reference angle analysis) and to calculator approximations.

Calculators must be in radian mode when finding circular function values.

Example 1 FINDING EXACT CIRCULAR FUNCTION VALUES

Find the exact values of $\sin\frac{3\pi}{2}$, $\cos\frac{3\pi}{2}$, and $\tan\frac{3\pi}{2}$.

Evaluating a circular function at the real number $\frac{3\pi}{2}$ is equivalent to evaluating it at $\frac{3\pi}{2}$ radians.



An angle of $\frac{3\pi}{2}$ intersects the circle at the point (0, –1).

Since $\sin s = y$, $\cos s = x$, and $\tan s = \frac{y}{x}$,

$\sin\frac{3\pi}{2} = -1$, $\cos\frac{3\pi}{2} = 0$, and $\tan\frac{3\pi}{2}$ is undefined.

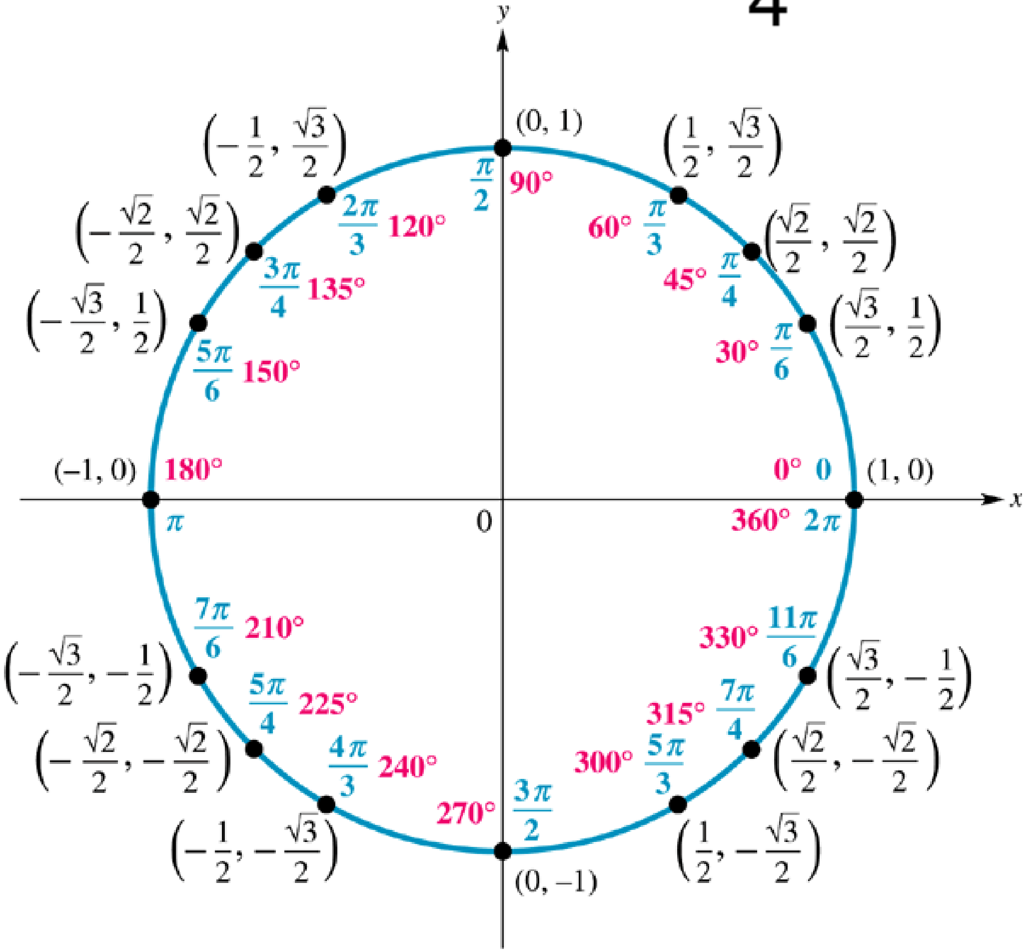
Example 2(a) FINDING EXACT CIRCULAR FUNCTION VALUES

Use the figure to find the exact values of $\cos\frac{7\pi}{4}$ and $\sin\frac{7\pi}{4}$.

