Introduction to rhythmic canons

1 Reexposition

Init canonCrawler

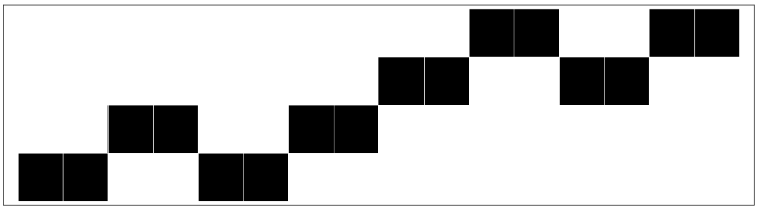
Definition

■ From cyclic to polynomial definition

What is a canon ?

■ Basic examples

```
pp[{0, 1, 4, 5}, {0, 2, 8, 10}];
```



The four ***voices*** are :

{0,1,4,5}, 2+{0,1,4,5}, 8+{0,1,4,5}, 10+{0,1,4,5} =

{{0,1,4,5},{2,3,6,7},{8,9,12,13},{10,11,14,15}}

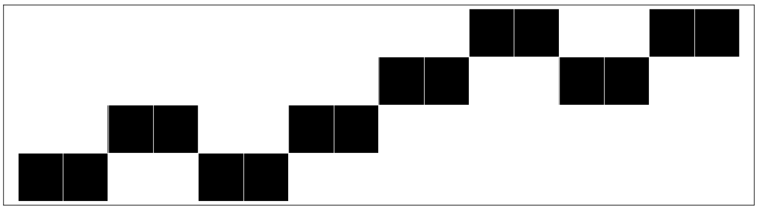
{0,1,4,5} is the motif (pattern) or ***inner rhythm***;

{0,2,8,10} is the pattern of entries, or ***outer rhythm.***

■ Repetition, gaps, period

A canon without gaps (tiling a ***range***) :

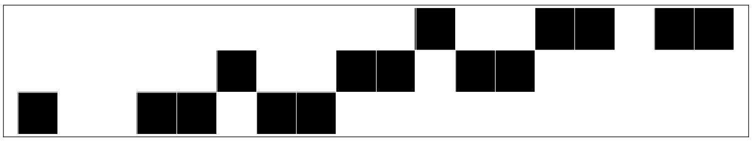
```
pp[{0, 1, 4, 5}, {0, 2, 8, 10}];
```



{0,1,4,5}⊕{0,2, 8, 10}= [0…15]

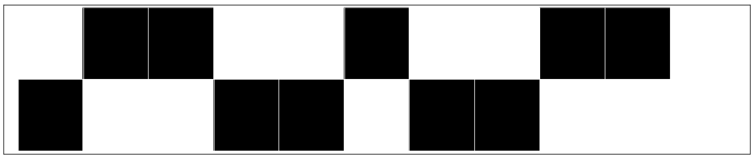
A canon with some gaps at first (like a fuga)

```
pp[{0, 3, 4, 6, 7}, {0, 5, 10}];
```



Which is really :

```
pp[{0, 3, 4, 6, 7}, {0, 5}, période → 10];
```



{0,3,4,6,7} ⊕ {0,5} ≡ {0,1,…,9} modulo 10

Both can be repeated indefinitely.

■ Definition

A 'canonical' canon is defined by:

$$A \oplus B = \mathbb{Z}_n$$

where *A* (the ***inner rhythm***) is the rhythmic motif, and *B* (the ***outer rhythm***) is the set of onsets, i.e. the moments when each voice begins.

Exponentiation of a set: **Plus @@ X^A** turns the set into a 0-1 polynomial

```
Plus @@ X^{0,1,4,5}
```

```
{1, X, X^4, X^5}
```

```
Expand[Plus @@ X^{0,1,4,5} × Plus @@ X^{0,2,8,10}]
```

```
1 + X + X^4 + X^5
```

The condition on sets turns into a condition on polynomials that I call condition (T_0):

$$A \oplus B = \mathbb{Z}_n \Leftrightarrow A(X) \times B(X) = 1 + X + X^2 + \ldots X^{n-1} \bmod X^n - 1$$

Here begins the mathematical study of rhythmic canons.

