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Computability and Fractal Dimension

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Abstrakt

Die vorliegende Arbeit verbindet Berechenbarkeitstheorie mit einigen Begriffen fraktaler Dimension. Ein algorithmischer Zugang zu Hausdorff-Maßen ermöglicht es, die Hausdorff-Dimension von einzelnen Punkten anstelle von Teilmengen eines metrischen Raumes zu definieren. Diese Idee wurde erstmals von Lutz (2000b) verwirklicht. Wir arbeiten hierbei im Cantorraum 2^ω, bestehend aus allen unendlichen Binärfolgen. Wir geben zunächst einen Überblick über die wichtigsten Definitionen und Eigenschaften der verschiedenen Begriffe fraktaler Dimension im Cantorraum. Danach entwickeln wir die Theorie der effektiven Dimension systematisch, auf der Grundlage des Zugangs zur algorithmischen Informationstheorie, welcher von Kolmogorov und seinen Schülern entwickelt wurde. Auf diese Weise können wir ein zentrales Resultat der effektiven Hausdorff-Dimension auf neue und einfache Weise herleiten: Die effektive Hausdorff-Dimension einer Folge entspricht ihrer unteren asymptotischen algorithmischen Entropie, definiert über Kolmogorov-Komplexität. Außerdem beweisen wir ein allgemeines Resultat hinsichtlich des Verhaltens von effektiver Hausdorff-Dimension unter r-expansiven Abbildungen, welche eine Verallgemeinerung von Hölder-Transformationen im Cantorraum darstellen. Wir untersuchen den Zusammenhang zwischen anderen effektiven Dimensionsbegriffen und algorithmischer Entropie. Darüberhinaus können wir zeigen, daß die Menge aller Folgen, welche effektive Hausdorff-Dimension s besitzen, Hausdorff-Dimension s sowie unendliches s-dimensionales Hausdorff-Maß hat (für $0 < s < 1$).

Es folgt eine Untersuchung der Hausdorff-Dimension (klassisch wie effektiv) von Objekten, welche in der Berechenbarkeitstheorie auftreten. Wir beweisen, daß der obere Kegel einer Folge bezüglich jeder der üblichen Reduzierbarkeiten Hausdorff-Dimension 1 besitzt und geben auf diese Weise ein Beispiel für eine Lebesgue-Nullmenge maximaler Dimension. Ferner benutzen wir die Resultate bezüglich effektiver r-expansiver Transformationen um zu zeigen, daß die effektive Dimension eines Grades der des zugehörigen unteren Kegels entspricht. Für Many-one-Reduzierbarkeit beweisen wir die Existenz eines unteren Kegels nicht-ganzzahliger effektiver Hausdorff-Dimension. Schließlich folgt der Beweis, daß effektiv-abgeschlossene Mengen $\mathcal{A} \subseteq 2^\omega$ positiver Hausdorff-Dimension eine berechenbare, surjektive Abbildung $\mathcal{A} \to 2^\omega$ erlauben.

Danach wenden wir uns einem genaueren Studium der komplexen Wechselbeziehung zwischen algorithmischer Entropie, Zufälligkeit, effektiver Hausdorff-Dimension und Reduzierbarkeit zu. Zu diesem Zweck führen wir eine verallgemeinerte Version der effektiven Hausdorff-Dimension ein, über den Begriff des starken effektiven Hausdorff-Maßes 0. Wir können zeigen, daß die Tatsache, daß eine Folge nicht starkes effektives Hausdorff-Maß 0 hat, nicht notwendig die Möglichkeit impliziert, aus dieser Folge eine Martin-Löf-zufällige Folge zu berechnen, eine Folge höchstmöglicher algorithmischer Entropie. Außerdem zeigen wir, daß eine Verallgemeinerung des effektiven Zufälligkeitsbegriffes auf nicht-berechenbare Maße einen sehr umfassenden Zufälligkeitsbegriff zur Folge hat: Jede nicht-berechenbare Folge ist zufällig bezüglich eines Maßes.

Es folgt die Einführung von Schnorr-Dimension, einem algorithmisch restriktiveren Dimensionsbegriff als der effektiven Dimension. Wir leiten eine Maschinencharakterisierung der Schnorr-Dimension her und zeigen, daß für rekursiv-aufzählbare Mengen Schnorr-Hausdorff-Dimension und Schnorr-Packing-Dimension nicht notwendig übereinstimmen.

stimmen müssen, im Gegensatz zur effektiven Dimension.

Desweiteren untersuchen wir subrekursive Dimensionsbegriffe. Unter der Verwendung von ressourcenbeschränkten Martingalen können wir die Höldertransformationstechniken auf den ressourcenbeschränkten Fall übertragen und können damit zeigen, daß das Small-Span-Theorem im Fall der Hausdorff-Dimension in Exponentialzeit E nicht gilt.

Schließlich studieren wir die effektive Hausdorff-Dimension von Folgen, gegen die keine berechenbare, nichtmonotone Wettstrategie gewinnt. Berechenbare, nichtmonotone Wettspiele sind eine Verallgemeinerung von berechenbaren Martingalen, und es ist ein offenes Problem, ob der darüber definierte Zufallsbegriff mit Martin-Löf-Zufälligkeit zusammenfällt. Wir zeigen, daß die Folgen, welche zufällig sind bzgl. berechenbarer, nichtmonotoner Wettspiele, effektive Hausdorff-Dimension 1 haben müssen, was impliziert, daß, unter dem Gesichtspunkt algorithmischer Entropie, solche Folgen recht nahe an Martin-Löf-Zufälligkeit sind.

Abstract

This thesis combines computability theory and various notions of fractal dimension, mainly Hausdorff dimension. An algorithmic approach to Hausdorff measures makes it possible to define the Hausdorff dimension of individual points instead of sets in a metric space. This idea was first realized by Lutz (2000b). Working in the Cantor space 2^ω of all infinite binary sequences, we study the theory of Hausdorff and other dimensions for individual sequences. After giving an overview over the classical theory of fractal dimension in Cantor space, we develop the theory of effective Hausdorff dimension and its variants systematically. Our presentation is inspired by the approach to algorithmic information theory developed by Kolmogorov and his students. We are able to give a new and much easier proof of a central result of the effective theory: Effective Hausdorff dimension coincides with the lower asymptotic algorithmic entropy, defined in terms of Kolmogorov complexity. Besides, we prove a general theorem on the behavior of effective dimension under r-expansive mappings, which can be seen as a generalization of Hölder mappings in 2^ω. Furthermore, we study the connections between other notions of effective fractal dimension and algorithmic entropy. Besides, we are able to show that the set of sequences of effective Hausdorff dimension s has Hausdorff dimension s and infinite s-dimensional Hausdorff measure (for every $0 < s < 1$).

Next, we study the Hausdorff dimension (effective and classical) of objects arising in computability theory. We prove that the upper cone of any sequence under a standard reducibility has Hausdorff dimension 1, thereby exposing a Lebesgue nullset that has maximal Hausdorff dimension. Furthermore, using the behavior of effective dimension under r-expansive transformations, we are able to show that the effective Hausdorff dimension of the lower cone and the degree of a sequence coincide. For many-one reducibility, we prove the existence of lower cones of non-integral dimension. After giving some 'natural' examples of sequences of effective dimension 0, we prove that every effectively closed set $\mathcal{A} \subseteq 2^\omega$ of positive Hausdorff dimension admits a computable, surjective mapping $\mathcal{A} \to 2^\omega$.

We go on to study the complex interrelation between algorithmic entropy, randomness, effective Hausdorff dimension, and reducibility more closely. For this purpose we generalize effective Hausdorff dimension by introducing the notion of strong effective Hausdorff measure 0. We are able to show that not having strong effective Hausdorff measure 0 does not necessarily allow to compute a Martin-Löf random sequence, a sequence of highest possible algorithmic entropy. Besides, we show that a generalization of the notion of effective randomness to noncomputable measures yields a very coarse concept of randomness in the sense that every noncomputable sequence is random with respect to some measure.

Next, we introduce Schnorr dimension, a notion of dimension which is algorithmically more restrictive than effective dimension. We prove a machine characterization of Schnorr dimension and show that, on the computably enumerable sets, Schnorr Hausdorff dimension and Schnorr packing dimension do not coincide, in contrast to the case of effective dimension.

We also study subrecursive notions of effective Hausdorff dimension. Using resource-bounded martingales, we are able to transfer the use of r-expansiveness to the resource-bounded case, which enables us to show that the Small-Span Theorem does not hold for

dimension in exponential time E.

Finally, we investigate the effective Hausdorff dimension of sequences against which no computable *nonmonotonic* betting strategy can succeed. Computable nonmonotonic betting games are a generalization of computable martingales, and it is a major open question whether the randomness notion induced by them is equivalent to Martin-Löf randomness. We are able to show that the sequences which are random with respect to computable nonmonotonic betting games have effective Hausdorff dimension 1, which implies that, from the viewpoint of algorithmic entropy, they are rather close to Martin-Löf randomness.

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Preface

This thesis brings together two important areas of mathematics that originated in the 20th century: Geometric measure theory and the theory of computability.

Geometric measure theory extends Lebesgue's ground-breaking work on measure theory and has become one of the mathematical foundations of fractal geometry, which in recent decades has received increased attention. Computability theory, on the other hand, laid the ground for accompanying theoretical studies of the development of computers and algorithms.

The combination of measure theory and computability is, of course, anything but a new idea. It has become a framework in which the theory of algorithmic randomness has developed into a rich subject. This theory arose from the problem of defining *individual random objects*. Modern probability theory, based on Kolmogorov's axiomatic formulation, does not allow for distinguishing single outcomes of chance experiments as random, although there is some intuitive appeal to the idea of a *typical outcome* of a sequence of unbiased coin tosses, for example. In the framework of probability theory, typicalness is expressed by exposing certain properties that hold *almost surely*, i.e., the set of outcomes possessing this property has measure one.

If one wants to define an object, usually an infinite binary sequence, as *random* or *typical* (with respect to a measure) by requiring it to be contained in every subset of measure 1, it is obvious that randomness in this *absolute sense does not exist*: The intersection of all measure 1 sets is empty.

However, if we restrict the class of *typical properties* (i.e. properties which hold almost surely) to a countable one, the intersection of these properties is not empty.

It was Per Martin-Löf's idea (1966) to obtain this countable restriction by admitting only properties that can be defined in an *algorithmically effective* way, by what we refer to today as a Martin-Löf test. Roughly speaking, a Martin-Löf test is a nullset that is defined in an effective way, by requiring that each level of the test (an open set of measure at most 2^{-n}) is uniformly recursively enumerable. This way, there are only countably many Martin-Löf tests, and an infinite sequence is random if it is not contained in any such nullsets.

It can be shown that most laws of probability (such as the Law of Large Numbers, the Law of the Iterated Logarithm, etc.) can be described via a Martin-Löf test, so typical sequences in the sense of Martin-Löf share the common laws of probability.

This approach, interpreting typicalness as algorithmic randomness, follows a tradition that started with Church's definition (1940) of stochasticity in terms of *computable selection functions*. (We will not dwell further on the history of randomness and stochasticity here, but refer the reader to Ambos-Spies and Kučera (2000) instead.)

Martin-Löf's concept of randomness found a significant complement in the theory of *algorithmic entropy* or *Kolmogorov complexity* (Kolmogorov, 1965, later Zvonkin and Levin, 1970, Chaitin, 1975, and Gács, 1974). Instead of following the paradigm of randomness as typicalness, one could see randomness as *chaoticness*. In algorithmic information theory, the degree of chaoticness is measured by the *descriptive complexity* of an object. An easy object will allow descriptions shorter than itself, the information contained in a chaotic (random) one cannot be compressed in this way.

In 1919, the same year that Richard von Mises undertook an effort to develop a theory of probability based on individual random objects, Felix Hausdorff introduced a generalization of Lebesgue measure nowadays known as *Hausdorff measure*. Hausdorff's idea was to supplement Lebesgue's translation invariant measure on Euclidean space $\mathbb{R}^n$ by a whole family of measures possessing similar qualities. He achieved this by varying the scaling factor by which the diameter of a set corresponds to its 'volume' in the underlying space (for Lebesgue measure, this is just the dimension of the underlying space).

By assigning each set the most suitable Hausdorff measure (the one with right 'scaling factor'), one can assign every set a dimension, the *Hausdorff dimension*, a not necessarily integral nonnegative real number, which coincides with topological dimension for a lot of 'regular' sets like open sets, smooth submanifolds, etc. However, there are sets, like the *middle-third Cantor set* in $[0, 1]$, that have non-integral dimension. Such sets are now called *fractal*.

Eggleston (1949) discovered that there is a close connection between (measure theoretic) *entropy* and Hausdorff dimension. In particular, he devised a whole class of fractal objects by studying limit frequencies of binary sequences. He showed that the Hausdorff dimension of the set of infinite binary sequences with limit frequency $p \in [0, 1]$ is equal to the entropy of the Bernoulli measure induced by p.

Kolmogorov complexity behaves in many respects similar to entropy. This is reflected in works of Ryabko (1984, 1986), Staiger (1989, 1993), and Cai and Hartmanis (1994), where a close link between algorithmic entropy and Hausdorff dimension is established.

It is possible to apply Martin-Löf 's effectivization of measure theory to Hausdorff measures. This was first done by Lutz (2000b, 2003). This way, it is possible to define a notion of *Hausdorff dimension for individual sequences*. Building on the earlier work cited above, it is possible to characterize the effective dimension of a sequence as its *lower asymptotic algorithmic entropy* (Mayordomo, 2002). Consequently, the dimension of an infinite sequence can not only be interpreted in terms of fractal geometry, but also as a *degree of randomness* of the sequence.

The goal of this thesis is to study the notion of Hausdorff dimension for individual sequences in a comprising fashion. We will develop the theory systematically and investigate the existence of fractal objects in the realm of the theory of computation. Special emphasis is put on the application of fractal geometric methods in the effective setting. Besides, we will try to shed further light on the complex and deep interplay between (Hausdorff) measures, randomness (in its various forms), and entropy.

Overview

In Chapter 1 we give a fairly self-contained introduction to Hausdorff measures and Hausdorff dimension. After shortly dealing with the general definitions, which work in arbitrary metric spaces, we turn our attention to the space of infinite binary sequences 2^ω, also known as the Cantor space. This space is a natural setting when dealing with algorithmic aspects, since subsets of natural numbers (for which notions of effectiveness are usually defined) can be identified with their characteristic function and hence with infinite binary sequences. Furthermore, 2^ω is compact and thus allows to employ a lot of useful techniques such as König's Lemma.

The Cantor space allows a particularly elegant presentation of Hausdorff measures and dimension. We present the cornerstones of the theory, some standard properties and examples, and a few generalizations of classical results that are possible when working in 2^ω. We also take a look at some other notions of fractal dimension such as box-counting and packing dimension. Finally, we explain some important correspondences between fractal dimensions and various notions of entropy, one of which carries over to the effective setting in a most striking fashion, as we will see in Chapter 2.

Chapter 2 develops the theory of effective Hausdorff dimension from scratch. We will follow the approach of Martin-Löf, who defined effective Lebesgue measure to obtain a notion of algorithmic randomness. We will contrast this with two equivalent formulations, one generalizing an approach of Solovay using simpler types of covers, the other based on enumerable semimeasures, which have been used by Levin and others to characterize algorithmic randomness.

The theory of semimeasures establishes, due to the fundamental Coding Theorem, a connection between effective measure theory and algorithmic information theory. This connection will lead to one of the main theorems on effective Hausdorff dimension, namely that the effective dimension of a sequence coincides with its lower normalized algorithmic entropy. We are able to give a new, particularly easy and elegant proof of this remarkable identity.

Next, we present effective versions of other notions of dimension as introduced in Chapter 2. Furthermore, we study to what extent the basic properties of the various concepts of fractal dimension carry over to the effective setting. We give some important examples of effective Hausdorff dimension, examples that serve as a paradigm throughout the text. Finally, we study the class of sequences of a fixed effective dimension (an analog to the class of Martin-Löf random sequences).

In Chapter 3 we investigate the relation between effective dimension and the principal notions of computability theory, such as the various reducibilities. We will prove that any upper cone (under any reducibility) has (classical) Hausdorff dimension 1, which contrasts a result by Sacks, who showed that the upper Turing cone of any non-recursive set has Lebesgue measure 0. This will be followed by a study of the dimension of joins of two sequences. Here we will consider a generalized notion of joins, since from the dimension viewpoint the arrangement in which order two sequences are coded into a new one is important, while it is certainly negligible from a purely computability theoretic perspective, as long as the coding is done computably.

The results on generalized joins allow us to devise a many-one lower cone of non-integral dimension. For weaker reducibilities, this problem turns out more intrinsic and will be dealt with in Chapter 4. After giving a number of examples of zero-dimensional sequences from the realm of computability theory, we devote the rest of the chapter to the generalization of a powerful result of Gács and Kučera, who independently proved that any Π^0_1-class of positive Lebesgue measure can be mapped onto 2^ω by means of an effective process. In particular, this means that any sequence is Turing reducible to a Martin-Löf random sequence. We generalize the result to Π^0_1-classes of positive Hausdorff dimension. We note that the Gács-Kučera argument works (under an accordant increase of redundancy in the coding) with Lebesgue measure replaced by any suitable computable measure. We make use of this fact by showing that any Π^0_1-class of positive Hausdorff dimension has positive measure with respect to some computable measure sufficiently 'similar' to Lebesgue measure.

The relations established in the first three chapters suggest to take a closer look at the general interplay between measures and entropy. This is done in Chapter 4. The results in the final part of Chapter 3 might lead to the hypothesis that any sequence of positive effective Hausdorff dimension, i.e., any sequence whose entropy can be bounded effectively from below, is random with respect to some computable measure. This would be a very strong property, since it would imply the ability to compute a Martin-Löf random sequence. This would in turn imply the non-existence of lower Turing-cones of non-integral effective Hausdorff dimension. However, we will see that the hypothesis is not true: For every Δ_2^0-computable real s there is a sequence of effective Hausdorff dimension s that is not random with respect to any computable measure.

On the other hand, being random with respect to a computable measure (in a non-trivial way, that is, the measure is not concentrated on the sequence) does not imply non-trivial entropy, as we construct a computable measure and a sequence random with respect to that measure such that the Kolmogorov complexity of the random sequence cannot be bounded from below by a nonconstant, nondecreasing, computable function.

Sequences whose complexity is bounded from below by such a function can be seen as having positive effective dimension in a generalized way. We will see that positive dimension in this generalized sense does not guarantee to compute a Martin-Löf random sequence.

Last, one could ask if sequences of positive dimension can be rendered random by allowing arbitrary measures instead of only computable ones. However, this generalization of Martin-Löf randomness turns out to be not restrictive enough, as we prove that *any nonrecursive sequence* is in fact random with respect to some measure in a non-trivial way.

Chapters 5 and 6 are devoted to the study of dimension notions one obtains by further restricting the computational resources available to detect the complexity of a sequence.

Chapter 5 uses the concept of Schnorr tests, which arose from Schnorr's criticism of Martin-Löf's notion of effective measure, to define accordant dimension notions. As in the case of Schnorr randomness, this concept will differ significantly from Martin-Löf randomness in some aspects. For instance, we will be able to show that for Schnorr dimension there can be computably enumerable sets whose characteristic sequence have high upper entropy, i.e. Schnorr packing dimension, which is impossible for effective packing dimension, due to a theorem of Barzdins.

Furthermore, it will be interesting to note that the restriction to Schnorr tests yields the same dimension concept as the restriction to computable martingales, in contrast to the corresponding randomness notions, which are known to differ.

If one further restricts the computational power available, one can define various resource bounded dimensions. They are defined via martingales that have to be computable within a given resource-bound, usually a time or space bound. We will deal with time bounded variants of Hausdorff dimension, as first introduced by Lutz (2000a) on the basis of time bounded martingales. In particular, we will restrict ourselves to the major two exponential time classes E and EXP.

We show that many results obtained in the effective case carry over to the resource bounded setting, although they will need new proofs due to the time bounds present. We will prove a dimension conservation theorem similar to the one obtained in Chapter 3. This allows us again to prove that the dimension of a polynomial time degree and a polynomial time lower cone coincide (with respect to various reducibilities). It follows that an analog to the Small Span Theorem of Juedes and Lutz (1995) does not hold for resource bounded dimension.

In the last chapter, we will deal with the relation between effective dimension and a randomness concept defined in terms of non-monotonic betting games. These are a generalization of martingales in the sense that the betting strategy underlying a martingale is no longer required to bet against a sequence in an increasing order of positions. If these betting strategies are required to be *computable*, one obtains a corresponding randomness concept, called *Kolmogorov-Loveland randomness* (KL-randomness, for short). KL-randomness is stricter than computable randomness, so the use of non-monotonicity does indeed make a difference. It is one of the major open questions in the study of algorithmic randomness whether KL-randomness is actually equivalent to Martin-Löf randomness. The main result of Chapter 7 will be that, to some extent, KL-random sequences are close to being Martin-Löf random, by showing that they must have effective dimension 1.

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Contents

CHAPTER 1

Measure and Dimension in the Cantor Space

1.1 The Cantor Space

The *Cantor space* 2^ω consists of all infinite sequences of zeros and ones. Formally, it is the product space of (countably) infinitely many copies of $\{0, 1\}$. An element of 2^ω will usually be called a *sequence* and be denoted by upper case letters like A, B, C, or X, Y, Z. We will refer to the nth bit ($n \geq 0$) in a sequence B by either B_n or $B(n)$, i.e. $B = B_0 B_1 B_2 \ldots = B(0)\, B(1)\, B(2) \ldots$.

As a sequence can also be seen as the characteristic function of a subset of $\mathbb{N}$, which is common in computability theory, in some contexts we will refer to sequences as *sets*. For the same reason, subsets of 2^ω will often be called *classes*. To distinguish them from sequences, subclasses of 2^ω will always be denoted by calligraphic capital letters like $\mathcal{A}$, $\mathcal{B}$, $\mathcal{X}$, $\mathcal{Y}$.

The *complement* of a class $\mathcal{A} \subseteq 2^\omega$ is denoted by $\mathcal{A}^{\complement}$, whereas, for a sequence $B \in 2^\omega$, the complement of B is the complement of B as a *subset of the natural numbers* and is denoted by $\overline{B}$, that is, $\overline{B}$ is the sequence obtained from B by flipping each bit in B.

A *string* is a finite sequence of 0s and 1s. We will use lower case letters from the end of the alphabet, u, v, w, x, y, z, to denote strings, along with some lower case Greek letters like σ and τ. $2^{<\omega}$ will denote the set of all strings. The *initial segment of length n*, $A\lceil_n$, of sequence A is the string of length n corresponding to the first n bits of A. For a class $\mathcal{A} \subseteq 2^\omega$, $\mathcal{A}\lceil_n$ will denote the set of all initial segments induced by sequences in $\mathcal{A}$: $\mathcal{A}\lceil_n = \{\sigma \in \{0,1\}^n : \sigma \sqsubset X \text{ for some } X \in \mathcal{A}\}$.

Given two strings v, w, v is called a *prefix* of w, $v \sqsubseteq w$ for short, if there exists a string x such that $v^\frown x = w$, where $^\frown$ denotes the concatenation of two strings. We will often suppress this and write simply vx, for example. If $v \sqsubseteq w$ and $v \neq w$, we will write $v \sqsubset w$. The same relation can be defined between strings and infinite sequences in an obvious way. Given a string σ and a class $\mathcal{A}$, we write $\sigma \sqsubset \mathcal{A}$ to denote that there exists a sequence $A \in \mathcal{A}$ such that $\sigma \sqsubset A$.

Furthermore, we define the *longest common initial segment* $v \sqcap w$ of two strings (or, analogously, of two sequences) to be the longest σ such that $\sigma \sqsubseteq v$ and $\sigma \sqsubseteq w$. Two strings v, w for which $v \sqcap w$ is one of v, w are called *comparable*. A set of strings is called *prefix free* if all its elements are pairwise incomparable.

We will often make use of a canonical correspondence between strings and natural numbers, given by the *length-lexicographical ordering* of $2^{<\omega}$: For two strings v, w, we say that $v <_{ll} w$ if $|v| < |w|$, or $|v| = |w| = n$ and $\sum_{i=0}^{n-1} v_i 2^{i+1} < \sum_{i=0}^{n-1} w_i 2^{i+1}$. Given $n \in \mathbb{N}$, s_n will denote the n-th string under this ordering, whereas, for any string $w \in 2^{<\omega}$, n_w denotes the position of w in the length-lexicographical chain of $2^{<\omega}$.

Initial segments induce a standard topology on 2^ω. The basis of the topology is formed by the *basic open cylinders* (or just *cylinders*, for short). Given a string $w = w_0 \dots w_{n-1}$ of length n, these are defined as

$$[w] = \{A \in 2^\omega : A\restriction_n = w\}.$$

On 2^ω this induces the *product topology* of the discrete topology on $\{0, 1\}$ (i.e. every subset of $\{0, 1\}$ is open).

A compatible *metric* is given by

$$d(A, B) = 2^{-N} \quad \text{where } N = \min\{n : A_n \neq B_n\}.$$

(If $A = B$, we set $d(A, B) = 0$, of course.) Note that this metric is actually an *ultrametric*, that is, it holds that

$$d(X, Z) \leq \max\{d(X, Y), d(Y, Z)\} \text{ for all } X, Y, Z \in 2^\omega.$$

1.1.1

2^ω as a metric space

In the following we list (without proof) some basic facts about the Cantor space 2^ω as a metric (topological) space. For proofs refer to Kechris (1995).

Theorem 1.1 *Every nonempty compact metrizable space is a continuous image of* 2^ω.

A metric space is called *Polish* if it is complete and separable, i.e. has a countable dense subset. A space is *perfect* if all its point are limit points.

Theorem 1.2 2^ω *can be embedded into every nonempty perfect Polish space, that is, there exists a subset which is homeomorphic to* 2^ω.

A topological space X is *connected* if there is no partition $X = U \cup V$, where U, V are disjoint, nonempty open sets. Equivalently, X is connected if the only *clopen* (i.e. open and closed) sets are $\emptyset$ and X. A space is called *zero-dimensional* if it is Hausdorff and has a basis of clopen sets.

The following topological characterization of 2^ω as a zero-dimensional space was shown by Brouwer.

Theorem 1.3 (Brouwer) *The Cantor space* 2^ω *is the unique (up to homeomorphism) non-empty, perfect, compact metrizable, zero-dimensional space.*

1.1.2

2^ω and the unit interval [0, 1]

It follows from Theorem 1.2 that there exists an embedding of 2^ω into the unit interval [0, 1], since this is obviously a perfect Polish space. Probably the most famous of such embeddings is the *middle third Cantor set* $\mathcal{C}_{1/3}$, also known as *Cantor's discontinuum*, consisting of all ternary expansions of real numbers between 0 and 1 having only 0 and 2 as coefficients:

$$\mathcal{C}_{1/3} = \left\{ y \in [0,1] : \; y = \sum_{i=0}^{\infty} \frac{y_i}{3^{i+1}}, \; y_i \in \{0,2\} \text{ for all } i \right\}$$

On the other hand, elements of 2^ω can be interpreted as binary expansions of real numbers from [0, 1] via the mapping $x \mapsto \sum_{i=0}^{\infty} x_i/2^{i+1}$. This mapping is continuous, thereby giving an example of a function whose existence is ensured by Theorem 1.1. It is not an homeomorphism for it is not injective: $1000\ldots$ and $0111\ldots$, for instance, both map to $1/2$.

1.2 Measure on the Cantor Space

Measure theory on 2^ω can be formulated quite conveniently, due to the special topological structure. For our purpose, it suffices to consider outer measures. Outer measures are set (class) functions defined on every subset of the Cantor space, satisfying a monotonicity and sub-additivity requirement. The Caratheodory approach to measure theory singles out a family of sets, the *measurable sets*.

A common starting point for constructing outer measures are special set functions, which occasionally are called *pre-measures*. Let $\mathfrak{F}$ be a family of subsets of 2^ω with $\emptyset \in \mathfrak{F}$, let $\rho : \mathfrak{F} \to [0,\infty]$ such that $\rho(\emptyset) = 0$.

Given $\mathcal{A} \subseteq 2^\omega$, a countable ensemble $\{\mathcal{C}_n\}_{n\in\mathbb{N}}$ is an $\mathfrak{F}$-*covering* of $\mathcal{A}$, if, for all n, $\mathcal{C}_n \in \mathfrak{F}$ and $\mathcal{A} \subseteq \bigcup_n \mathcal{C}_n$. Now define

$$\mu(\mathcal{A}) = \inf\left\{ \sum_{n=0}^{\infty} \rho(\mathcal{C}_n) : \; \{\mathcal{C}_n\} \text{ is an } \mathfrak{F}\text{-covering of } \mathcal{A} \right\} \tag{1.1}$$

The function μ satisfies the following properties:

(M1) $\mu(\mathcal{A}) \in [0,\infty]$ for every $\mathcal{A} \subseteq 2^\omega$.

(M2) $\mu(\emptyset) = 0$

(M3) $\mathcal{A} \subseteq \mathcal{B}$ implies $\mu(\mathcal{A}) \leq \mu(\mathcal{B})$

(M4) For a countable family $\{\mathcal{A}_n\}$ of subsets of 2^ω,

$$\mu\left(\bigcup_{n=0}^{\infty} \mathcal{A}_n\right) \leq \sum_{n=0}^{\infty} \mu(\mathcal{A}_n),$$

i.e., μ is *subadditive*.

A set function μ with the properties (M1)-(M4) is called an *outer measure*. That the construction in (1.1) actually yields an outer measure can be seen as follows: properties (M1)-(M3) are clear. To see that (M4) holds, let $\{\mathcal{A}_n\}$ be any sequence of subsets of 2^ω. (M4) trivially holds if

$$\sum_{n=0}^{\infty} \mu(\mathcal{A}_n) = \infty.$$

So suppose that $\sum \mu(\mathcal{A}_n)$ is finite, which, in particular, implies that every $\mu(\mathcal{A}_n)$ is finite. Let $\varepsilon > 0$. For each $n \in \mathbb{N}$ we can find sets $\{\mathcal{B}_j^{(n)}\}_{j\in\mathbb{N}}$ in $\mathfrak{F}$ such that, for all n,

$$\mathcal{A}_n \subseteq \bigcup_{j\in\mathbb{N}} \mathcal{B}_j^{(n)} \quad \text{and} \quad \sum_{j\in\mathbb{N}} \rho(\mathcal{B}_j^{(n)}) \leq \mu(\mathcal{A}_n) + \varepsilon/2^n.$$

Then, for $\mathcal{D}_{\langle i,j\rangle} = \mathcal{B}_j^{(i)}$, where $\langle\cdot,\cdot\rangle : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ is some standard pairing function, we have that $\mathcal{D}_k \in \mathfrak{F}$ for all k and

$$\bigcup_{n\in\mathbb{N}} \mathcal{A}_n \subseteq \bigcup_{n\in\mathbb{N}} \mathcal{D}_n$$

and

$$\mu\left(\bigcup_{n\in\mathbb{N}} \mathcal{A}_n\right) \leq \sum_{n\in\mathbb{N}} \rho(\mathcal{D}_n) = \sum_{i,j\in\mathbb{N}} \rho(\mathcal{B}_j^{(i)}) \leq \sum_{i\in\mathbb{N}} \mu(\mathcal{A}_i) + \varepsilon/2^i = \varepsilon + \sum_{i\in\mathbb{N}} \mu(\mathcal{A}_i),$$

from which the sub-additivity property follows directly, as ε is an arbitrary positive number.

An outer measure is defined for every subset of 2^ω. However, there is an important family of sets that behave particularly well under μ, the *measurable sets*: Given an outer measure μ, a set $\mathcal{A} \subseteq 2^\omega$ is called μ*-measurable*, if for every other set $\mathcal{D} \subseteq 2^\omega$ it holds that

$$\mu(\mathcal{D}) = \mu(\mathcal{A} \cap \mathcal{D}) + \mu(\mathcal{A} \setminus \mathcal{D}). \tag{1.2}$$

This definition of measurability is due to Carathéodory (1968).

A central result of measure theory says that the measurable sets form a σ-*algebra*, i.e they contain 2^ω, are closed under complementation and countable unions. Furthermore, on the family of measurable sets the outer measure μ is *additive*, i.e. for any countable family $\{\mathcal{A}_n\}$ of pairwise disjoint, measurable sets,

$$\mu\left(\bigcup_{n=0}^{\infty} \mathcal{A}_n\right) = \sum_{n=0}^{\infty} \mu(\mathcal{A}_n). \tag{1.3}$$

An outer measure μ restricted to the μ-measurable sets is simply called a *measure*. (In general, this term applies to any pair $(\mathfrak{F}, \mu)$ consisting of a σ-algebra $\mathfrak{F}$ in 2^ω and a countably additive set function $\mu : \mathfrak{F} \to [0, \infty]$ with $\mu(\emptyset) = 0$.)

Of course, the (outer) measure one obtains by (1.1) depends on the family of sets used for covering. It suggests itself to take, as the Cantor space is a topological space, some 'topologically respectable' family, such as the open sets. Any open set is a union of basic open cylinders, hence it suffices to consider coverings by basic open cylinders.

Lebesgue measure assigns every cylinder its usual 'geometrical size': An open cylinder $[w]$ corresponds to a binary interval of length $2^{-|w|}$. Therefore, by setting $\rho([w]) = 2^{-|w|}$ we obtain an outer measure extending the elementary geometrical measure of intervals to a larger family of sets. The resulting measure is denoted by λ.

Another important family of measures on Cantor space 2^ω are the (*generalized*) *Bernoulli measures*. Their definition is motivated by investigating certain random processes of independent trials (i.e. coin tosses – not necessarily fair ones). Let $\vec{p} = (p_0, p_1, p_2, \dots)$ be a sequence of real numbers such that $0 \leq p_i \leq 1$ for all i. This sequence induces a set function on the basic open cylinders in the following way: Let $[w]$, $|w| = n$, be an open cylinder. For $i \geq 0$, let $\rho_i(1) = p_i$, $\rho_i(0) = 1 - p_i$, and set

$$\rho([w]) = \prod_{i=0}^{n-1} \rho_i(w(i)) \tag{1.4}$$

The resulting measure μ is a probability measure on 2^ω (i.e. $\mu(2^\omega) = 1$), called a *generalized Bernoulli measure*. We denote it by $\mu_{\vec{p}}$. It is a *Bernoulli measure*, if all the p_i are identical, i.e. $p_i = p$ for all i. We denote this by μ_p . Note that the Lebesgue measure is a special Bernoulli measure with $p = 1/2$, i.e., $\lambda = \mu_{1/2}$.

The set functions of a (generalized) Bernoulli measure, defined on the basic open cylinders, are already additive: It holds that

$$\rho([w]) = \rho([w0]) + \rho([w1]) \tag{1.5}$$

In this case, the basic open cylinders are measurable by the outer measure obtained via (1.1), that is, the resulting σ-algebra of measurable sets includes the basic open cylinders, and hence, any set that can be obtained from open cylinders by the operations of countable unions and complementation. These sets form the *Borel σ-algebra*. It is the smallest σ-algebra containing the basic open cylinders.

An outer measure μ for which the Borel sets are measurable is called a *Borel measure*. Obviously, such a measure has to satisfy the additivity condition (1.5):

$$\mu([w]) = \mu([w0]) + \mu([w1]). \tag{1.6}$$

On the other hand, any function on basic open cylinders satisfying (1.6) induces a Borel measure. Namely, (1.6) generalizes inductively to finite unions of cylinders, that is, for any finite set of pairwise disjoint cylinders $W = \{[w_1], \dots, [w_n]\}$, it holds that

$$\mu\left(\bigcup_{i=1}^{n}[w_i]\right) = \sum_{i=1}^{n}\mu([w_i]).$$

The set of all finite unions of basic open cylinders forms an *algebra*, they are closed under finite unions and complementation, and they contain 2^ω. It is one of the fundamental results of measure theory that a measure on an algebra has an extension to the σ-algebra generated by it. (Under certain conditions, which are, in particular, fulfilled if the function μ is finite, this extension is unique.)

The Cantor space offers thus an easy method to specify Borel measures. This will prove particularly valuable when devising effective notions of measure.

For the sake of readability we will in the following often suppress the parentheses of a measure function and write $\mu\mathcal{X}$ instead of $\mu(\mathcal{X})$. In particular, we mostly write $\mu[w]$ for $\mu([w])$.

1.2.1 Measures on a metric space

If the underlying space is endowed with a metric, there is a method of constructing outer measures that ensures that the Borel sets are always measurable. Using the metric d on 2^ω, one can define the *diameter* d of a set $\mathcal{A} \subseteq 2^\omega$ by

$$d(\mathcal{A}) = \sup\{d(A, B) : A, B \in \mathcal{A}\} \tag{1.7}$$

(where $d(\emptyset) = 0$ by definition). We use the diameter to introduce a special type of coverings: Given $\delta > 0$, a δ-*covering* of a class $\mathcal{A} \subseteq 2^\omega$ is a covering $\{\mathcal{C}_i\}_{i\in\mathbb{N}}$ such that $d(\mathcal{C}_i) \le \delta$ for all $i \in \mathbb{N}$.

The construction of (1.1) can be modified to deal with δ-coverings: Let ρ be a set function as above. For $\mathcal{A} \subseteq 2^\omega$, define

$$\mu_\delta(\mathcal{A}) = \inf\left\{\sum_{n=0}^{\infty}\rho(\mathcal{C}_i) : \{\mathcal{C}_i\} \text{ is a } \delta\text{-covering of } \mathcal{A}\right\}. \tag{1.8}$$

When δ goes to 0 there are fewer and fewer δ-coverings available. Hence μ_δ is non-decreasing and therefore

$$\mu(\mathcal{A}) = \lim_{\delta\to 0}\mu_\delta(\mathcal{A}) = \sup_{\delta>0}\mu_\delta(\mathcal{A}) \tag{1.9}$$

It is the 'fine' covers, those with small diameter, that determine the value of $\mu(\mathcal{A})$. This gives μ a special property which on the other hand ensures that all Borel sets are μ-measurable: μ is a *metric* outer measure, i.e. it is additive on *positively separated* sets (sets with a positive distance $d(\mathcal{A}, \mathcal{D})$ defined as $\inf\{d(A, B) : A \in \mathcal{A}, B \in \mathcal{D}\}$). Note that two disjoint cylinders $[w]$, $[x]$ are always positively separated. One can prove that every metric outer measure renders the Borel sets measurable, so the construction underlying (1.8) and (1.9) always yields outer measures with some nice properties. (Proofs of the cited facts can be found in the monograph by Rogers, 1970.)

In the following, we will admit only open sets in a covering. It will turn out that this leaves the general theory unchanged, but render the situation particularly convenient in Cantor space, since every covering by open sets is a covering by basic open cylinders.

1.3 Martingales

Most generally, martingales are sequences of random variables $X_1, X_2, \ldots$ over nested σ-algebras $\mathfrak{F}_1 \subseteq \mathfrak{F}_2 \subseteq \ldots$ (also called a *filtration*) such that X_n (almost surely) is the conditional expectation of X_{n+1} with respect to $\mathfrak{F}_n$:

$$X_n = E[X_{n+1} \mid \mathfrak{F}_n]. \tag{1.10}$$

Once again, the Cantor space allows significant simplifications: As a filtration $\mathfrak{F}_n$ take the σ-algebra generated by $\{[w] : w \in \{0,1\}^n\}$. Then, given a Borel measure μ, a *martingale* on 2^ω is simply a function $d : 2^{<\omega} \to [0, \infty)$, and condition (1.10) becomes

$$\mu[w]d(w) = \mu[w0]d(w0) + \mu[w1]d(w1). \tag{1.11}$$

Martingales allow a nice interpretation as betting games. Think of d as function keeping track of a player's capital, while the player bets against a sequence of outcomes of a 0-1 experiment, or, formally speaking, a simple random variable with outcomes $\{0, 1\}$. Each round he might bet any percentage of his current capital on the next outcome being either 0 or 1. If his bet is correct, his stake is multiplied by $\mu[w^\frown(1-i)]/\mu[wi]$, where w is the finite sequence of outcomes so far and i is the bit he bet on, and is added to his capital (the game stopping if $\mu[wi] = 0$), if he is wrong he loses his stake. (1.11) reflects the presumption that the betting game is fair, e.g. there are no costs involved for betting and there are no outcomes which make the player lose his capital regardless of his bet (as there are in most casinos etc.).

Note that for Lebesgue measure, (1.11) takes the particularly nice form

$$d(w) = \frac{d(w0) + d(w1)}{2}. \tag{1.12}$$

In the following, we often have to deal with λ-martingales only, so, if the context is clear, we refer to a λ-martingale simply as a *martingale*.

It will often be quite convenient to devise a μ-martingale through its accordant *betting strategy*, a function b which for every possible state of the game defines the percentage of the capital that is bet in the next round and on which outcome it is to be bet.

Definition 1.4 A *betting strategy* b is a function

$$b : 2^{<\omega} \to [0,1] \times \{0,1\}.$$

Given a probability measure μ on 2^ω, the *(normed)* μ*-martingale*

$$d_b : \{0,1\}^* \to [0,\infty)$$

induced by a betting strategy b is inductively defined by $d_b(\lambda) = 1$ and

$$d_b(w^\frown i_w) = d_b(w)\left(1 + q_w \frac{\mu[w^\frown(1-i_w)]}{\mu[w^\frown i_w]}\right), \tag{1.13}$$

$$d_b(w^\frown(1-i_w)) = d_b(w)(1-q_w) \tag{1.14}$$

for $w \in 2^{<\omega}$ and $b(w) = (q_w, i_w)$. (This is only defined if $\mu[w^\frown i_w] > 0$. The other case has to be treated separately, but will not cause any obstacle to the general theory presented here, as we almost exclusively deal with λ-martingales.)

It is easy to check that a martingale induced by some betting strategy is indeed a μ-martingale, that is, equation (1.11) holds. On the other hand, every (normed) μ-martingale stems from some betting strategy.

A simple form of the fundamental martingale convergence theorem says that on almost all sequences of outcomes (with respect to μ), the player will not be able to increase his capital beyond any limit.