The real number $\frac{7\pi}{4}$ corresponds to the unit circle point $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

$$\cos\frac{7\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\sin\frac{7\pi}{4} = -\frac{\sqrt{2}}{2}$$

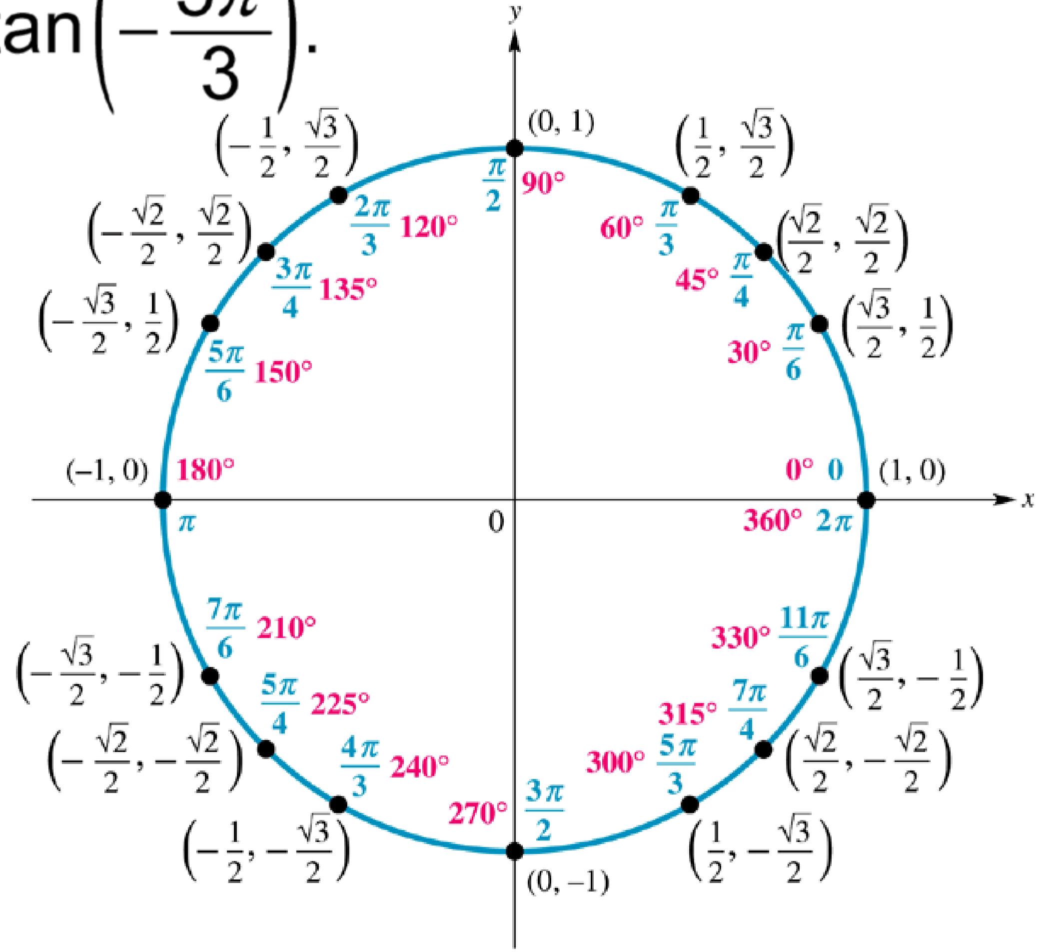


The unit circle $x^2 + y^2 = 1$

Example 2(b) FINDING EXACT CIRCULAR FUNCTION VALUES

Use the figure and the definition of the tangent to find the exact value of $\tan\left(-\frac{5\pi}{3}\right)$.

Moving around the unit circle $\frac{5\pi}{3}$ units in the negative direction yields the same ending point as moving around the circle $\frac{\pi}{3}$ units in the positive direction.