What is a canon ?

■ Cyclotomics

• **Cyclotomic polynomials** are the irreducible factors of $X^n - 1$ (in $\mathbb{Z}[X]$) : $X^n - 1 = \prod \Phi_d(X), d \mid n$.
• Any cyclotomic factor of $X^n - 1$is of the following form :
$\Phi_d(X) = \prod(X - \xi)$ where ξ are all roots of 1 of order exactly d (and of course d is a divisor of n).
• The degree of ϕ_d is given by Euler totient function.

For example,$\Phi_5(X) = 1 + X + X^2 + X^3 + X^4$.
But $\Phi_{216}(X) = X^{72} - X^{36} + 1$.

Regular beats as cyclotomic polynomials

◆ $\Phi_p(X)=1+X+\ldots+X^{p-1}$
◆ $\Phi_{p^\alpha}(X) = 1 + X^{p^{\alpha-1}} + X^{2\,p^{\alpha-1}} + \ldots X^{(p-1)\,p^{\alpha-1}}$

Other regular beats may be expressed simply with cyclotomic. Example : $\Phi_{49}(x^4)$

```
Cyclotomic[49, x^4]
```

```
1 + x^28 + x^56 + x^84 + x^112 + x^140 + x^168
```

Indeed regular beats are easiliy characterized by their cyclotomic factors :

ϕ_d is a factor of
the regular beat with k notes and N beats between each note, $1 + X^N + X^{2N} + X^{(k-1)N}$, if and only if d is a factor of $k\,N$ and *not* a factor of N.

Proof : multiply by $X^N - 1$, get as a result $X^{k N} - 1$ and compare the roots.

This is an example, with rhythm {0, 10, 20, 30, 40, 50}

```
recogListe[{10, 10, 10, 10, 10, 10}]
```

```
{3, 4, 6, 12, 15, 20, 30, 60}
```

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What is a canon ?

We notice at once the importance in canons of ***cyclotomic factors*** as they must divide $A(X)$ or $B(X)$:

$$\texttt{Factor}\left[\sum_{k=0}^{11} x^k\right]$$

$$(1 + x)\ (1 + x^2)\ (1 - x + x^2)\ (1 + x + x^2)\ (1 - x^2 + x^4)$$

Identifying the polynomial factors:

```
recog /@ List @@ %
```

```
{2, 4, 6, 3, 12}
```

In any rhythmic canon, cyclotomic factors of $A(X)$ and $B(X)$ should check some conditions (see below)

■ A word about procedure `recog`

Once a polynomial (0-1) is factored in ℤ[X], which of its irreducible factors are, or are not, a Φ_n, and which ?

Each factor is submitted to **`recog`**, which

first checks numerically whether ***all*** roots are on the unit circle,

(some numeric tolerance had to be fine-tuned to the polynomial's degree)

IF the answer is positive, then the relevant cyclotomic polynomial (it must be one, from a little known theorem) is found by means of a straightforward iteration, the index is returned;

All this can be directly applied to an interval list, i.e. a rhythmic motif, by factoring the associated polynomial and applying **`recog`** to all factors.

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Coven-Meyerowitz conditions (1998)

Let $A(X)$ be a 0-1 polynomial,
R_A the set of all k with Φ_k dividing $A(X)$
S_A the set of all p^α with Φ_{p^α} dividing $A(X)$
(T_1): $A(1) = \prod_{p^k \in S_A} p$.
(T_2): $p^k, q^l, \ldots \in S_A \Rightarrow p^k q^l \ldots \in R_A$.