Theorem 1.5 (Doob) *For any Borel measure μ and any μ-martingale d, the set of sequences $A \in 2^\omega$ such that*

$$\limsup_{n\to\infty} d(A\restriction_n) = \infty \tag{1.15}$$

has μ-measure zero.

If (1.15) holds for a sequence A, we say that the martingale d *succeeds on* A. One can also prove a converse of this theorem, implying that martingales yield an alternative characterization of null sets.

Theorem 1.6 (Levy, Ville, Doob) *Given a Borel measure μ on 2^ω, a set $\mathcal{A} \subseteq 2^\omega$ has μ-measure zero if and only if there is a μ-martingale d such that*

$$(\forall A \in \mathcal{A})\ d \text{ succeeds on } A.$$

Martingales were first defined by Paul Levy, before they were extended and generalized by Doob (1953) to become one of the most prominent tools of modern probability theory. For the effective setting, Theorem 1.6 will turn out to be very important. In this connection, it is often convenient not to work with martingales directly, but with a generalized form called *supermartingales*. Supermartingales are functions like martingales, except that they do not need to satisfy (1.11) with equality replaced by

$$d(w) \geq \frac{\mu[w0]d(w0) + \mu[w1]d(w1)}{\mu[w]}. \tag{1.16}$$

Like continuous semimeasures (which will occur later in the text), supermartingales can be seen as representing defective measures, but they suffice for describing nullsets, for, obviously, if there exists a martingale succeeding on a class, then there exists also a supermartingale which does so. On the other hand, we can extend any supermartingale (by augmenting it in a suitable fashion) to a martingale that succeeds on at least the same sequences as the original supermartingale.

We will mostly deal with λ-martingales. For this reason, we will usually refer to λ-martingales simply as martingales.

An important tool for working with martingales is the following lemma, sometimes referred to as *Kolmogorov's inequality*, but first shown by Ville (1939).

Lemma 1.7 *Let d be a (super)martingale. Then it holds for every $k > 0$,*

$$\lambda\{B \in 2^\omega : d(B\restriction_n) \geq k \text{ for some } n\} \leq \frac{d(\epsilon)}{k}.$$

Note that Lemma 1.7 immediately implies Theorem 1.6.

1.4 Hausdorff measures

Lebesgue measure reflects the geometrical nature of sets by treating translation invariant sets alike, e.g., intervals of identical length are assigned the same measure. One may generalize this notion with respect to the measures constructed via (1.8) and (1.9) by requiring that the function ρ depends only on the diameter of a set. This leads directly to the concept of Hausdorff measures.

Hausdorff measures were first defined by Hausdorff (1919). Properties of Hausdorff measures, of which we will present some here, have been developed by Besicovitch and his students. The books by Rogers (1970) and Falconer (1990) may serve as a reference here.

In this section we give an overview of Hausdorff measures as they can be defined on an arbitrary metric space. So, in the following, let X be a metric space with metric d. Let $\mathfrak{F}_d$ be the family of all functions $h : \mathbb{R} \to [0, \infty]$ that are increasing, continuous on the right with $h(0) = 0$ and $h(1) = 1$. For reasons that shall become obvious later, we will call h a *dimension function*. The most general construction of a corresponding Hausdorff measure now follows (1.8) and (1.9), assigning every set E in the metric space the pre-measure $h(d(E))$.

Let $E \subseteq X$. For $\delta > 0$, let

$$\mathcal{H}^h_\delta(E) = \inf \left\{ \sum_{n \in \mathbb{N}} h(d(C_n)) : \{C_n\}_{n \in \mathbb{N}} \text{ is a } \delta\text{-covering of } E \right\}. \tag{1.17}$$

As before, letting δ go to 0 yields an outer measure.

Definition 1.8 Given a dimension function h, the *h-outer Hausdorff measure* (or simply *h-measure*) $\mathcal{H}^h$ of $E \subseteq X$ is defined as

$$\mathcal{H}^h(E) = \lim_{\delta \to 0} \mathcal{H}^h_\delta(E) \tag{1.18}$$

The function h can be regarded as a 'scaling function', which may be chosen to reflect certain geometrical (or other) properties. Let us illustrate this by the most prominent group of dimension functions, $h(t) = t^s$ for some real $s \geq 0$. We denote the corresponding Hausdorff measure by $\mathcal{H}^s$.

For $s = 0$, we have $h(t) = 1$ for all t, and $\mathcal{H}^0(E)$ is equal to the number of elements of E, if this number is finite, or $+\infty$, if E is infinite, so $\mathcal{H}^0(E)$ is a counting measure. For positive s, however, $\mathcal{H}^s$ behaves quite differently.

In Euclidean space $\mathbb{R}^n$, the volume of many 'regular' geometric objects like circles, squares, etc. relates to their diameter by the exponential factor n, the dimension of the underlying space. For instance, the volume of an n-dimensional cube with edge length r is r^n. A sphere with diameter $2r$ in $\mathbb{R}^n$ has volume $r^n V_n$, where V_n is the volume of the unit sphere in $\mathbb{R}^n$. Therefore, an integral exponent n in $h(t) = t^n$ can be interpreted as a 'dimension' scaling. In fact, one can show that, for a Borel subset F of $\mathbb{R}^n$

$$\mathcal{H}^n(F) = c_n \operatorname{vol}^n(F), \tag{1.19}$$

where vol^n denotes the n-dimensional volume (given through Lebesgue measure), and c_n is the reciprocal of the volume of an n-dimensional sphere of diameter 1 (see Rogers, 1970).

As s can take non-integer values, too, $\mathcal{H}^s$-measure can be seen as a generalization of classical Lebesgue measure theory. A lot of properties of 'classical' measure theory carry over – for instance, behavior under transformations such as translations or scalings. This is reflected by the following proposition on Hölder transformations.

Definition 1.9 Let X, Y be metric spaces with metrics d_X and d_Y, respectively. A mapping $h : X \to Y$ satisfies a *Hölder condition* if there exist real numbers $\gamma, c > 0$ such that for all $x, y \in X$

$$d_Y(h(x), h(y)) \leq c d_X(x, y)^\gamma. \tag{1.20}$$

Obviously, functions that satisfy a Hölder condition are continuous. Therefore, such mappings are also called *Hölder continuous*.

Proposition 1.10 *Let X, Y be metric spaces with metrics d_X and d_Y, respectively. If $E \subseteq X$, and if $f : E \to Y$ is Hölder continuous with Hölder constants γ, c, then for each s*

$$\mathcal{H}^{s/\gamma}(f(E)) \leq c^{s/\gamma} \mathcal{H}^s(E) \tag{1.21}$$

If in particular $\gamma = 1$, then h is *Lipschitz continuous*, and in this case c is called the *Lipschitz constant*. If in addition to this $c = 1$ and f satisfies (1.20) with equality, f becomes an *isometry*. Proposition 1.10 (applying it to the inverse mapping f^{-1}, too) shows that isometries preserve Hausdorff measure.

Proof of Proposition 1.10. Let $\{C_i\}$ be a δ-covering of E. Then for each i

$$d_Y(f(C_i \cap E)) \leq c d_X(C_i \cap E)^\gamma \leq c d_X(C_i)^\gamma \leq c\delta^\gamma,$$

hence $\{f(C_i \cap E)\}$ is a $c\delta^\gamma$-covering of $f(E)$. It follows that

$$\mathcal{H}^{s/\gamma}_{c\delta^\gamma}(f(E)) \leq \sum_i d_Y(f(C_i \cap E))^{s/\gamma} \leq c^{s/\gamma} \sum_i d_X(C_i)^s.$$

And since this holds for any δ-covering of E, we have

$$\mathcal{H}^{s/\gamma}_{c\delta^\gamma}(f(E)) \leq c^{s/\gamma} \mathcal{H}^s_\delta(E).$$

Now $\delta \to 0$ implies $c\delta^\gamma \to 0$, and this yields the desired inequality. □

1.5 Hausdorff measures in Cantor space

When working in Cantor space 2^ω, considering open coverings leads to families of cylinders, which all have diameter equal to 2^{-n}, where n is the length of the initial segment inducing the cylinder. For convenience, from now on we will identify coverings by basic open cylinders with sets of finite strings inducing them. Equation (1.17) now becomes

$$\mathcal{H}^h_\delta(\mathcal{A}) = \inf \left\{ \sum_{w \in C} h(2^{-|w|}) : C \subseteq 2^{<\omega} \text{ is a } \delta\text{-covering of } \mathcal{A} \right\} \tag{1.22}$$

for $\mathcal{A} \subseteq 2^\omega$ and $\delta > 0$. Using the correspondence between strings and cylinders, one can give particularly easy descriptions of $\mathcal{H}^h$-nullsets.

Proposition 1.11 *A set $\mathcal{A} \subseteq 2^\omega$ has h-dimensional Hausdorff measure 0 if and only if for every $n \in \mathbb{N}$ there is a set $C_n \subseteq 2^{<\omega}$ that covers $\mathcal{A}$ and*

$$\sum_{w \in C_n} h(2^{-|w|}) < 2^{-n}.$$

As cylinders are either disjoint or one of them is contained in the other, we can always choose C_n to be a *prefix-free* collection of strings, i.e., no string in C_n extends another.

Another characterization of measure 0 sets, which will be helpful later on, replaces the sequence of sets C_n of Proposition 1.11 by a condition on a single set $C \subseteq 2^{<\omega}$. (A more general version of this theorem, though the proofs are mostly identical, can be found in Rogers, 1970, Theorem 32.)

Proposition 1.12 *A set $\mathcal{A} \subseteq 2^\omega$ has h-dimensional Hausdorff measure 0 if and only if there exists a set $C \subseteq 2^{<\omega}$ such that*

$$\sum_{w \in C} h(2^{-|w|}) < \infty \quad \textit{and} \quad (\forall A \in \mathcal{A})\, (\overset{\infty}{\exists} w \in C)\, w \sqsubset A. \tag{1.23}$$

Proof. ($\Rightarrow$) Suppose $\mathcal{H}^h(\mathcal{A}) = 0$. Then, for all n, there is a set $C_n \subseteq 2^{<\omega}$ that is a 2^{-n}-covering of $\mathcal{A}$ and for which $\sum_{w \in C_n} h(2^{-|w|}) < 2^{-n}$ holds (since $\mathcal{H}^h_{2^{-n}}(\mathcal{A}) = 0$). Set $C = \bigcup C_n$. Then we have $\sum_{w \in C} h(2^{-|w|}) < 1$ (absolute convergence) and each $A \in \mathcal{A}$ extends infinitely many $w \in C$.

($\Leftarrow$) Now suppose that there is a set $C = \{w_0, w_1, w_2, \dots\}$ with

$$\sum_{w \in C} h(2^{-|w|}) < \infty$$

and for all $A \in X$ there are infinitely many n such that $w_n \sqsubset A$. Let m be a natural number. It suffices to show that for any $\varepsilon > 0$ there exists a 2^{-m}-cover W of $\mathcal{A}$ such that $\sum_{w \in W} h(2^{-|w|}) < \varepsilon$.

Choose N large enough that

$$\sum_{n \geq N} h(2^{-|w_n|}) < \varepsilon \;\text{ and for all } n \geq N,\ |w_n| \geq m.$$

As each $A \in \mathcal{A}$ extends infinitely many w_n, for each $A \in \mathcal{A}$ there exists some $n \geq N$ (in fact, infinitely many) such that $w_n \sqsubset A$. Thus $\{w_n : n \geq N\}$ is a 2^{-m}-cover of $\mathcal{A}$ with the desired properties. □

In the following we will call a set C of strings satisfying (1.23) a *Solovay h-cover* of $\mathcal{A}$, due to Solovay (1975), who used covers of this type to define a randomness concept equivalent to Martin-Löf randomness. In addition to this, we will call any set of strings satisfying the second conjunct of (1.23) simply a *Solovay cover* of $\mathcal{A}$, and if the sum is finite, but no $\mathcal{A}$ specified, we will call C a *Solovay h-test*.

Proposition 1.12 can be seen from a different perspective: Given a set $\mathcal{A} \subseteq 2^\omega$, collect all finite initial segments of sequences in $\mathcal{A}$ in a set C. If $\mathcal{A}$ is rather large, then the sum

$$\sum_{w \in C} h(2^{-|w|})$$

will be infinite. (Imagine for instance $h = \mathrm{id}$, $\mathcal{A} = 2^\omega$.) Now Proposition 1.12 tells us that for any set of non-negligible $\mathcal{H}^h$-size, this sum will stay infinite even if we allow that C may contain not all but only infinitely many initial segments of any $A \in \mathcal{A}$. On the other hand, for $\mathcal{H}^h$-nullsets the sum can be made finite by picking, for any sequence in the set, an appropriate set of initial segments.

This fact is a special instance of the theory of semimeasures, to which we turn our attention now.

1.5.1 Semimeasures

Semimeasures allow for an alternative description of nullsets, a way that will become particularly useful when introducing an effective (i.e. algorithmically effective) version of Hausdorff measures.

Suppose a set is negligible with respect to a measure μ, then an integrable function may take the value ∞ at the points of this set and still have a finite μ-integral. Pick any function $f : 2^\omega \to \mathbb{R}_+^\infty$. The function f is called *lower semicontinuous*, if for any $t \in \mathbb{R}$, the set

$$\{A : f(A) > t\}$$

is open. Equivalently, there is a nondecreasing sequence of continuous functions converging pointwise to f. It is not hard to see that the supremum of any set of lower semicontinuous functions is again a lower semicontinuous function. Characteristic functions of open sets provide basic examples for lower semi-continuous functions. It will become clear in the next chapter why lower semi-continuous functions are of particular interest in this context.

Lower semicontinuous functions may serve as a *measure of impossibility* of a sequence with respect to a measure in the following sense: Let μ be a measure on 2^ω. If $\mathcal{A} \subseteq 2^\omega$ is a μ-nullset, then there exists a lower semicontinuous function $f : 2^\omega \to \mathbb{R}_+^\infty$ such that

$$\int f \, d\mu \leq 1 \quad \text{and} \quad f(\mathcal{A}) = \{\infty\}.$$

Thus, for a ‘typical’ sequence B regarding the measure μ, one may expect that $f(B) < \infty$.

This approach can be realized in a different fashion. As a measure on 2^ω is completely determined by the values it takes on the basic cylinders, one may replace the condition $f(B) < \infty$ by a discrete approximation, which later on, when the effective case is studied, makes things much easier to handle. We start with the basic definition of a discrete semimeasure.

Definition 1.13 A (discrete) *semimeasure* is a function $m : 2^{<\omega} \to \mathbb{R}$ such that

$$\sum_{w \in 2^{<\omega}} m(w) < \infty.$$

One can think of a discrete semimeasure as a distribution of some finite mass over the finite strings. Equivalently, since finite strings and natural numbers are in one to one correspondence, a semimeasure is just a converging series of real numbers.

The next theorem shows how semimeasures can be used to describe nullsets. We state this theorem only with respect to Hausdorff measures, but it can be easily adapted to handle other types of measures.

Theorem 1.14 *Let h be a dimension function. A set $\mathcal{A} \subseteq 2^\omega$ has $\mathcal{H}^h$-measure 0 if and only if there exists a discrete semimeasure m such that for any $A \in \mathcal{A}$,*

$$\limsup_{n \to \infty} \frac{m(A\restriction_n)}{h(2^{-n})} = \infty. \tag{1.24}$$

Proof. ($\Rightarrow$) Let $\mathcal{A} \subseteq 2^\omega$ be an $\mathcal{H}^h$-nullset and $A \in \mathcal{A}$. Let $C = \{w_0, w_1, \dots\}$ be an Solovay h-cover of $\mathcal{A}$, which exists according to Proposition 1.12. Assume that

$$\sum_{w \in C} h(2^{-|w|}) = M < \infty.$$

Define a sequence $0 = n_0 < n_1 < n_2 < \dots$ of natural numbers in such a way that for all $i \in \mathbb{N}$

$$\sum_{k \geq n_i} h(2^{-|w_k|}) \leq \frac{M}{2^i}.$$

For $i \geq 0$, let $C_i = \{w_k : k \geq n_i\}$ and $D_i = C_i \setminus C_{i+1}$. Let the function $f : C \to \mathbb{N}$ assign every $w \in C$ the index i such that $w \in D_i$. Since the D_i are disjoint, f is well defined. Hence also the function $m : 2^{<\omega} \to \mathbb{R}$ defined as

$$m(w) = \begin{cases} f(w)h(2^{-|w|}), & \text{if } w \in D_i \\ 0, & \text{otherwise.} \end{cases}$$

m is well defined. We claim that m is a semimeasure satisfying (1.24).

First, note that

$$\sum_{w \in 2^{<\omega}} m(w) = \sum_{i=0}^{\infty} \sum_{w \in D_i} m(w) = \sum_{i=1}^{\infty} i \sum_{w \in D_i} h(2^{-|w|}) \leq \sum_{i=1}^{\infty} \frac{iM}{2^i} < \infty.$$

Thus m is a semimeasure. Furthermore, let $A \in \mathcal{A}$. Then there exist infinitely many $w_n \sqsubset A$ from C. Choose a subsequence $w_{j_1}, w_{j_2}, w_{j_3}, \dots$ of C such that $w_{j_i} \sqsubset A$ for all $i \geq 1$. Now

$$\frac{m(w_{j_i})}{h(2^{-|w_{j_i}|})} \geq \frac{f(w_{j_i})h(2^{-|w_{j_i}|})}{h(2^{-|w_{j_i}|})} = f(w_{j_i}).$$

But each D_k is a finite set, so $\limsup_n m(A\restriction_n)/h(2^{-n}) = \infty$.

($\Leftarrow$) Let m be a semimeasure such that (1.24) holds. Define a set $C \subseteq 2^{<\omega}$:

$$C = \{w \in 2^{<\omega} : m(w) \geq h(2^{-|w|})\}.$$

From (1.24) it is clear that C is a Solovay cover for $\mathcal{A}$. Furthermore, we have that

$$\sum_{w \in C} h(2^{-|w|}) \leq \sum_{w \in 2^{<\omega}} m(w) < \infty.$$

Thus C is also a Solovay h-cover. □

To determine whether a set has $\mathcal{H}^h$-measure 0 will be important in the context of *Hausdorff dimension*, which we will turn to in the next section.

1.6 Hausdorff dimension

Of course, there are a lot of possible dimension functions, and each dimension function determines a measure on its own. This wide variety of measures allows to 'choose' for every set a measure which reflects best some geometrical or other properties of the set. Let us illustrate this by an analogon from Lebesgue measure. Consider a square of positive side length in the plane. Its two-dimensional Lebesgue measure is surely positive. However, if we embed this square into $\mathbb{R}^3$, for instance as a subset of a 2-dimensional subspace, its 3-dimensional measure is zero. In this sense, we might say that 2-dimensional Lebesgue measure is the 'right' measure to grasp objects living in $\mathbb{R}^2$. The concept of a Hausdorff measure allows a much wider variety of measures to choose from. They can capture objects that are in a certain sense 'in between', i.e., Lebesgue measure of integral dimension is too coarse to catch some of their features.

Hausdorff dimension indicates the appropriate Hausdorff measure for a set, i.e. the appropriate scaling function h. In order to avoid technical difficulties, we concentrate on the family of measures given by $h(x) = x^s$ for $s \geq 0$ with the corresponding measures $\mathcal{H}^s$. Again, the theory can be developed in a much broader context (as in Rogers, 1970, or, even going beyond that, Federer, 1996).

We first observe that for each set $\mathcal{A} \subseteq 2^\omega$, the family $\{\mathcal{H}^s\}_{s \geq 0}$ has a critical value s_0, at which the value of $\mathcal{H}^s(\mathcal{A})$ drops to zero.

Proposition 1.15 *Let $\mathcal{A} \subseteq 2^\omega$. For every $s \geq 0$,*

$$\mathcal{H}^s(\mathcal{A}) < \infty \ \textit{ implies that } \ \mathcal{H}^t(\mathcal{A}) = 0 \textit{ for all } t > s.$$

Proof. Let $\mathcal{H}^s(\mathcal{A}) < \infty$ and $t > s$. We may assume that there is some $M > 0$ such that for every $n \in \mathbb{N}$ there is a a 2^{-n}-covering $\{w_i^{(n)}\}$ of $\mathcal{A}$ satisfying

$$\sum_{i=0}^{\infty} 2^{-|w_i^{(n)}|s} \leq M.$$

The family $C = \{w_i^{(n)}\}_{i,n}$ surely defines a Solovay cover for $\mathcal{A}$. We claim that it is actually a Solovay t-cover:

$$\begin{aligned}
\sum_{w \in C} 2^{-|w|t} = \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} 2^{-|w_i^{(n)}|t} &= \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} 2^{-|w_i^{(n)}|s}\, 2^{-|w_i^{(n)}|(t-s)} \\
&\leq \sum_{n=0}^{\infty} 2^{-n(t-s)} \sum_{i=0}^{\infty} 2^{-|w_i^{(n)}|s} \\
&\leq M \sum_{n=0}^{\infty} 2^{-n(t-s)} < \infty.
\end{aligned}$$

It follows by Proposition 1.12 that $\mathcal{H}^t(\mathcal{A}) = 0$. □

In analogy to n-dimensional Lebesgue measure the critical value s_0 may be seen as a kind of dimension $\mathcal{A}$. It is called the *Hausdorff dimension* of $\mathcal{A}$.

Definition 1.16 For a class $\mathcal{A} \subseteq 2^\omega$, the *Hausdorff dimension* of $\mathcal{A}$, $\dim_{\mathrm{H}}(\mathcal{A})$, is defined as

$$\dim_{\mathrm{H}}(\mathcal{A}) = \inf\{s \geq 0 : \ \mathcal{H}^s(\mathcal{A}) = 0\}$$

1.6.1 Stability of Hausdorff dimension

In the following, we discuss some basic properties of Hausdorff dimension which show that it is a stable and reasonable concept to work with.

Lebesgue measure and Hausdorff dimension. For $s = 1$, the corresponding measure $\mathcal{H}^1$ scales the sets in a covering by a factor 1, that is, cylinders enter with their original diameter $2^{-|w|}$. Hence, $\mathcal{H}^1$ is the same as Lebesgue measure on 2^ω. From this it immediately follows that $\dim_{\mathrm{H}}(2^\omega) = 1$, as $\mathcal{H}^1(2^\omega) = 1$. Furthermore, if a set $\mathcal{A} \subseteq 2^\omega$ has positive Lebesgue measure, then it has dimension 1, too. (Note however that the middle-third Cantor set C, as a subset of the unit interval, has Hausdorff dimension $1/\log 3$, giving an example of a set with non-integral dimension. This shows that Hausdorff dimension is indeed not a pure topological notion, as it is not invariant under homeomorphisms.)

Open sets. By the remarks made in the previous paragraph, it is also obvious that $\dim_{\mathrm{H}}(\mathcal{U}) = 1$ for every open set $\mathcal{U} \subseteq 2^\omega$.

Monotonicity. If $\mathcal{A} \subseteq \mathcal{D} \subseteq 2^\omega$, the definition of $\mathcal{H}^s$ immediately gives $\mathcal{H}^s(\mathcal{A}) \leq \mathcal{H}^s(\mathcal{D})$ and thus $\dim_{\mathrm{H}}(\mathcal{A}) \leq \dim_{\mathrm{H}}(\mathcal{D})$. In particular, $\dim_{\mathrm{H}}(\mathcal{A}) \leq 1$ for all $\mathcal{A} \subseteq 2^\omega$.

Countable stability. For any sequence of classes $\mathcal{A}_1, \mathcal{A}_2, \dots$ in 2^ω it holds that

$$\dim_{\mathrm{H}}\left(\bigcup_{n=1}^{\infty} \mathcal{A}_n\right) = \sup\{\dim_{\mathrm{H}}(\mathcal{A}_n) : n \geq 1\}. \tag{1.25}$$

For a proof note that $\mathcal{H}^s$ is an (outer) measure and hence stable under countable unions of $\mathcal{H}^s$-nullsets. If $s > \sup\{\dim_{\mathrm{H}}(\mathcal{A}_n) : n \geq 1\}$, it follows that $\bigcup_{n=1}^{\infty} \mathcal{A}_n$ is $\mathcal{H}^s$-null, which implies $\dim_{\mathrm{H}}\left(\bigcup_{n=1}^{\infty} \mathcal{A}_n\right) \leq \sup\{\dim_{\mathrm{H}}(\mathcal{A}_n) : n \geq 1\}$. On the other hand, for any $s < \sup\{\dim_{\mathrm{H}}(\mathcal{A}_n) : n \geq 1\}$ there must be an n such that $\mathcal{A}_n$ is not $\mathcal{H}^s$-null, and therefore $\bigcup_{n=1}^{\infty} \mathcal{A}_n$ is not $\mathcal{H}^s$-null either.

Countable sets. If $\mathcal{A} \subseteq 2^\omega$ is countable, then $\dim_{\mathrm{H}}(\mathcal{A}) = 0$, since for any $A \in 2^\omega$ we have $\mathcal{H}^0(\{A\}) = 1$ and $\dim_{\mathrm{H}}$ is countably stable.

The discussion shows that on 2^ω Hausdorff dimension can also be seen as a ramification of Lebesgue nullsets. Sets of positive Lebesgue measure stay at dimension 1, whereas the countable sets drop to dimension 0. It is one of the primary tasks of fractal geometry to expose objects that lie in between.

1.6.2 Continuous transformations

The topological definition of continuity says that the preimages of an open set have to be open, too. This tells us that, in the Cantor space, the behavior of continuous functions is governed by the way they map basic open cylinders. Cylinders, again, can be represented by finite strings, so continuous mappings are constituted by functions mapping finite strings to finite strings. Call a function $\varphi : 2^{<\omega} \to 2^{<\omega}$ *monotone* if $x \sqsubseteq y$ implies $\varphi(x) \sqsubseteq \varphi(y)$. Every monotone function induces a partial mapping $\widehat{\varphi}$ on infinite sequences in the following way: if, for $B \in 2^{\omega}$, $\lim |\varphi(B\restriction_n)| = \infty$, let $\widehat{\varphi}(B)$ be the unique sequence extending all $\varphi(B\restriction_n)$. Continuity on 2^{ω} can be characterized this way, as the following proposition shows.

Proposition 1.17 *A function $f : 2^{\omega} \to 2^{\omega}$ is continuous if and only if there exists a monotone function $\varphi : 2^{<\omega} \to 2^{<\omega}$ such that $\widehat{\varphi} = f$.*

Proof. ($\Rightarrow$) Let $x \in 2^{<\omega}$. Let y be the longest string of length less or equal $|x|$ such that

$$[x] \subseteq f^{-1}([y]).$$

(Note that such y always exists since $f^{-1}([\epsilon]) = 2^{\omega}$.) Set $\varphi(x) = y$.

Now φ is monotone, for $x_1 \sqsubseteq x_2$ implies

$$[x_2] \subseteq [x_1] \subseteq f^{-1}([\varphi(x_1)]),$$

and hence

$$[x_2] \subseteq f^{-1}([\varphi(x_1)]) \cap f^{-1}([\varphi(x_2)]) = f^{-1}([\varphi(x_1)] \cap [\varphi(x_2)]),$$

from which it follows that

$$[\varphi(x_1)] \cap [\varphi(x_2)] \neq \emptyset,$$

which in turn implies

$$\varphi(x_1) \sqsubseteq \varphi(x_2) \;\text{ or }\; \varphi(x_2) \sqsubset \varphi(x_1).$$

But $\varphi(x_2) \sqsubset \varphi(x_1)$ is impossible since in this case we would have $|x_2| > |x_1|$ and $[x_2] \subseteq f^{-1}([\varphi(x_1)])$, contrary to the choice of $\varphi(x_2)$.

To show that $\widehat{\varphi} = f$, we have to show that for any A,

$$\lim_{n\to\infty} |\varphi(A\restriction_n)| = \infty,$$

and that for all n,

$$\varphi(A\restriction_n) \sqsubset f(A).$$

The latter follows easily, since we have, for all $n \in \mathbb{N}$, $A \in f^{-1}([\varphi(A\restriction_n)])$, so that $f(A) \in [\varphi(A\restriction_n)]$ and hence $\varphi(A\restriction_n) \sqsubset f(A)$.

To see that the other property of $\widehat{\varphi}$ holds, it suffices to show that, given $A \in 2^\omega$, for each $n \in \mathbb{N}$ there is an $m > n$ such that $|\varphi(A\restriction_m)| > |\varphi(A\restriction_n)|$. So fix n and some $k > |\varphi(A\restriction_n)|$. Since $A \in f^{-1}([f(A)\restriction_k])$, for all $m > n$

$$[A\restriction_m] \cap f^{-1}([f(A)\restriction_k]) \neq \emptyset.$$

As $f^{-1}([f(A)\restriction_k])$ is open and contains A, for some m_0 all $[A\restriction_m]$ with $m \geq m_0$ are contained in it. So some $A\restriction_m$ with $m \geq m_0$ must map to $f(A)\restriction_{k'}$ with $k' \geq k$ and thus $|\varphi(A\restriction_m)| > |\varphi(A\restriction_n)|$.

($\Leftarrow$) It suffices to show that for any $w \in 2^{<\omega}$ $f^{-1}([w])$ is open. But this is obvious since $f(A) \in [w]$ if and only if there is some n such that $w \sqsubseteq \varphi(A\restriction_n)$, so $f^{-1}([w]) = \bigcup_{\varphi(v) \sqsupseteq w}[v]$. □

In Proposition 1.10 we saw that Hölder mappings are an important set of transformations, as they behave well with respect to Hausdorff measure. We want to study their nature as transformations on 2^ω, using Proposition 1.17, thereby passing to a more general set of functions.

Let $f : 2^\omega \to 2^\omega$ be a Hölder transformation on the Cantor space, i.e.,

$$(\forall A, B \in 2^\omega)\; d(f(A), f(B)) \leq c\, d(A, B)^r$$

for some $r, c > 0$. This implies (recall the definition of metric d)

$$|f(A) \sqcap f(B)| \geq r\,|A \sqcap B| - \log c$$

The last formula suggests a generalization of Hölder mappings based on string functions.

Definition 1.18 A monotone mapping $\varphi : 2^{<\omega} \to 2^{<\omega}$ is *r-expansive*, $r > 0$, if for all $B \in \mathrm{dom}(\widehat{\varphi})$,

$$\liminf_{n\to\infty} \frac{|\varphi(B\restriction_n)|}{n} \geq r.$$

It is obvious that every Hölder function f is represented by some r-expansive φ with $r > 0$. (On the other hand, there are r-expansive φ such that $\widehat{\varphi}$ is not Hölder (not even continuous), so expansiveness is a more general notion.) A notion similar to r-expansiveness was studied by Cai and Hartmanis (1994), and by Staiger (2002b).

Proposition 1.19 *Let $\varphi : 2^{<\omega} \to 2^{<\omega}$ be r-expansive for some $r > 0$. Then for all $\mathcal{B} \subseteq \mathrm{dom}(\widehat{\varphi})$*

$$\dim_{\mathrm{H}}(\widehat{\varphi}(\mathcal{B})) \leq \tfrac{1}{r}\dim_{\mathrm{H}}(\mathcal{B}).$$

Proof. It suffices to show that for any $s > 0$ with $\mathcal{H}^s(\mathcal{B}) = 0$ it holds that

$$\mathcal{H}^{\frac{s}{r}+\varepsilon}(\hat{\varphi}(\mathcal{B})) = 0$$

for any $\varepsilon > 0$.

So let $C \subseteq 2^{<\omega}$ be such that

$$\sum_{w \in C} 2^{-s|w|} < \infty \quad \text{and} \quad (\forall A \in \mathcal{B})\,(\overset{\infty}{\exists} w \in C)\; w \sqsubset A, \tag{1.26}$$

i.e., C is a Solovay s-cover for $\mathcal{B}$. Obviously, $\varphi(C)$ is a Solovay cover for $\widehat{\varphi}(\mathcal{B})$. Let $\varepsilon > 0$. We can find $\varepsilon' > 0$ such that

$$\frac{s}{r} + \varepsilon = \frac{s}{r - \varepsilon'}.$$

Define

$$C_{\varepsilon'} = \{w \in C : \; |\varphi(w)|/|w| \geq r - \varepsilon'\}.$$

Observe that, by r-expansiveness and (1.26), $\varphi(C_{\varepsilon'})$ is an infinite Solovay cover for $\widehat{\varphi}(\mathcal{B})$, too. Furthermore,

$$\sum_{v \in \varphi(C_{\varepsilon'})} 2^{-|v|\frac{s}{r-\varepsilon'}} \leq \sum_{w \in C_{\varepsilon'}} 2^{-|\varphi(w)|\frac{s}{r-\varepsilon'}} \leq \sum_{w \in C_{\varepsilon'}} 2^{-(r-\varepsilon')|w|\frac{s}{r-\varepsilon'}} = \sum_{w \in C_{\varepsilon'}} 2^{-|w|s} < \infty,$$

so $\varphi(C_{\varepsilon'})$ is a Solovay $\frac{s}{r} + \varepsilon$-cover for $\widehat{\varphi}(\mathcal{B})$. □

Proposition 1.19 was obtained independently by Staiger (2002b). The case where $r = 1$ is especially important. As was already mentioned, such functions are called *Lipschitz*, and if the inverse mapping is Lipschitz, too, that is, if the distance between two points is not decreased too much, one speaks of a *bi-Lipschitz* function. An easy corollary of Proposition 1.10 yields that Hausdorff dimension is *invariant under bi-Lipschitz functions*. In view of Felix Klein's approach to geometry, one could therefore consider fractal geometry as the study of properties invariant under the group of bi-Lipschitz transformations.

1.7 Cantor Sets

Probably the most prominent and easiest example of fractal subsets of the real line are the *Cantor sets*, which can be seen as homeomorphic copies of 2^ω within the unit interval. For proofs and further details for the following statements refer to Mattila (1995, Chapter 4).

1.7.1 Constructing Cantor sets in [0, 1]

Choose a real number $0 < \gamma < 1/2$. Define two subintervals of [0, 1] as

$$I_{1,1} = [0, \gamma] \quad \text{and} \quad I_{1,2} = [1-\gamma, 1],$$

i.e. cancelling out a middle interval of length $1 - 2\gamma$. Iterate this by cutting out of each $I_{k,i}$, $1 \le i \le 2^k$, a middle interval of length $\gamma^k(1-2\gamma)$, yielding 2^{k+1} new intervals $I_{k+1,1}, \dots I_{k+1,2^{k+1}}$ of length γ^{k+1}.

Define a limit set of this construction by

$$\mathcal{C}_\gamma = \bigcap_{k=1}^{\infty} \bigcup_{i=1}^{2^k} I_{k,i} \tag{1.27}$$

$\mathcal{C}_\gamma$ is an uncountable compact set without interior points, therefore by Theorem 1.3 it is homeomorphic to 2^ω. One can show that for

$$s = -\frac{\log 2}{\log \gamma}$$

$1/4 < \mathcal{H}^s(\mathcal{C}_\gamma) \le 1$, and therefore $\dim_H \mathcal{C}_\gamma = s$. It is easy to see that $\mathcal{H}^s(\mathcal{C}_\gamma) \le 1$, since for every k, $\mathcal{C}_\gamma \subseteq \bigcup_i I_{k,i}$ and hence

$$\mathcal{H}^s_{\gamma^k}(\mathcal{C}_\gamma) \le \sum_{i=1}^{2^k} d(I_{k,i})^s = 2^k \gamma^{ks} = (2\gamma^s)^k,$$

and $(2\gamma^s)^k \to 0$ as $k \to \infty$ for any $s > -\log 2/\log\gamma$ (remember that $0 < \gamma < 1/2$). The lower bound on $\mathcal{H}^s(\mathcal{C}_\gamma)$ is much harder to obtain and will not be demonstrated here.

For $\gamma = 1/3$, one obtains the *middle third* Cantor set $\mathcal{C}_{1/3}$, introduced in Section 1.1.

1.7.2 Generalized Cantor sets

It is possible to generalize the construction given above by varying the length of the interval that gets cut out each stage.

Let $\Gamma = \{\gamma_1, \gamma_2, \dots\}$ be a sequence of real numbers such that $0 < \gamma_i < 1/2$ for all i. Divide an interval $I_{k,i}$, $1 \le i \le 2^k$, into two new intervals $I_{k+1,2i-1}$, $I_{k+1,2i}$ by removing a middle piece of length $(1-2\gamma_{k+1})|I_{k,i}|$, hence each of the 2^{k+1} intervals $I_{k+1,j}$ has length $\prod_{i=1}^k \gamma_i$. The resulting limit set $\mathcal{C}_\Gamma$ is defined as in (1.27).

If h is a dimension function such that $h(\gamma_1 \cdots \gamma_k) = 2^{-k}$, then it can be shown that

$$1/4 \le \mathcal{H}^h(\mathcal{C}_\Gamma) < \infty.$$

On the other hand, if h is a dimension function such that $h(2r) < 2h(r)$ for all $0 < r < \infty$, one can choose γ_i inductively to assure $h(\gamma_1 \cdots \gamma_k) = 2^{-k}$. This makes it possible to define Cantor sets of dimension 0 and 1, respectively, the first giving an example of an uncountable set of dimension 0, the second being a set of Lebesgue measure 0 but of dimension 1.

1.8 Hausdorff Dimension and Martingales

Theorem 1.6 yielded a possibility to describe nullsets by 'winning conditions' on martingales. As Hausdorff measures on 2^ω can be seen as a refinement of Lebesgue measure 0, one might expect that, for $s < 1$, Hausdorff s-nullsets should be related to imposing stricter winning conditions on the martingale, i.e. making it harder to win against a sequence. A connection between Hausdorff dimension and martingales was observed by Ryabko (1993, 1994) and Staiger (1998, 2000), before Lutz (2000a,b) was able to obtain the characterization given in Theorem 1.21. While Lutz used a generalization of martingales called *gales*, in our presentation we will stick to martingales and generalize the winning conditions instead.

Note that each stage a martingale can at most double the current capital. Therefore, $d(B\restriction_n) \leq 2^n$ holds trivially for any sequence B. On the other hand, a martingale successful on B is only required to increase its capital unboundedly, no matter how slowly. We might therefore try to measure the speed with which the increase takes place. This is reflected in the following definition.

Definition 1.20 Given $s \geq 0$, a martingale d is called *s-successful* on a sequence $B \in 2^\omega$ if

$$(\overset{\infty}{\exists} n)\ [d(B\restriction_n) \geq 2^{(1-s)n}]. \tag{1.28}$$

It turns out that, in terms of Hausdorff dimension, the relation between $\mathcal{H}^s$-nullsets and s-successful martingales is indeed very close.

Theorem 1.21 (Lutz) *For any $\mathcal{X} \subseteq 2^\omega$ it holds that*

$$\dim_{\mathrm{H}} \mathcal{X} = \inf\{s : \text{ some martingale } d \text{ is } s\text{-successful on all } B \in \mathcal{X}\}.$$

Parts of the following proof are adapted from Terwijn (2003).

Proof. ($\leq$) Suppose a martingale d is s-successful on all $B \in \mathcal{X}$. It suffices to show that for any $t > s$, $\mathcal{X}$ is $\mathcal{H}^t$-null via some cover $\{U_k^{(t)}\}_{k\in\mathbb{N}}$. By appropriate rescaling, we may assume that $d(\epsilon) = 1$.

Observe that for $t > s$,

$$(\forall B \in \mathcal{X})\ \limsup_{n\to\infty} \frac{d(B\restriction_n)}{2^{(1-t)n}} = \infty.$$

For each k, the sets

$$U_k^{(t)} = \left\{\sigma : \frac{d(\sigma)}{2^{(1-t)|\sigma|}} \geq 2^k\right\}$$

induce a cover of $\mathcal{X}$. By collecting only the shortest such σ, we may assume that $U_k^{(t)}$ is prefix-free. Furthermore, it holds that

$$\sum_{w\in U_k^{(t)}} 2^{-|w|s} \leq \sum_{w\in U_k^{(t)}} 2^{-|w|s} \frac{d(w)}{2^{(1-s)|w|}} 2^{-k} = 2^{-k} \sum_{w\in U_k^{(t)}} d(w) 2^{-|w|}.$$

Finally, it is easy to prove the following fact via induction: For every prefix-free set of strings C, $d(\epsilon) \geq \sum_{w \in C} d(w) 2^{-|w|}$. (This is essentially a generalization of the fairness condition (1.12).) Hence, the $U_k^{(t)}$ witness that $\mathcal{X}$ is $\mathcal{H}^s$-null.

($\geq$) Assume $\mathcal{X} \subseteq 2^\omega$ is $\mathcal{H}^s$-null via $\{V_k\}$, i.e. for every k, $\mathcal{X} \subseteq [V_k]$ and

$$\sum_{w \in V_k} 2^{-|w|s} \leq 2^{-k}.$$

Furthermore, we may assume each V_k is a prefix-free set. For every string σ and every $k \in \mathbb{N}$, define

$$d_k(\sigma) = \begin{cases} 2^{(1-s)|w|} & \text{if } \sigma \sqsupset w \text{ for some } w \in V_k, \\ \sum_{\sigma w \in V_k} 2^{-|w|+(1-s)(|\sigma|+|w|)} & \text{otherwise.} \end{cases}$$

We verify that d_k is a martingale. Given $\sigma \in 2^{<\omega}$, if there is a $w \in V_k$ such that $w \sqsubseteq \sigma$, we have

$$d_k(\sigma 0) + d_k(\sigma 1) = 2^{1+(1-s)|w|} = 2 d_k(\sigma).$$

If such w does not exist,

$$\begin{aligned} d_k(\sigma 0) + d_k(\sigma 1) &= \sum_{\sigma 0 w \in V_k} 2^{-|w|+(1-s)(|\sigma|+|w|+1)} + \sum_{\sigma 1 w \in V_k} 2^{-|w|+(1-s)(|\sigma|+|w|+1)} \\ &= \sum_{\sigma v \in V_k} 2^{(-|v|+1)+(1-s)(|\sigma|+|v|)} = 2 d_k(\sigma). \end{aligned}$$

Besides, $d_k(\epsilon) = \sum_{w \in V_k} 2^{-|w|+(1-s)|w|} = \sum_{w \in V_k} 2^{-s|w|} \leq 2^{-k}$, so the function

$$d = \sum_k d_k$$

defines a martingale as well (using additivity). Finally, note if $w \in V_k$, $d(w) \geq d_k(w) = 2^{(1-s)|w|}$. So if $B \in \bigcap_k [V_k]$, $d(B\restriction_n) \geq 2^{(1-s)n}$ infinitely often, which means that d is s-successful on all $B \in \mathcal{X}$. □

1.9 Other Notions of Dimension

Hausdorff dimension is by far not the only concept studied in geometric measure theory and fractal geometry. There is a wide variety of notions, and it is beyond the scope of this text to deal with all of them. Many of them can be found in the treatise by Federer (1996). We will give a short overview over some dimension concepts, especially those that are important in context of the topics dealt with here.