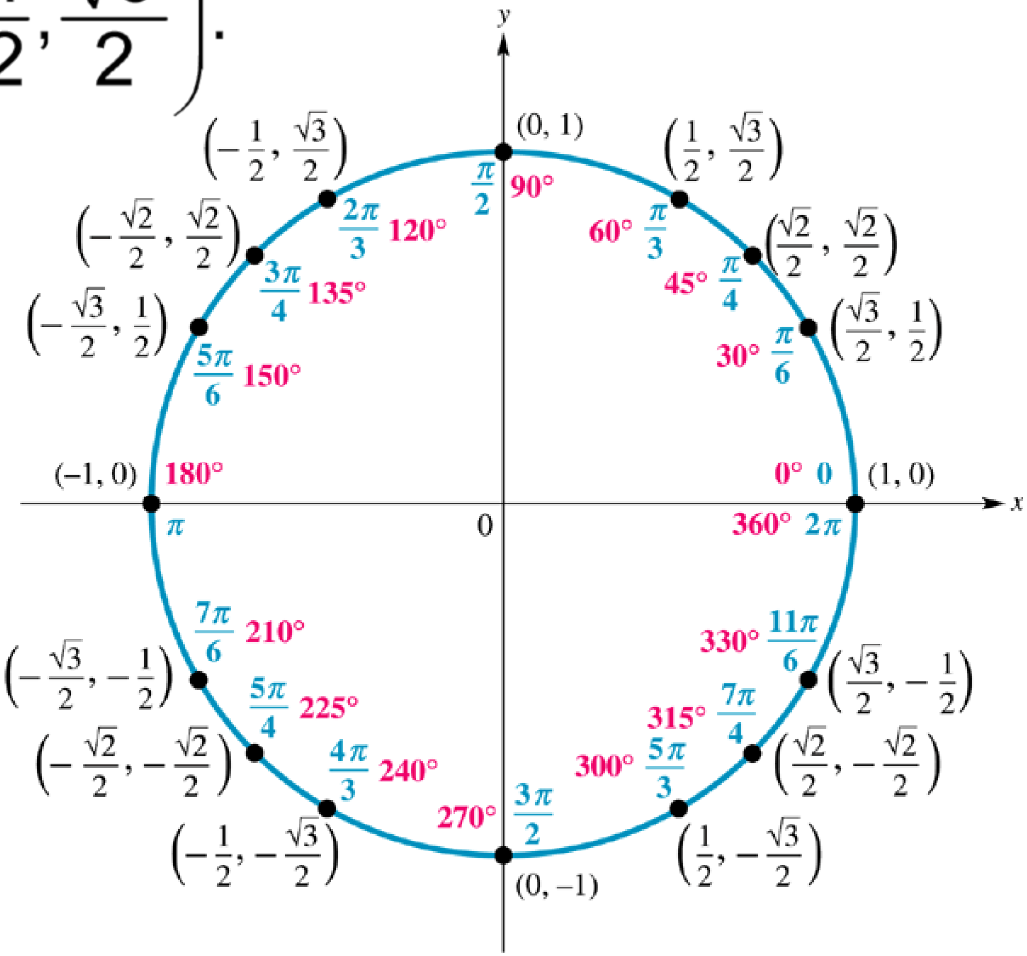
The unit circle $x^2 + y^2 = 1$

Example 2(b) FINDING EXACT CIRCULAR FUNCTION VALUES

$-\frac{5\pi}{3}$ corresponds to $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

$$\tan\theta = \frac{y}{x}$$

$$\tan\left(-\frac{5\pi}{3}\right) = \tan\frac{\pi}{3} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$



The unit circle $x^2 + y^2 = 1$

Example 2(c) FINDING EXACT CIRCULAR FUNCTION VALUES

Use reference angles and radian-to-degree conversion to find the exact value of $\cos\dfrac{2\pi}{3}$.

An angle of $\dfrac{2\pi}{3}$ radians corresponds to an angle of 120°.

In standard position, 120° lies in quadrant II with a reference angle of 60°.

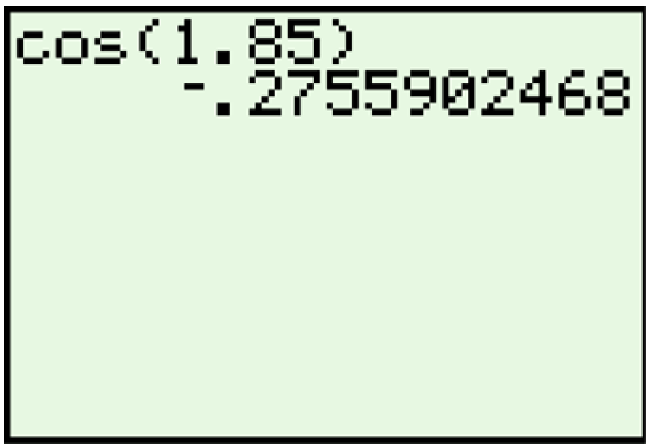
$$\cos\frac{2\pi}{3} = \cos 120° = -\cos 60° = -\frac{1}{2}$$

Cosine is negative in quadrant II.

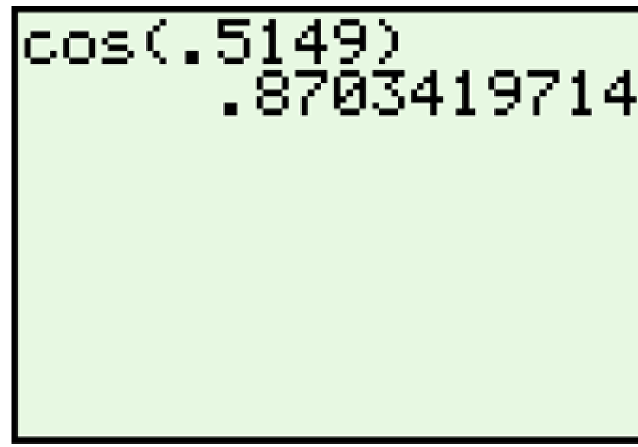
▶ Example 3 APPROXIMATING CIRCULAR FUNCTION VALUES

Find a calculator approximation for each circular function value.