Thms [C-M]
- **If A tiles, then (T_1),**
- **$(T_1) + (T_2) \Rightarrow A$ tiles,**
- **if the number of notes $\# A = A(1)$ has at most two prime factors, then $(T_1) + (T_2)$ $\Leftrightarrow$ A tiles**

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A few explanations

Integer $A(1)$ is the number of notes of the motif.

Cyclotomic factors of polynomial $A(X)$ taken for $X = 1$ give either prime factors of $A(1)$, or 1:

Lemma :
$\Phi_n(1) = 1$, except when n is a prime power.
In this case,
$\Phi_{p^\alpha}(X) = 1 + X^{p^{\alpha-1}} + X^{2p^{\alpha-1}} + \ldots X^{(p-1)p^{\alpha-1}}$ and hence $\Phi_{p^\alpha}(1) = p$.

The field generated over rationals by unit roots of order p^α, q^β … contains other roots, of order $p^\alpha \times q^\beta \times \ldots$.

Moreover, with two prime factors, one has more precisely

(Sands)
Let $\mathbb{Q}_n$ be the cyclotomic field generated by a n^{th} root of unity.
Then $\mathbb{Q}_{p^\alpha}\left[e^{2 i \pi / q^\beta}\right] = \mathbb{Q}\left[e^{2 i \pi / p^\alpha}, e^{2 i \pi / q^\beta}\right] = \mathbb{Q}\left[e^{2 i \pi / (p^\alpha q^\beta)}\right] = \mathbb{Q}_{p^\alpha q^\beta}$
Meaning the Galois group of the greatest field is the (direct) product of the groups of the smaller ones.

Sands follows with a capital lemma, stating that for such periods, in a rhythmic canon ($A \oplus B = \mathbb{Z}_n$) one of factors A, B is wholly divisible by p or q. This is the key to [C-M]'s most difficult theorem.We will soon identify this notion with ***reducibility***.

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Conservation of properties T_1, T_2

■ The Conservation Principle

Thm (Amiot, 2004-2005)
All the usual transformations on canons (affine multiplication, concatenation, stuttering, multiplexing) preserve conditions (T_1), (T_2) i.e. if $A(X)$ checks the conditions in the first place, it does still check them after transformation.

Idea of proof:
For every species of transformation, one looks at the associated polynomial of the transformed motif, and what happened to its cyclotomic factors.

For instance, ***stuttering*** a motif comes to

$$A(X) \to (1 + X + X^2 + \ldots X^{k-1})\, A(X^k)$$

and some unprepossessing lemmas about the Φ_n allow painless conclusion. For instance,

■ Some Unprepossessing Lemmas (on examples)

```
pol[x_] = 1 + x^3 + x^6 + x^12 + x^23 + x^27 + x^36 + x^42 + x^47 + x^48 + x^51 + x^71;
```

```
Factor[pol[x]];
(recog /@ (List @@ %)) // Sort
```

```
{0, 2, 8, 9, 18, 72}
```

```
Factor[pol[x^3]];
recog /@ (List @@ %) // Sort
```

```
{0, 2, 6, 8, 24, 27, 54, 216}
```

```
Factor[pol[x^5]]; recog /@ (List @@ %) // Sort
```

```
{2, 8, 10, 18, 9, 40, 72, 45, 90, 360, 0}
```

■ Ex: orbits under the affine group

Reminder : let us have a canon $A \oplus B = \mathbb{Z}_n$ if m is coprime with n,

another canon will be $m A \oplus B = \mathbb{Z}_n$, and [C-M] conditions are preserved.

This follows from preceding techniques. (I skip over a few technicalities),

♣ turning A into $m A$ means $A(X)$ becomes $A(X^m)$

◊ The roots of unit whose order divide n and which are roots of $A(X)$ are globally invariant under this transformation ($\xi \mapsto \xi^m$ being an automorphism of group μ_n)

♡ The Φ_d, $d \mid n$ factors of $A(X)$ are henceforth preserved, i.e. $R_{m A} = R_A$, hence (T_1), (T_2) too.