1.9.1 Box counting dimension

The definition of $\mathcal{H}^s_\delta$ allows only sets of diameter less or equal δ in a covering. In 2^ω, for any set $\mathcal{A} \subseteq 2^\omega$ and any $\delta = 2^{-n}$ there is an obvious δ-covering: Take the cylinders induced by

$$\mathcal{A}\restriction_n := \{w \in \{0,1\}^n : (\exists A \in \mathcal{A})\, w \sqsubset A\}.$$

Define the *upper* and *lower box counting dimension* of $\mathcal{A}$ as

$$\overline{\dim}_{\mathrm{B}}\mathcal{A} = \limsup_{n\to\infty} \frac{\log(|\mathcal{A}\restriction_n|)}{n} \quad \text{and} \quad \underline{\dim}_{\mathrm{B}}\mathcal{A} = \liminf_{n\to\infty} \frac{\log(|\mathcal{A}\restriction_n|)}{n}. \tag{1.29}$$

If $\overline{\dim}_{\mathrm{B}}$ and $\underline{\dim}_{\mathrm{B}}$ coincide, than this value is simply called the *box counting dimension*, sometimes also *Minkowski dimension* of $\mathcal{A}$. The name box counting obviously is related to the fact that, for each covering level δ, one simply counts the number of boxes of size δ needed to cover $\mathcal{A}$.

The following relations between Hausdorff and box counting dimension are obvious:

Proposition 1.22 *For any set $\mathcal{A} \subseteq 2^\omega$ it holds that*

$$\dim_{\mathrm{H}} \mathcal{A} \leq \underline{\dim}_{\mathrm{B}}\mathcal{A} \leq \overline{\dim}_{\mathrm{B}}\mathcal{A}. \tag{1.30}$$

(Lower) box counting dimension gives an easy upper bound on Hausdorff dimension, although this estimate may not be very exact. For instance, for $\mathbb{Q}\cap[0,1]$, identified with the set of all infinite binary sequence which are 0 from some point on, we have $0 = \dim_{\mathrm{H}}(\mathbb{Q}\cap[0,1]) < \underline{\dim}_{\mathrm{B}}(\mathbb{Q}\cap[0,1]) = 1$. In fact, this holds for any dense subset of 2^ω. This shows that, in general, box counting dimension is not a stable concept of dimension. Staiger (1989, 1998) has investigated some conditions when Hausdorff and box counting dimension coincide. Probably the most famous example of such sets arises in the context of dynamical systems (see Section 1.11).

One can modify box counting dimension to obtain a countably stable notion. This yields the concept of *modified box counting dimension*, denoted $\dim_{\mathrm{MB}}$, defined as follows:

$$\underline{\dim}_{\mathrm{MB}}\mathcal{X} = \inf\left\{\sup_i \underline{\dim}_{\mathrm{B}}\mathcal{X}_i : \mathcal{X} \subseteq \bigcup_{i\in\mathbb{N}} \mathcal{X}_i\right\}, \tag{1.31}$$

$$\overline{\dim}_{\mathrm{MB}}\mathcal{X} = \inf\left\{\sup_i \overline{\dim}_{\mathrm{B}}\mathcal{X}_i : \mathcal{X} \subseteq \bigcup_{i\in\mathbb{N}} \mathcal{X}_i\right\}, \tag{1.32}$$

(1.33)

(That is, we split up a set into countably many parts and look at the dimension of its 'most complicated' part. Then we optimize this by looking for the decomposition with the lowest such 'overall' dimension.)

The modified box counting dimensions behave more stable as their original counterparts, in particular all countable sets have dimension zero. However, they are usually hard to calculate, due to the extra inf / sup process involved.

1.9.2 Packing measures and packing dimension

Packing dimension can be seen as a dual to Hausdorff dimension. Whereas Hausdorff measures are defined in terms of economical coverings, that is, enclosing a set from outside, packing measures approximate from the inside, by packing it economically with disjoint sets of small size.

For this purpose, we say that a prefix free set $P \subseteq 2^{<\omega}$ is a *packing* in $\mathcal{X} \subseteq 2^{\omega}$, if for every $\sigma \in P$, $\sigma \sqsubset \mathcal{X}$. Geometrically speaking, a packing in $\mathcal{X}$ is a collection of mutually disjoint open balls with centers in $\mathcal{X}$. If the balls all have radius $\leq \delta$, we call it a δ-packing in $\mathcal{X}$.

Now one can try to find a packing as 'dense' as possible: Given $s \geq 0$, $\delta > 0$, let

$$\mathcal{P}^s_\delta(\mathcal{X}) = \sup\left\{\sum_{w\in P} 2^{-|w|s} : P \text{ is a } \delta\text{-packing in } \mathcal{X}.\right\}. \tag{1.34}$$

Again, as $\mathcal{P}^s_\delta(\mathcal{X})$ decreases with δ, the limit

$$\mathcal{P}^s_0(\mathcal{X}) = \lim_{\delta\to 0} \mathcal{P}^s_\delta(\mathcal{X})$$

exists. However, this definition leads to the same problems we encountered with box counting dimension: Taking, for instance, the rational numbers in the unit interval, we can find denser and denser packings yielding that for every $0 \leq s < 1$, $\mathcal{P}^s_0(\mathbb{Q} \cap [0, 1]) = \infty$, hence it lacks countable additivity, in particular it is not a measure. This can be overcome by applying a Caratheodory process to $\mathcal{P}^s_0$. Hence define

$$\mathcal{P}^s(\mathcal{X}) = \inf\left\{\sum \mathcal{P}^s_0(\mathcal{X}_i) : \mathcal{X} \subseteq \bigcup_{i\in\mathbb{N}} \mathcal{X}_i\right\}. \tag{1.35}$$

(The infimum is taken over arbitrary countable covers of $\mathcal{X}$.) $\mathcal{P}^s$ is an (outer) measure on 2^{ω}, and it is Borel regular. (This needs no longer be true if the dimension function $h(x) = x^s$ is replaced by more irregular functions not satisfying even weak continuity requirements.) $\mathcal{P}^s$ is called, in correspondence to Hausdorff measures, the *s-dimensional packing measure* on 2^{ω}. Packing measures were introduced by Tricot (1982) and Taylor and Tricot (1985). They can be seen as a dual concept to Hausdorff measures, and behave in many ways similar to them. In particular, one may define *packing dimension* in the same way as Hausdorff dimension.

Definition 1.23 The *packing dimension* of a class $\mathcal{X} \subseteq 2^{\omega}$ is defined as

$$\dim_{\mathrm{P}} \mathcal{X} = \inf\{s : \mathcal{P}^s(\mathcal{X}) = 0\} = \sup\{s : \mathcal{P}^s(\mathcal{X}) = \infty\}. \tag{1.36}$$

Packing dimension has stability properties similar to Hausdorff dimension, e.g. countable stability. With some effort, one can show that it *coincides with* $\overline{\dim}_{\mathrm{MB}}$ (see Falconer, 1990, Chapter 3). Generally, the following relations between the different dimension concepts hold: For any $\mathcal{X} \subseteq 2^\omega$,

$$\dim_{\mathrm{H}} \mathcal{X} \leq \underline{\dim}_{\mathrm{MB}} \mathcal{X} \leq \overline{\dim}_{\mathrm{MB}} \mathcal{X} = \dim_{\mathrm{P}} \mathcal{X} \leq \overline{\dim}_{\mathrm{B}} \mathcal{X}. \tag{1.37}$$

Whereas the traditional definition of packing dimension is rather complicated due to the additional decomposition/optimization step, there is a martingale characterization, discovered by Athreya et al. (2004), that exposes the dual nature of packing measures and packing dimension much more clearly.

Definition 1.24 (Athreya et al., 2004) Given $0 < s \leq 1$, a martingale $d : 2^{<\omega} \to [0, \infty)$ is *strongly s-successful* (or *s-succeeds strongly*) on a sequence A if

$$d(A\restriction_n) \geq 2^{(1-s)n} \quad \text{for all but finitely many } n. \tag{1.38}$$

Obviously, condition (1.38) implies $\lim_n d(A\restriction_n)/2^{(1-t)n} = \infty$ for all $t > s$. Since we are only concerned with packing dimension here, which is defined as an infimum, we will denote the latter condition as s-successful too. From a game-theoretical perspective, succeeding strongly means not only to accumulate arbitrary high levels of capital, but also to be able to guarantee that the capital stays above arbitrary high levels from a certain time on.

Theorem 1.25 (Athreya et al., 2004) *For any set $\mathcal{X} \subseteq 2^\omega$,*

$$\dim_{\mathrm{P}} \mathcal{X} = \inf\{s : \text{ some martingale } d \ s\text{-succeeds strongly on all } B \in \mathcal{X}\}.$$

Proof. ($\leq$) Suppose some martingale d is strongly s-successful on $\mathcal{X}$. By (1.37) it suffices to show that $\overline{\dim}_{\mathrm{MB}} \mathcal{X} \leq s$. Consider the set of strings

$$D_n = \{\sigma \in \{0,1\}^n : d(\sigma) > 2^{(1-s)n}\}.$$

Obviously, every $B \in \mathcal{X}$ is contained in all but finitely many $[D_n]$. Therefore,

$$\mathcal{X} \subseteq \bigcup_{i \in \mathbb{N}} \bigcap_{j \geq i} [D_j].$$

If we let $\mathcal{X}_i = \bigcap_{j \geq i} [D_j]$, it is enough to show that $\overline{\dim}_{\mathrm{B}} \mathcal{X}_i \leq s$ for all i. Note that $\mathcal{X}_i \subseteq [D_n]$ for all $n \leq i$, hence $|\mathcal{X}_i\restriction_n| \leq |D_n|$. Now it follows from Kolmogorov's inequality (Lemma 1.7) that

$$|D_n| \leq 2^{ns}.$$

Therefore,

$$\overline{\dim}_{\mathrm{B}}\mathcal{X}_i = \limsup_{n\to\infty} \frac{\log|\mathcal{X}_i\restriction_n|}{n} \leq \limsup_{n\to\infty}\frac{\log|D_n|}{n} \leq s.$$

($\geq$) We may assume $\overline{\dim}_{\mathrm{MB}}\mathcal{X} < 1$. Fix any s', s, t be such that $\overline{\dim}_{\mathrm{MB}}\mathcal{X} < s' < s < t < 1$. We show that there exists a martingale d which is strongly t-successful on $\mathcal{X}$. From the definition of modified box counting dimension we can infer the existence of sets $\mathcal{X}_i$ such that $\mathcal{X} \subseteq \bigcup_i \mathcal{X}_i$ and $\overline{\dim}_{\mathrm{B}}\mathcal{X}_i < s'$ for all i.

It follows that, for all i, there exists a number n_i,

$$(\forall n \geq n_i)\ \frac{\log|\mathcal{X}_i|}{n} < s'.$$

We show that for each i we can find a strongly t-successful martingale for $\mathcal{X}_i$. (This is actually enough: Using additivity of martingales, one could combine these into a single martingale which is $(t+\varepsilon)$-successful on $\mathcal{X}$, with $\varepsilon > 0$ arbitrary small.)

Fix an arbitrary $i \in \mathbb{N}$. Let $X_n = \mathcal{X}_i\restriction_n$. For each $n \geq n_i$, define a martingale d_n (inductively) as follows:

$$d_n(\sigma) = \begin{cases} 2^{|\sigma|}|X_n\restriction_{[\sigma]}|2^{-sn}, & \text{if } |\sigma| \leq n, \\ 2^{n-|\sigma|}d_n(\sigma\restriction_n), & \text{if } |\sigma| > n. \end{cases}$$

Note that for $\sigma \in X_n$, $d_n(\sigma) = 2^{(1-s)n}$. Let $d = \sum_{n\geq n_i} d_n$. The finiteness of d follows from

$$d(\epsilon) = \sum_{n\geq n_i} d_n(\epsilon) = \sum_{n\geq n_0}|X_n|2^{-sn} < \sum_{n\geq n_i} 2^{(s'-s)n} < \infty.$$

Finally, if $B \in \mathcal{X}_i$, we have that $B\restriction_n \in X_n$ for all n, thus, for $n \geq n_i$,

$$\frac{d(B\restriction_n)}{2^{(1-t)n}} \geq \frac{d_n(B\restriction_n)}{2^{(1-t)n}} \geq \frac{2^{(1-s)n}}{2^{(1-t)n}} = 2^{(t-s)n}.$$

As $t > s$, this completes the proof. □

1.10 Mass Distributions and Capacity

The definition of Hausdorff measure makes it often much easier to estimate the Hausdorff dimension of a set from above than from below. To give an upper bound s on the Hausdorff dimension of a set $\mathcal{X}$, one has to expose a single family of coverings that verifies $\mathcal{H}^s(\mathcal{X}) = 0$. On the other hand, to provide a lower bound on $\dim_{\mathrm{H}} \mathcal{X}$, one has to take into account all possible coverings of $\mathcal{X}$ and show that for some s, no covering witnesses that $\mathcal{H}^s(\mathcal{X})$ is zero.

One way to overcome this difficulty is to study mass distributions μ which render $\mathcal{X}$ non-negligible, $\mu(\mathcal{X}) > 0$, and study how the mass of sets $\mathcal{U}$ used in a covering compares to $d(\mathcal{U})^s$ for some given s. If $\mu(\mathcal{U}_i) \leq d(\mathcal{U}_i)^s$, $\sum d(\mathcal{U}_i)^s$ cannot be too small.

Definition 1.26 Let $\mathcal{X} \subseteq 2^\omega$. A *mass distribution* on $\mathcal{X}$ is a finite measure μ on 2^ω such that $0 < \mu(\mathcal{X})$.

The following method to provide a lower bound on the Hausdorff dimension is called the *mass distribution principle*.

Theorem 1.27 (Mass distribution principle) *Let μ be a mass distribution on $\mathcal{X}$ and suppose that for some $s \geq 0$ there are $c, \delta > 0$ such that*

$$\mu(\mathcal{U}) \leq c d(\mathcal{U})^s \tag{1.39}$$

for all $\mathcal{U} \subseteq 2^\omega$ with $d(\mathcal{U}) < \delta$. Then $\mathcal{H}^s(\mathcal{X}) \geq \mu(\mathcal{X})/c$ and thus

$$s \leq \dim_{\mathrm{H}}(\mathcal{X}).$$

Theorem 1.27 can be generalized by classifying mass distributions on $\mathcal{X}$ according to whether they satisfy (1.39) for some s. This approach is closely related to the notion of *capacity*, which arises normally in the context of potential theory.

Definition 1.28 Let μ be a mass distribution on 2^ω, $0 \leq t \leq 1$. The *t-potential at $A \in 2^\omega$ due to μ* is defined as

$$\phi_t(A) = \int d(A, B)^{-t} d\mu(B). \tag{1.40}$$

The *t-energy* of μ is given by

$$I_t(\mu) = \int \phi_t(A) d\mu(A) = \iint d(A, B)^{-t} d\mu(B) d\mu(A). \tag{1.41}$$

Using the special nature of 2^ω, (1.40) can be simplified and stated in terms of sums instead of integrals. For $A \in 2^\omega$, denote by $(A\upharpoonright_n)'$ the initial segment of A of length n with the last bit switched, i.e. $(A\upharpoonright_n)' = A_0 A_1 \dots A_{n-2}{}^\frown(1 - A_{n-1})$. Then it holds that

$$\phi_t(A) = \begin{cases} \sum_{n=0}^{\infty} 2^{nt} \mu((A\upharpoonright_{n+1})'], & \text{if } \mu(\{A\}) > 0, \\ \infty, & \text{if } \mu(\{A\}) = 0. \end{cases}$$

Inserting this in (1.41) yields

$$I_t(\mu) = 2\sum_{n=0}^{\infty} 2^{nt} \sum_{w\in\{0,1\}^n} \mu[w0]\mu[w1]. \tag{1.42}$$

Observe that if a mass distribution satisfies for some $c, s \in \mathbb{R}$

$$\mu[w] \le c2^{-|w|s} \quad \text{for all } w \in 2^{<\omega}, \tag{1.43}$$

it follows immediately that $\phi_t(A) \le$ const for all $t < s$, hence $I_t(\mu) < \infty$. On the other hand, if $I_t(\mu) < \infty$, (1.43) holds for a suitable restriction of μ.

Definition 1.29 Let $s > 0$. The *s-capacity* of a class $\mathcal{A} \subseteq 2^\omega$ is defined as

$$C_s(\mathcal{A}) = \sup\left\{\frac{1}{I_s(\mu)} : \ \mu \text{ mass distr. on } \mathcal{A} \text{ with } \mu(2^\omega) = 1\right\}.$$

(As potentials and capacities may be infinite, we adopt the convention that $1/\infty = 0$.) We note from the definition that a class has positive s-capacity if and only if there is a mass distribution μ on it such that $I_s(\mu) < \infty$. This suggests the following definition.

Definition 1.30 The *capacitary dimension* of a class $\mathcal{A} \subseteq 2^\omega$ is

$$\dim_c(\mathcal{A}) = \sup\{s : \ C_s(\mathcal{A}) > 0\}.$$

With little effort it can be shown that

$$\dim_c(\mathcal{A}) = \sup\{s : \ \exists\mu \text{ mass distr. on } \mathcal{A} \text{ with } \mu[w] \le 2^{-|w|s}\ \forall w \in 2^{<\omega}\}.$$

Furthermore, the capacitary dimension of a Borel class is equal to its Hausdorff dimension.

Theorem 1.31 (Frostman, 1935) *Let $\mathcal{A}$ be a Borel class in 2^ω. Then*

$$\dim_c(\mathcal{A}) = \dim_{\mathrm{H}}(\mathcal{A}).$$

We will not give a proof of this theorem here. It can be found in the textbooks by Falconer (1990) or Mattila (1995). Theorem 1.31 yields a method for obtaining lower bounds on the Hausdorff dimension of a class, a task which is usually much more difficult then giving upper bounds.

1.11 Entropy and Hausdorff Dimension

The theory of dynamical systems provides a lot of examples of fractal objects. The work of Mandelbrot (see for example Mandelbrot, 1976, 1980, 1982) brought to the attention of a large (even non-mathematical) audience that limit sets of dynamical systems are fractals. On the other hand, in dynamical systems theory a variety of entropy notions are used to classify the complexity of the underlying system. So it is not really surprising that a close relation between entropy and Hausdorff dimension exists. This relation turned out to be particularly close when the underlying space is a sequence space (such as 2^ω). We illustrate this by two examples, one of topological, the other of measure theoretic nature.

1.11.1 Topological entropy

There is a large family of sets studied in the theory of dynamical systems for which Hausdorff dimension equals box counting dimension.

In general, a *topological dynamical system* consists of a continuous transformation of a compact metric space. An important example is the *shift map* T on 2^ω, defined by

$$(T(A))_n = A_{n+1} \quad \text{for all } A \in 2^\omega, n \geq \mathbb{N}.$$

If $\mathcal{A} \subseteq 2^\omega$ is shift-invariant, i.e. if $T(\mathcal{A}) = \mathcal{A}$, the pair $(\mathcal{A}, T)$ constitutes a *symbolic dynamical system*. One might now study how complex this system behaves, for instance, how sets transform under repeated application of T from a topological point of view. *Topological entropy* $h_{\text{top}}(\mathcal{A})$ yields a measure of this complexity by following how open covers of $\mathcal{A}$ refine under T. The general definition of topological entropy for topological dynamical systems is quite involved and requires some preliminary work, so we will not present it here. (The monograph by Walters (1982) may serve as a reference for this section.) However, if $\mathcal{A}$ is a shift-invariant, closed subclass of 2^ω, it is just the box counting dimension of $\mathcal{A}$ (upper and lower box counting coincide for such sets). Furthermore, for those classes box counting and Hausdorff dimension are identical.

Theorem 1.32 *Let $\mathcal{A} \subseteq 2^\omega$ be closed and shift-invariant, i.e. $T(\mathcal{A}) = \mathcal{A}$. Then it holds that*

$$\dim_{\text{H}}(\mathcal{A}) = \lim_{n\to\infty} \frac{\log |\mathcal{A}\restriction_n|}{n} =: h_{\text{top}}(\mathcal{A}) \tag{1.44}$$

Of course, the theorem also asserts that the limit (1.44) exists (as mentioned before). We will not show this here. One way to prove the existence of this limit is to exploit the closure under the shift-transformation combinatorially, another (much shorter) proof is based on the powerful Perron-Frobenius theory (again, for details refer to Walters, 1982). The following proof of Theorem 1.32 is adapted from Furstenberg (1967).

Proof. Let $s > h_{\mathrm{top}}(\mathcal{A})$. Then for all s' such that $s > s' > \lim_n \log |\mathcal{A}\restriction_n|/n$ it holds that

$$s' > \frac{\log |\mathcal{A}\restriction_n|}{n} \tag{1.45}$$

for all sufficiently large n. Every set $\mathcal{A}\restriction_n$ induces a 2^{-n}-cover of $\mathcal{A}$ (consider $\{[w] : w \in \mathcal{A}\restriction_n\}$). Therefore,

$$\mathcal{H}^s_{2^{-n}}(\mathcal{A}) \leq |\mathcal{A}\restriction_n| 2^{-ns} < \delta^n \tag{1.46}$$

for some $\delta < 1$ and for all sufficiently large n. Hence, $\mathcal{H}^s(\mathcal{A}) = 0$, which implies $\dim_{\mathrm{H}}(\mathcal{A}) \leq s$ and thus, as s was arbitrary, $\dim_{\mathrm{H}}(\mathcal{A}) \leq h_{\mathrm{top}}(\mathcal{A})$.

Now let $s < h_{\mathrm{top}}(\mathcal{A})$. To obtain $h_{\mathrm{top}}(\mathcal{A}) \leq \dim_{\mathrm{H}}(\mathcal{A})$, it suffices to show that, for all sufficiently large n, for all prefix-free finite 2^{-n}-covers $\{w_1, \dots, w_m\}$ of $\mathcal{A}$, and for all s' such that $s < s' < h_{\mathrm{top}}(\mathcal{A})$, it holds that

$$\sum_{i=1}^{m} 2^{-|w_i|s} \geq 1. \tag{1.47}$$

(Then $\mathcal{H}^{s'}(\mathcal{A}) > 0$ and $\dim_{\mathrm{H}}(\mathcal{A}) > s$.) Note that finite covers suffice as $\mathcal{A}$ is closed and hence compact as a subset of a compact space.

So let $C = \{w_1, \dots, w_m\}$ be a prefix-free 2^{-n}-cover of $\mathcal{A}$. Denote by B the set of strings that occur as a block in some $A \in \mathcal{A}$. Since $\mathcal{A}$ is shift-invarinat, it holds that $\mathcal{A}\restriction_n = B \cap \{0,1\}^n$.

Suppose for a contradiction that $a = \sum 2^{-|w_i|s'} < 1$. As a geometric series $\sum_{b=1}^{\infty} b^n$ converges if $b < 1$, we have that

$$\sum 2^{-|w_{i_1} w_{i_2} \dots w_{i_l}|s'} < \infty \tag{1.48}$$

where the sum is taken over all possible concatenations of the strings w_i generating the cover. (The above series can be written as

$$\sum_{i_1} 2^{-|w_{i_1}|s'} + \sum_{i_1,i_2} 2^{-|w_{i_1} w_{i_2}|s'} + \sum_{i_1,i_2,i_3} 2^{-|w_{i_1} w_{i_2} w_{i_3}|s'} + \dots = \sum_{n=1}^{\infty} a^n \tag{1.49}$$

from which (1.48) follows.)

Now we claim that there exists a finite set of strings $\{v_1, \dots, v_k\}$ such that every string $w \in B$ (i.e. that occurs as a block in some $A \in \mathcal{A}$) can be written as a concatenation $w = w_{i_1} w_{i_2} \dots w_{i_l} v_j$, where $w_{i_j} \in C$. For, if w occurs as a block in some $A \in \mathcal{A}$, then, by the shift invariance of $\mathcal{A}$, it also occurs as an initial segment of some other $A' \in \mathcal{A}$. If $|w| \geq \max\{|w_i| : 1 \leq i \leq m\}$, there has to be some w_{i_1} such that $w = w_{i_1} w'$. Again, $w' \in B$, and the same argument as before yields that $w' = w_{i_2} w''$. Inductively we get that every $w \in B$ can be written as

$w = w_{i_1} w_{i_2} \dots w_{i_l} v$ with $|v| < \max\{|w_i| : 1 \le i \le m\}$. But there are only finitely many strings v with this property.

Combining this with (1.48), we get

$$\sum_{w \in B} 2^{-|w|s'} < \infty, \tag{1.50}$$

in other words

$$\sum_{n} |\mathcal{A}\restriction_n| 2^{-ns'} < \infty. \tag{1.51}$$

But $|\mathcal{A}\restriction_n| > 2^{ns'}$ for all sufficiently large n and therefore the series above must diverge. □

1.11.2 Measure-theoretic entropy

The concept of entropy is fundamental to many areas related to probability and information. Shannon (1948) introduced it into communication theory in his seminal paper on mathematical information theory. Ten years later, Kolmogorov (1958, 1959) devised it as a new invariant in ergodic theory, the measure-theoretic study of dynamical systems. In both fields, it developed into a indispensable notion. Most basically, entropy can be thought of as the amount of information gained (information-theoretically) or the amount of uncertainty/randomness (from a probability-theoretic point of view) when performing a chance experiment.

Measure-theoretic entropy is generally defined for (finite) partitions.

Definition 1.33 Let $P = \{p_1, \dots, p_n\}$ be a finite set of non-negative real numbers such that $\sum p_i = 1$. The *entropy* $H(P)$ of P is defined as

$$H(P) = -\left[\sum_{i=1}^{n} p_i \log p_i\right]. \tag{1.52}$$

So, mathematically, $H(P)$ is nothing but the expected value of $-\log p_i$. Note that, given a Borel probability measure μ on 2^ω, every finite partition of 2^ω given by a prefix free sets of strings induces a set P as in Definition 1.33.

It is easy to see that H is at most $\log n$ (if all the p_i are equal), and at least 0 (if one of the p_i is 1). Relating this to the introductory paragraph, randomness should be regarded highest for a n-sided dice for which every side has probability $1/n$, and lowest for a dice which totally prefers a single side, i.e., where $p_i = 1$ for some i and $p_j = 0$ for $j \neq i$.

The definition of entropy for a measure is more complicated and will not be presented here. However, for the case of (generalized) Bernoulli measures it turns out to be a direct analogue of Definition 1.33.

Definition 1.34 Let $\vec{p} = (p_0, p_1, p_2, \dots)$ be a sequence of real numbers such that $0 \le p_i \le 1$ for all i and $p_i \to p$. The *entropy* $H(\mu_{\vec{p}})$ of the generalized Bernoulli measure $\mu_{\vec{p}}$ is

$$H(\mu_{\vec{p}}) = -[p \log p + (1-p)\log(1-p)]. \tag{1.53}$$

So, among all Bernoulli measures Lebesgue measure is the one with the highest entropy. A fundamental result on information and measure-theoretic entropy is the *Shannon-McMillan-Breiman theorem.*

Theorem 1.35 (Shannon-Mcmillan-Breiman) *If $\mu_{\vec{p}}$ is a generalized Bernoulli measure induced by $\vec{p} = (p_0, p_1, p_2, \dots)$ with $p_i \to p$, then it holds that, for $\mu_{\vec{p}}$-almost every $A \in 2^\omega$,*

$$-\frac{1}{n} \log \mu_{\vec{p}}[A\restriction_n] \xrightarrow{n\to\infty} H(\mu_{\vec{p}}). \tag{1.54}$$

Eggleston (1949) proved one the first results relating measure-theoretic entropy to Hausdorff dimension. (The following theorem was first conjectured by Good (1941).) For a string w and $i \in \{0,1\}$, denote by $N_i(w)$ the number of occurences of i in w, that is, $N_1(w) = \sum_{i<|w|} w(i)$ and $N_0(w) = |w| - N_1(w)$.

Theorem 1.36 *For $p \in [0,1]$, let*

$$\mathcal{B} = \left\{ A \in 2^\omega : \lim_{n\to\infty} \frac{N_1(A\restriction_n)}{n} = p \right\}.$$

Then it holds that

$$\dim_{\mathrm{H}} \mathcal{B} = H(\mu_p).$$

We will not prove 1.36 here, but an effective version in section 2.5.3, thereby providing a canonical example of sequences having non-integral effective dimension.

Chapter 2

Effective Hausdorff Dimension

2.1 Effectivizing Hausdorff Measures and Dimension

There have been many attempts to define the notion of an *effective measure*. Part of those, such as Brouwer (see, for instance, Heyting, 1966) or Bishop (1967) (also Bishop and Cheng, 1972) were motivated by intuitionist or constructivist approaches to mathematics. Others, like Martin-Löf (1966), aimed for an appropriate characterization of *algorithmic randomness*. The key idea of Martin-Löf consisted in defining *effective nullsets*. An effective nullset is a nullset with the additional property that one is able to witness this by a sequence of uniformly enumerable covers. This means that one must be able to produce an algorithm such that for any n, the algorithm enumerates a covering such that the accumulated measure of it is less than 2^{-n}. Martin-Löf proved the existence of a *universal effective nullset*, one that contains all other effective nullsets. A *random sequence* in the sense of Martin-Löf is a sequence which is not contained in this maximal nullset.

A different approach was taken, among others, by Kolmogorov (1965), trying to characterize randomness by information theoretic incompressibility, nowadays known as *Kolmogorov complexity*. It turned out that Kolmogorov's original concept was not suitable for defining random infinite sequences, but a modification of Kolmogorov complexity, *prefix complexity*, introduced independently by Levin (1974), Gács (1974), and Chaitin (1975), yielded a definition of randomness that proved to be equivalent to Martin-Löf's concept.

A crucial link between both concepts is provided by the theory of enumerable semimeasures, studied in a seminal paper by Zvonkin and Levin (1970).

2.1.1 Effective Hausdorff measure

Martin-Löf (1966) effectivized the notion of a Lebesgue nullset. For a set to be of *effective measure* 0, he required that the sequence of coverings testifying that the measure is 0 can be given effectively, i.e there is an algorithm (that is, a computable function) that, given an input n, enumerates a cover of the set which has measure no more than 2^{-n}. Hausdorff measures are similar to ordinary Lebesgue measure in the sense that the small sets are the ones that can be approximated well in measure with respect to the function h. Therefore, the notion of an effective nullset can be extended to Hausdorff measures in a straightforward way. Using the martingale characterization of Hausdorff measures, as presented in Section 1.8, this effectivization was first introduced by Lutz (2000a,b, 2003).

Our generalized presentation will be based on Martin-Löf's measure-theoretic approach. For this purpose, we have to make sure to use dimension functions which are in some sense effective. Therefore, from now on we will restrict $\mathfrak{F}_d$ to the class of *computable* dimension functions.

The following definition presents a general effectivization of Hausdorff measures.

Definition 2.1 Let $h \in \mathfrak{F}_d$ be a computable dimension function. A set $\mathcal{X} \subseteq 2^\omega$ has *effective h-dimensional Hausdorff measure* 0 if there is a uniformly computably enumerable sequence $\{C_n\}_{n\in\mathbb{N}}$ of sets of finite strings such that for every $n \in \mathbb{N}$, the open set induced by $C_n \subseteq 2^{<\omega}$ covers $\mathcal{X}$ and

$$\sum_{w\in C_n} h(2^{-|w|}) \leq 2^{-n}. \tag{2.1}$$

Sometimes we will also write that $\mathcal{X}$ has Σ_1^0-$\mathcal{H}^h$*-measure* 0 (Σ_1^0-$\mathcal{H}^h(\mathcal{X}) = 0$) or simply say that $\mathcal{X}$ is Σ_1^0-$\mathcal{H}^h$*-null*. The computable sequence of c.e. sets C_n satisfying (2.1) generalizes Martin-Löf's approach to effective Lebesgue measure. In this spirit, we will call such a sequence of C_n an h-Martin-Löf test. Indeed, setting $h = \mathrm{id}$, we obtain the concept of *effective Lebesgue nullsets*, which we will denote as Σ_1^0-λ-null.

At this point it is useful to introduce some notation: if $C \subseteq 2^{<\omega}$, we write $[C]$ for the set of all sequences extending some $w \in C$, i. e.

$$[C] \stackrel{\text{def}}{=} \bigcup_{w\in C}[w].$$

So we might express Definition 2.1 as follows: Σ_1^0-$\mathcal{H}^h(\mathcal{X}) = 0$ if and only if there is a computable function f such that

$$\mathcal{X} \subseteq \bigcap_{n\in\mathbb{N}}[W_{f(n)}] \quad \text{and} \quad \sum_{w\in W_{f(n)}} h(2^{-|w|}) \leq 2^{-n} \text{ for all } n,$$

where $\{W_e\}_{n\in\mathbb{N}}$ is a standard enumeration of the computably enumerable (c.e.) subsets of $\mathbb{N}$ (which may, as mentioned before, be interpreted as sets of strings as well).

Observe how Definition 2.1 resembles Proposition 1.11 except for the additional effectivity constraints.

In Chapter 1 we derived two equivalent characterizations of $\mathcal{H}^h$-nullsets in terms of Solovay covers and semimeasures. It will turn out that these characterizations will *not* transfer seamlessly to the effective case. The main reason for this lies in the fact that, unless h is the identity function (which induces Lebesgue measure), the function $h(d([w])) = h(2^{-|w|})$ is not additive on prefixes. That is it does not hold that

$$h(2^{-|w|}) = h(2^{-|w0|}) + h(2^{-|w1|}).$$

(It is easy to see that any dimension function which satisfies (2.1.1) must be the identity on all numbers of the form 2^{-n}, $n \geq 0$.) It therefore, in general not possible to replace a string w in a covering by a set of longer strings inducing the same cover. This possibility, however, is crucial in the effective setting.

Definition 2.2 For any $s \geq 0$, a Solovay s-test C is *effective* if C is a computably enumerable set.

It is immediate that if a class $\mathcal{X}$ has effective h-dimensional Hausdorff measure 0, then $\mathcal{X}$ has an effective Solovay h-cover, simply by taking the union of all the sets of a Martin-Löf s-test covering $\mathcal{X}$.

The converse implication of Proposition 1.12, however, is no longer true in the effective case, as we shall see later.

Nevertheless, when defining an effective variant of Hausdorff dimension, both notions, Martin-Löf s-tests and effective Solovay s-tests can be used equivalently.

Proposition 2.3 *If $\mathcal{X} \subseteq 2^\omega$ is covered by an effective Solovay s-test, then $\mathcal{X}$ is t-Σ^0_1-$\mathcal{H}^s$-null for any $t > s$.*

Proof. Assume C is an effective Solovay s-cover for $\mathcal{X}$, and let $t > s$. Given $n \geq 0$, we define a c.e. set C_n by enumerating only those elements of C which have length greater than $n/(t-s)$. Then C_n obviously covers $\mathcal{X}$, and it holds that

$$\sum_{w \in C_n} 2^{-|w|t} = \sum_{w \in C_n} 2^{-|w|(t-s)} 2^{-|w|s} \leq 2^{-n} \sum_{w \in C_n} 2^{-|w|s} \leq 2^{-n}.$$

Hence, (C_n) is a Martin-Löf t-test for $\mathcal{X}$. □

Not only are Solovay tests often more easily to deal with, we will also see that they correspond to the effective nullsets defined via *enumerable* semimeasures.

Definition 2.4 A function $f : \mathbb{N} \to \mathbb{R}$ is *enumerable from below* or *left-enumerable*, if the left cut

$$\{(n, q) \in \mathbb{N} \times \mathbb{Q} : \; f(n) > q\}$$

is computably enumerable. f is *right-enumerable* if the right cut

$$\{(n, q) \in \mathbb{N} \times \mathbb{Q} : \; f(n) < q\}$$

is computably enumerable. f is *computable* if it is both left- and right-enumerable.

Equivalently, a function $f : \mathbb{N} \to \mathbb{Q}$ is enumerable from below (above) if it can be approximated by an increasing (decreasing) computable function. Thus, enumerable functions form an effective analog to the semicontinuous functions in the classical setting. We mostly deal with left-enumerable functions. We will also call them *left-computable*, or, if the context is clear, simply call them *enumerable*.

One of the cornerstones of Martin-Löf's effective theory of measure is the *presence of universal objects*. Just as a universal Turing machines can simulate all other Turing machines, a universal Martin-Löf h-test embraces all other Martin-Löf h-tests. This has two important consequences: On the one hand, studying Σ_1^0-$\mathcal{H}^h$-nullsets is greatly simplified since one has always at hand a 'greatest possible' nullset. On the other hand, a maximal nullset enables one to consider the objects outside this set as *random* with respect to the underlying measure. This was Martin-Löf's original intention when he introduced his notion of effective measure zero.

We collect the existence of universal tests for Hausdorff measures in the following theorem.

Theorem 2.5 *Let $h \in \mathfrak{F}_d$ be a computable dimension function.*

(a) There exists a Martin-Löf h-test $\{U_n\}$ such that for all $\mathcal{X} \subseteq 2^\omega$

$$\mathcal{X} \text{ is } \Sigma_1^0\text{-}\mathcal{H}^h\text{-null} \iff \mathcal{X} \subseteq \bigcap_{n\in\mathbb{N}} [U_n].$$

(b) There exists an enumerable semimeasure $\widetilde{m}$ such that for any other enumerable semimeasure m there is a constant c_m with

$$(\forall w \in 2^{<\omega})\,[m(w) \leq c_m \widetilde{m}(w)].$$

Note that part (c) of the theorem, the existence of a universal semimeasure, is in a certain sense a stronger statement than the other two as it is independent of h.

Proof. (a) *Construction of a universal Martin-Löf h-test:* There are several constructions of a universal Martin-Löf test with respect to Lebesgue measure (which corresponds to a Martin-Löf id-test in our diction). We will adopt the original construction given by Martin-Löf (1966), variants of which were exposed by Kučera (1985) or Terwijn (1998). It stresses the diagonalizational aspect of the construction. For a different construction which might serve as a starting point see the book by Li and Vitányi (1997).

The test we construct will have the further advantage that the covers are nested, i.e. $[U_0] \supseteq [U_1] \supseteq [U_2] \supseteq \ldots$.

Given $n \in \mathbb{N}$, consider all indices $e > n$. For each such e, enumerate all elements of $W_{\{e\}(e)}$ into U_n (where we understand that $W_{\{e\}(e)}$ is empty if $\{e\}(e)$ is undefined) as long as the condition

$$\sum_{w \in W_{\{e\}(e)}} h(2^{-|w|}) < 2^{-e}$$

is satisfied. Then

$$\sum_{w \in U_n} h(2^{-|w|}) \leq \sum_{e>n} 2^{-e} = 2^{-n}.$$

Thus, $\{U_n\}$ is a Martin-Löf h-test. To see that it is universal, let $\{e\}$ be the index of some Martin-Löf h-test $\{V_i\}$, i.e. $V_i = W_{\{e\}(i)}$. It is a known fact that each computable function possesses infinitely many indices (using padding). So, for every n there exists $i > n$ such that $\{e\} = \{i\}$, which means that $W_{\{e\}(i)} = W_{\{i\}(i)}$ will enter U_n completely. Conversely, for every i that is an index of the function $\{e\}$, $W_{\{i\}(i)}$ will enter U_m for all $m < i$. This suffices to conclude that

$$\bigcap_{i \in \mathbb{N}} [V_i] \subseteq \bigcap_{i \in \mathbb{N}} [U_i],$$

which implies that $\bigcap [U_i]$ contains all Σ^0_1-$\mathcal{H}^h$-nullsets.

(c) The existence of a universal enumerable semimeasure is a classic result from Algorithmic Information Theory, first established by Levin (Zvonkin and Levin, 1970).

□

Using fundamental results on $\widetilde{m}$ due to Chaitin (1976), we can prove an alternative characterization of effective $\mathcal{H}^h$-nullsets via enumerable semimeasures.

Theorem 2.6 *Let $h \in \mathfrak{F}_d$ computable, $\mathcal{X} \subseteq 2^{\omega}$. Then the following are equivalent:*

(i) $\mathcal{X}$ has effective h-dimensional Hausdorff measure 0.

(ii) For any $A \in \mathcal{X}$,

$$\limsup_{n\to\infty} \frac{\widetilde{m}(A\restriction_n)}{h(2^{-n})} = \infty, \tag{2.2}$$

where $\widetilde{m}$ denotes the universal, discrete semimeasure enumerable from below introduced in Theorem 2.5.

Proof. (i) ⇒ (ii): Assume $\mathcal{X}$ is a Σ_1-h-nullset. Thus there exists a computable sequence $C_1, C_2, C_3, \ldots$ of enumerable sets of strings such that for all n

$$(\forall A \in \mathcal{X})\,(\exists w \in C_n)\; w \sqsubset A \quad \text{and} \quad \sum_{w \in C_n} h(2^{-|w|}) \leq 2^{-n}. \tag{2.3}$$

Define functions $m_n : 2^{<\omega} \to \mathbb{Q}$ by

$$m_n(w) = \begin{cases} nh(2^{-|w|}) & \text{if } w \in C_n, \\ 0 & \text{otherwise,} \end{cases}$$

and let

$$m(w) = \sum_{n=1}^{\infty} m_n(w).$$

Obviously, all m_n and thus m are enumerable from below. Furthermore,

$$\sum_{w \in 2^{<\omega}} m(w) = \sum_{w \in 2^{<\omega}} \sum_{n=1}^{\infty} m_n(w) = \sum_{n=1}^{\infty} n \sum_{w \in C_n} h(2^{-|w|}) \leq \sum_{n=1}^{\infty} \frac{n}{2^n} < \infty,$$

so m is an enumerable semimeasure. Now let $A \in \mathcal{X}$ and let $c > 0$ be any constant. If we set $k = \lceil c \rceil + 1$, then, by (2.3), there is some $w \in C_k$ with $w \sqsubset A$, say $w = A\restriction_n$. This implies $m(A\restriction_n) \geq kh(2^{-n}) > ch(2^{-n})$ and therefore $\limsup m(A\restriction_n)/h(2^{-n}) = \infty$.