(a) cos 1.85 ≈ –0.2756



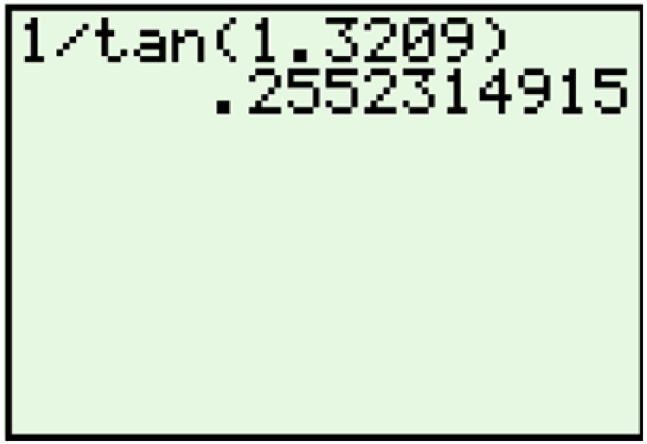
(b) cos 0.5149 ≈ 0.8703



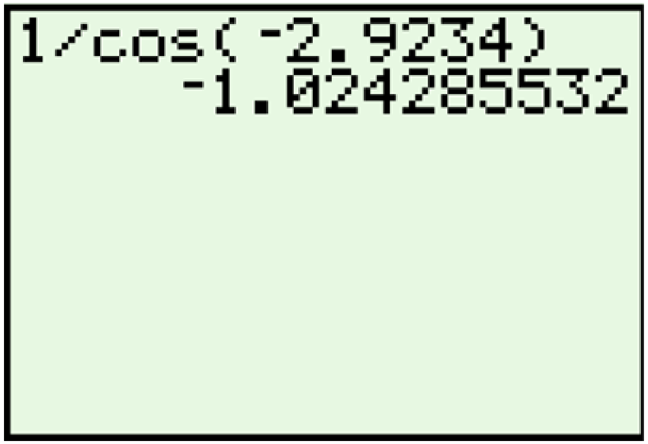
Example 3 APPROXIMATING CIRCULAR FUNCTION VALUES (continued)

Find a calculator approximation for each circular function value.

(c) cot 1.3209 ≈ 0.2552



(d) sec –2.9234 ≈ –1.0243



Caution

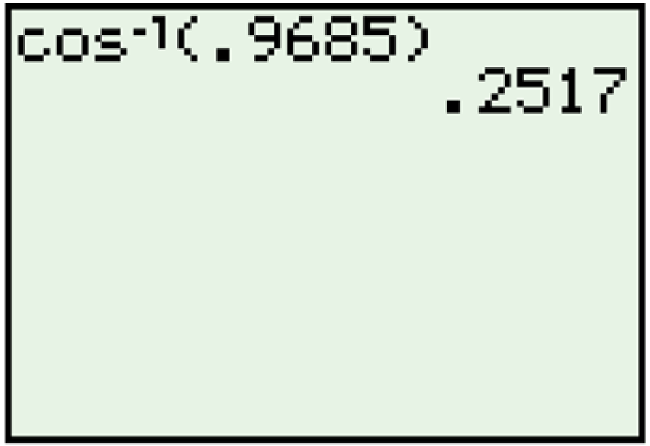
A common error is using a calculator in degree mode when radian mode should be used.

Remember, if you are finding a circular function value of a real number, the calculator must be in radian mode.

Example 4(a) FINDING A NUMBER GIVEN ITS CIRCULAR FUNCTION VALUE

Approximate the value of *s* in the interval $\left[0, \frac{\pi}{2}\right]$, if cos *s* = 0.9685.

Use the *inverse cosine* function of a calculator.

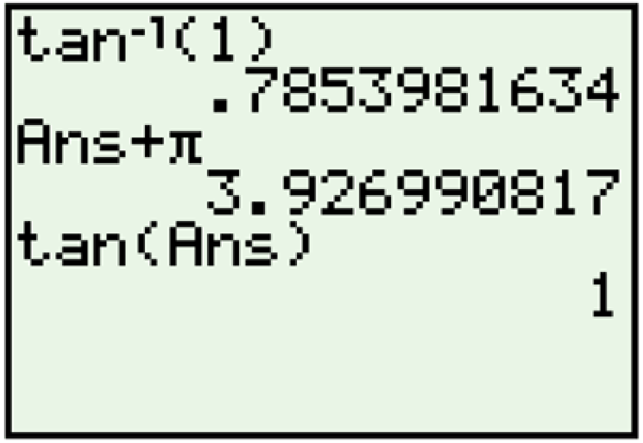


The screen indicates that the real number in $\left[0, \frac{\pi}{2}\right]$ whose cosine is 0.9685 is 0.2517.

Example 4(b) FINDING A NUMBER GIVEN ITS CIRCULAR FUNCTION VALUE

Find the exact value of *s* in the interval $\left[\pi, \frac{3\pi}{2}\right]$, if tan *s* = 1.