♠ From there it is easy to check that

- $A(X^m)$ is still a 0-1 polynomial (even reduced modulo $X^n - 1$)
- and it still tiles with B ($A(X) \times B(X)$ mod $X^n - 1$ being monic, of degree $< n$, and divisible by $1 + X + \ldots X^{n-1}$)

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Consequences

■ Difficult to evade (T_2)

From this general result, the condition (T_2) must be true for an overwhelming majority of rhythmic canons: any canon obtained from a (smaller) canon checking (T_2) must also check the same condition, and it is quite difficult to find a canon that is not reducible to a smaller one ! This also includes a recent statement from I. Laba about products of metronomes :

> **A metronome is a motif of genus {0,k,2k,... (m-1)k}**

> **Characterisation :**
> **$A(X) = 1 + X^k + \ldots X^{(m-1)k} = \frac{X^{m k}-1}{X^k-1}$ is a product of Φ_d where $d \mid m k$ but d does NOT divide k.**

G. Lygeti wrote a piece of music with such a canon (without any special constraint that I know of).

Also includes the cases of canons tiling without bothering to reduce modulo n (i.e. tiling a range, not a cycle), for after an old result of deBruijn *On British Number Systems,* such canons ARE reducible recursively by deconcatenation to the trivial canon (one voice, one note).

■ Reducibility of a canon

> **Say a canon $A \oplus B = \mathbb{Z}_n$ is *reducible* ⇔ it may be obtained from a smaller canon by one or another of the methods above.**

For instance, the famous Vuza canons (i.e. canons where neither A, B admits sub-cycles) are irreducible canons in the sense of concatenation.

Notice that if there exists a canon not checking (T_2), then (if necessary reducing, or using duality) one gets down to a Vuza canon, not checking (T_2) either. Let us stress the point:

Trouble makers (canons not checking (T_2)) must (essentially) be Vuza canons.

■ Fuglede's conjecture

A famous conjecture linking tilings (we are tiling the line i.e. dimension 1) and Fourier series :

> **Ω is *spectral* iff $L^2(\Omega)$ admits a Hilbert base of exponentials,**
> **e.g. of the form $\phi_j : \xi \in \mathbb{R}^n \mapsto e^{i\langle\lambda_j|\xi\rangle}$. $\{\lambda_j\} = \Lambda$ is the *spectrum* of Ω.**

> **Conjecture (Fuglede, 1974):**
> **Ω tiles $\mathbb{R}^n \Leftrightarrow$ it is spectral.**

As so many fascinating conjectures, this is ***almost true*** (proved in most cases, wrong in general, in high dimension; the first counter-example only occured in 2003).

What about rhythmic canons ? A gives a rhythmic canon iff $A + [0, 1[$ tiles $\mathbb{R}$. The conjecture is still undecided in dimension 1, but considering the above and this result from Izabella Laba (**(T_2)⇒ spectral,** 2000):

> **Thm (Amiot, juin 2004)**
> **The spectral conjecture is true in dim 1 iff it is true for all Vuza canons.**

A few hitherto undecided cases are thus added. Furthermore, Vuza canons are extrememely scarce (from H. Fripertinger: less than 1 out of 7.10^6 and only for special periods). Better still, all of the algorithms hitherto used by musicians to construct Vuza canons automatically produce canons verifiying (T_2)… Is Fuglede's conjecture at bay ???

Would that it be: for instance, it was noticed years ago that stuttering (for instance) gives Vuza canons that are NOT produced by Vuza's algorithm (nor *a fortiori* F.J.'s) , e.g. some Vuza canons exists that are not produced by any known algorithm.