(ii) $\Rightarrow$ (i): This part of the proof is an adaption of a standard proof that every Marti-Löf random sequence is incompressible with respect to prefix-free Kolmogorov complexity (see e.g. Downey and Hirschfeldt, 2004). It is based on a fundamental result by Chaitin (1976) which establishes that for any l,

$$|\{\sigma \in \{0,1\}^n : \widetilde{m}(\sigma) \geq \widetilde{m}(n)2^{-n+l}\}| \leq 2^{n-l+c}, \tag{2.4}$$

where c is a constant independent of l. (Remember that the natural numbers are identified with their binary representation.)

Assume (2.2) holds for every $A \in \mathcal{X}$. W.l.o.g. we may assume that

$$\sum_{\sigma \in 2^{<\omega}} \widetilde{m}(\sigma) \leq 1$$

(if necessary we can set a finite number of values to zero, which does not have an influence on property (2.2)). Choose a c for which 2.4 holds and let

$$V_n := \{\sigma \in 2^{<\omega} : \widetilde{m}(\sigma) \geq h(2^{-|\sigma|})2^{n+c}\}.$$

Then each V_n covers $\mathcal{X}$, due to (2.2). Furthermore, each V_n is c.e., since $\widetilde{m}$ is enumerable from below. Finally, using 2.4, we have for each n,

$$\sum_{w \in V_n} 2^{-|w|s} = \sum_{k=0}^{\infty} \sum_{\substack{w \in V_n \\ |w|=k}} h(2^{-|w|}) = \sum_{k=0}^{\infty} h(2^{-k})|\{0,1\}^k \cap V_n|$$

$$\leq 2^{-n} \sum_{k=0}^{\infty} \widetilde{m}(k) \leq 2^{-n}$$

□

2.1.2 Effective Hausdorff dimension

It is easy to see that the following 'effective' version of Proposition 1.15 holds.

Proposition 2.7 *Let $\mathcal{C} \subseteq 2^\omega$. Then for any rational $s \geq 0$ it holds that*

$$\Sigma^0_1\text{-}\mathcal{H}^s(\mathcal{C}) = 0 \quad \Rightarrow \quad \Sigma^0_1\text{-}\mathcal{H}^t(\mathcal{C}) = 0 \text{ for all rational } t \geq s.$$

The definition of effective Hausdorff dimension follows in a straightforward way.

Definition 2.8 (Lutz, 2000b) The *effective Hausdorff dimension* of a class $\mathcal{C} \subseteq 2^\omega$ is defined as

$$\dim^1_{\mathrm{H}}(\mathcal{C}) = \inf\{s \geq 0 : \Sigma^0_1\text{-}\mathcal{H}^s(\mathcal{C}) = 0\}.$$

Proposition 2.3 ensures that we can use effective Solovay tests, too, to define effective Hausdorff dimension: It holds that

$$\dim^1_{\mathrm{H}}(\mathcal{C}) = \inf\{s \geq 0 : \mathcal{C} \text{ is covered by an effective Solovay } s\text{-test}\}.$$

We check some basic properties of effective dimension.

Dimension Conservation. We have $\dim^1_{\mathrm{H}}(2^\omega) = 1$. Obviously, the trivial cover $C = 2^{<\omega}$ is c.e. and $\sum_{w \in 2^{<\omega}} 2^{-|w|s} < \infty$ if $s > 1$.

Monotonicity. $\dim^1_{\mathrm{H}}(\mathcal{C}) \leq \dim^1_{\mathrm{H}}(\mathcal{D})$ for $\mathcal{C} \subseteq \mathcal{D}$ follows just as in the non-effective case.

Refinement of effective Lebesgue measure zero. If $\mathcal{C}$ is not Σ^0_1-λ-null, it follows immediately that $\dim^1_{\mathrm{H}}(\mathcal{C}) = 1$. This is another straightforward analogy to the classical case.

Classical and effective Hausdorff dimension. It is obvious from the definition that

$$\dim^1_{\mathrm{H}}(\mathcal{C}) \geq \dim_{\mathrm{H}}(\mathcal{C}).$$

The other important properties of Hausdorff dimension, countable stability and invariance under bi-Lipschitz transformations, require more intensive treatment.

The existence of a maximal effective $\mathcal{H}^h$-nullset, i.e., one that contains all other effective nullsets, yields the countable stability of effective dimension. Besides, now (i.e., in the effective setting) it makes sense to consider the effective dimension of individual sequences (viewed as a singleton class), as these have not automatically effective dimension 0. (In the following, we write $\dim^1_{\mathrm{H}}(B)$ or simply $\dim^1_{\mathrm{H}} B$ for $\dim^1_{\mathrm{H}}(\{B\})$, $B \in 2^\omega$.)

An example are the *Martin-Löf random sequences*. These are precisely the sequences not contained in the maximal Σ^0_1-λ-nullset. Every single Martin-Löf random sequence has effective dimension 1.

Furthermore, the effective dimension of a class can be characterized in terms of the effective dimension of its members. The following theorem has first been observed by Lutz (2000b).

Theorem 2.9 (Lutz) *For any $\mathcal{C} \subseteq 2^\omega$,*

$$\dim^1_{\mathrm{H}}(\mathcal{C}) = \sup_{B \in \mathcal{C}} \dim^1_{\mathrm{H}}(B).$$

Proof. The theorem is an easy consequence of the existence of a universal Martin-Löf s-test, s computable, denoted by $\{U^s_n\}$: It holds that

$$\mathcal{X} \text{ is } \Sigma^0_1\text{-}\mathcal{H}^s\text{-null} \quad \Longleftrightarrow \quad (\forall B \in \mathcal{X}) \left[B \in \bigcap_n [U^s_n] \right].$$

□

Theorem 2.9 will be quite useful in the study of effective dimension. It allows us to pass from the study of classes to the investigation of single sequences. For this purpose Kolmogorov complexity will prove an indispensable and elegant tool.

2.2 Kolmogorov Complexity and Effective Dimension

Theorem 2.6 in combination with Theorem 2.5 (c) permits us to characterize effective Hausdorff measure zero by a condition on a single enumerable semimeasure. This semimeasure is closely connected to a concept of algorithmic entropy known as *Kolmogorov complexity*.

In Section 1.11 we saw a close relationship between entropy (topological and measure theoretic) and Hausdorff dimension. It should not surprise, therefore, if a similar connection arose between effective dimension and algorithmic entropy.

Recall that the local entropy of a measure μ was defined as $-\log \mu[w]$ for all $w \in 2^{<\omega}$. Similarly, we define the entropy K of the universal semimeasure.

Definition 2.10 Given a universal semimeasure $\widetilde{m}$, we define its *algorithmic entropy K* as

$$\mathrm{K}(x) = -\log \widetilde{m}(x) \tag{2.5}$$

K is known under various terms, such as *prefix complexity* or *Kolmogorov-Chaitin complexity*. This is due to the fact that it can be defined using a different, algorithmic approach, yielding the advantage of reasoning about entropy from yet a different perspective. We briefly sketch it here, referring to the comprehensive volume by Li and Vitányi (1997) for an exhaustive treatment of this important theory.

(Plain) Kolmogorov complexity can be seen as a *description complexity* of individual objects. A *binary interpreter* V is a partial computable function $V : 2^{<\omega} \times 2^{<\omega} \to 2^{<\omega}$. Given a binary interpreter V, define for any pair of strings x, y,

$$C_V(x|y) = \min\{|p| : \ p \in 2^{<\omega}, \ V(p, y) = x\},$$

called the *conditional Kolmogorov complexity of x given y, with respect to the interpreter V*. $C_V(x|y)$ is the length of the shortest program (for V) giving x as output with y as additional input. We write $C_V(x|\, y_1, \dots, y_n)$ for $C_V(x|\langle y_1, \dots, y_n\rangle)$.

If $y = \epsilon$, we simply write $C_V(x)$ for $C_V(x|\epsilon)$ and call it the *unconditional Kolmogorov complexity of x with respect to V* or simply the *V-complexity of x*.

Using the existence of universal partial computable functions, one can show that there is an *optimal* interpreter U in the sense that

$$(\forall V \text{ interpreter})\, (\exists c)\, (\forall x, y)\, C_U(x|y) \le C_V(x|y) + c. \tag{2.6}$$

Therefore, it makes sense to fix one such optimal interpreter U and speak of

$$\mathrm{C}(x|y) \stackrel{\text{def}}{=} \mathrm{C}_U(x|y)$$

simply as the *Kolmogorov complexity of x given y*. Furthermore, since this notion is invariant up to an additive constant, it makes sense to introduce the following notation: Given two functions $f, g : \mathbb{N} \to \mathbb{N}$, we write that $f \stackrel{+}{\le} g$ if there exists a constant $c \in \mathbb{Z}$ such that for all n, $f(n) \le g(n) + c$. We write $f \stackrel{\pm}{=} g$ if $f \stackrel{+}{\le} g$ and $g \stackrel{+}{\le} f$.

In order to have a notion of algorithmic complexity closer to the concept of entropy used in information theory or symbolic dynamics, for instance, one which is *subadditive*, Levin (1974), Gács (1974), and Chaitin (1975) independently introduced a variant of Kolmogorov complexity.

Instead of looking at arbitrary interpreters, one may restrict the theory to *prefix free interpreters*, that is, partial computable functions V for which an additional condition holds:

$$\forall y\, (V(p, y)\downarrow \Rightarrow \ (\forall \sigma \sqsupset p)\, V(\sigma, y)\uparrow),$$

that is, no two halting inputs are prefixes of one another. It can be shown that there exist optimal prefix free interpreters for which (2.6) holds with respect to all prefix free interpreters. Given such an optimal interpreter V, one can show that

$$\mathrm{K}(x) \stackrel{\pm}{=} \min\{|p| : \ p \in 2^{<\omega}, \ V(p) = x\}, \qquad \text{for all } x \in 2^{<\omega}. \tag{2.7}$$

This remarkable identity is known as the *Coding Theorem*. One of the key ingredients of its proof is the *Kraft-Chaitin Theorem*, a most important cornerstone of algorithmic information theory.

Theorem 2.11 (Kraft-Chaitin Theorem) *(1) If $W \subseteq 2^{<\omega}$ is prefix free, then*

$$\sum_{\sigma \in W} 2^{-|\sigma|} \leq 1.$$

(2) If $\{l_i, l_2, \dots\}$ is a sequence of natural numbers ('lengths') such that

$$\sum_{i \in \mathbb{N}} 2^{-l_i} \leq 1,$$

then there exists a prefix free set $V = \{v_1, v_2, \dots\}$ such that $|v_i| = l_i$ for all i.

(3) If a c.e. set $D = \{(w_1, l_1), (w_2, l_2), \dots\} \subseteq 2^{<\omega} \times \mathbb{N}$ (often called axiom set *or* Kraft-Chaitin set*) satisfies $\sum_{i \in \mathbb{N}} 2^{-l_i} \leq 1$, one can construct (primitive recursively) a prefix-free Turing machine M and strings $\{\tau_i\}_{i \in \mathbb{N}}$, such that*

$$|\tau_i| = l_i \quad \textit{and} \quad M(\tau_i) = \sigma_i.$$

The Coding Theorem makes it easy to prove a characterization of Martin-Löf random sequences through prefix complexity (shown by Schnorr, see Li and Vitányi, 1997, section 3.10):

$$\{B\} \text{ is not } \Sigma_1^0\text{-}\lambda\text{-null} \quad \Longleftrightarrow \quad (\exists c)\ (\forall n)\ \mathrm{K}(B\restriction_n) \geq n - c. \tag{2.8}$$

In the following, we will see that a characterization of effective Hausdorff dimension in the same spirit is possible. The relation between Hausdorff dimension and Kolmogorov complexity turns out to be quite close, even closer than the results in Section 1.11 promise. Consequently, a lot of research has been done exploring this particular connection. The next section reviews some of it.

2.2.1 Characterizing dimension via algorithmic entropy

Theorem 2.6 and Theorem 2.5 ensure that for any $B \in 2^\omega$,

$$\dim_{\mathrm{H}}^1(B) < s \quad \Longleftrightarrow \quad \limsup_{n\to\infty} \frac{\widetilde{m}(B\restriction_n)}{2^{-ns}} = \infty.$$

Using Definition 2.10, from $\dim_{\mathrm{H}}^1(B) < s$ it follows that

$$(\overset{\infty}{\exists} n)[\mathrm{K}(B\restriction_n)/n < s].$$

This in turn implies

$$\liminf_{n\to\infty} \frac{\mathrm{K}(B\restriction_n)}{n} \leq s.$$

Thus we have

$$\liminf_{n\to\infty} \frac{\mathrm{K}(B\upharpoonright_n)}{n} \leq \dim^1_{\mathrm{H}}(B).$$

On the other hand, suppose that $\liminf_{n\to\infty} \mathrm{K}(B\upharpoonright_n)/n < s$, i.e., there exist infinitely many n such that $\mathrm{K}(B\upharpoonright_n) < ns$. Define

$$C = \{w \in 2^{<\omega} : \mathrm{K}(w) < |w|s\}.$$

Then, obviously, C is a Solovay cover for $\{B\}$. Furthermore,

$$\sum_{w\in C} 2^{-|w|s} < \sum_{w\in C} 2^{-\mathrm{K}(w)} = \sum_{w\in C} 2^{\log \widetilde{m}(w)} = \sum_{w\in C} \widetilde{m}(w) < 1,$$

since $\widetilde{m}$ is a semimeasure. Hence, $\{B\}$ is Σ^0_1-$\mathcal{H}^s$-null, as witnessed by C (C is a Solovay s-cover for $\{B\}$.) It follows that $\dim^1_{\mathrm{H}}(B) \leq s$ and thus, by assumption,

$$\dim^1_{\mathrm{H}}(B) \leq \liminf_{n\to\infty} \frac{\mathrm{K}(B\upharpoonright_n)}{n}.$$

Summing up, we proved the following theorem.

Theorem 2.12 *For any $B \in 2^\omega$ it holds that*

$$\dim^1_{\mathrm{H}}(B) = \liminf_{n\to\infty} \frac{\mathrm{K}(B\upharpoonright_n)}{n}. \tag{2.9}$$

Theorem 2.12 was first explicitely stated and proved by by Mayordomo (2002), but much of it was already implicit in earlier works by Ryabko (1984, 1986, 1993, 1994), Staiger (1993, 1998), and Cai and Hartmanis (1994). Observe how the different characterizations of Σ^0_1-$\mathcal{H}^s$-nullsets (and the use of the Coding Theorem) made the proof of Theorem 2.12 easy.

We set $\underline{\mathrm{K}}(B) = \liminf_{n\to\infty} \mathrm{K}(B\upharpoonright_n)/n$ and $\overline{\mathrm{K}}(B) = \limsup_{n\to\infty} \mathrm{K}(B\upharpoonright_n)/n$ to denote the *lower* and *upper entropy*, respectively, of a sequence B.

In Theorem 2.12 it does not matter what version of Kolmogorov complexity we use. The plain and the prefix version are asymptotically equivalent – namely, it holds that for all x that

$$\mathrm{C}(x) \overset{+}{\leq} \mathrm{K}(x) \overset{+}{\leq} \mathrm{C}(x) + 2\log \mathrm{C}(x).$$

So we have

$$\dim^1_{\mathrm{H}}(B) = \liminf_{n\to\infty} \frac{\mathrm{C}(B\upharpoonright_n)}{n},$$

too, which might serve useful in some calculations, for plain Kolmogorov complexity is sometimes easier to handle.

The resemblance of the formula in Theorem 2.12 with what has been laid out in Section 1.11 is striking. We try to illuminate this further in the next sections.

2.2.2 Characterizing nullsets via algorithmic entropy

Applying the Coding Theorem to Theorem 2.12, one can immediately infer the following result.

Theorem 2.13 *Let h be a computable dimension function. A sequence $A \in 2^\omega$ is not Σ_1^0-$\mathcal{H}^h$-null if and only if*

$$\mathrm{K}(A\upharpoonright_n) \overset{+}{\geq} \lfloor -\log h(n) \rfloor$$

For $h(x) = x^s$, this was observed by Tadaki (2002), too. He called sequences not being Σ_1^0-$\mathcal{H}^s$-null *weakly Chaitin s-random*. Calude et al. (2004) introduced those sequences as *Martin-Löf s-random*. Following Chaitin (1987), Tadaki also introduced the notion of *strongly Chaitin s-random* sequences, which are defined as sequences A satisfying

$$\lim_{n\to\infty} (\mathrm{K}(A\upharpoonright_n) - sn) = \infty.$$

Note that in the case $s = 1$, weak and strong Chaitin randomness coincide (Chaitin, 1987). The weakly Chaitin 1-random sequences are precisely the Martin-Löf random sequences.

For $s < 1$ however, things are different. Given any rational $0 < s < 1$, it is possible to construct a sequence A such that, for some constant c,

$$\forall n\, (ns - c \leq \mathrm{K}(A\upharpoonright_n) \leq ns + c).$$

The existence of such sequences was independently observed by Lutz (2003) and Miller (2004). The basic idea for the proof is also present in Cai and Hartmanis (1994) (see also Theorem 2.29).

It follows that for positive, rational $s < 1$, there are weakly Chaitin s-random sequences which are not strongly Chaitin s-random. Calude et al. (2004) showed that strong Chaitin s-randomness is captured by effective Solovay s-tests.

Proposition 2.14 (Calude et al. 2004) *For any positive, rational $s < 1$, a sequence $A \in 2^\omega$ is not strongly Chaitin s-random if and only if it is covered by an effective Solovay s-test.*

2.3 Effective Box Counting and Packing Dimension

Theorem 2.12 tells us that the effective Hausdorff dimension of a sequence can be seen as the minimum entropy rate the sequence obtains. What about the maximum entropy rate, i.e. $\limsup_n \mathrm{K}(B\upharpoonright_n)/n$, $B \in 2^\omega$?

We will introduce effective notions of box counting dimension. As we are concerned mainly with the dimension of single sequences, we can disregard the modified version of box counting dimension needed to obtain countably stable notions (see Section 1.9).

2.3.1 Effective box counting dimension

The following simple but very useful observation is attributed to Kolmogorov (see Li and Vitányi, 1997, Theorem 2.1.3).

Proposition 2.15 *Let $A \subseteq \mathbb{N} \times 2^{<\omega}$ be computably enumerable. Suppose $A_m = \{x : (m, x) \in A\}$ is finite. Then, for all $x \in A_m$,*

$$\mathrm{C}(x|m) \overset{+}{\leq} \log |A_m|.$$

We may use this as a starting point to define effective box counting dimension.

Definition 2.16 Given a sequence $B \in 2^\omega$, call a c.e. set $C \subseteq 2^{<\omega}$ an *effective box cover* (or, if the effective context is clear, just *box cover*) of B, if

$$(\overset{\infty}{\forall} n)[B\upharpoonright_n \in C].$$

Equivalently, an effective box cover of a sequence is nothing but a computably enumerable tree, which has the sequence as an infinite path in it. So, in the following, box covers are identified with trees. Effective box counting dimension measures how efficient the initial segments of a sequence can be 'wrapped' in an c.e. tree. We fix the following notation: Given a set $D \subseteq 2^{<\omega}$, let $D^{[n]} = \{w \in D : |w| = n\}$.

Definition 2.17 For a sequence $B \in 2^\omega$, we define the *effective lower* and *upper box counting dimension* as

$$\underline{\dim}^1_{\mathrm{B}} B = \inf\left\{\liminf_{n\to\infty} \frac{\log |C^{[n]}|}{n} : C \text{ is an effective box cover of } B\right\},$$
$$\overline{\dim}^1_{\mathrm{B}} B = \inf\left\{\limsup_{n\to\infty} \frac{\log |C^{[n]}|}{n} : C \text{ is an effective box cover of } B\right\}.$$

It is a trivial observation that, as in the classical case, effective box counting dimension always bounds effective Hausdorff dimension from above, in particular $\dim^1_{\mathrm{H}} B \leq \underline{\dim}^1_{\mathrm{B}} B$ for any $B \in 2^\omega$.

Concerning lower box counting dimension, an effective version of Theorem 1.32 holds. A class $\mathcal{A} \subseteq 2^\omega$ is called *effectively closed* or Π^0_1, if it is the complement of an effectively open class. A class is *effetively open* if can be represented as $[U]$, where $U \subseteq 2^{<\omega}$ is a computably enumerable set. It can be shown that Π^0_1-classes are precisely the subsets of 2^ω that can be obtained as the infinite paths through a computable *tree*: A set $T \subseteq 2^{<\omega}$ is a *tree* if $\sigma \in T$ implies $\tau \in T$ for all $\tau \sqsubset \sigma$. The *infinite paths* through T are the sequences X for which every initial segment is in T.

Theorem 2.18 *Let $\mathcal{C} \subseteq 2^\omega$ be a shift-invariant Π^0_1-class. Then it holds that*

$$\dim^1_{\mathrm{H}} \mathcal{C} = \lim_{n\to\infty} \frac{|\mathcal{C}\restriction_n|}{n} \tag{2.10}$$

Theorem 2.18 can be proved by a straightforward effectivization the proof given in Section 1.11. For upper algorithmic entropy, a close connection between Kolmogorov complexity and effective box counting dimension holds.

Theorem 2.19 *For any sequence $B \in 2^\omega$,*

$$\overline{\dim}^1_{\mathrm{B}} B = \overline{K}(B) = \limsup_{n\to\infty} \frac{\mathrm{K}(B\restriction_n)}{n}.$$

Note that, due to the asymptotic equivalence of plain and prefix complexity (see the remarks following Theorem 2.12), it does not matter which version of complexity we use.

Proof. ($\leq$) Assume $\overline{\dim}^1_{\mathrm{B}} B < s$ with s rational. It follows immediately from Proposition 2.15 that $\overline{C}(B) \leq s$.

($\geq$) Suppose now $\overline{C}(B) < s$. We show this implies $\overline{\dim}^1_{\mathrm{B}} B \leq s$. Define an c.e. set D by letting

$$D = \{w \in 2^{<\omega} : \mathrm{C}(w) < s|w|\}.$$

By assumption, D is a box cover of B. An easy combinatorial argument yields that the number of programs of length strictly less than sn is less than $2^{sn} - 1$. Hence

$$D^{[n]} \leq 2^{sn},$$

and therefore

$$\overline{\dim}^1_{\mathrm{B}} B \leq \limsup_{n\to\infty} \frac{\log |D^{[n]}|}{n} \leq s,$$

which completes the proof. □

2.3.2 Effective measures and martingales

Schnorr (1971) made the fundamental observation that the Martin-Löf nullsets correspond to those sets on which a left-enumerable (super-)martingale is successful.

Theorem 2.20 (Schnorr) *A sequence $B \in 2^\omega$ is Martin-Löf random if and only if no left-computable (super-)martingale succeeds on B.*

Using an argument similar to the proof of Theorem 1.21, one may generalize this result to s-successful martingales, as done by Lutz (2000b).

Theorem 2.21 *A class $\mathcal{X} \subseteq 2^\omega$ is Σ_1^0-$\mathcal{H}^s$-null if and only if there exists a left-computable (super-)martingale that s-succeeds on $\mathcal{X}$.*

Consequently, one can characterize effective Hausdorff dimension in terms of martingales.

Corollary 2.22 (Lutz, 2000b) *For any sequence $B \in 2^\omega$,*

$$\dim_{\mathrm{H}}^1 B = \inf\{s : \text{ some left-computable martingale } d \text{ } s\text{-succeeds on } B\}. \quad (2.11)$$

2.3.3 Effective packing dimension

Note that the somewhat involved definition of packing measures (see Section 1.9) with the extra optimization renders a direct Martin-Löf style effectivization in terms of enumerable covers difficult. This obstacle can be overcome by using the martingale characterizations of measure zero sets, given in Sections 1.5 and 1.9.

In view of Theorem 1.25 and Corollary 2.22, the definition of effective packing dimension is a straightforward affair.

Definition 2.23 (Athreya et al., 2004) Given $\mathcal{X} \subseteq 2^\omega$, define the *effective packing dimension* of $\mathcal{X}$ as

$$\dim_{\mathrm{P}}^1 \mathcal{X} = \inf\{s : \exists \text{ left-comp. martingale } d \text{ str. } s\text{-successful on all } B \in \mathcal{X}\}.$$

In Section 1.9 we stated the fact that packing dimension equals upper modified box counting dimension, see (1.37). However, as regards individual sequences, we can disregard the modified version of box counting dimension. Namely, a careful effectivization of the proof of Theorem 1.25 yields the following.

Proposition 2.24 *For every sequence $B \in 2^\omega$, $\dim_{\mathrm{P}}^1 B = \overline{\dim}_{\mathrm{B}}^1 B$.*

Combining this with Theorem 2.19 gives an easy proof that effective packing dimension and upper algorithmic entropy coincide.

Corollary 2.25 (Athreya et al., 2004) *For every sequence $B \in 2^\omega$,*

$$\dim_{\mathrm{P}}^1 B = \overline{K}(B).$$

2.4 Effective Transformations

In this section we describe the behavior of effective dimension under computable transformations. It establishes an effective counterpart to the behavior of Hausdorff dimension under Hölder mappings, as presented in Proposition 1.19. The result yields a powerful coding technique, as one can insert bits into a sequence to alter its dimension leaving its computational power untouched. It will be used frequently throughout Chapter 3.

2.4.1 Transformations and Turing functionals

To study effective transformations of 2^ω, one could simply consider (partial) mappings $2^\omega \to 2^\omega$ induced by *computable* monotone functions $\varphi : 2^{<\omega} \to 2^{<\omega}$. However, with regard to later investigations and applications, it is useful to introduce effective transformations from a more general point of view, based on *Turing functionals*. Turing functionals define oracle computations formally, and thus serve as a starting point for the definition of Turing and other reducibilities.

Definition 2.26 A *Turing functional* Φ is a computably enumerable set of triples (n, i, σ) such that $n \in \mathbb{N}$, $i \in \{0, 1\}$, $\sigma \in 2^{<\omega}$, and such that the following *consistency condition* holds: If $(n, i, \sigma), (n, j, \tau) \in \Phi$, and σ and τ are comparable, then $i = j$ and $\sigma = \tau$.

The relation $(n, i, \sigma) \in \Phi$ can be read as $\Phi(n, \sigma) \downarrow= i$. The Turing functionals we consider are required to be *use monotone*, that means they have to satisfy two further properties: First, if $(n_1, i_1, \sigma_1), (n_2, i_2, \sigma_2) \in \Phi$ and $\sigma_1 \sqsubset \sigma_2$, then $n_1 < n_2$. Second, for all n_1, n_2, i_2 and σ_2, if $n_2 > n_1$ and $(n_2, i_2, \sigma_2) \in \Phi$, then there is an i_1 and a $\sigma_1 \sqsubseteq \sigma_2$ such that $(n_1, i_1, \sigma_1) \in \Phi$.

Given a use monotone Turing functional Φ, we write that $\Phi(n, \sigma) = i$ if there is some $\tau \sqsubseteq \sigma$ such that $(n, i, \tau) \in \Phi$. In this case $|\tau|$ is called the *use of* $\Phi(n, \sigma)$. For a sequence A, $\Phi(n, A) = i$ means that $\Phi(n, A\restriction_l) = i$ for some $l \in \mathbb{N}$.

Furthermore, we say that $\Phi(\sigma) = \tau$ if for all $n < |\tau|$, $\Phi(n, \sigma) = \tau(n)$ and $\Phi(m, \sigma) \uparrow$ for all $m \geq |\tau|$. Thus, every use monotone Turing functional induces a partial monotone mapping $2^{<\omega} \to 2^{<\omega}$. For sequences, $\Phi(A) = \epsilon$ if $\Phi(A\restriction_n)$ is undefined for all n; otherwise, it is defined to be the longest binary sequence C (possibly infinite) such that $\Phi(n, A) \downarrow= C(n)$ for all $n < |C|$.

A sequence A is *Turing reducible to* a sequence B (or simply *computable in* B), written $A \leq_T B$, if there is a Turing functional Φ such that $\Phi(B) = A$. We say that a Turing reduction from A to B via Φ is *weak truth-table*, written $A \leq_{wtt} B$, if there is a computable function f such that the use of $\Phi(n, B)$ is bounded by $f(n)$ for all n. The reduction is *truth-table*, $A \leq_{tt} B$, if the functional Φ is *total*, i.e., for all $X \in 2^\omega$, $\Phi(X) \in 2^\omega$.

In the following, we will assume that computable, monotone functions $\phi : 2^{<\omega} \to 2^{<\omega}$ are given by Turing functionals. We will call the function $\hat{\Phi} : 2^\omega \to$

2^ω induced by a Turing functional Φ a *process*.

2.4.2 Effective Hölder transformations and dimension

The Kolmogorov complexity characterization of effective Hausdorff dimension (Theorem 2.12) along with the stability of dimension for individual sequences allows to prove an effective version of Proposition 1.19.

Theorem 2.27 *Let $\varphi : 2^{<\omega} \to 2^{<\omega}$ be a computable r-expansive mapping for some real $r > 0$. Then it holds that, for any $\mathcal{A} \subseteq \operatorname{dom} \hat{\varphi}$,*

$$\dim^1_{\mathrm{H}}(\widehat{\varphi}(\mathcal{A})) \leq \tfrac{1}{r} \dim^1_{\mathrm{H}}(\mathcal{A}). \tag{2.12}$$

Proof. It suffices to show that, for any $A \in \mathcal{A}$,

$$\dim^1_{\mathrm{H}} \hat{\varphi}(A) \leq \tfrac{1}{r} \dim^1_{\mathrm{H}} A.$$

As $\varphi : 2^{<\omega} \to 2^{<\omega}$ is r-expansive, for every $\varepsilon > 0$ there exists some n_0 such that $|\varphi(A\upharpoonright_n)| \geq (r-\varepsilon)n$ for all $n \geq n_0$. Hence, for large enough n,

$$\mathrm{K}(\widehat{\varphi}(A)\upharpoonright_{(r-\varepsilon)n}) \stackrel{+}{\leq} \mathrm{K}(\varphi(A\upharpoonright_n) \stackrel{+}{\leq} \mathrm{K}(A\upharpoonright_n).$$

Therefore,

$$\liminf_{n\to\infty} \frac{\mathrm{K}(\hat{\varphi}(A)\upharpoonright_{(r-\varepsilon)n})}{(r-\varepsilon)n} \leq \liminf_{n\to\infty} \frac{\mathrm{K}(A\upharpoonright_n)}{(r-\varepsilon)n}.$$

It follows with Theorem 2.12 that

$$\dim^1_{\mathrm{H}} \hat{\varphi}(A) = \liminf_{n\to\infty} \frac{\mathrm{K}(\hat{\varphi}(A)\upharpoonright_n)}{n} \leq \liminf_{n\to\infty} \frac{1}{r}\frac{\mathrm{K}(A\upharpoonright_n)}{n} = \dim^1_{\mathrm{H}} A.$$

This completes the proof. □

Note that the *symmetry of algorithmic information* for prefix complexity says that

$$\mathrm{K}(x,y) := \mathrm{K}(\langle x,y\rangle) \stackrel{+}{=} \mathrm{K}(x) + \mathrm{K}(y|x,\mathrm{K}(x)) \tag{2.13}$$

A proof of this identity can be found in Li and Vitányi (1997). Rewriting this in two different ways and replacing y by $\varphi(x)$, where φ maps strings to strings, we get

$$\mathrm{K}(\varphi(x)) \stackrel{+}{=} \mathrm{K}(x) + \mathrm{K}(\varphi(x)|x,\mathrm{K}(x)) - \mathrm{K}(x|\varphi(x),\mathrm{K}(\varphi(x))). \tag{2.14}$$

Now suppose f satisfies a Hölder condition from below:

$$(\exists r, c > 0)\ (\forall A, B \in 2^\omega)\ c\, d(A,B)^r \leq d(f(A), f(B)). \tag{2.15}$$

Note that this implies that f is injective. Suppose further that f has a computable monotone representation $\varphi : 2^{<\omega} \to 2^{<\omega}$ with $\hat{\varphi} = f$ that is injective, too. This means that $\mathrm{K}(x|\varphi(x), \mathrm{K}(\varphi(x))) = \mathrm{O}(1)$, since we can always simply scan through all possible strings for a preimage of a given $\varphi(x)$.

Therefore, we get $\mathrm{K}(\varphi(x)) \stackrel{+}{=} \mathrm{K}(x)$, and using the lower bound on the length of $\varphi(x)$, we see that, for any A,

$$\underline{\mathrm{K}}(\varphi(A\restriction_n)) \geq \tfrac{1}{r}\underline{\mathrm{K}}(A\restriction_n).$$

Combining this observation with Theorem 2.27, the invariance of effective dimension under computable bi-Lipschitz mappings follows.

Corollary 2.28 *Let $f : 2^\omega \to 2^\omega$ be a bi-Lipschitz transformation such that there exists a computable, 1-expansive, injective mapping $\varphi : 2^{<\omega} \to 2^{<\omega}$ with $\hat{\varphi} = f$. Then, for any $\mathcal{X} \subseteq 2^\omega$,*

$$\dim^1_{\mathrm{H}}(f(\mathcal{X})) = \dim^1_{\mathrm{H}}(\mathcal{X}).$$

2.5 Examples of Effective Dimension

In this section we present two basic classes of sequences having non-integral dimension. Both support the intuition that the effective dimension of a sequence reflects its *degree of randomness*.

2.5.1 Diluted randomness

One method for obtaining sequences of non-integral dimension consists in 'diluting' a random sequence with redundant (easy to describe) information, e.g. strings of zeroes. This can be seen as an effective analog to constructing Cantor sets within 2^ω. This was studied by Daley (1974) (see also Staiger, 1993, 2002a). The Kolmogorov complexity characterization of randomness immediately tells us that the Hausdorff dimension of such sequences can be very different from 1, depending on the degree of the dilution.

For example, if $A \in 2^\omega$ is a Martin-Löf random sequence, it is obvious that the diluted sequence $\widetilde{A} = A_0\,0\,A_1\,0\,A_2\,0\ldots$ has dimension $1/2$.

We prove a general theorem of this kind in Section 3.2. Note that we can use the diluting technique to prove the following results.

Theorem 2.29 *(1) For any Δ^0_2-computable number δ there is a sequence $X \in \Delta^0_2$ such that $\dim^1_{\mathrm{H}} X = \delta$.*

(2) For any $s \in [0,1]$ there exists a sequence $B \in 2^\omega$ such that $\dim^1_{\mathrm{H}} B = s$.

The first assertion was shown by Lutz (2003). It is an easy consequence of Theorem 2.9 and Daley's and Staiger's observations. A proof of the second assertion was given by Cai and Hartmanis (1994). The idea is the following: Given $0 < s < 1$, let B equal a random sequence till $\mathrm{K}(B\restriction_n) \geq sn$. Then append a string of zeroes so long that $\mathrm{K}(B\restriction_n) \leq sn$. Now repeat the process with smaller oscillation.

Besides, Cai and Hartmanis study geometrical and topological properties of the set

$$\Gamma_K = \{(X, \underline{\mathrm{K}}(X)) : X \in 2^\omega\} \tag{2.16}$$

as a subset of $[0, 1] \times [0, 1]$. They show that $\dim_{\mathrm{H}} \Gamma_K = 2$, and that the topological dimension (in the sense of Urysohn-Meyer, also known as small inductive dimension, see Hurewicz and Wallman, 1941) of Γ_K is 1, so the set is a fractal in the sense of Mandelbrot (its Hausdorff dimension is greater than its topological dimension).

2.5.2 Generalized Ω-numbers

Another example of effective dimension is a generalization of Chaitin's halting probability Ω (Chaitin, 1975).

Theorem 2.30 (Mayordomo, 2002; Tadaki, 2002) *Let U be a universal, prefix-free machine. Given a computable real number $0 < s \leq 1$, the binary expansion of the real number*

$$\Omega_s = \sum_{\sigma \in \mathrm{dom}(U)} 2^{-\frac{|\sigma|}{s}}$$

has effective Hausdorff dimension as well as effective packing dimension s.

In particular, Ω_1 is identical to Chaitin's Ω, which is Martin-Löf random and hence has dimension 1.

2.5.3 Dimension, entropy and randomness

One can show that Theorem 1.36 is indeed an effective law of randomness, i.e., it holds that for every sequence A random with respect to measure μ_p, $\dim^1_{\mathrm{H}} A = H(\mu_p)$. A sequence is μ-random (with respect to a measure μ) if it cannot be covered by a uniformly computable sequence of Σ^0_1-classes of smaller and smaller μ-measure. The most intuitive definition is a straightforward adaption of the definition of Σ^0_1-$\mathcal{H}^h$-nullsets, but it will work only for measures that are in some sense effective (for instance, computable) themselves. Attempts to define randomness with respect to arbitrary measures was undertaken, among others, by Martin-Löf (1966) and Levin (1973, 1976, 1984), and, more recently, by Gács (2003).

In this section, we restrict ourselves to computable measures, because for the examples to be presented here they are sufficient. Later we will consider more general randomness notions.

Definition 2.31 A measure μ on 2^ω is *computable*, if the function $g_\mu : 2^{<\omega} \to \mathbb{R}_0^+$ given by $g_\mu(\sigma) = \mu[\sigma]$ is computable.

The following definition is due to Martin-Löf (1966).

Definition 2.32 Let μ be a computable measure. A class $\mathcal{A} \subseteq 2^\omega$ has *effective μ-measure* 0 if and only if there is a computable sequence $\{C_n\}_{n\in\mathbb{N}}$ of c.e. sets of finite strings such that for every $n \in \mathbb{N}$, $C_n \subseteq 2^{<\omega}$ covers $\mathcal{A}$ and

$$\sum_{w\in C_n} \mu[w] \leq 2^{-n}. \tag{2.17}$$

A sequence $A \in 2^\omega$ is *μ-random* if $\{A\}$ does not have effective μ-measure 0.

It is not hard to show that the class of μ-random sequences has μ-measure 1. The following result is an effective version of Eggleston's Theorem (Theorem 1.36).

Theorem 2.33 *Let $\vec{p} = (p_0, p_1, \dots)$ be a computable sequence of rational numbers with $0 < p_i < 1$ for all i and $p_i \to p$ for $i \to \infty$. It holds for every $\mu_{\vec{p}}$-random sequence $A \in 2^\omega$ that*

$$\dim_{\mathrm{H}}^1 A = H(\mu_{\vec{p}}) \tag{2.18}$$

Theorem 2.33 was first proved by Lutz (2000b), based on the martingale characterization of effective dimension. It was generalized by Athreya et al. (2004). We give an alternative proof, employing a more general theorem of Billingsley (see Billingsley, 1965, section 14), which deals with non-Bernoulli measures, too.

For this purpose, it is necessary to extend Hausdorff dimension from metric outer measures to arbitrary measures.

Definition 2.34 Let μ be a computable measure and let $s \in [0, 1]$ be rational. A set $\mathcal{A} \subseteq 2^\omega$ has *effective μ-s-dimensional Hausdorff measure* 0, $\Sigma_1^0\text{-}\mathcal{H}_\mu^s(\mathcal{A}) = 0$, if there is a computable sequence $\{C_n\}_{n\in\mathbb{N}}$ of c.e. sets of finite strings such that for every $n \in \mathbb{N}$, $C_n \subseteq 2^{<\omega}$ covers $\mathcal{A}$ and

$$\sum_{w\in C_n} \mu[w]^s < 2^{-n}.$$

Thus, Σ_1^0-$\mathcal{H}_\mu^s$-measure is obtained by replacing in (2.1) $2^{-|w|} = \lambda[w]$ by $\mu[w]$. Note that Σ_1^0-$\mathcal{H}^s$-measure corresponds to Σ_1^0-$\mathcal{H}_\lambda^s$-measure. As with Hausdorff s-measure, one can prove that every set has a 'critical point' (see Proposition 1.15) with respect to Σ_1^0-$\mathcal{H}_\mu^s$-measure. Therefore, the following definition is sound.

Definition 2.35 For a computable probability measure μ and a set $\mathcal{A} \subseteq 2^\omega$ define the *effective μ-dimension* of $\mathcal{A}$ as

$$\dim_\mu^1 \mathcal{A} = \inf\{s \geq 0 : \Sigma_1^0\text{-}\mathcal{H}_\mu^s(\mathcal{A}) = 0\}.$$

The concept of μ-dimension is also referred to as *Billingsley dimension*. Cajar (1981) has written a monograph on Billingsley dimension in probability spaces. We now present an effective version of Billingsley's result relating (local) entropy of measures to the corresponding dimension.

Theorem 2.36 *Let μ, ν be computable measures. If, for $A \in 2^\omega$ and $\delta \in \mathbb{Q}$,*

$$\liminf_{n\to\infty} \frac{\log \nu[A\restriction_n]}{\log \mu[A\restriction_n]} \geq \delta, \tag{2.19}$$

then

$$\dim_\mu^1 A \geq \delta \dim_\nu^1 A.$$

Proof. Suppose $q > 1/\delta$ and $\dim_\mu^1 A < t$, q, t rational. It suffices to show that $\dim_\nu^1 A \leq qt$.