Recall that $\tan\frac{\pi}{4} = 1$, and in quadrant III, tan *s* is positive.



$$\tan\left(\pi + \frac{\pi}{4}\right) = \tan\frac{5\pi}{4} = 1, \text{ so } s = \frac{5\pi}{4}.$$

Example 5 MODELING THE ANGLE OF ELEVATION OF THE SUN

The angle of elevation θ of the sun in the sky at any latitude L is calculated with the formula

$$\sin\theta = \cos D \cos L \cos\omega + \sin D \sin L$$

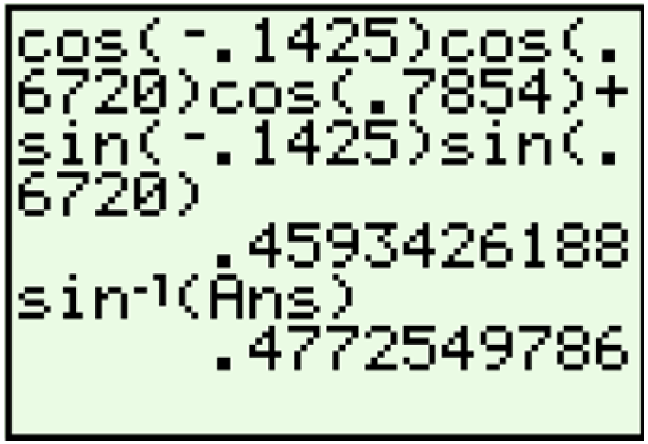
where $\theta = 0$ corresponds to sunrise and $\theta = \frac{\pi}{2}$ occurs if the sun is directly overhead. ω is the number of radians that Earth has rotated through since noon, when $\omega = 0$. D is the declination of the sun, which varies because Earth is tilted on its axis.

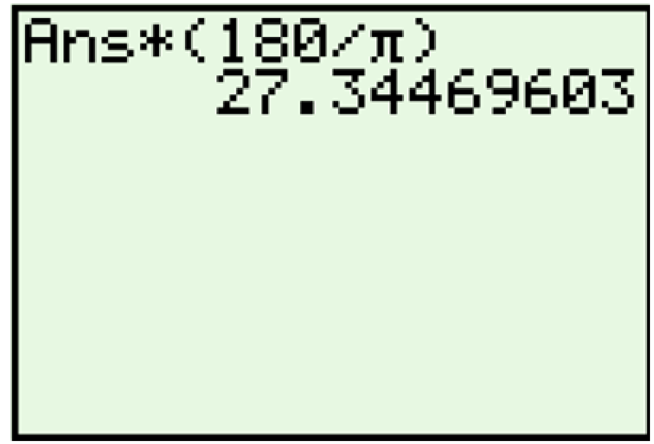
Example 5 MODELING THE ANGLE OF ELEVATION OF THE SUN (continued)

Sacramento, California, has latitude L = 38.5° or 0.6720 radian. Find the angle of elevation θ of the sun at 3 P.M. on February 29, 2012, where at that time, $D \approx -0.1425$ and $\omega \approx 0.7854$.

$$\begin{aligned}
\sin\theta &= \cos D \cos L \cos\omega + \sin D \sin L \\
&= \cos(-.1425)\cos(.6720)\cos(.7854) \\
&\qquad + \sin(-.1425)\sin(.6720) \\
&\approx .4593426188
\end{aligned}$$

$$\sin\,\theta \approx .4593426188 \Rightarrow \theta \approx .4773 \text{ radian}$$

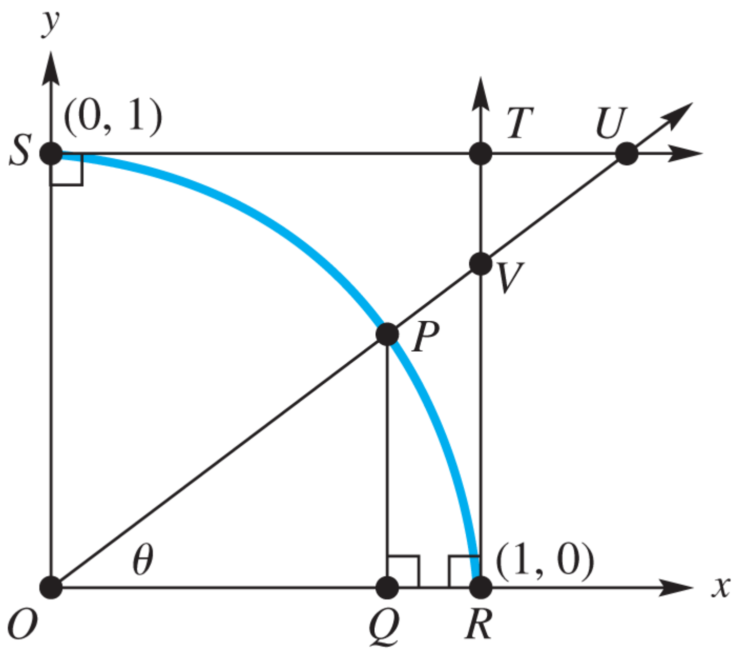




The angle of elevation of the sun is about 0.4773 radian or 27.3°.