■ Raising fuzzy hopes

As it happens, ***all Vuza canons known today*** are ***reducible*** by demultiplexing :

> **Reminder : multiplexing**
> **Let $A_1, \dots A_k$ be several 'inner rhythms' (motifs) tiling a canon of period n with the *same* 'outer rhythm' B :**
>
> $$\forall\ i = 1 \dots k \quad A_i \oplus B = \mathbb{Z}_n$$
>
> **Then $A = \bigcup (k\, A_i + i)$ tiles, with outer rhythm $k\, B$.**
>
> **Thm [C-M]:** *Any canon whence 'outer rhythm' B is divisible by some (prime) number p is the product of such a combination of smaller canons.*

If the fact is general and can be proved, then Fuglede's conjecture is over in dimension 1 !!! (at least tiling ⇒ spectral). Keeping alive the suspense, I'll sadistically change the subject.

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2 Canons modulo *p* & Galoisian properties of finite fields

■ Principle

Reminder :

> **A canon modulo *p* is a rhythmic canon where on each beat one allows a number of notes *congruent* to 1 mod *p*.**

A simple case: $p = 2$, we allow any ***odd*** number of notes on each beat. The result is the simplest possible:

> **Thm (Amiot, april 2004)**
> **For *ANY* (prime) *p* , for *ANY* motif, there exists a canon modulo *p*.**

My initial motivation was mainly mathematical there: looking for ***local*** conditions for a motif to tile (arithmetically speaking).

Indeed, a kind of chinese remainder theorem spells that

> **Thm (Amiot, 2002)**
> **If condition (T_0) holds modulo any p (prime) then it holds in $\mathbb{Z}[X]$.**

This (rather easy) condition ***is not equivalent*** to tiling for all p: in this latter case, 'outer rhythm' B may change with p !!! Indeed this is what happens for most motifs A.

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■ Sketch of proof

It is all about polynomials. The core of the matter is the following theorem, not often mentioned in the literature. I rediscovered it while exploring the area, and could only find one old little book of A. Warusfel for a confirmation:

> **Let $A(X)$ be polynomial with coefficients in a finite field $\mathbb{F}_q$, non vanishing in 0. Then there exists an integer $n > 1$ such that $A(X) \mid X^n - 1$ in $\mathbb{F}_q[X]$**

Essentially this holds because any root of $A(X)$ lives in some greater finite field of same characteristic; and any element (≠0) of a finite field is of finite order

Example :

Let $J(X) = 1 + X + X^4$ in $\mathbb{F}_2[X]$.

It factors in the field $\mathbb{F}_{16}$, hence all roots α of J are elements of the group $\mathbb{F}_{16}^*$ and hence verify $\alpha^{15} = 1$.

I have fond memories of this polynomial, which set me on the track of Galois theory for rhythmical canons.

One must master a few sordid details (multiple roots — yes, $X^n - 1$ now admits multiple roots ! I have found a strange formula for the multiplicity of 1 in a 0-1 polynomial $A(X)$, based on the digits of the p −adic expression of $A(1)$) which cannot resist the secret weapon:

> **Lemma : with characteristic *p*, for any polynomial *P*,**
> $$P(X^p) = (P(X))^p$$

■ Details (for the Thomasses)

Consider an ***irreducible*** factor $A(x)$ of $P(x)$. It is completely decomposed in a greater field, say $\mathbb{F}_p[x]/A(x)$. This field is finite and of characteristic p, it is a $\mathbb{F}_q$ (indeed, $q = p^{d^\circ A}$). As all elements of this field verify $a^{q-1} = 1$ by Lagrange's thm, and $A(x)$ has only simple roots (a finite field is perfect) we have $A(x) \mid x^{q-1} - 1$.