(2.19) implies that

$$(\exists n_0)\,(\forall n \geq n_0)\,(\nu[A\restriction_n])^q \leq \mu[A\restriction_n]].$$

As $\dim_\mu^1 A < t$, there is a Solovay μ-t-cover C of A. We define a c.e. set D as follows: Enumerate w into D if $w \in C$ and $(\nu[w])^q \leq \mu[w]$. Then D is obviously a Solovay cover of A, too. Furthermore,

$$\sum_{v\in D} \nu[v]^{qt} \leq \sum_{v\in D} \mu[v]^t < \infty,$$

so D is indeed a Solovay ν-qt-cover of A. Hence $\dim_\nu^1 A \leq qt$. □

We now show how to derive Theorem 2.33 from Theorem 2.36. Given a computable sequence $\vec{p} = (p_0, p_1, \dots)$ as in the statement of Theorem 2.33, the associated generalized Bernoulli measure $\mu_{\vec{p}}$ is computable, too. It can be shown that every $\mu_{\vec{p}}$-random sequence satisfies the *law of large numbers*, that is, for $\mu_{\vec{p}}$-random A,

$$\frac{N_1(A\restriction_n)}{n} \xrightarrow{n\to\infty} p \quad \text{and} \quad \frac{N_0(A\restriction_n)}{n} \xrightarrow{n\to\infty} 1-p.$$

An easy calculation shows that for such A,

$$\frac{\log \mu_{\vec{p}}[A\restriction_n]}{\log \lambda[A\restriction_n]} \xrightarrow{n\to\infty} H(\mu_{\vec{p}}).$$

Applying Theorem 2.36 twice (with $\mu_{\vec{p}}$ and λ interchanged), we get

$$\dim^1_{\mathrm{H}} A = \dim^1_{\lambda} A = H(\mu_{\vec{p}}) \dim^1_{\mu_{\vec{p}}} A.$$

The result now follows easily from the following theorem, which is easily proved as well.

Theorem 2.37 *If $A \in 2^\omega$ is μ-random, then* $\dim^1_\mu A = 1$.

Proof. Every μ-random sequence is by definition not Σ^0_1-$\mathcal{H}^1_\mu$-null, hence $\dim^1_\mu A = 1$, as $\dim^1_\mu A \leq 1$ for all $A \in 2^\omega$ and all probability measures μ. □

We may exploit Theorem 2.36 further to show that no sequence with limit frequency p can have effective dimension larger than $H(\mu_p)$.

Theorem 2.38 *For $A \in 2^\omega$, if*

$$\frac{N_1(A\restriction_n)}{n} \xrightarrow{n\to\infty} p,$$

then

$$\dim^1_{\mathrm{H}} A \leq H(\mu_p).$$

2.6 The Sequences of Dimension s

What is the classical Hausdorff dimension of all sequences of effective dimension s? In the following, we will call this set $\mathcal{D}_s$, i.e.

$$\mathcal{D}_s = \{A : \dim^1_{\mathrm{H}} A = s\}.$$

(Accordingly, we will use $\mathcal{D}_{\leq s}$ to denote the set $\{A : \dim^1_{\mathrm{H}} A \leq s\}$.) It follows from the definition that the set of all Martin-Löf random sequences has Lebesgue-measure 1. However, due to the definition of Hausdorff dimension as a limit value this might not carry over directly. We will show that the dimension of $\mathcal{D}_s$ is indeed s, and we will determine the s-Hausdorff measure of $\mathcal{D}_s$.

Results from Staiger (1998) (see also Hitchcock, 2002) yield that for some topologically easy classes, such as arbitrary unions of Π^0_1-classes, classical and effective dimension coincide. This, however, is not applicable here, since the set $\mathcal{D}_s$ is too complicated (Hitchcock et al., 2003).

2.6.1 The Hausdorff dimension of $\mathcal{D}_s$

That the Hausdorff dimension of $\mathcal{D}_s$ is actually s was first seen by Cai and Hartmanis (1994). However, their proof appears incomplete (they only deal with computable s, and skip some arguments). We therefore give a new and complete proof (see also Staiger, 1993).

Theorem 2.39 *For every real* $0 \leq s \leq 1$ *it holds that* $\dim_{\mathrm{H}} \mathcal{D}_s = s$.

Proof. Consider the Cantor set $\mathcal{C}_\gamma$ defined in Section 1.7, where $\gamma = 2^{-1/s}$. We know that $\mathcal{C}_\gamma$ has positive finite $\mathcal{H}^s$-measure and therefore Hausdorff dimension s. We show that $\mathcal{C}_\gamma$ also has effective dimension s.

Suppose t is rational and $t > s$. Choose numbers m, n such that

$$\gamma = 2^{-\frac{1}{s}} < \frac{m}{2^n} < 2^{-\frac{1}{t}}.$$

Define intervals $J_{k,i}$ of length $(m/2^n)^k$ as in the construction of the Cantor set $\mathcal{C}_{m/2^n}$. Then each interval $J_{k,i}$ is a binary interval (cylinder), and the union $\bigcup_{i=1}^{2^k} J_{k,i}$ covers $\mathcal{C}_\gamma$, for every k. It is obvious that this defines an effective covering.

It follows that

$$\mathcal{H}^t_{(m/2^n)^k}(\mathcal{C}_\gamma) \leq \sum_{i=1}^{2^k} (m/2^n)^{kt} < 2^k 2^{-\frac{1}{t}kt} = 1.$$

As t can be chosen arbitrary close to s, it follows from the properties of Hausdorff measure that $\mathcal{C}_\gamma$ is Σ^0_1-$\mathcal{H}^t$-null for all rational $t > s$. Hence $\dim^1_{\mathrm{H}} \mathcal{C}_\gamma = s$.

Now it holds that $\mathcal{H}^s$-almost every X has effective dimension s. Otherwise there would be an $s' < s$ such that $\mathcal{H}^s(\mathcal{D}_{\leq s'} \cap \mathcal{C}_\gamma) > 0$, which is impossible, since it would imply that $\dim^1_{\mathrm{H}}(\mathcal{D}_{\leq s'} \cap \mathcal{C}_\gamma) \geq s$, so, by Theorem 2.9, $\mathcal{D}_{\leq s'} \cap \mathcal{C}_\gamma$ would contain a B with $\dim^1_{\mathrm{H}} B > s'$. Thus, $\mathcal{H}^s(\mathcal{D}_s \cap \mathcal{C}_\gamma) > 0$, and therefore $\mathcal{H}^s(\mathcal{D}_s) > 0$, which implies $\dim_{\mathrm{H}} \mathcal{D}_s \geq s$.

On the other hand, $\dim_{\mathrm{H}} \mathcal{D}_s \leq s$ follows immediately from the relation $\dim_{\mathrm{H}} \leq \dim^1_{\mathrm{H}}$ and Theorem 2.9. □

We can actually determine the Hausdorff measure of $\mathcal{D}_s$ for $s < 1$, thanks to a nice observation due to Jarník (1930).

Proposition 2.40 *Let* $\mathcal{C}$ *be a Lebesgue nullset in* $[0, 1]$ *and* $0 \leq s \leq 1$. *Suppose that for any interval* $(a, b) \subseteq [0, 1]$,

$$\mathcal{H}^s(\mathcal{C} \cap (a, b)) \leq \gamma (b - a)\mathcal{H}^s(\mathcal{C})$$

for some constant $\gamma > 0$. *Then* $\mathcal{H}^s(\mathcal{C}) = 0$ *or* ∞.

Note that effective dimension is invariant under changing prefixes, i.e.

$$\dim_{\mathrm{H}}^{1}(wA) = \dim_{\mathrm{H}}^{1}(vA)$$

for all $v, w \in 2^{<\omega}$, $A \in 2^{\omega}$.

In particular, $\mathcal{H}^s(\mathcal{D}_s \cap [v]) = \mathcal{H}^s(\mathcal{D}_s \cap [w])$ for all v, w with $|v| = |w|$. Since $\mathcal{H}^s$ is a Borel measure (and $\mathcal{D}_s$ is a Borel set), it holds that

$$\mathcal{H}^s(\mathcal{D}_s \cap [w]) = 2^{-|w|}\mathcal{H}^s(\mathcal{D}_s).$$

Note that the proof of Theorem 2.39 also shows that $\mathcal{H}^s(\mathcal{D}_s)$ cannot be 0. Transfering Jarnik's observation to the Cantor space, we may conclude that the s-dimensional Hausdorff measure of $\mathcal{D}_s$ is indeed infinite for every $s < 1$ (in contrast to the case $s = 1$, where it is 1).

Theorem 2.41 *For every $s < 1$, $\mathcal{H}^s(\mathcal{D}_s) = \infty$.*

Theorem 2.41 enables us to employ a lot of results from geometric measure theory to the set $\mathcal{D}_s$. A particular important result here is the existence of subsets of finite measures; see the books by Falconer (1990); Mattila (1995).

Theorem 2.42 *Let $s < 1$. For every real number $\gamma > 0$, there is a subset $\mathcal{D} \subseteq \mathcal{D}_s$ such that*

$$\mathcal{H}^s(\mathcal{D}) = \gamma.$$

CHAPTER 3

Effective Dimension and Computability

This chapter is devoted to the study of Hausdorff dimension, effective and classical, of some of the principal objects arising in computability theory, that is, for example, cones and degrees of sets induced by reducibilities. We assume the reader to be familiar with the basic notions of computability theory such as computability, computable enumerability, reducibilities, etc. Refer to the books by Odifreddi (1989), Soare (1987), or Rogers (1987) for unexplained notions.

We use the following notation for degrees and cones: Given a set $A \in 2^\omega$ and a reducibility r, the lower r-cone $\{B : B \leq_r A\}$ of A is denoted by ${}^{\leq_r}A$. Likewise, $A^{\leq_r}$ denotes the upper r-cone $\{B : A \leq_r B\}$ of A. Finally, $A^{\equiv_r}$ denotes the r-degree of A, i.e. $A^{\equiv_r} = {}^{\leq_r}A \cap A^{\leq_r}$.

3.1 The Dimension of Upper Cones

We start with a theorem contrasting a basic result in the measure theoretical study of Turing degrees. Sacks (1963) showed that, for any non-computable set A, its upper Turing-cone $A^{\leq_T}$ has Lebesgue measure 0. Sacks argued that, if the upper cone of A has positive measure, there has to be a cylinder in which more than $3/4$ of the sequences compute A. (This follows from the *Lebesgue density theorem*, see the proof in Terwijn, 1998.) But this enables one to actually compute A by waiting for a majority of oracles to compute the same value, hence A must be computable.

We show that Sacks' result is contrasted by the fact that every Turing (even many-one) upper cone has highest possible Hausdorff dimension.

Theorem 3.1 *For every set $A \in 2^\omega$, $\dim_H A^{\leq_m} = 1$.*

Proof. We use the mass distribution principle (1.27). If f_e denotes the eth many-one reduction, we let

$$f_e^{-1}(A) = \{B : x \in A \iff f_e(x) \in B\}$$

to be the part of the upper cone induced by Φ_e. Since $A^{\leq_T} = \bigcup_e f_e^{-1}(A)$, by countable stability (1.25) it suffices to show that, for any $s < 1$, there is some

m-reduction e for which $\dim_H f_e^{-1}(A) \geq s$. In fact, a single m-reduction has this property: consider the (one-one) reduction f which has $n \in A$ if and only if $f(n) := 2^n \in B$. Now consider the generalized Bernoulli measure μ on 2^ω induced by the following biases p_n:

$$p_n = \begin{cases} A(n), & \text{if } n = 2^m \text{ fo some } m, \\ 1/2, & \text{otherwise.} \end{cases}$$

It holds that $\mu(f_e^{-1}(A)) = 1$, so it remains to verify that μ satisfies a condition as given in (1.39).

But it is easy to see that there is $c > 0$ such that for all sufficiently large $w \in 2^{<\omega}$ and any $s < 1$,

$$\mu[w] \leq 2^{-(|w|-\lfloor \log |w| \rfloor)} \leq c2^{-|w|s}.$$

$\square$

3.2 Joins, Degrees and Lower Cones

With the classical dimension of upper cones at the largest possible value, their study from an effective point of view does not make much sense. Therefore, we turn our attention to degrees and lower cones, which are of countable cardinality and thus promising objects for investigation by effective dimension.

3.2.1 Generalized joins

In computability theory the *join* $A \oplus B$ of two sets is used to give a set representing the combined computational power of A and B. We want to find out how this operation interacts with Hausdorff dimension. As dimension is also concerned with the *density* with which information is coded, it makes sense to look at a generalized notion of join.

Definition 3.2 Let $Z \subseteq \mathbb{N}$ be a computable, infinite, co-infinite set of natural numbers. The *Z-join* of two sequences $A, B \in 2^\omega$, $A \oplus_Z B$, is the unique sequence X which satisfies

$$X \upharpoonright_Z = A \quad \text{and} \quad X \upharpoonright_{\overline{Z}} = B.$$

Obviously, from the computability theoretic perspective, the generalized join does not yield anything new, any Z-join is Turing equivalent to the standard join, which is represented by the set $Z = \{2n : n \in \mathbb{N}\}$.

One could expect now that the dimension of a Z-join of two sequences is determined by the dimension of the sequences relative to the other, and by the density of Z.

The relativization of algorithmic entropy and effective dimension with respect to a given oracle is straightforward and can be done in a standard manner. When dealing with relativized notions, $\dim_{\mathrm{H}}^{1,A}$ will denote effective dimension relative to A, C^A and K^A will denote relativized Kolmogorov and prefix free complexity with oracle A, respectively. In particular, using a relativized version of the Coding Theorem it can be showm that the characterization of effective dimension in terms of complexity, as given in Theorem 2.12, holds in a relativized world, too, that is

$$\dim_{\mathrm{H}}^{1,A} B = \liminf_{n\to\infty} \frac{\mathrm{K}^A(B\restriction_n)}{n}.$$

This identity allows us to resort once more to the symmetry of algorithmic information, which, for ease of reference, we state again.

$$\mathrm{K}(x,y) := \mathrm{K}(\langle x,y\rangle) \stackrel{+}{=} \mathrm{K}(x) + \mathrm{K}(y|x,\mathrm{K}(x)). \tag{3.1}$$

Note the addition of $\mathrm{K}(x)$ in the second term of the right hand side is essential. The *classical symmetry* of information holds only up to a logarithmic term, i.e.

$$\mathrm{K}(x,y) = \mathrm{K}(x) + \mathrm{K}(y|x) + \mathrm{O}(|x|). \tag{3.2}$$

For a set $Z \subseteq \mathbb{N}$, denote by $Z\restriction_n$ the finite subset $Z \cap \{0,\dots,n-1\}$ of Z. Suppose $Z \subseteq \mathbb{N}$ is computable, infinite, co-infinite. Note that for such Z,

$$(A \oplus_Z B)\restriction_n = A\restriction_{|Z\restriction_n|} \oplus_Z B\restriction_{|\overline{Z}\restriction_n|},$$

if one generalizes the notion of Z-join to finite strings in the obvious way. Furthermore, it is easy to see that

$$\mathrm{K}((A \oplus_Z B)\restriction_n) \stackrel{+}{=} \mathrm{K}(A\restriction_{|Z\restriction_n|}, B\restriction_{|\overline{Z}\restriction_n|}).$$

Define the *density* δ_Z of Z as

$$\delta_Z = \lim_{n\to\infty} \frac{|Z \cap \{0,\dots,n-1\}|}{n},$$

if the limit exists. Note that in this case $\delta_{\overline{Z}} = 1 - \delta_Z$.

We can now formulate a first theorem on the dimension of joins.

Theorem 3.3 *Suppose $Z \subseteq \mathbb{N}$ is computable, infinite, co-infinite, with density $\delta = \delta_Z$. Then it holds for any $A, B \in 2^\omega$,*

$$\dim_{\mathrm{H}}^1 A \oplus_Z B \geq \delta \dim_{\mathrm{H}}^1 A + (1-\delta)\dim_{\mathrm{H}}^{1,A} B. \tag{3.3}$$

Proof. We assume $\delta > 0$. (If $\delta = 0$, the proof is almost identical.) Given $\delta > \varepsilon > 0$, choose n_ε large enough that for all $n \geq n_\varepsilon$,

$$\left| \frac{|Z\restriction_n|}{n} - \delta \right| < \varepsilon.$$

With n large enough we have, using (classical) symmetry of information (3.2),

$$\begin{aligned} \mathrm{K}((A \oplus_Z B)\restriction_n) &\stackrel{+}{=} \mathrm{K}(A\restriction_{|Z\restriction_n|}, B\restriction_{|\overline{Z}\restriction_n|}) \\ &= \mathrm{K}(A\restriction_{|Z\restriction_n|}) + \mathrm{K}(B\restriction_{|\overline{Z}\restriction_n|} | A\restriction_{|Z\restriction_n|}) + \mathrm{O}(\log|Z\restriction_n|) \\ &\geq \mathrm{K}(A\restriction_{(\delta-\varepsilon)n}) + \mathrm{K}(B\restriction_{(1-(\delta+\varepsilon))n} | A\restriction_{(\delta+\varepsilon)n}) + \mathrm{O}(\log(\delta+\varepsilon)n) \\ &\geq \mathrm{K}(A\restriction_{(\delta-\varepsilon)n}) + \mathrm{K}^A(B\restriction_{(1-(\delta+\varepsilon))n}) + \mathrm{O}(\log(\delta+\varepsilon)n). \end{aligned}$$

(Note that, for any A and any n, $\mathrm{K}^A(x) \stackrel{+}{\leq} \mathrm{K}(x|A\restriction_n)$.) Now it follows that

$$\begin{aligned} \frac{\mathrm{K}((A \oplus_Z B)\restriction_n)}{n} &\geq \frac{\mathrm{K}(A\restriction_{(\delta-\varepsilon)n})}{n} + \frac{\mathrm{K}^A(B\restriction_{(1-(\delta+\varepsilon))n})}{n} + \frac{\mathrm{O}(\log(\delta+\varepsilon)n)}{n} \\ &= (\delta-\varepsilon)\frac{\mathrm{K}(A\restriction_{(\delta-\varepsilon)n})}{(\delta-\varepsilon)n} + (1-(\delta+\varepsilon))\frac{\mathrm{K}^A(B\restriction_{(1-(\delta+\varepsilon))n})}{(1-(\delta+\varepsilon))n} \\ &\quad + \frac{\mathrm{O}(\log(\delta+\varepsilon)n)}{n}. \end{aligned}$$

It is easy to show that for bounded, positive sequences (a_n), (b_n) of real numbers, $\liminf_n(a_n + b_n) \geq \liminf_n a_n + \liminf_n b_n$. Therefore

$$\underline{\mathrm{K}}(A \oplus_Z B) \geq (\delta-\varepsilon)\underline{\mathrm{K}}(A) + (1-(\delta+\varepsilon))\underline{\mathrm{K}}^A(B).$$

As ε was arbitrary, the result follows. □

A symmetric argument shows that $\dim^1_{\mathrm{H}} A \oplus_Z B \geq \delta \dim^{1,B}_{\mathrm{H}} A + (1-\delta)\dim^1_{\mathrm{H}} B$. Does equality hold? We construct a counterexample to show that this is not the case.

Theorem 3.4 *There exist sequences A_0, A_1 such that*

$$\dim^1_{\mathrm{H}} A_0 = \dim^1_{\mathrm{H}} A_1 = 0, \ \textit{but} \ \dim^1_{\mathrm{H}}(A_0 \oplus A_1) = 1/2.$$

Proof. Let B be a Martin-Löf random sequence and split it into sequences B_0, B_1 such that $B = B_0 \oplus B_1$. It is a result by Van Lambalgen (1987) that B_0 is random relative to B_1 and vice versa.

Choose a strictly increasing, computable sequence (i_n) of natural numbers such that $i_n/i_{n+1} \to 0$. Partition the natural numbers into intervals I_n defined by

$$I_0 = [0, i_0) \qquad I_{n+1} = [i_n, i_{n+1}).$$

Define sets

$$Z_0 = \bigcup_{i \in \mathbb{N}} I_{2i} \qquad Z_1 = \bigcup_{i \in \mathbb{N}} I_{2i+1},$$

and let A_0, A_1 such that

$$A_0 = B_0 \oplus_{Z_0} 0 \quad \text{and} \quad A_1 = B_1 \oplus_{Z_1} 0,$$

respectively, where 0 here denotes the infinite sequence of 0s.

Using the *subadditivity* of K, namely $\mathrm{K}(xy) \overset{+}{\leq} \mathrm{K}(x) + \mathrm{K}(y)$ for all $x, y \in 2^{<\omega}$ (a proof is in Li and Vitányi, 1997), we get that

$$\begin{aligned}
\mathrm{K}(A_0 \upharpoonright_{i_{2n+1}}) &\overset{+}{\leq} \mathrm{K}(A_0 \upharpoonright_{i_{2n}}) + \mathrm{K}(0^{i_{2n+1} - i_{2n}}) \\
&\overset{+}{\leq} i_{2n} + 2 \log i_{2n} + \mathrm{K}(i_{2n+1} - i_{2n}) \\
&\leq i_{2n} + 2 \log i_{2n} + \log i_{2n+1} + 2 \log \log i_{2n+1}.
\end{aligned}$$

Using the assumption on the i_n it follows that

$$\frac{\mathrm{K}(A_0 \upharpoonright_{i_{2n+1}})}{i_{2n+1}} \xrightarrow{n \to \infty} 0,$$

and hence $\dim^1_{\mathrm{H}} A_0 = 0$. An analogous argument yields $\dim^1_{\mathrm{H}} A_1 = 0$.

However, after deleting a (computable) subsequence of density 0, $A = A_0 \oplus A_1$ is a sequence of the form

$$\widetilde{X} = X(0)\, 0\, X(1)\, 0\, X(2)\, 0 \ldots$$

(apart from the with X being a Martin-Löf random sequence (which is easily verified). Therefore, by the examples from Section 2.5, $\dim^1_{\mathrm{H}} A = 1/2$. □

Can we obtain a useful upper bound on the dimension of Z-joins? We can employ the following fact on limits of sequences: If (a_n), (b_n) are bounded sequences of real numbers, then

$$\liminf_n (a_n + b_n) \leq \limsup_n a_n + \liminf_n b_n. \tag{3.4}$$

Resorting to a line of thought similar to that in the proof of Theorem 3.3, one can deduce an upper bound.

Proposition 3.5 *Suppose $Z \subseteq \mathbb{N}$ is computable, infinite, co-infinite, with density $\delta = \delta_Z$. Then it holds for any $A, B \in 2^\omega$,*

$$\dim^1_{\mathrm{H}} A \oplus_Z B \leq \delta \dim_{\mathrm{H}} A + (1 - \delta) \limsup_{n \to \infty} \mathrm{K}(B \upharpoonright_{\lceil (1-\delta) n \rceil} \mid A \upharpoonright_{\lfloor \delta n \rfloor}). \tag{3.5}$$

We can combine this with Theorem 3.3 to get the following easy but important observation.

Theorem 3.6 *Suppose $Z \subseteq \mathbb{N}$ is computable, infinite, co-infinite, with density $\delta_Z = 1$. Then it holds for any $A, B \in 2^\omega$,*

$$\dim^1_{\mathrm{H}} A \oplus_Z B = \dim^1_{\mathrm{H}} A. \tag{3.6}$$

Theorem 3.6 can be interpreted in a geometrical way, in the light of Proposition 1.19. Consider the "insertion mapping" $g : A \to A \oplus_Z B$. This mapping g satisfies a Hölder condition: Define $\varphi : 2^{<\omega} \to 2^{<\omega}$ by

$$\varphi(A\upharpoonright_n) = (A \oplus_Z B)\upharpoonright_{z_n},$$

where z_n denotes the nth element of Z. Then g is the limit of φ, and φ has bounded distension from above and below, because for each $\varepsilon > 0$ there exist constants $c_1, c_2 > 0$ such that

$$\frac{1}{\delta + \varepsilon} - c_1 \le \frac{|\varphi(A\upharpoonright_n)|}{n} = \frac{z_n}{n} \le \frac{1}{\delta - \varepsilon} + c_2$$

for all n.

3.2.2 Lower cones and degrees

Theorem 3.6 has some interesting consequences regarding the dimension of degrees and lower cones. Namely, we can code information into sequences preserving dimension (by coding at very distant positions). Therefore the effective dimension of a degree and a lower cone coincide.

Corollary 3.7 *For any set $A \subseteq \mathbb{N}$, it holds that*

$$\dim^1_{\mathrm{H}} A^{\equiv_{\mathrm{T}}} = \dim^1_{\mathrm{H}} {}^{\le_{\mathrm{T}}}A$$

Proof. For any $B \in {}^{\le_{\mathrm{T}}}A$, we expose some set $C \in A^{\equiv_{\mathrm{T}}}$ such that

$$\dim^1_{\mathrm{H}} C = \dim^1_{\mathrm{H}} B.$$

For this purpose, choose any computable, infinite, co-infinite set Z with density $\delta_Z = 1$ and let $C = B \oplus_Z A$. Theorem 3.6 ensures that the dimension of B and C is the same, and since $B \le_T A$, $C \equiv_T A$. □

The reduction from A to C is actually one-one, so the result holds for Turing reducibility replaced by other standard reducibilities (m, 1-tt, tt, wtt). In Chapter 6 we are going to see that a resource-bounded version of Corollary 3.7 holds, too (with Turing reducibility replaced by many-one).

Corollary 3.7 allows us to exhibit an interesting example of an effective Lebesgue null-class that nevertheless has effective dimension 1.

It is a result by Terwijn (1998) that the lower tt-span of the halting problem $\emptyset'$ contains a Martin-Löf random sequence, hence it does not have effective measure 0, i.e., $\Sigma^0_1\text{-}\lambda(^{\leq_{\mathrm{tt}}}\emptyset') \neq 0$. On the other hand, $\emptyset'$ does not tt-reduce to a Martin-Löf random sequence (Bennett, 1988; Juedes et al., 1994), which implies $\Sigma^0_1\text{-}\lambda(\emptyset'^{\equiv_{\mathrm{tt}}}) = 0$. (For details on these results refer to Terwijn (1998).) Therefore, we have the following corollary.

Corollary 3.8 *$\emptyset'^{\equiv_{\mathrm{tt}}}$ is Σ^0_1-λ-null but* $\dim^1_{\mathrm{H}} \emptyset'^{\equiv_{\mathrm{tt}}} = 1$.

Note that Corollary 3.8 holds for truth-table reducibility only. The lower btt-span of $\emptyset'$ is known to have effective measure 0, and we can strengthen this result.

Theorem 3.9 *It holds that* $\dim^1_{\mathrm{H}} {}^{\leq_{\mathrm{btt}}}\emptyset' = 0$*, and hence* $\dim^1_{\mathrm{H}} \emptyset'^{\equiv_{\mathrm{btt}}} = 0$.

Proof. It is sufficient to prove $\dim^1_{\mathrm{H}} {}^{\leq_{\mathrm{btt}}}\emptyset' = 0$ as the degree is a subset of the lower cone. We use the stability theorem and show that $\dim^1_{\mathrm{H}} X = 0$ for any sequence X btt-reducible to $\emptyset'$. There is a constant c such that every $X(n)$ depends only on c places of $\emptyset'$, and these places can be computed without querying $\emptyset'$. Therefore, one can compute for given n the up to cn places which are necessary to compute $X\restriction_n$ from a code for n. Furthermore, one can enumerate $\emptyset'$ at the queried places until all elements have shown up provided one knows how many will eventually do so. These two numbers can be codes with $(2c+2)\log(n)$ many bits and so one has that the overall number of bits needed to compute $X\restriction_n$ is in $\mathrm{O}(\log(n))$. It follows that $\dim^1_{\mathrm{H}}(X) = 0$ and thus $\dim^1_{\mathrm{H}}(^{\leq_{\mathrm{btt}}}\emptyset') = 0$. □

3.3 Many-One Reducibility

Many-one reducibility may well increase entropy, that is, it can be the case that $A \leq_{\mathrm{m}} B$ and $\dim^1_{\mathrm{H}} A > \dim^1_{\mathrm{H}} B$. The ‘diluted’ sequence

$$\widetilde{X} = X(0)\,0\,X(1)\,0\,X(2)\,\ldots$$

with X Martin-Löf random may serve as an example – in this case we have $X \leq_{\mathrm{m}} \widetilde{X}$ and $\dim^1_{\mathrm{H}} X = 1 > \frac{1}{2} = \dim^1_{\mathrm{H}} \widetilde{X}$. However, if B is already random with respect to a non-pathological measure (which excludes the above situation), we can show that many-one reducibility cannot increase the entropy of that measure, yielding the existence of cones and degrees of non-integral dimension with respect to many-one reducibility (using Theorem 2.33). In fact, such cones (degrees) exist for any dimension that is computably approximable as a real, i.e. Δ^0_2-computable.

There are two ingredients to prove the desired result. First, a sequence random with respect to some generalized Bernoulli measure $\mu_{\vec{p}}$ has dimension $H(\mu_{\vec{p}})$. Second, a many-one reduction increases the redundancy in this sequence, so the algorithmic density (and with it the Hausdorff dimension) will decrease.

Theorem 3.10 (Reimann and Terwijn) *Let $\mu_{\vec{p}}$ be a computable generalized Bernoulli measure induced by $\vec{p} = (p_0, p_1, \dots)$, $p_i \to p$. If A is $\mu_{\vec{p}}$-random, then*

$$\dim^1_{\mathrm{H}} {}^{\leq_{\mathrm{m}}} A = H(\mu_{\vec{p}}).$$

Proof. Theorem 2.33 ensures that $\dim^1_{\mathrm{H}} A = H(\mu_{\vec{p}})$. Hence it suffices to prove that $\dim^1_{\mathrm{H}} {}^{\leq_{\mathrm{m}}} A \leq H(\mu_{\vec{p}})$. So, let A be $\mu_{\vec{p}}$-random, and $B \leq_m A$ via a many-one reduction f. We show that $\dim^1_{\mathrm{H}} B \leq H(\mu_{\vec{p}})$.

Consider the computable set

$$F = \{n : (\forall m < n) f(m) \neq f(n)\},$$

so F is the set of all positions of B, where an instance of A is queried for the first time. F induces a specific *Kolmogorov-Loveland place selection rule*. It has been shown that such a selection rule, when applied to a $\mu_{\vec{p}}$-random sequence, yields a new sequence with the same limit frequency as the random sequence (see Uspensky et al., 1990; Muchnik et al., 1998). So, by Theorem 2.38, we may conclude that for $X = B\restriction_F$, $\dim^1_{\mathrm{H}} X \leq H(\mu_{\vec{p}})$.

If we let Z be such that

$$B = X \oplus_F Z,$$

it is obvious that there is an algorithm that, if

$$B\restriction_n = X\restriction_k \oplus_F Z\restriction_l$$

with $n = k + l$, computes $Z\restriction_l$ from $X\restriction_k$. Therefore,

$$\frac{\mathrm{K}(B\restriction_n)}{n} \leq \frac{\mathrm{K}(X\restriction_k)}{n} + \frac{\mathrm{O}(1)}{n} \leq \frac{\mathrm{K}(X\restriction_k)}{k} + \frac{\mathrm{O}(1)}{n}.$$

Hence, $\dim^1_{\mathrm{H}} B \leq \dim^1_{\mathrm{H}} X \leq H(\mu_{\vec{p}})$. □

It can be shown (see Lutz, 2003) that if $s \in [0, 1]$ is a Δ^0_2-computable real, then there exists a Δ^0_2-computable real p such that $H(\mu_p) = s$. Therefore, combining Theorems 2.33 and 3.10, we get the following corollary.

Corollary 3.11 *If s is a Δ^0_2-computable real number, then there exists a lower many-one cone of effective Hausdorff dimension s.*

3.3.1 Other reducibilities

Concerning other reducibilities, the question on the existence of lower cones of non-integral dimension seems to be much harder to answer. We will return to it in Chapter 4, here we give an easy argument which settles the problem for left-computable sequences.

Proposition 3.12 *If $A \in 2^\omega$ is left-computable and $\dim^1_H A > 0$, then*

$$\dim^1_H({}^{\leq_T}A) = 1.$$

Proof. Given a left-computable sequence A such that $\dim^1_H A = s > 0$, pick a rational number $0 < q < s$ and construct a left-computable sequence $C = B \oplus_Z 0$ of effective dimension q by filling zeroes into a Martin-Löf random sequence, i.e., formally, by choosing some left-computable Martin-Löf random B (such as Chaitin's Ω) and devising a computable set $Z \subseteq \mathbb{N}$ such that $\dim^1_H B \oplus_Z 0 = q$. Obviously, for this sequence we even have $\dim^1_H C = \dim^1_P C$.

Stephan (see Downey and Hirschfeldt, 2004) observed that if for two left-computable sequences C, A it holds that

$$(\exists c)\,(\forall n)\;\; \mathrm{C}(C\restriction_n) \leq \mathrm{C}(A\restriction_n) + c, \tag{3.7}$$

then $C \leq_T A$.

It is easy to see that, for A and $C = B \oplus_Z 0$, (3.7) holds (since $\dim^1_H C = \dim^1_P C$). Furthermore, C is clearly Turing equivalent to a Martin-Löf random sequence (B), so $B \leq_T A$, which, by stability of effective dimension, implies $\dim^1_H({}^{\leq_T}A) = 1$. □

Nevertheless, it is not clear whether Proposition 3.12 holds in general, i.e. whether it holds that for arbitrary sequences A of dimension $s > 0$, $\dim^1_H({}^{\leq_T}A) = 1$. We will address this question further in the next chapter, but the general problem remains open.

> **Open Question:** Does there exist a Turing lower cone of non-integral effecive Hausdorff dimension?

3.4 Examples of zero-dimensional sequences

We give some examples of sequences occurring in computability theory which are not only non-random but rather orthogonal to randomness in every sense. *Genericity* can be seen as a topological counterpart to randomness, and thus it is no surprise that generic sequences appear small with respect to the family of Hausdorff measures, too. *Semirecursiveness* is essentially an algorithmic concept, however, using some combinatorial facts it may be shown that semirecursive sequences are well approximable by c.e. (even computable sets). Hence their effective dimension is trivial as well.

3.4.1 Computably enumerable sets

A theorem of Barzdin' (1968) maintains that the complexity of the characteristic sequence of a computably enumerable set is of trivial asymptotic complexity, namely for such a set A it holds that

$$\mathrm{C}(A\restriction_n) \leq \log n + c, \tag{3.8}$$

for some constant c and all $n \in \mathbb{N}$. Namely, to determine the initial segment of length n of (the characteristic sequence of) A, it suffices to know the number of 1s contained in the segment. But this number can be encoded by a binary string of length $\log n$.

Consequently, all c.e. sets have *effective Hausdorff and packing dimension zero*.

3.4.2 Generic sequences

Generic sequences can be seen as a topological counterpart to random sequences, which arise in the context of measures. As random sequences can be seen (from an effective point of view) as typical instances of (effective) measure 1 classes, generic sequences are contained in any effectively topologically large class, namely those of second category in the sense of Baire.

A class $\mathcal{A} \subseteq 2^\omega$ is *nowhere dense* if its complement contains a dense open class. $\mathcal{A}$ is *meager* or of *first category* if it is the countable union of nowhere dense classes. Complements of meager classes are called *co-meager* or of *second category*.

It is easy to see that the meager classes in 2^ω form, just as the measure 0 classes with respect to some measure μ, a σ-ideal, i.e., they are closed under subsets, countable unions and contain the empty set. Kechris (1995) and Oxtoby (1980) provide background on Baire category.

Category can be effectivized in the same manner as measure, by allowing only effective classes to witness a class being meager. This makes it possible for a single sequence to be non-meager. Such sequences are considered *generic*. A precise definition is as follows.

Definition 3.13 A sequence $Y \in 2^\omega$ is 1-*generic* if for every c.e. set $U \subseteq 2^{<\omega}$ it holds that

$$\exists x \sqsubset Y \; (x \in U \text{ or no extension of } x \text{ is in } U). \tag{3.9}$$

1-generic sequences were intensively studied by Jockusch (1980). They can be thought of sets possessing all the properties that can be obtained by Kleene-Post (finite) extension arguments. Genericity behaves orthogonal to randomness, in the sense that no 1-random sequence is 1-generic and vice versa.

We strengthen this result by showing that the effective dimension of 1-generic sequences is zero.

Proposition 3.14 *For every* 1*-generic sequence* Y *it holds that*

$$\dim^1_{\mathrm{H}}(Y) = 0.$$

Proof. We will give, for any $s > 0$, a Solovay s-cover for Y. Given $s > 0$, let $c = \lceil 1/s \rceil$ and define

$$C_s = \{v0^{c|v|} : v \in 2^{<\omega}\}.$$

We claim that C_s is a Solovay cover for Y. First note that C_s is obviously c.e., so there has to be some $x \sqsubset Y$ such that (3.9) is satisfied. But surely some extension of x is in C_s (namely $x0^{c|x|}$), hence $x \in C_s$. By applying the same argument to a variant of C_s from which all strings of length at most $|x|$ have been deleted, we see inductively that there are infinitely many $x \in C_s$ such that $x \sqsubset Y$.

It remains to show that C_s is an s-cover:

$$\sum_{w \in C_s} 2^{-|w|s} = \sum_{n \geq 0} 2^n 2^{-(n+cn)s} = \sum_{n \geq 0} 2^{n(1-s(1+c))} < \infty,$$

since $(c+1)s > 1$. □

We can even show more, namely, that no 1-generic set can bound a sequence of positive effective dimension. The proof uses a result by Demuth and Kučera (1987). Recall that a set $S \subseteq \mathbb{N}$ (and analogously, by the usual correspondence between strings and natural numbers, a set of strings) is called *simple* if it is c.e. and its complement is infinite but does not contain any infinite c.e. subset.

Theorem 3.15 (Demuth-Kucera) *If* X *is* 1*-generic and* $B \leq_{\mathrm{T}} X$*, then any simple set* $S \subseteq 2^{<\omega}$ *contains a string* w *such that* $w \sqsubset B$*.*

Since simple sets are closed under finite variants, it follows that any simple set $S \subseteq 2^{<\omega}$ constitutes a Solovay cover for a sequence that is Turing reducible to a 1-generic sequence.

Corollary 3.16 *If $X \in 2^\omega$ is 1-generic, then* $\dim^1_H(\leq_T X) = 0$.

Proof. By Theorem 3.15 it suffices to show that for any $n \geq 1$, there is a simple set $S \subseteq 2^{<\omega}$ such that S is a Solovay $(1/n)$-test.

This can be shown by modifying Post's construction (Post, 1944) of a simple set: Enumerate each c.e. set W_e until the first element σ with $|\sigma| > ne$ appears and put it into S. □

3.4.3 Semirecursive sets

The use of coding in 'thin' places as in Theorem 3.6 shows that every Turing degree contains a sequence of dimension 0 (simply fill in enough zeroes). However, we can present more 'natural' evidence for this (from a computability theoretic point of view).

A set $A \subseteq \mathbb{N}$ is *semirecursive*, if there is a computable function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that

(a) $f(m, n) \in \{m, n\}$ for all $m, n \in \mathbb{N}$,

(b) $m \in A \vee n \in A$ implies $f(m, n) \in A$.

Semirecursive sets were introduced by Jockusch (1968) and have been used to give a structural solution to *Post's Problem* (for details refer to Odifreddi (1989)). Note that there are semirecursive sets which are not computably enumerable (and hence not computable). In fact, every tt-degree contains a semirecursive set (Jockusch, 1968). Nevertheless, from the point of view of effective dimension, semirecursive sets are not very complex.

Theorem 3.17 *The class of all semirecursive sets has effective packing dimension* 0 *(and hence effective Hausdorff dimension* 0*).*

Theorem 3.17 will immediately follow from two lemmas.

If X is the characteristic sequence of a semirecursive set, then, for any two numbers m, n, we can computably exclude one of four possible values of the two-bit string $X(m)X(n)$. This observation was used by Beigel et al. (1995), who studied semirecursive sets in the broader context of *frequency computability*, to prove the following lemma.

Lemma 3.18 *If $X \in 2^\omega$ is the characteristic sequence of a semirecursive set, then there exists a computable set $C \subseteq 2^{<\omega}$ such that, for any n, $|C \cap \{0,1\}^n| \leq n + 1$ and $X\restriction_n \in C \cap \{0,1\}^n$.*

The proof of the lemma, which occurs in Beigel et al. (1995) in a much more general form, uses the so-called *Sauer-Perles-Shelah-Lemma* from extremal combinatorics.

On the other hand, we can use Proposition 2.15 to deduce the following *covering principle*:

Lemma 3.19 *Suppose for some $X \in 2^\omega$ there is a c.e. set $A \subseteq 2^{<\omega}$ such that for almost every n it holds that $X\restriction_n \in A^{[n]} = A \cap \{0,1\}^n$. Then*

$$\dim_{\mathrm{P}}^1 A \leq \limsup_{n\to\infty} \frac{\log |A^{[n]}|}{n}.$$

3.5 Perfect Subsets

The properties of sequences having non-integral dimension as well as the examples given so far seem to indicate that such a fractal sequence possesses some degree of randomness. One of the most important results in the study of the computational power of algorithmic randomness is that every sequence Turing-reduces to a Martin-Löf random one. This was independently shown by Kučera (1985) and Gács (1986) (see also Merkle and Mihailovic, 2002). Using coding in thin places (Theorem 3.6), it is easy to extend this result to positive dimension, namely, given any Δ^0_2-computable real number s, $0 < s \leq 1$, every Turing degree containing a random sequence contains also a sequence of dimension s. So every sequence is Turing-reducible to some sequence of dimension s.

The proof of the Gács-Kučera result actually reveals an interesting alternative perspective on sets of random sequences.

If we consider the full Cantor space 2^ω as a fully random space, because everything that can happen may eventually occur (it contains all possible outcomes of an infinite chance experiment), we may ask for the conditions under which a given subset of 2^ω contains a homeomorphic copy of 2^ω, that is, a *perfect subset*. We could then ascribe to such subsets a certain amount of randomness.

The Cantor-Bendixson Theorem asserts that every uncountable, closed subset of 2^ω contains a perfect subset. However, it may, from an algorthmic point of view, be rather difficult to expose such a subset. For effectively closed sets of random sequences, on the other hand, this can be done effectively, by means of a *process*. Furthermore, this process can be chosen to be 1-expansive in terms of Definition 1.18, as shown by Gács (1986).

Observe that his implies the reducibility theorem, as effective processes describe Turing reductions.

In this section, we show that the efficient exposition of a perfect subset is possible for effectively closed sets of positive dimension, too.