Expressing Function Values as Lengths of Line Segments

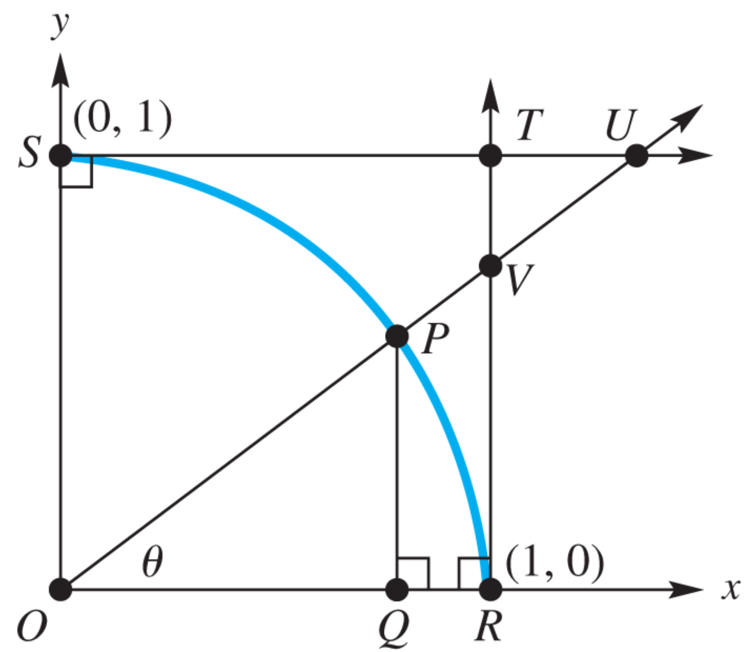
The figure illustrates a correspondence that ties together the right triangle ratio definitions of the trigonometric functions and the unit circle interpretation.



Expressing Function Values as Lengths of Line Segments

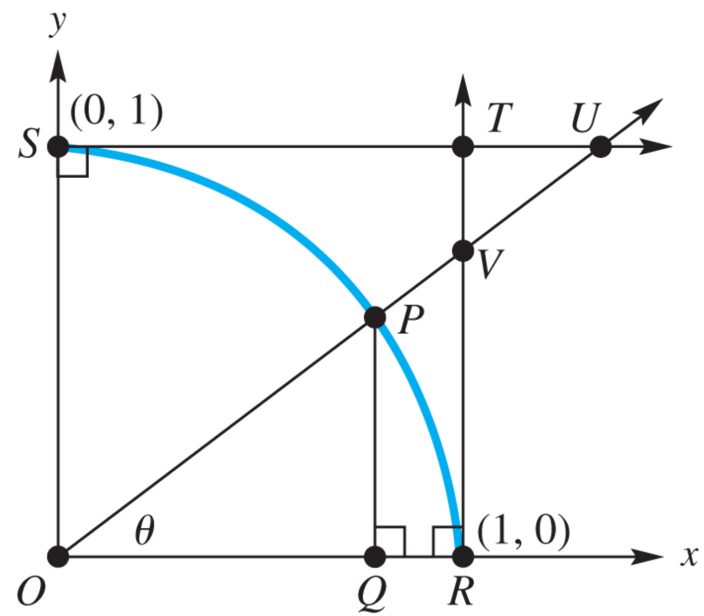
The arc *SR* is the first-quadrant portion of the unit circle, and the standard-position angle *POQ* is designated θ. By definition, the coordinates of *P* are $(\cos\theta, \sin\theta)$.

The six trigonometric functions of θ can be interpreted as lengths of line segments found in the figure.



Expressing Function Values as Lengths of Line Segments

For cos *θ* and sin *θ*, use right triangle *POQ* and right triangle ratios.