This enables to decompose all factors of $P(x)$. But there is the question of multiple roots. Now if $A(x) \mid x^n - 1$then we get

$(A(x)^k \mid (x^n - 1)^k \mid (x^n - 1)^{p^\alpha}$ for $p^\alpha \geq k$. But this is a power of the FROBENIUS automorphism, and $(x^n - 1)^{p^\alpha} = x^{n.p^\alpha} - 1 = x^m - 1$. By getting the LCM of such $m' s$ for all irreducible factors of a given polynomial, we have found an exponent N with $P(x) \mid x^N - 1$. If we want $P(x) \mid \frac{x^N - 1}{x-1}$ then replace P with $P(x) \times (x - 1)$.

■ Finale, giocoso assai

From there, one easily builds for any set A (beginnning with 0) a polynomial $C(X)$ such that

A(X) × C(X) = $1 + X + X^2 + \ldots X^{n-1}$ modulo p

Objection ? Yes, there is no law compelling C to be 0-1 (except when p=2…).

Solution: one ***unfolds*** $C(X)$ starting with a representative of $C(X) \in \mathbb{N}[X]$:

> If $m > 1$ then replace $m\, X^k$ with $(m - 1)\, X^k + X^{n+k}$.

Thus $C(X)$ is unchanged… modulo $X^n - 1$, neither is equation (T_0), and in finite time one gets a 0-1 polynomial, qed.

3 Back to canonical canons

Down in flames

Kolountzakis & Matolcsi wrote their paper as a refutation of a foolhardy conjecture (Lagarias & Wang, 2001) even stronger than mine.

It was stillborn, not unlike a conjecture of Tijdeman (1996) who did not know Szabò's idea….

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Canon alla hungarese

Szabò's method

Szabò is a (conjectures) killer, he conjured up a twisted algorithm for building canons, refuting a conjecture of a Sands (great specialist of tilings of abelian groups). The resulting canon is

- irreducible by deconcatenation — i.e. a Vuza canon
- irreducible by «de-stuttering»
- irreducible by demultiplexing, contrariwise to all hitherto constructedVuza canons !

So I'm dead.

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An example

This construction is guaranteed to work but with rather huge periods. Szabò's demonstration is valid for values upwards of n = 30030. I found (by trial and error) a much smaller one, that still works :

A Vuza-Szabò for n=900

- **Pick some u_i, v_i and their products :**

```
Evaluate[Table[u_i, {i, 3}]] = {2, 3, 5};
Evaluate[Table[v_i, {i, 3}]] = {2, 3, 5};
Do[m_i = u_i v_i, {i, 3}]
n = m_1 m_2 m_3
```

```
900
```

- **Compute $g_i = \prod_{k \neq i} u_k v_k$ and sequences of their multiples $\{0, g_i, 2 g_i, \ldots (u_i - 1) g_i\}$**

```
Do[g_i = n/m_i, {i, 3}]
Table[(k - 1) g_i, {i, 3}, {k, u_i}]
```

```
{{0, 225}, {0, 100, 200}, {0, 36, 72, 108, 144}}
```

- **Adding these sequences constitutes the inner rhythm A**

```
A = Union[Flatten[Outer[Plus, %[[1]], %[[2]], %[[3]]]]]
```

```
{0, 36, 72, 100, 108, 136, 144, 172, 200, 208, 225, 236, 244, 261, 272,
  297, 308, 325, 333, 344, 361, 369, 397, 425, 433, 461, 469, 497, 533, 569}
```

- **We build B from the same sequences, times v_i.**

```
Table[(k - 1) u_i g_i, {i, 3}, {k, v_i}]
```

```
{{0, 450}, {0, 300, 600}, {0, 180, 360, 540, 720}}
```

```
B = Union[Flatten[Outer[Plus, %[[1]], %[[2]], %[[3]]]] mod n]
```

```
{0, 30, 60, 90, 120, 150, 180, 210, 240, 270, 300, 330, 360, 390, 420, 450,
 480, 510, 540, 570, 600, 630, 660, 690, 720, 750, 780, 810, 840, 870}
```

■ Esay to check we have a canon :

```
Union @@ (Outer[Plus, A, B] mod n) = Range[0, n - 1]
```

```
True
```