Theorem 3.20 *Let $\mathcal{A} \subseteq 2^\omega$ be Π^0_1 with $\dim_{\mathrm{H}}(\mathcal{A}) > s > 0$, s rational. Then there exists a surjective process $F : \mathcal{A} \to 2^\omega$.*

The proof of the Kucera-Gacs Theorem is based on the following observation: If we can bound the (Lebesgue) measure of a Π^0_1-class effectively from below, say by 2^{-n+1}, we know that the pruned tree associated with the Π^0_1-class must branch before or at level n, thereby yielding an opportunity to code the first two branches of a perfect tree.

This observation has been used by Hertling (1997) to give a combinatorial condition for the existence of a surjective mapping from a class onto 2^ω.

Definition 3.21 A class $\mathcal{A}$ is a *computably growing Cantor set* if there exists a computable function f such that

$$(\forall w \in \mathcal{A}\lceil_{f(n)})\ |\{x \in \{0,1\}^{f(n+1)} :\ w \sqsubset x \wedge \ \sqsubset \mathcal{A}\}| \geq 2. \tag{3.10}$$

Theorem 3.22 (Hertling, 1997) *If $\mathcal{A} \subseteq 2^\omega$ is Π^0_1 and contains a computably growing Cantor set, then there is a process $F : 2^\omega \to 2^\omega$ with $F(\mathcal{A}) = 2^\omega$.*

We are going to show that every Π^0_1-class of positive Hausdorff dimension contains a computably growing Cantor set. The key ingredient to the proof will be to show that the class is non-negligible with respect to a (probability and hence finite) measure sufficiently similar to Lebesgue measure.

Theorem 3.23 *Let $s > 0$. If $\mathcal{A} \subseteq 2^\omega$ is Π^0_1 with $\dim_{\mathrm{H}}(\mathcal{A}) > s > 0$, then there exists a computable measure ν on 2^ω such that $0 < \nu(\mathcal{A}) \leq 1$ and for some $c > 0$*

$$(\forall w \in 2^{<\omega})\ \nu[w] \leq c2^{-ns}. \tag{3.11}$$

The theorem can be seen as an effective version of an important result from geometric measure theory called *Frostman's Lemma* (Frostman, 1935). The proof given here is inspired by the presentation of the classical result in the book by Mattila (1995).

Proof. If $\dim_{\mathrm{H}}(\mathcal{A}) > s > 0$, then $\mathcal{H}^s(\mathcal{A}) = \infty$, so there is a constant $b > 0$ such that for any cover $C \subseteq 2^{<\omega}$,

$$\sum_{w \in C} 2^{-|w|s} \geq b. \tag{3.12}$$

Let $T \subseteq 2^{<\omega}$ be a computable tree such that $\lceil T \rceil = \mathcal{A}$. We first define a sequence of computable measures $\{\mu^n\}$. Each μ^n can be seen as an approximation to ν, knowing only the paths of T up to length n.

Given $n \in \mathbb{N}$, define a computable measure μ_n^n such that for all $w \in 2^n$

$$\mu_n^n \upharpoonright_{[w]} = \begin{cases} (2^{(1-s)n})\, \lambda \upharpoonright_{[w]}, & \text{if } w \in T, \\ 0, & \text{if } w \notin T. \end{cases}$$

Now we modify μ_n^n downward in order to ensure (3.11). Define μ_{n-1}^n by requiring that for all $w \in 2^{n-1}$,

$$\mu_{n-1}^n \upharpoonright_{[w]} = \begin{cases} \mu_n^n \upharpoonright_{[w]}, & \text{if } \mu_n^n[w] \le 2^{-(n-1)s}, \\ 2^{-(n-1)s}(\mu_n^n[w])^{-1}\mu_n^n \upharpoonright_{[w]}, & \text{if } \mu_n^n[w] > 2^{-(n-1)s}. \end{cases}$$

Obviously, μ_{n-1}^n is computable, too. We continue in this fashion: μ_{n-k-1}^n is obtained from μ_{n-k}^n in such a way that for $w \in 2^{n-k-1}$,

$$\mu_{n-k-1}^n \upharpoonright_{[w]} = \gamma(w)\mu_{n-k}^n \upharpoonright_{[w]}$$

where $\gamma(w) = \min\{1, 2^{-(n-k-1)s}(\mu_{n-k}^n[w])^{-1}\}$. We stop as soon as $T^{[n-k_0]} = \{w\}$ for some $w \in 2^{n-k_0}$ and some $k_0 \ge 0$, and define $\mu^n = \mu_{n-k_0}^n$. Again, k_0 can be determined effectively and hence μ^n is a computable measure.

Picking for each $w \in T^{[n]}$ the shortest $v \sqsubseteq w$ with $\mu^n[v] = 2^{-|v|s}$ we obtain a finite, prefix-free set $\{v_1, \dots, v_l\}$ of strings sucht that $\mathcal{A} \subseteq [T^{[n]}] \subseteq \bigcup [v_i]$. It follows that

$$\mu^n(2^\omega) = \sum_{i=1}^{l} \mu^n[v_i] = \sum_{i=1}^{l} 2^{-|w|s} \ge b, \tag{3.13}$$

using (3.12). We now define

$$\nu = \sum_{n=1}^{\infty} 2^{-n}(\mu^n(2^\omega))^{-1}\mu^n.$$

Then ν is a computable measure: to know ν with precision 2^{-n} it suffices to compute $\mu^1, \dots, \mu^{n+1}$. Futhermore, (3.11) is satisfied as for any $w \in 2^{<\omega}$,

$$\nu[w] = \sum_{n=1}^{\infty} 2^{-n}(\mu^n(2^\omega))^{-1}\mu^n[w] \le b^{-1}\sum_{n=1}^{\infty} 2^{-n}2^{-|w|s} \le b^{-1}2^{-|w|s}.$$

It remains to show that $\nu(\mathcal{A}) > 0$. A reasoning like that in (3.13) yields that for each n, $\mu^n(\mathcal{A}) \ge b$. It follows that

$$\nu(\mathcal{A}) = \sum_{n=1}^{\infty} 2^{-n}(\mu^n(2^\omega))^{-1}\mu^n(\mathcal{A}) \ge b > 0,$$

since $\mu^n(2^\omega) \le 1$ for all n. □

Next we generalize an argument by Kučera (1985) to ensure a computable, non-zero lower bound on the μ-measure of sections of Π^0_1-classes that contain μ-random sequences with respect to a computable measure μ. We use the following notation: Given a (computable) measure μ and a universal μ-Martin-Löf test $(U_n)_{n\in\mathbb{N}}$, we denote by $\mathcal{P}^\mu_n$ the complement of the nth level of (U_n), i.e. $\mathcal{P}^\mu_n = [U_n]^\complement$.

Lemma 3.24 *Let μ be a computable measure on 2^ω. If $\mathcal{A} \subseteq 2^\omega$ is Π^0_1, then there exists a computable function $g : 2^{<\omega} \times \mathbb{N} \to \mathbb{Q}^{>0}$ such that for any $w \in 2^{<\omega}$ and any $n \in \mathbb{N}$ it holds that*

$$\mathcal{P}^\mu_n \cap \mathcal{A} \cap [w] \neq \emptyset \quad \Rightarrow \quad \mu(\mathcal{A} \cap [w]) \geq g(w, n).$$

Note that it follows directly from the definiton of effective measure that $\mu(\mathcal{A} \cap [w]) \neq 0$. The lemma however asserts that we can bound the measure of $\mathcal{A} \cap [w]$ effectively from below by a postive value.

Proof. Assume that

$$\mathcal{P}^\mu_n \cap \mathcal{A} \cap [w] \neq \emptyset$$

Since $\mathcal{A}$ is Π^0_1, there an index a such that $\mathcal{A}^\complement \subseteq [W_a]$. We define the following program: On input j we start to enumerate W_a until at some time s we find that

$$\mu([w] \setminus [W_{a,s}]) < 2^{-j}$$

(which may of course never happen). If such s exists, let V_j be a minimal (finite) cover of $[w] \setminus [W_{a,s}]$, otherwise we leave V_j empty. Obviously, $\{V_j\}$ forms a Martin-Löf μ-test. Let e be the least index $> n$ of the program just defined.

Suppose now $\{e\}(e) \downarrow$ is defined. This means that V_e will contribute to the universal Martin-Löf μ-test $\{U^\mu_j\}$ at level n, so $[V_e] \subseteq \mathcal{U}^\mu_n$. Hence there is a time t such that $[w] \setminus [W_{a,t}] \subseteq \mathcal{U}^\mu_n$. But this would imply

$$\mathcal{P}^\mu_n \cap \mathcal{A} \cap [w] \;\subseteq\; \mathcal{P}^\mu_n \cap [W_{a,t}] \cap [w] \;=\; \emptyset,$$

contradicting our assumption. So $\{e\}(e) \uparrow$ and we have

$$\mu(\mathcal{A} \cap [w]) \geq \mu(\mathcal{A} \cap [W_{a,t}]) \geq 2^{-e} \stackrel{\text{def}}{=} g(w, n).$$

□

Finally, we can combine Theorem 3.23 and Lemma 3.24 to show that any Π^0_1-class of positive Hausdorff dimension contains a computably growing Cantor set.

Theorem 3.25 *Every Π^0_1-class $\mathcal{A}$ of positive Hausdorff dimension contains a computably growing Cantor set.*

Proof. We define a computable function f witnessing that $\mathcal{A}$ contains a computably growing Cantor set. If $\dim_{\mathrm{H}} \mathcal{A} > s > 0$, there is a computable measure μ such that $\mu(\mathcal{A}) \leq 1$ and for some $c > 0$

$$(\forall w \in 2^{<\omega})\ \mu[w] \leq c2^{-ns}.$$

The existence of such a μ is guaranteed by Theorem 3.23. Let $g : 2^{<\omega} \times \mathbb{N} \to \mathbb{Q}^{>0}$ be the computable function given by Lemma 3.24. Since $\mu(\mathcal{A}) > 0$, $\mathcal{A}$ has to contain a μ-random sequence, in fact, there has to be an n such that $\mu(\mathcal{A} \cap \mathcal{P}_n^{\mu}) > 0$.

Let T be a computable tree such that $\mathcal{B} := \mathcal{A} \cap \mathcal{P}_n^{\mu})$ corresponds to the set of infinite paths in T. We define f inductively and set $f(0) = 0$. To define $f(1)$, compute $g(\epsilon, n)$ and pick the smallest m such that

$$c2^{-ms} < g(\epsilon, n).$$

Then it must hold that $|\{w \in \{0,1\}^m : w \sqsubset \mathcal{B}\}| \geq 2$.

Given $f(k)$, set

$$M_k = \min\{g(w, n) : w \in T \cap \{0,1\}^{g(k)}\}.$$

and define $f(k+1)$ to be the smallest m such that $c2^{-ms} < M_k$. By hypothesis, there are at least 2^k strings $w \in \{0,1\}^{g(k)}$ for which $\mu([w] \cap \mathcal{B}) > 0$, so for these strings it holds that

$$|\{v \in \{0,1\}^{f(k+1)} : v \sqsupseteq w \quad \text{and} \quad v \sqsubset \mathcal{B}\}| \geq 2.$$

□

CHAPTER 4

Generalized Dimension, Entropy, and Measures

The results about effective dimension in the preceding chapters seem to suggest that sequences of positive dimension behave in many respects like random (that is, Martin-Löf random) ones. Indeed, the canonical example given in 2.5 of a sequence of non-integral dimension was a 'diluted' Martin-Löf random sequence. From sequences of this type, however, it is easy to recover the original random content effectively. So it may be said that these sequences have a *random content*.

The other examples encountered so far are the sequences random to some generalized Bernoulli measure. How about recovering Martin-Löf randomness from them? Von Neumann (1963) gave a nice trick how to turn a biased coin into an unbiased one: Toss the coin twice. If the two outcomes are equal, discard and toss anew. Otherwise, interpret HEADS-TAILS as HEADS, and TAILS-HEADS as TAILS. Can this procedure be generalized to other measures?

The general question that arises in this context is whether we are able to recover randomness from sequences of positive dimension. Mathematically speaking, we ask if every sequence of positive dimension computes a Martin-Löf random sequence.

Note that a positive answer to this question would rule out the existence of Turing lower cones of nonintegral dimension.

In the following, we are going to investigate the interplay between randomness, entropy, and reducibility.

4.1 Natural Sequences

Sequences that are random with respect to some computable probability measure are called *proper* (Zvonkin and Levin, 1970) or *natural* (Muchnik et al., 1998). Of course, we can make any computable sequence random with respect to a computable measure by assigning a positive amount of measure to this sequence. Such measures are called atomic.

Definition 4.1 A measure μ is called *atomic* if there exists a sequence A such that $\mu\{A\} > 0$. A is called an *atom* for μ. A nonatomic measure (i.e. one that does not have atoms) is also called *continuous*.

Obviously, any atom of a measure is random with respect to it. On the other hand, such atomic parts of computable measures are necessarily computable sequences.

Proposition 4.2 *If μ is a computable measure and $\mu\{A\} > 0$ for some $A \in 2^\omega$, then A is computable.*

Proof. Suppose $\mu(\{A\}) > c > 0$ for some computable μ and rational c. Define a computably enumerable tree T by letting $w \in T$ if and only if $\mu[w] > c$. By definition of T and the fact that μ is a (probability) measure, it holds that

$$(\forall m)\ |T^{[m]}| \leq 1/c.$$

But this means that every infinite path through T is isolated, i.e. if A is an infinite path through T, there exists a string σ such that for all $\tau \sqsupseteq \sigma$, $\tau \in T$ implies $\tau \sqsubset A$. Now it is not hard to see that such an isolated path must be computable. □

As regards noncomputable sequences, Levin proved that, from a computability theoretic point of view, random sequences with respect to computable probability measures are computationally as powerful as Martin-Löf random sequences (with respect to Turing reductions). This was independently shown by Kautz (1991).

Theorem 4.3 (Levin, 1970) *A natural sequence is either computable or Turing equivalent to a ML-random sequence.*

The proof of Theorem 4.3 uses the fact that reductions induce continuous (partial) mappings from 2^ω to 2^ω. Such mappings transform measures.

4.1.1 Transformation of Measures

Let μ be a Borel measure on 2^ω, and let $f : 2^\omega \to 2^\omega$ be continuous. f induces a new measure μ_f, often referred to as the *image measure*, on 2^ω by letting

$$\mu_f(\mathcal{A}) = \mu(f^{-1}(\mathcal{A})).$$

An important theorem of Oxtoby (1970) asserts that any nonatomic measure can be transformed into Lebesgue measure λ this way.

Theorem 4.4 (Oxtoby, 1970) *If μ is a nonatomic probability measure on 2^ω, then there exists a continuous mapping $f : 2^\omega \to 2^\omega$ such that $\mu_f = \lambda$.*

Levin's idea to prove Theorem 4.3 is to show an effective version of Oxtoby's theorem. Roughly speaking, a computable transformation transforms a computable measure to another computable measure. At the same time, a computable transformation should not affect randomness properties, so a sequence random with respect to measure should, when transformed by a computable mapping, be random with respect to the transformed measure. This is the fundamental *conservation of randomness* property.

Now we look in detail at the way a computable process transforms computable measures. Let f be a process, represented by some machine M which computes a monotone function ϕ such that $\widehat{\phi} = f$. Note that a process may be partial, i.e., there may be sequences X that are not mapped to infinite sequences but to finite strings. To define the transformed measure, for $w \in 2^{<\omega}$ let

$$V_w = \{v \in 2^{<\omega} : \phi(v) \sqsupseteq w\}.$$

Now set

$$\mu_f[w] = \mu[V_w].$$

Note that μ_f is not necessarily a measure. It only satisfies the inequality

$$\mu_f[w] \geq \mu_f[w0] + \mu_f[w1] \tag{4.1}$$

for all $w \in 2^{<\omega}$. In analogy to the discrete case, where semimeasures are 'defective' probability distributions over $\mathbb{N}$, we call set functions satisfying (4.1) *continuous semimeasures*.

Important for the effective theory is that each enumerable semimeasure can be effectively transformed into Lebesgue measure λ as described above. The following theorem was first proved by Zvonkin and Levin (1970). (See also Section 4.5 in the book by Li and Vitányi, 1997.)

Theorem 4.5 *If $\mu : 2^{<\omega} \to [0, \infty)$ is an enumerable continuous semimeasure, then there exists a process f such that*

$$\mu = \lambda_f,$$

where λ is the Lebesgue measure on 2^ω.

If the process transforming λ maps almost every (in the sense of measure) infinite sequence to an infinite sequence again, the resulting measure is computable. Indeed, Levin and Zvonkin showed that every computable measure can be obtained from λ this way.

Definition 4.6 Let μ be a measure. A process $f : 2^\omega \to 2^\omega \cup 2^{<\omega}$ is *μ-regular*, if the set

$$\{X \in 2^\omega : f(X) \in 2^\omega\}$$

has μ-measure one.

Theorem 4.7 (Levin, 1970) *Let μ be a computable measure.*

(i) If f is a μ-regular process, then μ_f is computable measure.

(ii) There exists a λ-regular process g such that $\mu = \lambda_g$ and a μ-regular process h such that $\lambda = \mu_h$, h is the inverse process of g (restricted to the domain of $g \circ h$), and if $h(X)$ is not an infinite sequence, then X is either computable or lying in an interval of μ-measure zero.

The proof uses a simple observation on distribution functions. For this purpose, we identify 2^ω with the unit interval as in Section 1.1. If F is the distribution function of μ, i.e.

$$F(\alpha) = \mu[0, \alpha],$$

F is a nondecreasing, right-continuous function mapping [0, 1] to [0, 1] such that $F(0) = 0$ and $F(1) = 1$. Let us define

$$G(\alpha) = \inf\{\beta : \alpha \leq F(\beta)\}.$$

F is nondecreasing and right-continuous, so $\{\beta : \alpha \leq F(\beta)\}$ is always an interval closed on the left. Therefore, $\{\beta : \alpha \leq F(\beta)\} = [G(\alpha), 1]$, so $G(\alpha) \leq \beta$ if and only if $\alpha \leq F(\beta)$, so G can be seen as an inverse to F. Since F is the distribution function of μ, we know that $\mu(\gamma, \delta] = F(\delta) - F(\gamma)$ for any $\gamma, \delta \in [0, 1]$. If we map a real $\alpha \in [0, 1]$ to $G(\alpha)$, then $\mu(G(\alpha), G(\beta)] = \beta - \alpha$. Hence to prove (ii) it suffices to construct a λ-regular process Φ that computes G.

A careful analysis of Levin's and Kautz's proofs yields that, if a computable measure μ is nonatomic, it is possible to strengthen the result to truth-table equivalence.

Theorem 4.8 *Let μ be a computable, nonatomic measure. Then every μ-random sequence is truth-table equivalent to some Martin-Löf random sequence.*

4.1.2 Unnatural sequences of high dimension

It might be tempting, especially in the light of Theorem 3.23, to solve the initial question by showing that every sequence of positive dimension is random with respect to some computable measure.

However, this is not the case, as we show in the following, generalizing a result by Muchnik (Muchnik et al., 1998).

Theorem 4.9 *If $X \in 2^\omega$ is 1-generic, then, for any $Y \in 2^\omega$ and any computable, infinite, co-infinite set $Z \subseteq \mathbb{N}$, $X \oplus_Z Y$ is not a natural sequence.*

Proof. Let μ be a computable measure. We may assume $\mu[w] > 0$ for all $w \in 2^{<\omega}$. Define a computable function $f : 2^{<\omega} \to \{0, 1\}$ by

$$f(w) = \begin{cases} 0, & \text{if } \mu[w0] \leq \mu[w1], \\ 1, & \text{otherwise.} \end{cases}$$

Further, define a betting strategy b by

$$b(w) = \begin{cases} (1/2, f(w)), & \text{if } |w| \in Z, \\ (0, 0), & \text{otherwise,} \end{cases}$$

that is, b bets only on positions in Z. Since X is 1-generic, it must have infinitely many initial segments $X\lceil_{3k}$ in the computable set

$$U = \{g^{2|w|}(w) : w \in 2^{<\omega}\},$$

where $g(w) = w\frown f(w)$. For each such k, the capital of d_b after round $n_Z(3k)$ is at least

$$\left(\frac{1}{2}\right)^k \left(\frac{3}{2}\right)^{2k} = \left(\frac{9}{8}\right)^k$$

(note that a winning bet will multiply the current capital by at least $3/2$, since f was chosen such that $\mu[w\frown(1 - i_w)]/\mu[w\frown i_w] \geq 1/2$), and hence grows unboundedly. □

Using Theorems 2.39 and 3.6, we can immediately infer that there are unnatural sequences of arbitrary high dimension.

Corollary 4.10 *For each rational $s \in [0, 1]$, there exists an unnatural sequence of dimension s.*

Proof. In Theorem 4.9, choose Y to be a Martin-Löf random sequence and $Z \subseteq \mathbb{N}$ such that $\delta_{\overline{Z}} = s$. □

4.2 Randomness, Entropy, and Strong Hausdorff Measure

In the previous section we saw that a sequence which is random with respect to a nonatomic, computable measure computes a Martin-Löf random sequence via a truth-table reduction. Since the use of a truth-table reduction is bounded by a computable function, we obtain a computable lower bound on the complexity of the sequence.

Proposition 4.11 *If μ is a nonatomic computable measure and A is random with respect to μ, then there exists a computable, nondecreasing, unbounded function $h : \mathbb{N} \to \mathbb{N}$ such that, for all n,*

$$\mathrm{K}(A\restriction_n) \geq h(n).$$

Note that the bound h is not uniform, i.e. there is no function h that works for all μ-random sequences alike.

Sequences which have a computable lower bound on its entropy have been called *complex* by Kjos-Hanssen et al. (2004). They showed an interesting connection between complex sequences and sequences which compute a *diagonally nonrecursive* (*dnr*) *function* (see Ambos-Spies and Kučera (2000) for definitions and properties): A sequence is truth-table above a dnr function if and only if it is complex. (Miller, 2002, had introduced *hyperavoidable* sequences, a notion which Kjos-Hanssen et al. (2004) have shown to be equivalent to the concept of complex sequences.)

Bounding entropy from below by a computable function h has an interesting measure-theoretic analogy, regarding *Hausdorff measures*. In Chapters 1 and 2 we introduced (effective) Hausdorff measures with respect to general dimension functions. Recall that a dimension function is a function $h : \mathbb{R} \to [0, \infty]$ that is increasing, continuous on the right with $h(0) = 0$. In the definition of Hausdorff measures, this function h is applied to the diameter of an (open) set. Since in Cantor space the diameters of basic open cylinders are precisely the rational numbers of the form 2^{-n}, one may pass from h to a function $\widetilde{h} : \mathbb{N} \to [0, \infty)$ which is nondecreasing and unbounded, by defining

$$\widetilde{h}(n) = -\log h(2^{-n}).$$

(For instance, if $h(x) = x^{1/2}$, $\widetilde{h}(n) = (1/2)n$.)

In contrast to dimension functions, we will call a function $h : \mathbb{N} \to \mathbb{N}$ which is unbounded and nondecreasing an *order function* or sometimes simply an *order* (after Schnorr, 1971). For technical reasons, we usually assume that $h(0) = 0$.

The definition of $\mathcal{H}^h$-measure zero then reads as follows: $\mathcal{A} \subseteq 2^\omega$ is $\mathcal{H}^h$-null if and only if

$$(\forall n \in \mathbb{N})\ (\exists C_n \subseteq 2^{<\omega})\ \mathcal{A} \subseteq \bigcup_{\sigma \in C_n} [\sigma] \ \text{ and } \ \sum_{\sigma \in C_n} 2^{-\widetilde{h}(|\sigma|)} \leq 2^{-n}. \tag{4.2}$$

On the other hand, if $\widetilde{h} : \mathbb{N} \to \mathbb{N}$ is an order function, classes $\mathcal{A} \subseteq 2^\omega$ satisfying (4.2) will be $\mathcal{H}^h$-null for an accordant dimension function h.

Therefore, we will use the notion of having $\mathcal{H}^h$-measure zero for order functions $h : \mathbb{N} \to \mathbb{N}$, too, where (4.2) is understood. It is obvious that for *computable* such h, the notion of $\mathcal{H}^h$-measure zero can be effectivized as well. This suggests the following definition:

Definition 4.12 A sequence $A \in 2^\omega$ has *strong effective Hausdorff measure zero* if for all computable order functions $h : \mathbb{N} \to [0, \infty)$, $\{A\}$ has effective $\mathcal{H}^h$-measure zero.

In Section 2.1 we showed a connection between lower bounds on entropy and generalized Hausdorff measures, which we restate here in terms of order functions.

Theorem 4.13 *Let $h : \mathbb{N} \to \mathbb{N}$ be a computable order function. A sequence $A \in 2^\omega$ is not effectively $\mathcal{H}^h$-zero if and only if there exists a constant c such that for all n,*

$$\mathrm{K}(A\restriction_n) \geq h(n) - c. \tag{4.3}$$

We can infer that the complex sequences and the sequences not having strong effective Hausdorff measure 0 coincide.

Corollary 4.14 *A sequence $A \in 2^\omega$ is complex if and only if it does not have strong effective Hausdorff measure* 0*.*

For later use, we note that, for a sequence to be complex, it suffices to be complex at an infinite, computable set of positions.

Proposition 4.15 *A sequence $A \in 2^\omega$ is complex if and only if there is a computable, unbounded, strictly increasing function $g : \mathbb{N} \to \mathbb{N}$ such that*

$$(\forall n)\ \mathrm{K}(A\restriction_{g(n)}) \geq n. \tag{4.4}$$

Proof. ($\Rightarrow$) Suppose there is a computable order function h such that for all n, $\mathrm{K}(A\restriction_n) \geq h(n)$. Then the function

$$g(n) = \mu\, m\, [h(m) \geq n]$$

is obviously computable and satisfies (4.4).

($\Leftarrow$) Let g be computable, unbounded, nondecreasing such that (4.4) holds. Define a computable function h as follows: First, let $Z = \{n : g(n) < g(n+1)\}$. Then $g\restriction_Z$ is injective, so there exists a computable, infinite set $Y \subseteq \mathbb{N}$ and an unbounded, nondecreasing, computable function h such that $\mathrm{K}(A\restriction_n) \geq h(n)$ for all $n \in Y$.

Suppose now A is not complex, that is, for every computable order function f there are infinitely many n such that $\mathrm{K}(A\upharpoonright_n) < f(n)$. In particular, this will hold for every function $f_c(n) = h(n) - c$, so for infinitely many $n \in Y$ we will find an extension $A\upharpoonright_m$ of $A\upharpoonright_n$ such that $\mathrm{K}(A\upharpoonright_m) < h(n) - c$. A simple use of the Recursion Theorem yields that there must be infinitely many $n \in Y$ such that $\mathrm{K}(A\upharpoonright_n) < h(n)$, a contradiction. □

4.3 Randomness and Strong Hausdorff Measure Zero

We are going to show that randomness with respect to a computable measure does not imply weak incompressibility, even if the random sequence is required to be noncomputable, and hence the measure nonatomic at this point.

Theorem 4.16 (Reimann and Slaman) *There exist a computable measure μ and a noncomputable sequence A such that A is μ-random but not complex.*

A proof of this theorem can be found in Reimann and Slaman (2004).

The preceding theorem says that randomness with respect to computable measure, even though it guarantees to compute a Martin-Löf random sequence (by Levin's result, Theorem 4.3), does not imply a computable lower bound on entropy.

Next, we shall see that entropy and randomness behave, in a certain sense, incompatible, despite the coincidence between Martin-Löf randomness and incompressibility. Namely, we are going to show that there exist complex sequences (i.e. sequences which do not have strong Hausdorff measure 0) that do not compute a Martin-Löf random sequence.

Theorem 4.17 (Reimann and Slaman) *There exists a complex sequence $A \in 2^\omega$ such that no sequence $B \leq_{\mathrm{T}} A$ is Martin-Löf random.*

Proof. We give an extended sketch of the contruction. We will construct a class of sequences $\mathcal{M}$ such that for the eth Turing functional Φ_e, the requirement

$$R_e: \quad \forall X \in \mathcal{M}(\Phi_e(X) = A \;\Rightarrow\; A \text{ is not Martin-Löf random}) \tag{4.5}$$

is satisfied. Furthermore, $\mathcal{M}$ will contain a perfect subclass whose branching levels are bounded by a computable function. We show that this implies that there exists a computable order h such that $\mathcal{M}$ is not $\mathcal{H}^h$-null, and therefore, by Theorem 4.13, contains a complex sequence.

Each requirement R_e can use two strategies to ensure its satisfaction: Either it will try to stay undefined for the oracles in $\mathcal{M}$, or, if it is defined for some oracle X, the prefix complexity of infinitely many initial segments σ of $\Phi_e(X)$ will be bounded by $|\sigma| + c_e$ for some constant c_e. Chaitin (1987) showed that a sequence A is Martin-Löf random if and only if $\lim_n(\mathrm{K}(A\restriction_n - n) = \infty$. Therefore, $\Phi(X)$ cannot be Martin-Löf random.

We will ensure the non-randomness of $\Phi_e(X)$ by enumerating an accordant Kraft-Chaitin set. As such a set has to be computably enumerable, it is useful to split each requirement into a countable number of subrequirements, each of which is based on an enumerable outcome. Assign to each functional Φ_e infinitely many lengths $(e_n)_{n\in\mathbb{N}}$. Then require

$$R_{\langle e,n\rangle}: \quad \forall X \in \mathcal{M}\ (\Phi_e(X)\downarrow = Y \in 2^\omega \Rightarrow (\forall n)\,\mathrm{K}(Y\restriction_{l_n}) \leq l_n + c_e), \tag{4.6}$$

where c_e is a constant independent of n.

We will approximate $\mathcal{M}$ in stages. Each requirement R_i will receive a certain amount of mass M_i to work with. Furthermore, we have to make sure $\mathcal{M}$ contains a perfect subclass. This will be done in a Cantor-set-like construction. Each M_i will, at each stage of the construction, be given as a clopen subclass of 2^ω. These clopen classes will be split sufficiently to ensure that $\mathcal{M}$ contains a perfect subclass.

In course of the construction, requirements with higher priority (i.e. those with lower index) may shrink the mass available for lower priority requirements. However, for each j, we ensure *a priori* how much measure will be left at minimum after all requirements $R_0, \dots, R_{j-1}$ have acted for the last time. Furthermore, this amount of measure will, at any time, be bounded from below by a computable function, thereby yielding a lower bound on the initial segments of a certain length contained in M_j.

We now describe the construction of $\mathcal{M}$.

Define $l_i = i + 2$, $n_0 = 0$ and $n_{i+1} = n_i + l_i + 3$, and $r_i = 2^{-n_i - l_i - 3}$.

We initialize a *tree of available masses* by setting up a Cantor scheme in 2^ω. Set $\mathcal{M}_\epsilon = 2^\omega$ and for each $\sigma \in 2^{<\omega}$, let $\mathcal{M}_\sigma$ be such that

(1) $\lambda(\mathcal{M}_\sigma) \geq 2^{-n_i}$ for all σ of length i,

(2) $\sigma \sqsubseteq \tau$ implies $\mathcal{M}_\sigma \supseteq \mathcal{M}_\tau$,

(3) If $|\sigma| = |\tau| = i$, then $d(\mathcal{M}_\sigma, \mathcal{M}_\tau) \geq r_i$ (where $d(\mathcal{M}_\sigma, \mathcal{M}_\tau)$ denotes the distance between $\mathcal{M}_\sigma$ and $\mathcal{M}_\tau$).

Conditions (1)-(3) will be preserved throughout the construction, with the slight variation of (1):

(1*) $\lambda(\mathcal{M}_\sigma) \geq 2^{-n_i - l_i - 1}$ for all σ of length i.

This ensures that

$$\mathcal{M} = \bigcap_{i \geq 0} \bigcup_{|\sigma|=i} \mathcal{M}_\sigma$$

will contain a perfect subset which (due to the control over the minimum distances r_i) will not be $\mathcal{H}^h$-null for some computable order h.

Each requirement $R_i = R_{\langle e,n \rangle}$ will control all classes $\mathcal{M}_\sigma$ with $|\sigma| = i$. Within each $\mathcal{M}_\sigma$, it will pursue one of the following two strategies:

(I) As long as

$$\lambda(\Phi_e^{-1}[\{0,1\}^{e_n}] \cap \mathcal{M}_\sigma) \leq \lambda(\mathcal{M}_\sigma)\left(1 - \frac{1}{2^{l_i}}\right),$$

i.e. as long as Φ_e is still undefined on a sufficient amount of measure within $\mathcal{M}_\sigma$, restrict $\mathcal{M}_\sigma$ to the part where Φ_e does not compute a string of length l_i.

(II) If, however, at some stage

$$\lambda(\Phi_e^{-1}[\{0,1\}^{e_n}] \cap \mathcal{M}_\sigma) > \lambda(M_\sigma)\left(1 - \frac{1}{2^{l_i}}\right),$$

pick some $w \in \{0,1\}^{l_i}$ such that $\lambda(\Phi_e^{-1}([w]) \cap \mathcal{M}_\sigma)$ is largest among all values $\lambda(\Phi_e^{-1}([v]) \cap \mathcal{M}_\sigma)$ for $v \in \{0,1\}^{l_i}$. Set $\mathcal{M}_\sigma := \mathcal{M}_\sigma \cap \Phi_e^{-1}([w])$. Enumerate the pair $(w, l_i + c_i)$ into D (unless another combination (w, l) has been enumerated into D before), increase c_i by one, and restrict all $\mathcal{M}_\tau$ with $\tau \sqsupset \sigma$ to $\mathcal{M}_\sigma$. If necessary, reinitialize the $\mathcal{M}_\tau$ according to (1*)-(3).

Every action by a requirement may injure actions already taken by lower priority requirements. However, since, as an easy calculation shows, at every stage of the construction the measure of $\mathcal{M}_\sigma$, $|\sigma| = i$, is at least $2^{-n_i-l_i-1}$, we can always find two clopen subsets $\mathcal{M}_{\sigma 0}$ and $\mathcal{M}_{\sigma 1}$ which satisfy $\lambda(\mathcal{M}_{\sigma 0}), \lambda(\mathcal{M}_{\sigma 1}) \geq 2^{-n_i-l_i-3}$ and $d(\mathcal{M}_{\sigma 0}, \mathcal{M}_{\sigma 1}) \geq 2^{-n_i-l_i-3}$.

It is clear that the described actions are enumerable events in the sense that if (II) is to occur it will so after finitely many steps and can be detected computably. Hence D will be computably enumerable. Furthermore, by the way that the l_i and c_i are chosen and updated, it is obvious that

$$\sum_{(w,k_w) \in D} 2^{-k_w} \leq 1,$$

and hence that D is a Kraft-Chaitin set.

□

Theorem 4.17 may be regarded as a first negative answer to the initial question. Having positive dimension (in a generalized sense) does not necessarily allow to extract a random content efficiently.

4.4 Randomness with Respect to Arbitrary Measures

In Section 4.1 we saw that sequences of positive dimension are not necessarily random with respect to a computable measure. We might therefore try to evaluate the random content of such sequences by generalizing the notion of randomness to arbitrary measures. This was done by several authors before – Martin-Löf (1966) studied randomness for arbitrary Bernoulli measures, Levin (1973, 1976, 1984) studied arbitrary measures on 2^ω, while Gács (2003) generalized Levin's approach to a large class of topological spaces.

We will restrict ourselves to (probability) measures on 2^ω. The idea is to follow the definition of *relative randomness* and extend it to measures. Given two sequences $A, B \in 2^\omega$, A is called *Martin-Löf random relative to* B if no Martin-Löf test $\{U_n\}$ that is *uniformly enumerable in* B covers A.

As we saw in Section 1.2, these measures allow a nice representation, in particular they can be specified by a mapping from $2^{<\omega} \to [0, \infty)$. This can be turned into a *Cauchy representation* (approximating the measure by simple rational valued measures with respect to a suitable metric on the space of measures), which in turn can be encoded by an infinite sequence.

Therefore, the basic idea for defining randomness with respect to arbitrary measures consist in endowing Martin-Löf tests with an oracle through which they have access to a representation of a measure.

4.4.1 Representing measures

In order to access measures as an oracle, we must fix an appropriate representation of measures as binary sequences.

Denote by $M(2^\omega)$ the set of probability measures on 2^ω. The *weak topology* is induced by defining $\mu_n \to \mu$ if $\mu_n(\mathcal{B}) \to \mu(\mathcal{B})$ for all Borel sets $\mathcal{B}$. The weak topology can be metrized using the *Prokhorov distance* Prokhorov, 1956. This is defined as

$$\rho(\mu, \nu) = \inf\{\varepsilon > 0 : \ \mu\mathcal{B} \leq \mathcal{B}^\varepsilon + \varepsilon \text{ for all Borel sets in } 2^\omega\},$$

where $\mathcal{B}^\varepsilon := \{y \in \mathcal{X} : \ d(y, B) \leq \varepsilon\}$. $(M(2^\omega), \rho)$ is a seperable, complete metric space. A countable, dense subset $D \subseteq M(2^\omega)$ is given by the set of measures which assume positive, rational values on a finite number of rationals, i.e. D is the set of measures of the form

$$\mu = \sum_{i=1}^{n} q_i \delta_{Q_i},$$

where $q_1, \dots, q_n$ are positive rationals with $\sum_i q_i = 1$ and $\{Q_1, \dots, Q_n\}$ is a set of sequences representing rational numbers. Every measure ν can be represented by a *Cauchy sequence* $(\nu_n)_{n\in\mathbb{N}}$ in D, where $\rho(\nu_n, \nu_{n+1}) \leq 2^{-(n+1)}$ for all n. Since $(M(2^\omega), \rho)$ is complete, every such sequence also represents a measure on 2^ω.

Given a measure $\mu \in M(2^\omega)$, we can devise a measure ν_n^μ such that $\rho(\nu_n^\mu, \mu) \leq 2^{-n}$ by approximating the distribution function of μ by dyadic rationals. Let $w_0 < \cdots < w_{2^n-1}$ denote the strings of length n in the usual lexicographic order. The measure ν_n^μ will reside on atoms of the form $w_k 0^\omega$. For $n = 0, \dots, 2^n - 1$, define inductively

$$\nu_n^\mu(\{w_n 0^\omega\}) = \max \left\{ \frac{m}{2^n} : \sum_{k=1}^{n-1} \nu_n^\mu(\{w_k 0^\omega\}) + \frac{m}{2^n} \leq \sum_{k=1}^{n} \mu([w_k]) \right\}.$$

Definition 4.18 Let μ be a probability measure on 2^ω. The *Cauchy representation* $\rho_C(\mu)$ of μ is defined as the sequence of measures $(\nu_n^\mu)_{n\in\mathbb{N}}$.

The Cauchy representation $\rho_C(\mu) = (\nu_n^\mu)$ of a measure μ can be coded into a sequence $B \in \omega^\omega$ by letting $d(n) := \lfloor \log(n+1) \rfloor$ and

$$B(n) = m \quad \Leftrightarrow \quad (\nu_{d(n)}^\mu(\{w_{n-2^{d(n)}} 0^\omega\}) = m.$$

Using a standard coding procedure, any Cauchy representation of a measure can be encoded by an infinite binary sequence, too.

On the other hand, every sequence in ω^ω which satisfies the corresponding consistency requirements (a dyadic interval of length 2^n sums up to 2^n, and all extensions correspond to measures ν_k of distance at most 2^{-n}) represents a measure in $M(2^\omega)$ (recall that $(M(2^\omega), \rho)$ is complete).

Therefore, the space $M(2^\omega)$ of probability measures on 2^ω can be identified with the set of infinite paths in a computably bounded subtree T_M of $\omega^{<\omega}$. (In particular, we see that $M(2^\omega)$ is compact.)

The Cauchy representation enables us to define Martin-Löf tests relative to arbitrary measures.

Definition 4.19 Let μ be a probability measure on 2^ω with Cauchy representation $\rho_C(\mu)$.

(1) A *Martin-Löf test for* μ is a sequence $(U_n)_{n\in\mathbb{N}}$ of subsets of $2^{<\omega}$ such that (U_n) is uniformly computably enumerable in $\rho_C(\mu)$ and for each n,

$$\sum_{w\in U_n} \mu[w] \leq 2^{-n}.$$

(2) A sequence $A \in 2^\omega$ is *random with respect to* μ, or simply μ*-random*, if for every Martin-Löf test $(U_n)_{n\in\mathbb{N}}$ for μ,

$$A \notin \bigcap_{n\in\mathbb{N}} [U_n]$$

It is clear that for computable measures, this definition of randomness agrees with the original one given in Definition 2.32.

It turns out that general randomness is too coarse to capture the random content of positive dimension. In fact, it can only separate computable from noncomputable sequences.

Theorem 4.20 (Reimann and Slaman) *For any sequence $A \in 2^\omega$, the following conditions are equivalent:*

(i) There exists a probability measure μ such that A is not a μ-atom and A is μ-random.

(ii) A is not computable.

The implication (i) $\Rightarrow$ (ii) follows immediately from Proposition 4.2. To prove the reverse implication, the idea of the proof is the following: Kucera's proof (Kučera, 1985) that every sequence is reducible to a Martin-Löf random sequence actually yields that every Turing degree $\geq \mathbf{0}'$ contains a Martin-Löf random sequence (so for every sequence A, $A \oplus \emptyset'$ is Turing equivalent to a random sequence). It is not hard to see that Kucera's result relativizes, i.e. every sequence $\geq_\mathrm{T} B'$ relative to B is Turing equivalent relative to some Martin-Löf random sequence X. We will use the Posner-Robinson Theorem (Posner and Robinson, 1981) to obtain such a sequence B for given A.

The (relative) Turing equivalence to a random sequence allows us to transform the uniform measure λ in a sufficiently controlled manner. In particular, we can obtain a Π^0_1 class of representations of measures, all of which are good candidates for a measure that renders A random. We will use a compactness argument to show that at least one member m of the class has the property that the Martin-Löf random sequence X is still λ-random *relative to m*. Then, A has to be random with respect to measure defined by m, since otherwise an Martin-Löf m-test could be effectively transformed to a Martin-Löf λ-test relative to m which X would fail.