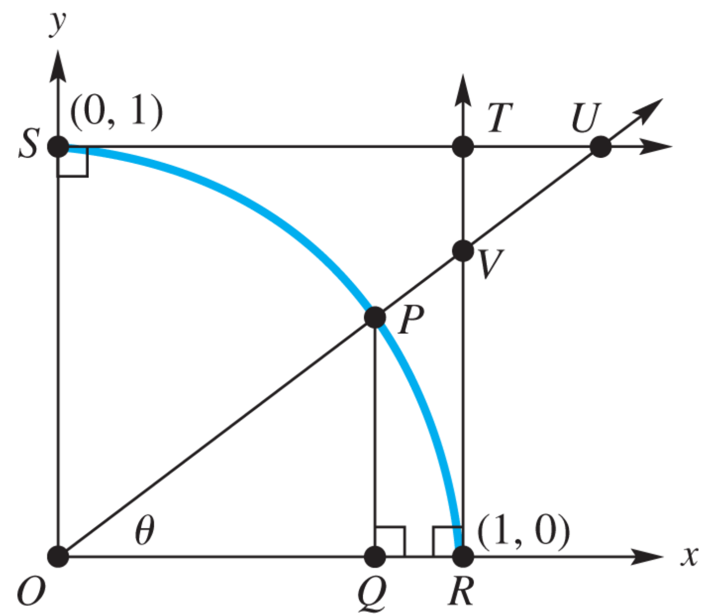


$$\cos\theta = \frac{\text{side adjacent to } \theta}{\text{hypotenuse}} = \frac{OQ}{OP} = \frac{OQ}{1} = OQ$$

$$\sin\theta = \frac{\text{side opposite } \theta}{\text{hypotenuse}} = \frac{PQ}{OP} = \frac{PQ}{1} = PQ$$

Expressing Function Values as Lengths of Line Segments

For tan θ and sec θ, use right triangle *VOR* and right triangle ratios.

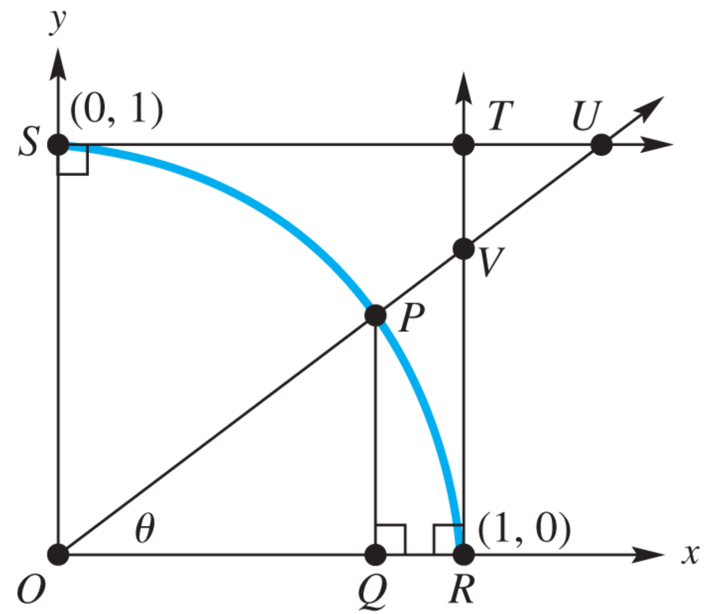


$$\tan\theta = \frac{\text{side opposite } \theta}{\text{side adjacent to } \theta} = \frac{VR}{OR} = \frac{VR}{1} = VR$$

$$\sec\theta = \frac{\text{hypotenuse}}{\text{side adjacent to } \theta} = \frac{OV}{OR} = \frac{OV}{1} = OV$$

Expressing Function Values as Lengths of Line Segments

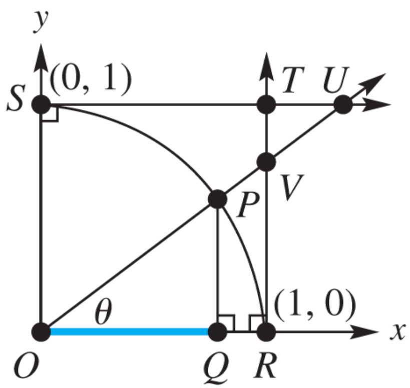
For csc θ and cot θ, first note that *US* and *OR* are parallel. Thus angle *SUO* is equal to θ because it is an alternate interior angle to angle *POQ*, which is equal to θ. Use right triangle *USO* and right triangle ratios.



$$\csc SUO = \csc\theta = \frac{\text{hypotenuse}}{\text{side opposite } \theta} = \frac{OU}{OS} = \frac{OU}{1} = OU$$

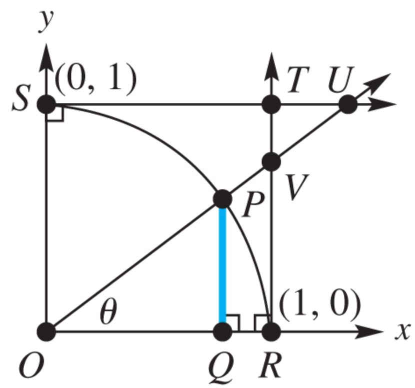
$$\cot SUO = \cot\theta = \frac{\text{side adjacent to } \theta}{\text{side opposite } \theta} = \frac{US}{OS} = \frac{US}{1} = US$$