■ One must admit *B* is extremely simple !

```
basicForm(B, n)
```

```
{30, 30, 30, 30, 30, 30, 30, 30, 30, 30, 30, 30, 30,
  30, 30, 30, 30, 30, 30, 30, 30, 30, 30, 30, 30, 30, 30, 30, 30, 30}
```

■ Now we *derange B* : we replace all elements of the form $k\, u_i\, g_i + u_{\sigma[i]}\, g_{\sigma[i]}$ with itself, + g_i. viz :

I define a permutation (circular here, it must move at least three indexes) :

```
σ[i_] := 1 + Mod[i + 1, 3]
```

Elements to be replaced are

```
Table[Mod[(k - 1) u_i g_i + u_σ[i] g_σ[i] , n], {i, 3}, {k, v_i}]
```

```
{{180, 630}, {450, 750, 150}, {300, 480, 660, 840, 120}}
```

The transformed ones are

```
Table[Mod[(k - 1) g_i u_i + g_σ[i] u_σ[i] + g_i , n],
     {i, 3}, {k, v_i}]
```

```
{{405, 855}, {550, 850, 250}, {336, 516, 696, 876, 156}}
```

Let's do it:

```
Do[x = Mod[(k - 1) uᵢ gᵢ + u_σ[i] g_σ[i] , n];
 B = ReplacePart[B, x + gᵢ , Position[B, x]],
 {i, 3}, {k, vᵢ}]
B = Sort[B]
```

```
{0, 30, 60, 90, 156, 210, 240, 250, 270, 330, 336, 360, 390, 405, 420, 510,
 516, 540, 550, 570, 600, 690, 696, 720, 780, 810, 850, 855, 870, 876}
```

- **Still a canon; furthermore, it is a Vuza :**

```
Mod[Flatten[Outer[Plus, A, B]], n] // Sort // Union // Length
```

```
900
```

```
{basicForm[B, n], basicForm[A, n]}
```

```
{{6, 24, 10, 20, 30, 90, 6, 24, 60, 30, 40, 5, 15, 6, 24,
  30, 30, 30, 66, 54, 30, 10, 20, 60, 6, 24, 30, 15, 15, 90},
 {36, 36, 28, 8, 28, 8, 28, 28, 8, 17, 11, 8, 17, 11, 25, 11,
  17, 8, 11, 17, 8, 28, 28, 8, 28, 8, 28, 36, 36, 331}}
```

```
vuzaQ[{6, 24, 10, 20, 30, 90, 6, 24, 60, 30, 40, 5, 15, 6, 24, 30,
  30, 30, 66, 54, 30, 10, 20, 60, 6, 24, 30, 15, 15, 90},
 {36, 36, 28, 8, 28, 8, 28, 28, 8, 17, 11, 8, 17, 11, 25, 11, 17, 8,
  11, 17, 8, 28, 28, 8, 28, 8, 28, 36, 36, 331}]
```

```
True
```

- **Sortie MIDI :**

```
A = {0, 36, 72, 100, 108, 136, 144, 172, 200, 208, 225, 236, 244,
  261, 272, 297, 308, 325, 333, 344, 361, 369, 397, 425, 433, 461,
  469, 497, 533, 569};
B = {0, 30, 60, 90, 156, 210, 240, 250, 270, 330, 336, 360, 390, 405,
   420, 510, 516, 540, 550, 570, 600, 690, 696, 720, 780, 810, 850,
   855, 870, 876};
```

```
Mod[oPlus[A, B], 900];
```

```
ExportMIDI[melodieToMidi[%, Table[i, {i, 36, 65}],
     timbre → ProgramChange[109, 1], tempo → 244],
    "unSzabo.mid"]
```

```
unSzabo.mid
```

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Conclusion

Thanks for enduring these difficult arguments. I do hope some of these methods will provide quality tools for today and tomorrow composers.