We start with the relativized version of Kucera's result. Given a sequence $C \in 2^\omega$, we write $\equiv_{\mathrm{T}(C)}$ to denote Turing equivalence relative to C (similarly for $\leq_{\mathrm{T}(C)}$ and $\geq_{\mathrm{T}(C)}$).

Theorem 4.21 (Kučera, 1985) *If for a sequence $B \in 2^\omega$ it holds that $B \geq_{\mathrm{T}(C)} C'$ for some $C \in 2^\omega$, then A is Turing equivalent relative to C to some sequence X that is Martin-Löf random relative to C.*

Proof. Let $(U^C_n)_{n\in\mathbb{N}}$ be a universal Martin-Löf test relative to C. We will construct a perfect tree $T : 2^{<\omega} \to 2^{<\omega}$ (which will be computable in C') such that all its infinite paths lie in $\mathcal{P}_0 := [U^C_0]^\complement$ (i.e. are all random sequences relative to C) and every path codes a sequence $B \in 2^\omega$. This coding will be effective in C, due to a relativized version of Lemma 3.24, which allows us to compute (relative to C) an effective lower bound for the measure of $\mathcal{P}_0$.

We start with $\mathcal{P}_0$. We describe how to code B into an infinite path of $\mathcal{P}_0$. To initialize, set $T(\epsilon) = \epsilon$. Assume now for $n \in \mathbb{N}$, $T(\sigma)$ where $\sigma = B\restriction_n$ has been constructed such that $T(\sigma) \sqsubset \mathcal{P}_0$. To define $T(B\restriction_{n+1})$, compute (computably in C') the smallest number n_σ such that the leftmost and the rightmost path of $[T(\sigma)] \cap \mathcal{P}_0$ differ (such an n_σ has to exist since a path in $\mathcal{P}_0$ cannot be isolated). Denote the latter by L_σ and R_σ, respectively. Choose $T(B\restriction_{n+1}) = L_\sigma\restriction_{n_\sigma}$ if $B(n) = 0$, $T(B\restriction_{n+1}) = R_\sigma\restriction_{n_\sigma}$, otherwise.

We claim that if $B \geq_{\mathrm{T}(C)} C'$, then $B \equiv_{\mathrm{T}(C)} T(B)$. $B \geq_{\mathrm{T}(C)} T(B)$ follows immediately from the construction, which is computable in C'. To prove $B \leq_{\mathrm{T}(C)} T(B)$, we employ a relativized version of Lemma 3.24.

Lemma 4.22 *Let μ be a computable measure on 2^ω. If $\mathcal{A} \subseteq 2^\omega$ is Π^0_1 relative to some sequence $C \in 2^\omega$, then there exists a C-computable function $g : 2^{<\omega} \times \mathbb{N} \to \mathbb{Q}^{>0}$ such that for any $w \in 2^{<\omega}$ and any $n \in \mathbb{N}$ it holds that*

$$\mathcal{P}^\mu_n \cap \mathcal{A} \cap [w] \neq \emptyset \quad \Rightarrow \quad \mu(\mathcal{A} \cap [w]) \geq g(w, n).$$

The proof of this is completely analogous to the proof of Lemma 3.24, a straightforward relativization.

Now, to compute $B(0)$, Lemma 4.22 gives us a C-computable lower bound on $\lambda\mathcal{P}_0$, say 2^{-b_0}, $b_0 \in \mathbb{N}$. We know then that the leftmost and the rightmost path of $\mathcal{P}_0\restriction_{b_0}$ must differ (the tree must branch because its measure is too large). Given $T(B)\restriction_{b_0}$ we compute $\mathcal{P}_0$ till it turns out to be the left- or rightmost path. Obviously, using Lemma 4.22, this decision procedure can be continued inductively to decide $B(n)$ for any $n \in \mathbb{N}$. □

As described above we will use the Posner-Robinson Theorem to obtain a sequence C relative to which the given noncomputable A is equivalent to the jump of C.

Theorem 4.23 (Posner and Robinson, 1981) *If $A \in 2^\omega$ in noncomputable, the there is a $C \in 2^\omega$ such that $A \oplus C \equiv_{\mathrm{T}} C'$.*

Finally, for the proof Theorem we need some sort of basis theorem concerning relative randomness.

Theorem 4.24 *Let $C \in 2^\omega$, and let $T \subseteq \omega^{<\omega}$ be an infinite tree which is computable in C and which has a finite, C-computable branching width. Then, for every sequence A which is Martin-Löf random relative to C, there is an infinite path X in T such that A is Martin-Löf random relative to $C \oplus X$.*

Proof. Given $C \in 2^\omega$ and some $\tau \in \omega^{<\omega}$, let $(U_n^{C,\tau})_{n \in \mathbb{N}}$ denote a universal Martin-Löf test relative to C and τ (which is still uniformly enumerable in C). We enumerate a Martin-Löf test $(V_n)_{n\in\mathbb{N}}$ computable in C as follows: enumerate a string σ into V_n if $[\sigma]$ is contained in $[U_n^{C,\tau}]$ for all $\tau \in T$ with $|\tau| = |\sigma|$ (note that there are only finitely many such τ).

If A is Martin-Löf random relative to C, there has to be some n such that $X \notin [V_n]$. This means that for every m, $A\restriction_m$ is not enumerated in V_n, hence for every m there is a $\tau \in T$ of length m such that $[\tau]$ is not contained in $[U_n^{C,\tau}]$. Consequently, there is an infinite subtree of T of nodes τ which do not enumerate an initial segment of A into $[U_n^{C,\tau}]$. Applying König's Lemma yields an infinite path X through this subtree.

Obviously, A is random relative to $C \oplus X$, because otherwise, due to the Use Principle, for every n there would be an initial segment τ of X such that $A \in [U_n^{C,\tau}]$, a contradiction. □

We can now give the proof of Theorem 4.20.

Proof of Theorem 4.20. Let A be a noncomputable sequence. Using Theorem 4.21 and the the Posner-Robinson Theorem 4.23, we obtain a sequence X which is Martin-Löf random relative to some $C \in 2^\omega$ and which is $\mathrm{T}(C)$-equivalent to A. There are Turing functionals Φ and Ψ computable in C such that

$$\Phi(X) = A \quad \text{and} \quad \Psi(A) = X.$$

We will use the functionals to define a class of measures which will render A random. If Φ was total and invertible, there would be no problem to define the desired measure, as one could simply 'pull back' Lebesgue measure using Φ^{-1}. In our case we have to use Φ and Ψ to control the measure. We are guaranteed that this will work *locally*, since Φ and Ψ are inverses on A and X. Therefore, given a string σ (a possible initial segment of A) we will single out strings which appear to be candidates for initial segments of an inverse sequence. To be precise, given $\sigma \in 2^\omega$, define the set $\mathrm{Pre}(\sigma)$ to be the set of minimal elements of

$$\{\tau \in 2^\omega : \ \Phi(\tau) \sqsupseteq \sigma \quad \& \quad \Psi(\sigma) \sqsubseteq \tau\}.$$

If we want to define a measure μ with respect to which A is random, we have to satisfy two requirements:

(1) The measure μ will *dominate* an image measure induced by Φ. This will ensure that any Martin-Löf random sequence is mapped by Φ to a μ-random sequence.

(2) The measure must *not be atomic* on A.

To meet these requirements, we restrict the values of μ in the following way (if Φ and Ψ are defined):

$$\lambda[\mathrm{Pre}(\sigma)] \leq \mu[\sigma] \leq \lambda[\Psi(\sigma)]. \tag{4.7}$$

The first inequality ensures that (1) is met, whereas the second guarantees that μ is nonatomic on the domain of Ψ.

We now show that there is a Π^0_1 class $\mathcal{M}$ in 2^ω computable in C such that every element encodes the representation of a measure relative to which A is random.

Note that each configuration implied by (4.7) kills off one or more branches in the computably bounded tree of measures T_M. Since the configurations (4.7) are a Σ^0_1 event, there is a computable subtree of T_M consisting of 'admissible' measures. Let $\mathcal{M}$ be the set of infinite paths through this subtree, which is Π^0_1.

Furthermore, the class $\mathcal{M}$ is not empty, as can be seen as follows. As $\Psi(A) = X$, there are inifinitely many $\sigma_n = A\upharpoonright_{l_n}$ such that $\Psi(\sigma_n) \sqsubset X$ and $|\Psi(\sigma_n)| \to \infty$. Note, too, that for each n, $[\mathrm{Pre}(\sigma_n)] \subseteq [\Psi(\sigma_n)]$ (by the definition of Pre), and that $\mathrm{Pre}(\sigma_n)$ is not empty for almost every n. Now $\lambda[\Psi(\sigma_n)]$ goes to zero, and since $[\mathrm{Pre}(\sigma_n)] \subseteq [\Psi(\sigma_n)]$, there exists a set of infinitely many, mutually different, *compatible* requirements of type (4.7) which cannot be enumerated into the Σ^0_1 class described above. Thus, $\mathcal{M}$ is not empty. It also follows from this argument that for any $\rho_C(\mu) \in \mathcal{M}$, $\mu\{A\} = 0$, so A is not a μ-atom.

Using Theorem 4.24, we obtain an element $\rho_C(\mu)$ of $\mathcal{M}$ such that X is Martin-Löf random relative to $C \oplus \rho_C(\mu)$.

It remains to show that A is Martin-Löf μ-random. Assume there was a Martin-Löf μ-test $(V_n)_{n\in\mathbb{N}}$ that covers A. Hence there must be infinitely many $\sigma_n = A\upharpoonright_{l_n}$ such that, for all n, $\sigma_n \in V_n$. We define a new test U_n by enumerating for every string σ that is enumerated into V_n, we enumerate the strings in $\mathrm{Pre}(\sigma)$ into U_n. By the definition of Pre, U_n covers X. Furthermore, μ satisfies the measure condition (4.7), so the Lebesgue measure of $[U_n]$ is bounded by $\mu[V_n]$, and hence (U_n) is a Martin-Löf λ-test relative to $C \oplus \rho_C(\mu)$. But this contradicts the fact that X is Martin-Löf random relative to $C \oplus \rho_C(\mu)$. □

CHAPTER 5

Schnorr Dimension

Schnorr (1971) issued a fundamental criticism on the notion of effective nullsets as introduced by Martin-Löf and presented in Chapter 2. He argued that although we know how fast a Martin-Löf test (and all s- or μ-tests likewise) converges to zero, it is not effectively given, in the sense that the measure of the test sets U_n is not computable, but only enumerable, so in general we cannot decide whether a given cylinder belongs to the nth level of some test.

Schnorr presented two alternatives, both clearly closer to what one would call a *computable approach to randomness*. One is based on the idea of randomness as an unpredictable event in the sense that it should not be possible to win in a betting game (martingale) against a truly random sequence of outcomes. The other sticks to Martin-Löf's approach, however, requires the tests defining a nullset to be *a uniformly computable sequence* of open sets *having computable measure*, not merely a computable sequence of computably enumerable sets of measure less than 2^{-n}.

Schnorr was able to show that both approaches yield reasonable notions of randomness, i.e. random sequences according to his concepts expose most of the robust properties one would expect from a random object. However, his suggestions have some serious drawbacks, on the other hand. They are harder to deal with technically, which is mainly due to the absence of universal tests. Besides, a machine characterization of randomness like the elegant coincidence of Martin-Löf-random sequences with those incompressible by a universal prefix free machine is much harder to obtain (only recently by Downey and Griffiths, 2004).

We will see that the same difficulties are met when generalizing Schnorr's concepts to Hausdorff measures and subsequently to Hausdorff dimension. However, we will see that for dimension, Schnorr's two approaches coincide, in contrast to Schnorr randomness and computable randomness. Furthermore, it turns out that, with respect to Schnorr dimension, computably enumerable sets can expose a complex behavior, to some extent. Namely, we will show that there are c.e. sets of high Schnorr packing dimension, which is impossible in the effective case, due to Barzdin'' result (see Section 3.4).

5.1 Schnorr null sets

The definition of Schnorr s-nullsets differs from Σ_1^0-$\mathcal{H}^s$-nullsets only by the type of coverings that are allowed.

Definition 5.1 Let $s \in [0, 1]$ be a rational number.

(a) A *Schnorr s-test* is a computable sequence $(S_n)_{n\in\mathbb{N}}$ of c.e. sets of finite strings which satisfies, for all n, the following conditions:

$$\sum_{w\in S_n} 2^{-|w|s} \leq 2^{-n}, \text{ and} \tag{5.1}$$

$$\sum_{w\in S_n} 2^{-|w|s} \text{ is a uniformly computable real number.} \tag{5.2}$$

(b) A class $\mathcal{A} \subseteq 2^\omega$ is *Schnorr s-null* if there exists a Schnorr s-test $\{S_n\}$ such that

$$\mathcal{A} \subseteq \bigcap_{n\in\mathbb{N}} [S_n].$$

The *Schnorr random sequences* are those which are (as a singleton class in 2^ω) not Schnorr 1-null.

Downey and Griffiths (2004) observe that, by adding elements, one can replace any Schnorr 1-test by an equivalent one (i.e., one detecting the same Schnorr nullsets) where each level of the test has measure exactly 2^{-n}. We can apply the same argument in the case of arbitrary rational s, and hence we may, if appropriate, assume that (5.1) holds with equality.

Note further that, for rational s, each set S_n in a Schnorr s-test is actually computable, since to determine whether $w \in S_n$ it suffices to enumerate S_n until the accumulated sum given by $\sum 2^{-|v|s}$ Exceeds $2^{-N} - 2^{|W|S}$ (Assuming The measure of the n-th level of the test is in fact 2^{-n}). If w has not been enumerated so far, it cannot be in S_n. (Observe, too, that the converse does not hold.)

One can describe Schnorr s-nullsets also in terms of Solovay tests. Call an effective Solovay s-test $C \subseteq 2^{<\omega}$ *total*, if the sum

$$\sum_{w\in C} 2^{-|w|s}$$

is a computable real number. An easy adaption of the proof of Theorem 2.6 shows that a class $\mathcal{A} \subseteq 2^\omega$ is Schnorr s-null if and only if there is a total Solovay s-cover for $\mathcal{A}$.

5.1.1 Schnorr Hausdorff dimension

Like in the classical and Σ_1^0-case each classhas a critical value with respect to Schnorr s-measures.

Proposition 5.2 *Let $\mathcal{A} \subseteq 2^\omega$. Then for any rational $s \geq 0$, if $\mathcal{A}$ is Schnorr s-null then it is also Schnorr t-null for any rational $t \geq s$.*

This follows from the fact that every Schnorr s-test is also a Schnorr t-test. The definition of Schnorr Hausdorff dimension can now be given in a straightforward way.

Definition 5.3 The *Schnorr Hausdorff dimension* of a class $\mathcal{A} \subseteq 2^\omega$ is defined as

$$\dim_{\mathrm{H}}^{\mathrm{S}}(\mathcal{A}) = \inf\{s \geq 0 : \ \mathcal{A} \text{ is Schnorr } s\text{-null}\}.$$

As usual, for $A \in 2^\omega$ we write $\dim_{\mathrm{H}}^{\mathrm{S}} A$ for $\dim_{\mathrm{H}}^{\mathrm{S}}\{A\}$.

5.2 A machine characterization of Schnorr dimension

From a technical point of view, one of the biggest advantages of Martin-Löf's approach to randomness is the existence of a *universal test*, i.e. one that comprises all other Martin-Löf tests.

Although all Martin-Löf tests are to some extent effective (they are uniformly enumerable), they also expose some highly non-effective behavior. Kučera and Slaman (2001) showed that the measure of a level a universal Martin-Löf test is a random real number (i.e. its binary expansion is a Martin-Löf random sequence). Similarly, the measure of the domain of a universal prefix free Turing machine is a random real, too – Chaitin's Ω (see Section 2.5).

To obtain a machine characterization of Schnorr Hausdorff dimension, we have to restrict the admissible machines to those with domains having computable measure.

Definition 5.4 A prefix free machine M is *computable* if

$$\sum_{w \in \mathrm{dom}(M)} 2^{-|w|} \tag{5.3}$$

is a computable real number.

Note that, as in the case of Schnorr tests, if a machine is computable then its domain is computable (but not vice versa). To determine whether $M(w)\downarrow$, enumerate $\mathrm{dom}(M)$ until $\sum_{w \in \mathrm{dom}(M)} 2^{-|w|}$ is approximated by a precision of 2^{-N}, where $N > |w|$. If $M(w)\downarrow$, w must have been enumerated up to this point.

Theorem 5.5 (Downey and Griffiths, 2004) *A sequence A is Schnorr random if and only if for every computable machine M,*

$$(\exists c)\,(\forall n)\; K_M(A\restriction_n) \geq n - c.$$

Building on this characterization, we can go on to describe Schnorr Hausdorff dimension as asymptotic entropy with respect to computable machines.

Theorem 5.6 *For any sequence A it holds that*

$$\dim_{\mathrm{H}}^{\mathrm{S}} A = \inf_M \underline{\mathrm{K}}_M(A) \overset{\mathrm{def}}{=} \inf_M \left\{ \liminf_{n\to\infty} \frac{K_M(A\restriction_n)}{n} \right\},$$

where the infimum is taken over all computable prefix free machines M.

Proof. ($\geq$) Let $s > \dim_{\mathrm{H}}^{\mathrm{S}} A$. We show that this implies $s \geq \underline{\mathrm{K}}_M(A)$ for some computable machine M, which yields $\dim_{\mathrm{H}}^{\mathrm{S}} A \geq \inf_M \underline{\mathrm{K}}_M(A)$.

As $s > \dim_{\mathrm{H}}^{\mathrm{S}} A$, there exists a Schnorr s-test $\{U_i\}$ such that $A \in \bigcap_i [U_i]$. Assume each set in the test is given as $U_n = \{\sigma_{n,1}, \sigma_{n,2}, \dots\}$. Note that the Kraft-Chaitin Theorem is applicable to the set of axioms

$$\langle \lceil s|\sigma_{n,i}|\rceil - 1, \sigma_{n,i}\rangle, \quad n \geq 2, i \geq 1.$$

Hence there exists a prefix-free machine M such that

$$(\forall n \geq 2)(\forall i)\; K_M(\sigma_{n,i}) = \lceil s|\sigma_{n,i}|\rceil - 1.$$

Furthermore, M is computable since $\sum 2^{-\lceil s|\sigma_{n,i_n}|\rceil - 1}$ is computable.

We know that for all n there is an i_n such that $\sigma_{n,i_n} \sqsubset A$, and it is easy to see that the length of these σ_{n,i_n} goes to infinity. Hence there must be infinitely many n such that

$$K_M(A\restriction_n) \leq \lceil s|\sigma_{n,i}|\rceil - 1 \leq sn,$$

which in turn implies that

$$\liminf_{n\to\infty} \frac{K_M(A\restriction_n)}{n} \leq s.$$

($\leq$) Suppose $s > \inf_M \underline{\mathrm{K}}_M(A)$. So there exists a computable prefix-free machine M such that $s > \underline{\mathrm{K}}_M(A)$. Define the set

$$S_M = \{w \in 2^{<\omega} :\; K_M(w) < |w|s\}.$$

We claim that this is a total Solovay s-cover for A. It is obvious that the set covers A infinitely often, so it remains to show that

$$\sum_{w\in S_M} 2^{-|w|s}$$

is a finite, computable real number. The finiteness follows automatically from

$$\sum_{w \in S_M} 2^{-|w|s} < \sum_{w \in S_M} 2^{-K_M(w)} \leq 1,$$

by Kraft's inequality and the fact that M is a prefix-free machine. To show computability, given ε compute the measure induced by $\mathrm{dom}(M)$ up to precision ε, so all strings not enumerated by that stage (call it s) will add in total at most ε to the measure of $\mathrm{dom}(M)$, which means they will also add at most ε to $\sum_{S_M} 2^{-|w|s}$, hence

$$\sum_{w \in S_{M_s}} 2^{-|w|s} \leq \sum_{w \in S_M} 2^{-|w|s} \leq \sum_{w \in S_{M_s}} 2^{-|w|s} + \varepsilon,$$

since a v contributes to S_M only if $\mathrm{K}(v) < |v|s$. But obviously, this only happens if $v \in \mathrm{dom}(M)$. □

5.2.1 Schnorr packing dimension

Building on Corollary 2.25, we can define a packing dimension analog of Schnorr dimension.

Definition 5.7 Given a sequence $A \in 2^\omega$, we define the *Schnorr packing dimension* $\dim_{\mathrm{P}}^{\mathrm{S}}$ of A as

$$\dim_{\mathrm{P}}^{\mathrm{S}} A = \inf_M \overline{\mathrm{K}}_M(A) \stackrel{\mathrm{def}}{=} \inf_M \left\{ \limsup_{n \to \infty} \frac{K_M(A\upharpoonright_n)}{n} \right\}.$$

Schnorr packing dimension is implicitely defined by Athreya et al. (2004). They define various notions of effective packing dimension using martingales. As we shall see below, Schnorr dimension coincides with a notion of dimension defined via computable martingales.

5.3 Schnorr Dimension and Martingales

Looking at the characterization of Martin-Löf nullsets through enumerable martingales (Theorem 2.20), one might be tempted to derive a similar relation between Schnorr null sets and successful *computable* martingales. However, Schnorr (1971) pointed out that the increase in capital of a successful martingale can be so slow it cannot be computably detected. Therefore, he introduced *orders* (“Ordnungsfunktionen”), which allow to ensure an effective control of the capital infinitely often. We saw in Section 4.2 that, in 2^ω, orders can be seen as an analog to (computable) dimension functions. In this section we will call regard as an *order* any positive, real, unbounded, nondecreasing function. (It should be remarked that, in Schnorr's terminology, an “Ordnungsfunktion” is always computable.)

Definition 5.8 Let $g : \mathbb{N} \to \mathbb{R}$ be a computable order. A martingale is g-successful on a sequence $B \in 2^\omega$ if

$$d(B\restriction_n) \geq g(n) \text{ for infinitely many } n.$$

Schnorr showed that Schnorr nullsets can be characterized via computable martingales successful against computable orders.

Theorem 5.9 (Schnorr) *A set $\mathcal{X} \subseteq 2^\omega$ is Schnorr 1-null if and only if there exists a computable martingale d and a computable order g such that d is g-successful on all $B \in \mathcal{X}$.*

Observe that, in the light of Theorem 1.21, a martingale being s-successful means it is g-successful for order $g(n) = 2^{(1-s)n}$. These are precisely what Schnorr calls *exponential orders*, so much of effective dimension is already, though apparently without explicit reference, present in Schnorr's treatment of algorithmic randomness (Schnorr, 1971).

If one drops the requirement of being g-successful for some computable g, one actually obtains a different concept of randomness, usually referred to as *computable randomness*. Wang (1999) showed that the concepts of computable randomness and Schnorr randomness do not coincide. There are Schnorr random sequences on which some computable martingale succeeds. However, the differences vanish if it comes to dimension.

Theorem 5.10 *For any sequence $B \in 2^\omega$,*

$$\dim^{\mathrm{S}}_{\mathrm{H}} B = \inf\{s \in \mathbb{Q} : \textit{some computable martingale } d \textit{ is } s\textit{-successful on } B\}.$$

Proof. ($\leq$) Suppose a martingale d is s-successful on B. (We may assume that $s < 1$. The case $s = 1$ is trivial.) It suffices to show that for any $1 > t > s$ we can find a Schnorr t-test which covers B.

The strategy to define the test will be the same as in the proof of Theorem 1.21, that is, we define

$$U_k^{(t)} = \left\{\sigma : \frac{d(\sigma)}{2^{(1-t)|\sigma|}} \geq 2^k\right\}$$

Since d is computable, the cover is effective. The only thing that is left to prove is that $\sum_{w \in U_k^{(t)}} 2^{-s|w|}$ is a computable real number.

To approximate $\sum_{w \in U_k^{(t)}} 2^{-s|w|}$ within 2^{-r}, effectively find a number n such that $2^{(1-t)n} \geq 2^r d(\epsilon)$. If we enumerate only those strings σ into $U_k^{(t)}$ for which $|\sigma| \leq n$, we may conclude for the remaining strings $\tau \in U_k^{(t)}$ that $d(\tau) \geq 2^{(1-t)n}2^k \geq 2^{r+k}d(\epsilon)$. By Kolmogorov's inequality (Lemma 1.7) the set of these strings has measure at most $2^{-(r+k)}$.

(≥) Suppose $\dim_H^S B < s < 1$. (Again the case $s = 1$ is trivial.) We show that for any $t > s$, there exists a computable martingale d which is s-successful on B.

Again we define d as in the proof of Theorem 1.21: Let

$$d_k(\sigma) = \begin{cases} 2^{(1-s)|w|} & \text{if } \sigma \sqsupset w \text{ for some } w \in V_k, \\ \sum_{\sigma w \in V_k} 2^{-|w|+(1-s)(|\sigma|+|w|)} & \text{otherwise,} \end{cases}$$

(where the V_k are defined as in the proof of Theorem 1.21) and set $d = \sum_k d_k$. Since each $d_k(\epsilon) \le 2^{-k}$, the computability of d follows easily from the computability of each d_k, which is easily verified based on the fact that the measure of the V_k is uniformly computable. (Note that each σ can be in at most finitely many V_k.) □

With little effort, one can obtain the accordant characterization for Schnorr packing dimension.

So, in contrast to randomness, the approach via Schnorr tests and the approach via computable martingales to dimension yield the same concept.

5.4 Schnorr Dimension and Computable Enumerability

It is easy to see that no computably enumerable set can be Schnorr random.

Proposition 5.11 *No computably enumerable set is Schnorr random.*

Proof. Every infinite c.e. set contains an infinite computable subset. So, given an infinite c.e. set $A \subseteq \mathbb{N}$, choose some computable infinite subset B. Assume $B = \{b_1, b_2, \dots\}$, with $b_i < b_{i+1}$.

Define a Schnorr test $\{V_n\}$ for A as follows: At level n, put all those strings v of length $b_n + 1$ into V_n for which

$$v(b_i) = 1 \quad \text{for all } i \le n + 1.$$

Then surely $A \in [V_n]$ for all n, and $\lambda[V_n] = 2^{-n}$. □

It does not seem clear how to improve the preceding result to Schnorr dimension zero. Indeed, defining coverings from the enumeration of a set directly might not work, because due to the dimension factor in Hausdorff measures, longer strings will be weighted higher. Depending on how the enumeration is distributed, this might not lead to a Schnorr s-covering at all.

However, one might exploit the somewhat predictable nature of a c.e. set to define a computable martingale which is, for any $s > 0$, s-successful on the characteristic sequence of the enumerable set, thereby ensuring that each c.e. set has computable dimension 0.

Theorem 5.12 (Merkle and Reimann) *Every computably enumerable set $A \subseteq \mathbb{N}$ has Schnorr Hausdorff dimension zero.*

Proof. Given rational $s > 0$, we show that there exists a computable martingale d such that d is s-successful on A.

First, partition the natural numbers into disjoint intervals I_n such that $|I_n| \ll |I_{n+1}|$, for instance, $|I_n| = 2^{|I_0|+\dots+|I_{n-1}|}$. Set $i_n = |I_n|$ and $j_n = i_0 + i_1 + \dots i_n$.

Denote by δ the upper density of A on I_n, i.e.

$$\delta = \limsup_{n\to\infty} \frac{|A \cap I_n|}{i_n}.$$

W.l.o.g. we may assume that $\delta > 0$. For any $\varepsilon > 0$ with $\varepsilon < \delta$ there is a rational number r such that $\delta - \varepsilon < r < \delta$. Given such an r, there must be infinitely many n_k for which

$$|A \cap I_{n_k}| > r i_{n_k}.$$

Define a computable martingale d by describing an accordant betting strategy as follows. At stage 0, initialize with $d(\epsilon) = 1$. At stage $k+1$, assume d is defined for all τ with $|\tau| \leq l_k$ for some $l_k \in \mathbb{N}$. Enumerate A until we know that for some interval I_{n_k} with $j_{n_k-1} > l_k$ (i.e. I_{n_k} has not been bet on before),

$$|A \cap I_{n_k}| > r i_{n_k}.$$

For all strings σ with $l_k < |\sigma| \leq j_{n_k-1}$, bet nothing (i.e. d remains constant here). Fix a (rational) stake $\gamma > 2^{1-s} - 1$. On I_{n_k}, bet γ on the mth bit being 1 ($j_{n_k-1} < m \leq j_{n_k}$) if m has already been enumerated into A. Otherwise bet γ on the mth bit being 0. Set $l_{k+1} = j_{n_k}$.

When betting against A, obviously this strategy will lose at most $\lceil 2\varepsilon \rceil |I_{n_k}|$ times on I_{n_k}. Thus, for all sufficiently large n_k,

$$\begin{aligned} d(A \upharpoonright_{l_{k+1}}) &\geq d(A \upharpoonright_{l_k})(1+\gamma)^{i_{n_k} - \lceil 2\varepsilon \rceil |I_{n_k}|}(1-\gamma)^{\lceil 2\varepsilon \rceil |I_{n_k}|} \\ &= d(A \upharpoonright_{l_k})(1+\gamma)^{i_{n_k}} \left(\frac{1-\gamma}{1+\gamma}\right)^{\lceil 2\varepsilon \rceil |I_{n_k}|} > 2^{(1-s) i_{n_k}} \left(\frac{1-\gamma}{1+\gamma}\right)^{\lceil 2\varepsilon \rceil |I_{n_k}|}. \end{aligned}$$

Choosing ε small and n large enough we see that d is s-successful on A. □

On the other hand, concerning upper entropy, c.e. sets may exhibit a rather complicated structure, in sharp contrast to the effective case of 1-dimension, where Barzdins' Theorem ensures that all c.e. sets have effective packing dimension 0 (Section 3.4). As the proof of the following theorem shows, this is due to the requirement that all machines involved in the determination of Schnorr dimension are total.

Theorem 5.13 *There exists a computably enumerable set $A \subseteq \mathbb{N}$ such that*

$$\dim_{\mathrm{P}}^{\mathrm{S}} A = 1.$$

Proof. Partition the natural numbers into disjoint intervals

$$\mathbb{N} = I_0 \cup I_1 \cup I_2 \cup \dots$$

such that $|I_n| \ll |I_{n+1}$ for all n, for instance, $|I_{n+1}| = 2^{|I_0|+\dots+|I_n|}$. Let $i_n = |I_n|$ and $j_n = i_0 + \dots + i_n$. Furthermore, let $M_0, M_1, \dots$ be a standard enumeration of all prefix-free (not necessarily computable) Turing machines.

The set A to be constructed will satisfy the following *requirements*:

$$R_{\langle e,n\rangle} : \ M_e \text{ has a domain of measure } 1 \ \Rightarrow \ K_{M_e}(A\restriction_{I_{\langle e,n\rangle}}) \geq i_{\langle e,n\rangle},$$

that is, we make A incompressible with respect to M_e on all intervals $I_{\langle e,n\rangle}$.

To see that this ensures Schnorr packing dimension 1, suppose

$$\dim_{\mathrm{P}}^{\mathrm{S}} A < 1.$$

Then there exists a computable machine M, an $\varepsilon > 0$ and some $n_\varepsilon \in \mathbb{N}$ such that

$$(\forall n \geq n_\varepsilon)\ K_M(A\restriction_n) \leq (1-\varepsilon)n.$$

We define another total machine $\widetilde{M}$ with the same domain as M: Given x compute $M(x)$. If $M(x)\downarrow$, check whether $|M(x)| = i_0 + i_1 + \dots + i_k = j_k$ for some k. If so, output the last i_k bits, otherwise output 0.

Let e be an index of $\widetilde{M}$. Obviously, $A\restriction_{I_{\langle e,n\rangle}}$ has $\widetilde{M}$-complexity

$$K_{\widetilde{M}}(A\restriction_{I_{\langle e,n\rangle}}) \leq K_M(A\restriction_{j_{\langle e,n\rangle}}) \leq (1-\varepsilon)j_{\langle e,n\rangle},$$

with n large enough.

On the other hand, $A\restriction_{I_{\langle e,n\rangle}}$ is M_e-incompressible (that is, $\widetilde{M}$-incompressible) by construction, i.e.

$$K_{M_e}(A\restriction_{I_{\langle e,n\rangle}}) \geq i_{\langle e,n\rangle}$$

But since $j_n = i_n + \log i_n$, this yields a contradiction for large enough n.

In order to construct A, at stage $s = 0$ initialize all $A\restriction_{I_n} = \emptyset$. At stage $s > 0$, say $R_{\langle e,n\rangle}$ *requires attention* if

$$\lambda(M_{e,s}) = \sum_{M_{e,s}(w)\downarrow} 2^{-|w|} \geq 1 - 2^{-i_{\langle e,n\rangle}} \quad \text{and} \quad K_{M_{e,s}}(A\restriction_{I_{\langle e,n\rangle}}) < i_{\langle e,n\rangle}.$$

Pick the least $m = \langle e,n\rangle$ such that R_m requires attention. Observe that any string not computed by M_e at this stage must have complexity at least i_m. (There is only 2^{-i_m} mass left.) Since by the pigeonhole principle there is at least one M_e-incompressible string of length i_m, and since $R_m = R_{\langle e,n\rangle}$ requires attention, pick any string σ of length i_m that has not been computed yet by M_e and set $A\upharpoonright_{I_{\langle e,n\rangle}} = \sigma$. ($R_m$ *receives attention* at s.)

Note that all the requirements work on disjoint intervals, so A changes on each I_k at most once. Furthermore, if M_e is a computable machine, all $R_{\langle e,n\rangle}$ will require (and hence, by the priority ordering, receive) attention at some stage. □

CHAPTER 6

Hausdorff Dimension in Exponential Time

In the previous chapter we saw how it is possible to obtain a stricter notions of effective dimension by strengthening the condition imposed on the coverings.

Lutz (2000a) has extended the theory of effective dimension to complexity theory by introducing resource-bounded dimension. As it is considerably difficult to handle resource bounded covers, his approach uses the martingale characterization of dimension.

Like for effective Hausdorff dimension, Lutz's approach yields a generalization of resource-bounded measure and a refinement of measure zero classes. Hence resource-bounded dimension might help to obtain a more complete picture of quantitative aspects of structural properties.

Here we focus on the exponential time classes $\mathrm{E} = \bigcup_{k\in\mathbb{N}} \mathrm{DTIME}(2^{kn})$ and $\mathrm{EXP} = \bigcup_{k\in\mathbb{N}} \mathrm{DTIME}(2^{n^k})$, and on the corresponding concepts of p-dimension and dimension in E, and p_2-dimension and dimension in EXP. This chapter can be seen as the continuation of the investigation of the dimension of measure-0-classes in E started by Lutz (2000a), where we concentrate on relations between reducibility and dimension. We show that most results of Chapter 3 carry over to the resource bounded case. In accordance with the usual notation in complexity theory, in this chapter we often regard sequences as *problems*, which in turn we sometimes identify with subsets of the natural numbers. As such problems are denoted by capital letters such as $A, B, C, \ldots$.

By proving an invariance theorem similar to Theorem 3.7 for resource-bounded dimension we show that (in contrast to p-measure), for any problem A in E, the class of problems m-reducible to A in polynomial time has the same dimension in E as the class of problems that are p-m-equivalent to A. In particular this shows that the measure-0-class of the p-m-complete problems for E has dimension 1 in E, which in turn implies that the small-span-theorem of Juedes and Lutz (1995) for measure in E cannot be extended to dimension in E.

The above investigations are supplemented by results on the p-dimension of some other interesting structural properties like different types of autoreducibility and immunity.

6.1 Martingales and Resource Bounded Dimension

In contrast to covers, martingales easily allow to impose subrecursive resource bounds. So we might, for example, require that a martingale (or rather its betting strategy) can be computed in polynomial time. Using Theorem 1.21, this should lead to a notion of resource bounded dimension in analogy to effective dimension, initially defined here in terms of effective coverings.

Definition 6.1 Let $t : \mathbb{N} \to \mathbb{N}$ be a computable function. A $t(m)$-*betting strategy* is a rational valued betting strategy b such that $b(w)$ can be computed in $\mathrm{O}(t(|w|))$ steps for all strings w. A $t(m)$-*martingale* d is a martingale induced by a $t(m)$-betting strategy.

Following Lutz (2000a) we define a resource-bounded version of Hausdorff dimension in terms of betting games, in obvious analogy to effective dimension.

Definition 6.2 Let $\mathcal{C} \subseteq 2^\omega$ and Δ be a class of computable functions. The Δ-dimension of $\mathcal{C}$, written $\dim_{\mathrm{H}}^{\Delta} \mathcal{C}$, is defined to be the infimum of all real numbers s such that there is a function $t \in \Delta$ and a t-betting strategy that is s-successful on $\mathcal{C}$.

Given a function $t(n)$, we will use the notation $\dim_{\mathrm{H}}^{t(n)}$ for $\dim_{\mathrm{H}}^{\mathrm{DTIME}(t(n))}$. And as before, given $\alpha \in 2^\omega$ we write $\dim_{\mathrm{H}}^{\Delta} \alpha$ for $\dim_{\mathrm{H}}^{\Delta}\{\alpha\}$.

Although the definition allows to deal with dimension for arbitrary classes of computable functions, we will focus on the common polynomial-time classes

$$\mathrm{P} = \bigcup_{k \in \mathbb{N}} \mathrm{DTIME}(n^k) \quad \text{and} \quad \mathrm{P}_2 = \bigcup_{k \in \mathbb{N}} \mathrm{DTIME}(2^{(\log n)^k}).$$

If we let Δ be the class of all computable functions, we obtain the concept of Schnorr dimension, as we saw in the previous chapter.

We briefly restate the definition of the polynomial-time dimensions.

Definition 6.3 Let $\mathcal{C} \subseteq 2^\omega$.

1. The p-*dimension* of $\mathcal{C}$, written $\dim_{\mathrm{H}}^{\mathrm{p}} \mathcal{C}$, is the infimum of all s such that there is a p-betting strategy that is s-successful on $\mathcal{C}$.
   The *dimension of* $\mathcal{C}$ *in* E, written $\dim_{\mathrm{H}}^{\mathrm{p}}(\mathcal{C} \,|\, \mathrm{E})$, is the p-dimension of $\mathcal{C} \cap \mathrm{E}$.

2. The p_2-*dimension* of $\mathcal{C}$, written $\dim_{\mathrm{H}}^{\mathrm{p}_2} \mathcal{C}$, is the infimum of all s such that there is a p_2-betting strategy that is s-successful on $\mathcal{C}$.
   The *dimension of* $\mathcal{C}$ *in* EXP, written $\dim_{\mathrm{H}}^{\mathrm{p}_2}(\mathcal{C} \,|\, \mathrm{EXP})$, is the p_2-dimension of $\mathcal{C} \cap \mathrm{EXP}$.

Lutz (2000a) established some basic properties of p- and p_2-dimension that show that these concepts are reasonable generalizations of Hausdorff dimension and might prove useful for investigating the structure of the classes E and EXP. He showed that every slice DTIME(2^{kn}) of E has p-dimension 0 while E itself has p-dimension 1 (an analogous result holds for EXP and p_2-dimension).

Moreover, for all classes $\mathcal{C}$ and $\mathcal{B}$

$$\dim_{\mathrm{H}}^{\mathrm{p}}(\mathcal{C} \cup \mathcal{B} |\, \mathrm{E}) = \max\{\dim_{\mathrm{H}}^{\mathrm{p}}(\mathcal{C} |\, \mathrm{E}), \dim_{\mathrm{H}}^{\mathrm{p}}(\mathcal{B} |\, \mathrm{E})\}$$

and in fact the latter assertion extends – with max replaced by sup – to unions of countably many classes that have an appropriate uniform representation. (Again, an analogous assertion holds for $\dim_{\mathrm{H}}^{\mathrm{p}_2}$.) It would be nice to have a characteriaztion of polynomial-time dimension of classes in terms of individual sequences, similar to Theorem 2.9. However this was based on the existence of universal nullclasses (or, equivalently, a universal semicomputable martingale). Since there are no universal p-martingales, this observation has no exact counterpart for p-dimension. Still, using the fact that, for any number k, there is a p-martingale that is universal for the class of m^k-martingales, we obtain a result of similar flavor which will serve as a useful tool in our investigation of p-dimension.

Proposition 6.4 *For any class $\mathcal{C} \subseteq 2^\omega$,*

$$\dim_{\mathrm{H}}^{\mathrm{p}} \mathcal{C} = \inf_{k\geq 1} \sup_{A\in\mathcal{C}} \dim_{\mathrm{H}}^{n^k} A \quad \textit{and} \quad \dim_{\mathrm{H}}^{\mathrm{p}_2} \mathcal{C} = \inf_{k\geq 1} \sup_{A\in\mathcal{C}} \dim_{\mathrm{H}}^{2^{(\log n)^k}} A. \tag{6.1}$$

Proof. (*Sketch*) We give a proof for the first assertion, the second one is proved completely analogously.

Given a class $\mathcal{C}$, let $s_1 = \dim_{\mathrm{H}}^{\mathrm{p}}(\mathcal{C} |\, \mathrm{E})$, $s_2 = \inf_{k\geq 1} \sup_{A\in\mathcal{C}} \dim_{\mathrm{H}}^{n^k} A$. Then $s_2 \leq s_1$, as easily follows from the definition of p-dimension. For a proof of $s_1 \leq s_2$, given any real $s > s_2$, it suffices to show that there is a p-betting strategy b which s-succeeds on $\mathcal{C}$.

Fix a rational t with $s > t > s_2$. By choice of s_2 there is a number k such that that for every sequence in $\mathcal{C}$ there is an n^k-betting strategy that t-succeeds on it. Now, if we let b be the standard betting strategy which is universal for the class of n^k-betting strategies, then b is a p-betting strategy and, for any n^k-betting strategy b' there is a rational number $c > 0$ such that the gain of b is at least c times the gain of b'. Thus b s-succeeds on every sequence in $\mathcal{C}$. □

On the other hand, the definition of Hausdorff dimension as an infimum makes it possible to deal only with betting strategies of a rather simple nature, namely those that are restricted to a finite set of non-zero betting ratios.