Expressing Function Values as Lengths of Line Segments



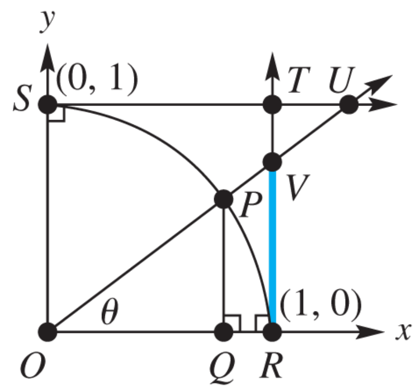
$\cos\theta = OQ$

(a)



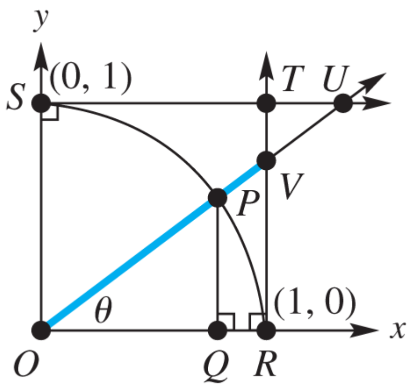
$\sin\theta = PQ$

(b)



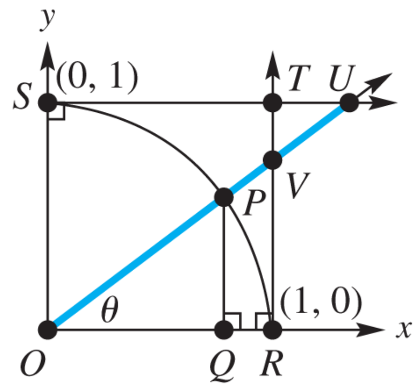
$\tan\theta = VR$

(c)



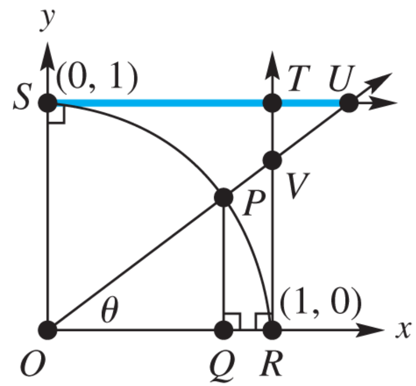
$\sec\theta = OV$

(d)



$\csc\theta = OU$

(e)



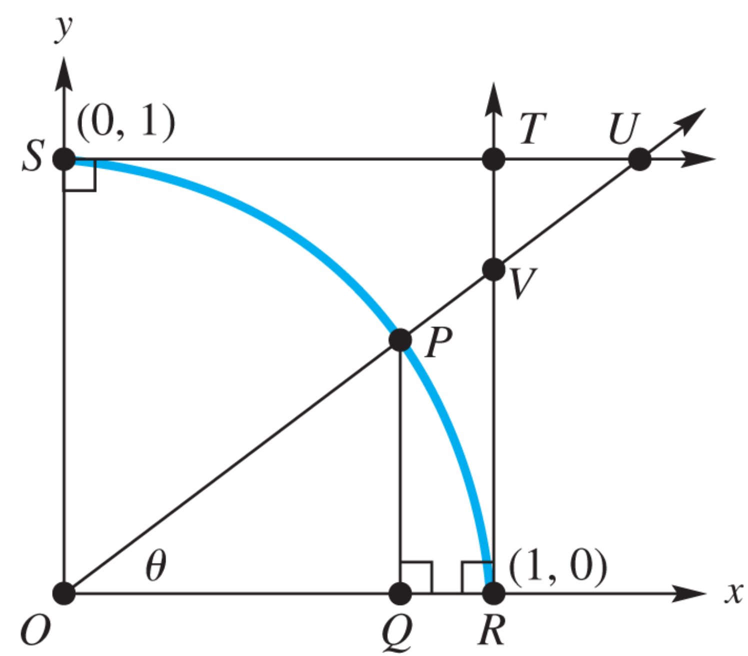
$\cot\theta = US$

(f)

Example 6 FINDING LENGTHS OF LINE SEGMENTS

Suppose that angle *TVU* measures 60º. Find the exact lengths of segments *OQ*, *PQ*, *VR*, *OV*, *OU*, and *US*.

Angle *TVU* has the same measure as angle *OVR* because they are vertical angles. Therefore, angle *OVR* measures 60º. Because it is one of the acute angles in right triangle *VOR*, *θ* must be its complement, measuring 30º.



Example 6 FINDING LENGTHS OF LINE SEGMENTS (continued)

Since $\theta = 30°$,

$$OQ = \cos 30° = \frac{\sqrt{3}}{2}$$

$$PQ = \sin 30° = \frac{1}{2}$$

$$VR = \tan 30° = \frac{\sqrt{3}}{3}$$

$$OV = \sec 30° = \frac{2\sqrt{3}}{3}$$

$$OU = \csc 30° = 2$$

$$US = \cot 30° = \sqrt{3}$$