Definition 6.5 The *set of weights* of a betting strategy b is defined as

$$W_b = \{q \in [0, 1] : b(w) = (q, i) \text{ for some string } w \text{ and } i \in \{0, 1\}\}.$$

A betting strategy b is *simple* if W_b is a finite set of rational numbers, and b is *strict* if $0 \notin W_b$.

The following lemma asserts that strict and simple betting strategies indeed suffice to define n^k-dimension (or $2^{(\log n)^k}$-dimension, respectively). This will also help when working with p- (or p_2-) dimension.

Lemma 6.6 *Let $k \geq 2$ and $s \in [0, 1]$ and assume that the class $\mathcal{C} \subseteq 2^\omega$ has n^k-dimension s. Then for any $\varepsilon > 0$ there is a strict and simple n^k-betting strategy b that $(s + \varepsilon)$-succeeds on $\mathcal{C}$.*

Proof. Fix a rational $\varepsilon > 0$. By assumption, there is an n^k-betting strategy b_0 that $(s + \varepsilon/2)$-succeeds on $\mathcal{C}$. Let $\gamma = 2^{\varepsilon/2}$ and fix rationals $\gamma_1, \ldots, \gamma_{l+1}$ such that $\gamma = \gamma_1 < \gamma_2 < \ldots < \gamma_{l+1} = 1$ and $\gamma_{j+1} < \gamma\gamma_j$ for $j = 1, \ldots, l$. Next define a betting strategy b that basically works like b_0 except that on a string w with $b_0(w) = (\alpha_w, i)$ the betting ratio α_w is adjusted to some γ_j according to

$$b(w) = \begin{cases} (\gamma_1, i) & \text{if } \alpha_w \leq \gamma_1 \\ (\gamma_j, i) & \text{if } \gamma_j < \alpha_w \leq \gamma_{j+1} \end{cases}$$

Then a simple case analysis shows that when betting by either b or b_0, on any single bet the two strategies differ at most by the factor γ. Thus, for any α and m,

$$\frac{d_b(\alpha\restriction_m)}{2^{(1-(s+\varepsilon))m}} \geq \frac{\gamma^{-m} d_{b_0}(\alpha\restriction_m)}{2^{(1-(s+\varepsilon))m}} = \frac{2^{-\frac{\varepsilon}{2}m} d_{b_0}(\alpha\restriction_m)}{2^{(1-(s+\varepsilon))m}} = \frac{d_{b_0}(\alpha\restriction_m)}{2^{(1-(s+\frac{\varepsilon}{2}))m}}.$$

Hence b $(s + \varepsilon)$-succeeds on $\mathcal{C}$ since by choice of b_0, for any sequence α in $\mathcal{C}$, $d_{b_0}(\alpha\restriction_m)2^{-(1-(s+\frac{\varepsilon}{2}))m}$ is unbounded in m. □

6.2 The Dimension of Upper and Lower Spans

In this section we will begin the investigation of the dimension of upper and lower p-m-spans of sets in E (where 'span' is synonym with 'cone', but more frequent in complexity theory). We will see that a resource-bounded version of Corollary 3.7 holds, that is, we will show that, for any set A, the p-m-degree and the lower p-m-span of A have the same p-dimension.

We will first prove a general invariance theorem for p-dimension, which is of a slightly different nature than the corresponding result presented in Chapter 3. Some further applications of this theorem will be given at the end of this section. Before stating our invariance theorem we have to introduce some notation.

We start with a definition of a special version of generalized joins, put to work in a resource-bounded environment.

Definition 6.7 Let $\mathcal{C}_0, \mathcal{C}_1$ be classes in 2^ω. $\mathcal{C}_1$ *contains a stretched version of* $\mathcal{C}_0$ if there is a number $k \in \mathbb{N}$ such that for every $A \in \mathcal{C}_0$ there are sets Z and B computable in time $O(2^{kn})$ such that Z has density 1, $\delta_Z = 1$, and

$$A \oplus_Z B \in \mathcal{C}_1.$$

The classes $\mathcal{C}_0$ and $\mathcal{C}_1$ are *close* if each contains a stretched version of the other.

Comparing this definition with Theorem 3.6, one would expect that stretching should not affect dimension, which is the assertion of the next theorem.

Theorem 6.8 *Let* $\mathcal{C}_0, \mathcal{C}_1 \subseteq 2^\omega$ *be such that* $\mathcal{C}_1$ *contains a stretched version of* $\mathcal{C}_0$. *Then* $\dim_{\mathrm{H}}^{\mathrm{p}} \mathcal{C}_1 \geq \dim_{\mathrm{H}}^{\mathrm{p}} \mathcal{C}_0$ *and* $\dim_{\mathrm{H}}^{\mathrm{p}}(\mathcal{C}_1|\,\mathrm{E}) \geq \dim_{\mathrm{H}}^{\mathrm{p}}(\mathcal{C}_0|\,\mathrm{E})$. *Hence, in particular, any two close classes have identical* p*-dimension, as well as identical dimension in* E.

Proof. As one can easily show, for any class $\mathcal{C}_1$ that contains a stretched version of a class $\mathcal{C}_0$, the intersection $\mathcal{C}_1 \cap \mathrm{E}$ of $\mathcal{C}_1$ with E contains a stretched version of $\mathcal{C}_0 \cap \mathrm{E}$. Hence it suffices to prove the assertion on p-dimension.

Let $s_i = \dim_{\mathrm{H}}^{\mathrm{p}} \mathcal{C}_i$. We show that $s_1 \geq s_0$. By Proposition 6.4, it suffices to show that for any rational $t > s_1$ there is a number $k \in \mathbb{N}$ such that every set $A_0 \in \mathcal{C}_0$ has n^k-dimension at most t.

So fix such a t. Pick s with $s_1 < s < t$ and let b_1 be a p-betting strategy that s-succeeds on every set in $\mathcal{C}_1$. Then for every $C_1 \in \mathcal{C}_1$, there are infinitely many numbers m such that

$$d_{b_1}(C_1 \restriction m) \;>\; 2^{(1-s)m}. \tag{6.2}$$

Next fix a number k_0 that witnesses that $\mathcal{C}_1$ contains a stretched version of $\mathcal{C}_0$, let k_1 be such that b_1 is an n^{k_1}-betting strategy, and let $k = \max(k_0, k_1) + 1$. Finally, let C_0 be any set in $\mathcal{C}_0$. Then it suffices to show that there is an n^k-betting strategy b_0 that t-succeeds on C_0.

By choice of k_0 choose sets Z and B computable in time $O(2^{k_0 n})$ such that Z has density 1 and $C_0 \oplus_Z B$ is in $\mathcal{C}_1$, and define a betting strategy b_0 by

$$b_0(X\restriction_m) = b_1(X \oplus_Z B\restriction_{n_Z(m)}), \tag{6.3}$$

where $n_Z(i)$ denotes the $i + 1$-st element of Z ($i \geq 0$).

Roughly speaking, b_0 mimics the bets of b_1 but b_0 skips all bets on places which are not in Z. The bet of b_0 on $X(m)$ corresponds to the bet of b_1 on place $n_Z(m)$ of $X \oplus_Z B$, i.e., to the bet of b_1 on the element of Z at which $X(m)$ has been coded into $X \oplus_Z B$.

To show that the betting strategy b_0 has the required properties, we first show that b_0 t-succeeds on C_0. Let $\varepsilon = t - s > 0$. Since the density of Z is 1 fix m_0 such that, for all $m > m_0$, there are less than εm elements of $\overline{Z}$ that are smaller than m. Moreover, for $m \geq \min Z$ let m' be the greatest number in $\{i < m : i \in Z\}$.

Then while betting on $C_0(0)$ through $C_0(m')$, b_0 gains up to a factor of at most $2^{\varepsilon m}$ the same capital as b_1 gains by betting on $(C_0 \oplus_Z B)(0)$ through $(C_0 \oplus_Z B)(m)$. Thus, by choice of ε, for every $m > m_0$ that satisfies (6.2) for $C_1 = C_0 \oplus_Z B$ we obtain

$$\begin{aligned} d_{b_0}(C_0 \restriction_{m'}) &\geq 2^{-\varepsilon m} d_{b_1}(C_0 \oplus_Z B \restriction_m) \\ &\geq 2^{(1-s-\varepsilon)m} = 2^{(1-t)m} > 2^{(1-t)m'} \end{aligned}$$

(where d_i is the martingale that corresponds to the betting strategy b_i). Since there are infinitely many such numbers m', b_0 t-succeeds on C_0.

It remains to show that b_0 is an n^k-betting strategy. In order to compute $b_0(X \restriction_m)$, we first compute $n_Z m$ and $w_m = X \oplus_Z B \restriction_{n_Z(m)}$. As Z has density 1, $|w_m|$ is at most $2m$ for almost all m. Hence w_m can be computed by successively computing for $i = 0, \ldots, 2m$ the values $|\{j \in Z : j \leq i\}|$, $|\{j \notin Z : j \leq i\}|$, and, in case of $i = n_Z(j)$, the value $B(j)$. Hence, except for updating some counters and some other negligible computations, it suffices to evaluate Z and B for all arguments up to $2m$. Since these sets are computable in time $\mathrm{O}(2^{k_0 n})$, this can be done in time $\mathrm{O}(4m2^{k_0|2m|})$ and hence in time $\mathrm{O}(m^{k_0+1})$. Finally, we have to compute $b_1(w_m)$. Since b_1 is an n^{k_1}-betting strategy and since $|w_m| \leq 2m$ this can be done in $\mathrm{O}(m^{k_1})$ steps, hence the total time required for computing $b_1(X \restriction_m)$ is bounded by $\mathrm{O}(m^{\max(k_0+1,k_1)})$. So b_1 is an n^k-betting strategy by choice of k. □

From the preceding theorem it is now easy to infer a resource-bounded version of Corollary 3.7.

Corollary 6.9 *Let A be in* E. *Then the lower* p-m-*span of A and the* p-m-*degree of A have the same p-dimension, as well as the same dimension in* E.

Proof. By Theorem 6.8, we are done if we can show that the lower p-m-span and the p-m-degree of A are close. As the latter class is contained in the former one, it suffices to show that the p-m-degree of A contains a stretched version of the lower p-m-span of A. So let $\overline{Z} = \{0^{|y|}y : y \in \mathbb{N}\}$ and – by $A \in \mathrm{E}$ – fix a set $\widetilde{A}$ in the p-m-degree of A that is computable in time $\mathrm{O}(2^n)$. Then Z has sublinear density and for every set $X \leq_{\mathrm{m}}^{\mathrm{p}} A$, the set $\widetilde{A} \oplus_Z X$ is in the p-m-degree of A.

The second assertion about dimension in E is shown by an almost identical argument where, however, we don't consider the p-m-degree and the lower p-m-span of A but the intersection of these classes with E. Moreover, we use that if X is in E, then so is $\widetilde{A} \oplus_Z X$. □

In Corollary 6.9 we can replace p-m-reducibility by most of the commonly studied polynomial-time-bounded reducibilities. Corollary 6.9 yields that the p-dimension of the degree of a set is growing with the relative complexity of the set.

Corollary 6.10 *Let A, B be sets in* E *with $A \leq_{\mathrm{m}}^{\mathrm{p}} B$. Then*

$$\dim_{\mathrm{H}}^{\mathrm{p}} A^{\equiv_{\mathrm{m}}^{p}} \leq \dim_{\mathrm{H}}^{\mathrm{p}} B^{\equiv_{\mathrm{m}}^{p}}$$

and

$$\dim_{\mathrm{H}}^{\mathrm{p}}(A^{\equiv_{\mathrm{m}}^{p}} | \mathrm{E}) \leq \dim_{\mathrm{H}}^{\mathrm{p}}(B^{\equiv_{\mathrm{m}}^{p}} | \mathrm{E})$$

Other interesting consequences of Corollary 6.9 include the following.

Corollary 6.11 *(a) The class of* p-m-*complete sets for* E *has dimension* 1 *in* E.

(b) The upper p-m-*span of any set in* E *has dimension* 1 *in* E.

(c) The class of p-m-*complete sets for* NP *has the same dimension in* E *as* NP.

Proof. (a) For any set A in the class of p-m-complete sets for E, this class coincides with $A^{\equiv_{\mathrm{m}}^{p}} \cap \mathrm{E}$. Hence, by Corollary 6.9, it suffices to show that the lower p-m-span of A has dimension 1 in E. But this is immediate since the lower p-m-span of A contains E.

(b) The assertion follows from (a) since the class of p-m-complete problems for E is contained in the upper p-m-span of any set in E.

(c) Fix any NP-complete set A. Then the p-m-degree of A coincides with the class of NP-complete problems, while the lower p-m-span of A coincides with NP. □

Mayordomo (1994) has shown that the class of p-m-complete sets has measure 0 in E, hence this class is an interesting example of a measure 0 class in E that has dimension 1 in E. Mayordomo's result has been extended by Juedes and Lutz (1995) who have shown that for any set $A \in \mathrm{E}$, either the lower p-m-span of A or the upper p-m-span of A has measure 0 in E. Corollary 6.11 shows that this small-span theorem fails with measure replaced by dimension.

We conclude this section by giving two more examples of dimension 1 results that can be derived from Theorem 6.8. First we show that the property of p-immunity yields a partition of E into four classes each having dimension 1 in E.

Corollary 6.12 *The following classes have dimension* 1 *in* E.

$$\begin{aligned} \mathcal{C}_1 &= \{A : A \textit{ is } \text{p-}\textit{immune and } \bar{A} \textit{ is } \text{p-}\textit{immune}\}, \\ \mathcal{C}_2 &= \{A : A \textit{ is } \text{p-}\textit{immune and } \bar{A} \textit{ is not } \text{p-}\textit{immune}\}, \\ \mathcal{C}_3 &= \{A : A \textit{ is not } \text{p-}\textit{immune and } \bar{A} \textit{ is } \text{p-}\textit{immune}\}, \\ \mathcal{C}_4 &= \{A : A \textit{ is not } \text{p-}\textit{immune and } \bar{A} \textit{ is not } \text{p-}\textit{immune}\}. \end{aligned}$$

Proof. Mayordomo (1994) has shown that $\mathcal{C}_1$ has measure 1 in E, hence dimension 1 in E. To show that $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_4$ have dimension 1 in E, too, it suffices to show that these classes contain stretched versions of $\mathcal{C}_1$. But this is witnessed by the pairs (Z, B_i), $i = 1, 2, 3$, where $Z = \{0^{|y|}y : y \in \mathbb{N}\}$ and B_i is any infinite and coinfinite set in $\mathcal{C}_i$ that is computable in time $\mathrm{O}(2^n)$. □

Recall that for any reducibility r, a set A is r-autoreducible if there is an r-reduction from A to itself that does not allow to query the oracle on the input. The measure of the p-T-autoreducible sets in E is not known (see Buhrman et al., 2000). For more restrictive reducibilities, however, the class of autoreducible sets has measure 0 in E. Examples are the classes of the p-m-autoreducible sets and of the sets that are p-T-autoreducible via order-decreasing reductions, i.e., by reductions that on input x can only query their oracle at places $y < x$. Dimension in E allows us to distinguish the size of these two measure-0 classes in E.

Corollary 6.13 *The class of* p-m-*autoreducible sets has dimension* 1 *in* E *while the class of sets that are* p-T-*autoreducible via order-decreasing reductions has dimension* 0 *in* E.

Proof. To show that the class of the p-m-autoreducible sets has dimension 1, we show that this class contains a stretched version of E. Given any set $X \in$ E, let $Z = \{0^{|y|}y : y \in \mathbb{N}\}$ and let $\widetilde{X}$ be any set in DTIME(2^n) that is p-m-equivalent to X. Then the set $X \oplus_Z \widetilde{X}$ is p-m-autoreducible.

On the other hand, to show that the class of sets that are p-T-autoreducible via order-decreasing reductions has dimension 0 in E, fix a set A in E and a polynomially time-bounded oracle Turing machine M such that $A(x) = M^{A\restriction_x}(x)$ for all x. By Proposition 6.4 it suffices to define an n^2-betting strategy b that s-succeeds on A for all $s > 0$. As one can easily check, the strategy b defined by $b(X\restriction_m) = (1, M^{X\restriction_m}(m))$ will do. □

CHAPTER 7

Hausdorff Dimension and Betting Games

7.1 Non-monotonic Betting Games

In the previous chapters we saw a close connection between Hausdorff dimension and martingales. One of the major criticisms one could bring forward against the notion of Martin-Löf randomness is that, while it captures almost all important probabilistic laws, it is not completely intuitive, since it is characterized by *computably enumerable* martingales (or an equivalent c.e. test notion), not by *computable* ones.

This point was issued first by Schnorr (1971), who asserted that Martin-Löf randomness was to strong to be regarded as an *effective notion* of randomness. He proposed two alternatives, one defined via coverings the measure of which is a computable real number (not merely one enumerable from below), leading to the concept today known as *Schnorr randomness* (see Chapter 5). The other concept is based on the paradigm of the nonexistence of a successful *computable* gambling system, that is, no *computable* martingale should win against a random sequence. This notion is commonly referred to as *computable randomness* (see Chapter 5).

If one is interested in obtaining stronger notions of randomness, closer to Martin-Löf randomness, without abandoning Schnorr's paradigm, one might stay with computable betting strategies and think of more general ways those strategies are allowed to bet. One possibility is to remove the requirement that the betting strategy is to bet on a given sequence in an order that is *monotone on the prefixes of that sequence*, that is, the strategy itself determines which place of the sequence it wants to bet against next. The resulting concept of *non-monotonic betting strategies* is a generalization of the concept of monotonic betting strategies that underly martingales. Infinite binary sequences against which no computable martingale (i.e., monotonic betting strategy) wins are called computably random. A sequence against which no computable non-monotonic betting strategy is successful is called *Kolmogorov-Loveland random*, or KL-random, for short. The concept is named after Kolmogorov (1998) and Loveland (1966), who studied non-monotonic selection rules to define accordant stochasticity concepts.

The concept of KL-randomness is robust in so far as it remains the same if one defines it in terms of partial computable in place of computable non-monotonic betting strategies (Merkle, 2003).

KL-randomness was introduced by Muchnik, Semenov, and Uspensky (1998). As they were able to show, Martin-Löf randomness implies KL-randomness, but it is not known whether the two concepts are different; a proof that they are the same would give a striking argument against Schnorr's criticism of Martin-Löf randomness. This question was raised by Muchnik et al. (1998) and also in Ambos-Spies and Kučera (2000), and is now a major open problem in the area.

Most researchers conjecture the notions are different. However, a result of Muchnik (Muchnik et al., 1998) indicates that KL-randomnes is rather close to Martin-Löf randomness.

Recall that it is possible to characterize Martin-Löf randomness as incompressibility with respect to prefix-free Kolmogorov complexity: A sequence A is random if and only if there is a constant c such that $\mathrm{K}(A\restriction_n) \geq n - c$ for all n. It follows that a sequence A cannot be Martin-Löf random if there is an unbounded function g such that $\mathrm{K}(A\restriction_n) \leq n - g(n)$ for every n.

Muchnik, on the other hand, showed that a sequence A cannot be KL-random if there is *computable* unbounded function h such that for all k, the prefix of A of length k has prefix-free Kolmogorov complexity of at most $k - h(k)$. So, the difference between Martin-Löf randomness and KL-randomness appears, from this viewpoint, rather small. Not being Martin-Löf random means that there are infinitely many initial segments for which the compressibility exceeds a given, constant bound. If, moreover, we are able to detect these initial segements efficiently (by means of a computable functions), then the sequence cannot even be KL-random.

In this chapter we give some more evidence that KL-random behaves very closely to Martin-Löf randomness.

We refine a splitting technique used by Muchnik. We show that, if A is KL-random and Z is a computable, infinite and co-infinite set of natural numbers, either the bits of A whose position is in Z or the remaining bits form a Martin-Löf random sequence. In fact both do if A is Δ^0_2. Moreover, in that case, for each computable, nondecreasing, and unbounded function g and almost all n, $\mathrm{K}(A\restriction_n) \geq n - g(n)$.

Using the dimension formulas obtained in Chapter 3, we can immediately infer that KL-random sequences have effective dimension 1. Note that this is a strengthening of Muchnik's result.

7.2 Non-monotonic Betting Strategies

Intuitively speaking, a betting strategy defines a process that places bets on bits of a given sequence $A \in 2^\omega$. More precisely, the betting strategy determines a sequence of mutually distinct places $n_0, n_1, \ldots$ at which it bets a certain portion of the current capital on the value of the respective bit of A being 1. (Note that, by betting none of the capital, the strategy may always choose to only 'inspect' the next bit.) The place n_{i+1} and the bet which is to be placed solely depends on the previously scanned bits $A(n_0)$ through $A(n_i)$.

Hence, a (non-monotonic) betting strategy is a partial function that receives as input the information $x = (r_0, b_0)\ldots(r_{n-1}, b_{n-1})$ consisting of positions (r_i) and corresponding values of the sequence (b_i), and outputs a pair (r, q), where r describes the next place to bet on, and q determines the portion of the current capital to be used in this bet as well the outcome (0 or 1) to bet upon.

As the formal definition is somewhat tedious, we present it in a sequence of definitions.

Definition 7.1 An (*ordered*) *finite assignment* (f.a.) is a sequence

$$x = (r_0, b_0)\ldots(r_{n-1}, b_{n-1}) \in (\mathbb{N} \times \{0, 1\})^*$$

of pairs of natural numbers and bits, where the r_i are pairwise different. The set of all finite assignments is denoted by FA.

Finite assignments can be thought of as partial values of an infinite binary sequence $A = A(0)\,A(1)\,A(2)\cdots \in 2^\omega$, in the sense that $A(r_i) = b_i$ for $i < n$. If this is the case for some f.a. x, we write $x \sqsubset A$. Given $x = (r_0, b_0)\ldots(r_{n-1}, b_{n-1})$, we call the subset of $\mathbb{N}$ induced by the r_i the *domain* of x, $\mathrm{dom}(x)$. (Note that we can interpret an f.a. as a partial function from $\mathbb{N}$ to $\{0, 1\}$.)

In a betting strategy, the player will successively gain more and more information on the sequence he bets against. Depending on his current knowledge of the sequence, he will determine the next place to bet on. We call the function which does this a *scan rule*.

Definition 7.2 A *scan rule* is a partial function $S : \mathrm{FA} \to \mathbb{N}$ such that

$$(\forall w \in \mathrm{FA})\ S(w) \notin \mathrm{dom}(w). \tag{7.1}$$

(7.1) ensures that no place is scanned (and bet on) twice. A betting strategy endowes each place selected with a bet.

Definition 7.3 A (non-monotonic) *betting strategy* is a pair $B = (S, Q)$, where S is a scan rule and $Q : \mathrm{dom}(S) \to [0, 2]$, the *stake function*.

Given an infinite sequence A, a betting strategy $B = (S, Q)$ defines a *betting strategy* played against A in the following sense: Define a *capital function* V. Start with a capital $V(\epsilon) = 1$. Given $x \in \mathrm{dom}(S)$, $x \sqsubset A$, the strategy picks $S(x)$ to be the next place to bet on. If $Q(x) < 1$ it bets that $A(S(x)) = 1$, if $Q(x) > 1$, it bets that $A(S(x)) = 0$, and if $Q(x) = 1$, the strategy refrains from making a bet.

If $A(S(x)) = 0$, the current capital is multiplied by $Q(x)$, else it is multiplied by $2 - Q(x)$, that is, if the strategy makes a right guess, it retrieves its stake doubled, otherwise the stake is lost.

Note at this point that it is not really necessary to define a betting strategy on finite assignments. It is sufficient to give a binary string $w \in 2^{<\omega}$ representing the values $b_0, \dots, b_{n-1}$ of an f.a. If the sequence was obtained by a scan rule S, the places selected can be recovered completely from this information. Therefore, it it suffices to consider betting strategies $B : 2^{<\omega} \to \mathbb{N} \times [0, 2]$ which satisfy condition (7.1) for the scan rule induced by keeping track of the first component of the accordant f.a..

7.2.1

Running a betting strategy on a sequence

We now describe the game that takes place when a betting strategy is applied to an infinite binary sequence. Formally, this induces a functional which transforms sequences (or even assignments) into assignments. So, in the following, assume Y is a sequence and $B = (S, Q)$ is a betting strategy.

The most important functional is the *sequence of scanned places* D_B^Y. This only depends on the scan rule S and is defined as follows: Set $D_B^Y(0) = \epsilon$, and, if $x_n = D_B^Y(n)$ is defined, let

$$D_B^Y(n+1) = x_n \,{}^\wedge\, (S(x_n), Y(S(x_n))),$$

if $S(x_n)$ is defined ($D_B^Y(n+1)$ is undefined otherwise).

The payoff already described above can now be defined as a functional P_B^Y, where $P_B^Y(0) = 1$ and

$$P_B^Y(n+1) = \begin{cases} Q(D_B^Y(n)), & \text{if } Y(S(D_B^Y(n))) = 0, \\ 2 - Q(D_B^Y(n)), & \text{if } Y(S(D_B^Y(n))) = 1. \end{cases}$$

The capital function V_B^Y is now easily described:

$$V_B^Y(n) = \prod_{i=0}^{n} P_B^Y(n).$$

Finally, we can define the randomness notion induced by non-monotonic betting strategies.

Definition 7.4 Let $A \in 2^{\omega}$.

(1) A (non-monotonic) betting strategy B *succeeds on* A if

$$\limsup_{n\to\infty} V_B^A(n) = \infty.$$

(2) A class $\mathcal{A} \subseteq 2^{\omega}$ is a KL-*nullclass* if there is a computable betting strategy that succeeds on all $A \in \mathcal{A}$.

(3) $A \in 2^{\omega}$ is KL-*random* if $\{A\}$ is not a KL-nullclass.

To illustrate how non-monotonic betting strategies work, we give a first, easy example.

Proposition 7.5 *No computable non-monotonic betting strategy can succeed on all c.e. sets.*

Proof. Let $B = (S, Q)$ be a computable betting strategy. We show that there exists a c.e. set W such that B does not succeed on A. For this purpose, we compute a sequence (x_n) of finite assignments, $x_n = (r_0, b_0)\dots(r_n, b_n)$. Start with $x_0 = \epsilon$, and set $r_{n+1} = S(x_n)$ and

$$b_{n+1} = \begin{cases} 1, & \text{if } Q(x_n) > 1, \\ 0, & \text{if } Q(x_n) \le 1. \end{cases}$$

Enumerate r_{n+1} into W if $b_{n+1} = 1$. (If $S(x_n)$ is undefined at some stage, the enumeration process will get stuck here as well and the resulting set W will be finite.) Obviously, W is defined in a way that B does not win a single bet against it, in particular, it does not succeed on W. □

Hence, there are Martin-Löf nullclasses which are not KL-nullclasses. We cannot conclude, however, that the corresponding randomness notions differ. A Martin-Löf nullclass could still be covered by a union of KL-nullclasses. We are going to see next that this is the case for the class of c.e. sets.

7.2.2 The Power of Two

Non-monotonic betting strategies exhibit a behavior quite different from other randomness concepts when studying the combined capabilities of two or more strategies. This section will present some evidence.

The following proposition contrasts Proposition 7.5.

Proposition 7.6 *There exist betting strategies $B_0 = (S_0, Q_0)$ and $B_1 = (S_1, Q_1)$ such that for every c.e. set W, at least one of B_0, B_1 will succeed on W.*

Proof. Define B_0 to be the following simple betting strategy: Let $S_0(\epsilon) = 0$, for $x_n = (0, b_0) \dots (n-1, b_{n-1})$ let $S_0(x_n) = n$ (undefined for all other f.a.), and set for all such x_n $Q_0(x_n) = 5/3$. Hence, B_0 is a *monotone* betting strategy that always bets $2/3$ of its current capital on the next bit being 0. An easy calculation shows that this betting strategy succeeds in particular on all sequences A for which the density δ_A of A is less than $1/4$, which obviously includes the characteristic sequences of finite sets.

To define B_1, fix a computable partition

$$\mathbb{N} = I_0 \cup I_1 \cup I_2 \cup \dots$$

of the natural numbers into pairwise disjoint intervals I_k such that

$$|I_k| \geq ck$$

with $c > 9$.

For every number e reserve a share of 2^{-e-1} of the initial capital 1. The intention is that the share 2^{-e-1} is used to bet on W_e, and if W_e is quite dense, i.e. if

$$(\overset{\infty}{\exists} k)[|W_e \cap I_k| \geq k+1] \tag{7.2}$$

then this share grows unboundedly, letting B_1 succeed on W_e.

We define B_1 as an oblivious strategy, only depending on the places visited so far, not on the bets and outcomes at this places. First, divide the capital function $V_{B_1}^X$ into infinitely many parts V_e^X, for it will always hold that $\sum V_e^X(n) = V_{B_1}^X(n)$. We start with $V_e^X(0) = 2^{-e-1}$ given a f.a. $x_n = (r_0, b_0) \dots (r_{n-1}, b_{n-1})$, define $S_1(x_n)$ as follows: Say e *requires attention* for $\langle k, m, s \rangle$, if the following conditions hold:

1. m is in I_k and is enumerated in W_e by stage s, i.e. $n \in W_{e,s} \cap I_k$.
2. m has not been bet on before ($S_1(x_i) \neq m$ for $i < n$) and no other element from $W_{e,s} \cap I_k$ has been bet on yet with stake $Q_1(x_i) = 1 - 2^{-e-1}$ for $i < n$.
3. $e \leq k \leq s$.

Pick the least s such that some e requires attention for some $\langle k, m, s \rangle$. Pick the least such e and let

$$S_1(x_n) = m \quad \text{and} \quad Q_1(x_n) = \frac{1}{2} - \frac{V_e^X(n-1)}{V_{B_1}^X(n-1)},$$

hence betting all the current capital obtained by V_e^X on the outcome that the mth position in the infinite sequence revealed during the application of the strategy is 1.

Now, if the infinite sequence revealed is a c.e. set W_e satisfying (7.2), then the share 2^{-e} will be doubled infinitely often during the game. Another calculation shows that (7.2) is satisfied by sequences having an upper density of ones of at least $1/4$. □

We can immediately deduce that KL-nullsets are not closed under finite union.

Proposition 7.7 *The KL-nullsets are not closed under finite union, that is, if a betting strategy B succeeds on $\mathcal{X} \subseteq 2^\omega$ and another betting strategy B' succeeds on $\mathcal{Y} \subseteq 2^\omega$, there does not necessarily exist a betting strategy $\widetilde{B}$ that succeeds on $\mathcal{X} \cup \mathcal{Y}$.*

7.3 The Dimension of KL-random sequences

Muchnik et al. observed that Martin-Löf randomness implies KL-randomness.

Theorem 7.8 (Muchnik-Semenov-Uspensky, 1998) *If $A \in 2^\omega$ is Martin-Löf random, then it is also KL-random.*

Since non-monotonic betting games are a generalization of martingales, every KL-random sequence is computably random. We will see below that the reverse implication does not hold (also shown by Muchnik et al., 1998).

As mentioned above, the question whether KL-randomness is actually equivalent to Martin-Löf randomness is still open.

Proposition 7.7 suggests that KL-nullsets behave very different from Martin-Löf nullsets, which are all covered by a universal Martin-Löf test (and hence trivially closed under finite unions). On the other hand, KL-random sequences expose some properties which makes them appear quite 'close' to Martin-Löf randomness. Muchnik's result (Muchnik et al., 1998) on the complexity of KL-random sequences gives evidence.

Theorem 7.9 (Muchnik) *Let g be a computable, unbounded function. If for some sequence A it holds that*

$$(\overset{\infty}{\forall} n)\ \mathrm{K}(A\upharpoonright_n) \leq n - g(n),$$

then A is not KL-random.

We can immediately conclude that KL-random sequences have high effective packing dimension.

Corollary 7.10 *If $A \in 2^\omega$ is KL-random, then* $\dim^1_{\mathrm{P}} A = 1$.

Proof. Suppose $\dim^1_{\mathrm{P}} A < 1$, that is, there is some rational s such that for almost every n, $\mathrm{K}(A\restriction_n) \leq sn = n-(1-s)n$. As the computable function $g(n) = (1-s)n$ is unbounded, A cannot be KL-random. □

In the following, we are going to strengthen Muchnik's result by showing that the KL-random sequences have effective dimension 1. To do so, we show that KL-random sequences possess a certain splitting property, which also stresses the importance of non-monotonicity in betting strategies. (The corresponding result does not hold for computable randomness.)

7.3.1 Splitting properties of KL-random sequences

It is not hard to see that KL-randomness, like the other concepts presented here, relativizes, by using betting strategies which have access to an oracle. Thus, a sequence is KL^B-random if no non-monotonic betting strategy which is computable in B succeeds on it.

Proposition 7.11 *Let Z be a computable, infinite and co-infinite set of natural numbers, and let $A = A_0 \oplus_Z A_1$ be KL-random. Then it holds that*

$$A_0 \textit{ is } \mathrm{KL}^{A_1}\textit{-random} \quad \textit{and} \quad A_1 \textit{ is } \mathrm{KL}^{A_0}\textit{-random}.$$

Proof. Suppose a betting strategy B^{A_1} (computable in A_1) succeeds on A_0. We devise a new (computable) strategy which succeeds on A. Of course, the idea is as follows: Scan the $\overline{Z}$-positions of A (corresponding to A_1) until we find an initial segment of A_1 which allows to compute a new value of B^{A_1}. Consequently, bet on A_0 according to B^{A_1}.

Formally, given an f.a. x_n, split it into two sub-f.a. x_n^0 and x_n^1, where (r_k, b_k) is a part of x_n^i if and only if $Z(r_k) = i$. Now define

$$B(x_n) = \begin{cases} B^{x_n^1}(x_n^0) & \text{if } B^{x_n^1}(x_n^0)\downarrow \text{ in } |x_n^1| \text{ steps,} \\ (\mu i \geq |x_n^1|\ \overline{Z}(i) = 1,\, 1) & \text{otherwise.} \end{cases}$$

(Here we identify x_n^1 with an initial segment of A_1.) □

This rather simple observation has some interesting consequences. One is that splitting a KL-random sequence by a computable set yields at least one part that is Martin-Löf random.

Theorem 7.12 (Stephan) *Let Z be a computable, infinite and co-infinite set of natural numbers. If the sequence $A = A_0 \oplus_Z A_1$ is KL-random, then at least one of A_0 and A_1 is Martin-Löf random.*

Proof. Suppose neither A_0 nor A_1 is Martin-Löf random. Then there are Martin-Löf tests $\{U_n^0\}$ and $\{U_n^1\}$ with $U_n^i = \{\sigma_{n,0}^i, \sigma_{n,1}^i, \dots\}$, such that for $i = 0, 1$

$$A_i \in \bigcap_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} [\sigma_{n,k}^i].$$

Define functions f_0, f_1 by $f_i(n) = \mu k\ \sigma_{n,k}^i \sqsubset A_i$. Obviously the following must hold:

$$(\exists i)\ (\overset{\infty}{\exists} m)\ f_i(m) \geq f_{1-i}(m).$$

We define a new test $\{V_n\}$ by

$$V_n = \bigcup_{m>n} \bigcup_{k=0}^{f_i(m)} [\sigma_{n,k}^{1-i}].$$

Then $\{V_n\}$ is a Schnorr test computable in A_i (a Schnorr^{A_i}-test) and covers A_{1-i}, so A_{1-i} is not Schnorr^{A_i}-random. KL-randomness implies Schnorr-randomness (for relativized versions, too), hence it follows that A_{1-i} is not KL^{A_i}-random, contradicting Proposition 7.11. □

If one half of a splitting of a KL-random sequence is Δ_2^0, it forces the other half to be Martin-Löf random.

Theorem 7.13 *Let Z be a computable, infinite and co-infinite set of natural numbers and let $A = A_0 \oplus_Z A_1$ be KL-random where A_1 is in Δ_2^0. Then A_0 is Martin-Löf random.*

Proof. We modify the proof of the previous theorem. For a proof by contradiction, assume that A_0 is not Martin-Löf random, witnessed by a test U_0 and define f_0 as before.

Let f_1 be a modulus of A_1, i.e., $f_1(m)$ is the least $s > n$ such that some fixed effective approximation $\{A_{1,s}\}$ to A_1 agrees after s steps with A on the first n places:

$$m(n) = \min\{s > n :\ A_{1,s}\restriction_n = A_1\restriction_n\}\ .$$

It is known that A_1 is computable in any function g that majorizes m (see Odifreddi, 1989, I, V.5.3 d).

In case f_0 majorized f_1, the sequence A_1 were computable in f_0 and hence in A_0, contradicting the assumption that A is KL-random. Otherwise we argue as before that A_0 is not Schnorr-random relative to A_1, again contradicting the assumed KL-randomness of A. □

Theorem 7.13 allows us to strengthen Muchnik's result considerably.

Corollary 7.14 *Suppose A is Δ^0_2 and KL-random. Then, for each computable, nondecreasing, unbounded function g,*

$$(\overset{\infty}{\forall} n)\ \mathrm{K}(A\restriction_n) \geq n - g(n).$$

Observe that this implies immediately that any Δ^0_2 KL-random sequence has effective Hausdorff dimension 1, a result that we shall later obtain for arbitrary KL-random sequences.

Proof. Let Z be a computable co-infinite set that for all n contains at least $n - g(n)/2$ of the first n natural numbers. Let A_0 and A_1 be the sequences such that $A = A_0 \oplus_Z A_1$. Then

$$\mathrm{K}(A_0\restriction_{n-g(n)/2} \leq^+ \mathrm{K}(A\restriction_n)\ ,$$

because the first $n - g(n)/2$ bits of A_0 can be effectively recovered from the first n bits of A. So if $\mathrm{K}(A\restriction_n) \leq n - g(n)$ for infinitely many n, for each such n the prefix of length $n - g(n)/2$ of A_0 would be compressible by at least $g(n)/2$ bits, hence A_0 would not be Martin-Löf random. Since A and hence also A_1 is in Δ^0_2, this contradicts Theorem 7.13. □

We use the method used in the proof of Theorem 7.12 to give an example of a computably random set where relative randomness of parts, in the sense of Proposition 7.11, fails. Here Z is the set of even numbers, and we write $A \oplus B$ instead of $A \oplus_Z B$. The same example works for Schnorr randomness.

Proposition 7.15 *There is a computably random (and hence Schnorr random) sequence $A = A_0 \oplus A_1$ such that for some $i \in \{0, 1\}$, A_i is not Schnorr random relative to A_{1-i}.*

Proof. Merkle (2003) has shown that there is a computably random set $A = A_0 \oplus A_1$ such that, for each n, $\mathrm{K}(A\restriction_n) \leq n/3$. Then, by Schnorr's characterization, neither A_0 nor A_1 are Martin-Löf random. Now the construction in the proof of Theorem 7.12 shows that for some $i \in \{0, 1\}$, A_i is not Schnorr random relative to A_{1-i}. □

However, it turns out that properties like the one in Theorem 7.12 do not necessarily imply Martin-Löf randomness.

Theorem 7.16 *There is a sequence A which is not computably random such that for each computable infinite and co-infinite set Z, $A\restriction_Z$ is Martin-Löf random.*

A proof of this theorem can be found in Merkle et al. (2004).

7.3.2 Dimension, multiple splittings and relative randomness

The idea of proving that all KL-random sequences have effective dimension 1 is the following: Use Theorem 7.12 to obtain a Martin-Löf random half of A (i.e. a subsequence of density $1/2$). The other half must still be KL-random, so we can apply the theorem again to obtain a Martin-Löf random subsequence of it. Show that these two combine to a single Martin-Löf random subsequnce of density $3/4$ and iterate this construction to obtain Martin-Löf random subsequences of arbitrary high density. Now apply the dimension formulas for splittings from Chapter 3.

In order to combine two Martin-Löf random sequences to a single Martin-Löf random one we will make use of a result by Van Lambalgen (1987).

Theorem 7.17 (Van Lambalgen) *Let Z be a computable, infinite and co-infinite set of natural numbers. The sequence $A = A_0 \oplus_Z A_1$ is Martin-Löf random if and only if A_0 is Martin-Löf random and A_1 is Martin-Löf random relative to A_0. (Furthermore, this equivalence remains true if we replace Martin-Löf randomness by Martin-Löf randomness relative to some oracle.)*

Note the subtle difference to Proposition 7.11: in the case of Martin-Löf randomness, one merely needs A_0 to be random, not random relative to A_1.

Theorem 7.18 *Let A be a KL-random sequence and let $\delta < 1$ be a rational number. Then there is a computable set Z of density at least δ such that $A\restriction_Z$ is Martin-Löf random.*

Proof. Let $Z_0 = \{2n : n \geq 0\}$ and set $A_0 = A\restriction_{Z_0}$ and $B_0 = A\restriction_{\overline{Z_0}}$. By Theorem 7.12, w.l.o.g. we may assume that A_0 is Martin-Löf random. Now split B_0 again: set $A_1 = B_0\restriction_{Z_0}$, $B_1 = B_0\restriction_{\overline{Z_0}}$. A straightforward relativization of the proof of Theorem 7.12 yields that one of A_1, B_1 is Martin-Löf random relative to A_0. Again, w.l.o.g., we assume that this is the case for A_1. Now using Van Lambalgen's Theorem above, we can conclude that $A_1 \oplus_{Z_1} A_0$, where $Z_1 = \{3n+1 : n \geq 0\}$, is Martin-Löf random.

Continuing inductively, for each $n \geq 1$ we obtain a splitting $A = A' \oplus_Z B$, where A' is Martin-Löf random and $\delta_Z = 1 - 2^{-n}$. So it suffices to choose $n \geq 1/(1-\delta)$. □

Corollary 7.19 *If $A \in 2^\omega$ is KL-random, then* $\dim^1_{\mathrm{H}} A = 1$.

Proof. Theorem 7.18 and Theorem 3.3 yield that $\dim^1_{\mathrm{H}} A \geq \delta$ for each rational number $\delta < 1$, from which the result follows. □

Recently, this result has been strengthened by showing that the class of KL-stochastic sequences has effective Hausdorff dimension 1. The class of KL-stochastic sequences properly contains the class of KL-random sequences. Details can be found in Merkle et al. (2004).

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